Rather Classless, Highly Saturated Models of Peano Arithmetic

J.H. Schmerl

Every saturated model of Peano Arithmetic having cardinality λ has 2^λ classes. Therefore, no saturated model of PA is rather classless. In other words, if $\kappa = \lambda$, then there are no rather classless, λ-saturated models of PA having cardinality κ. However, as long as λ is regular and $\kappa > \lambda$, there are no obstacles to the existence of rather classless, λ-saturated models of PA of cardinality κ other then there being no λ-saturated models of PA of cardinality κ at all. This is the content of the following theorem, which is the main result of this paper.

Theorem *If λ is regular, $\mathcal{N} \models$ PA is λ-saturated and $\lambda < |N|$, then there is a rather classless, λ-saturated $\mathcal{M} \succ \mathcal{N}$ such that $|M| = |N|$.*

The first rather classless, highly saturated models of Peano Arithmetic were constructed by Keisler [5]. His general theorem, specialized to models of PA, yields that whenever T is a consistent completion of PA, $\lambda^{<\lambda} = \lambda \geq \aleph_1$, and the combinatorial principle $\diamondsuit_{\lambda^+}$ holds, then there are rather classless, λ-saturated models of T of cardinality λ^+ (which, moreover, are λ^+-like). More rather classless, highly saturated models of PA can be obtained from a general theorem of Shelah (Theorem 12 of [8]) which, when specialized to models of PA, yields the following: If T is a consistent completion of PA, κ is the successor of a regular cardinal, and λ is a regular cardinal such that $\aleph_1 \leq \lambda < \kappa = \kappa^{<\lambda}$, then T has a rather classless, λ-saturated model of cardinality κ.

Kaufmann [3], assuming the combinatorial principle $\diamondsuit$, proved that there are $\aleph_1$-like, rather classless, recursively saturated models of each consistent completion of PA. Subsequently, this dependence on $\diamondsuit$ was eliminated by Shelah [8].

Rather classless, recursively saturated models of PA of each uncountable cardinality were constructed in Schmerl [7]. These models could be made

[1]Received september 26, 1996; revised version september 22, 1997.

$\aleph_0$-saturated by requiring that their standard systems contain all subsets of ω. The following, which is essentially Corollary 3.5 of [7], is that part of our main theorem in which $\lambda = \aleph_0$.

Theorem 1 *Let $\mathcal{N} \models \mathsf{PA}$, and suppose $\kappa \geq |N| + 2^{\aleph_0}$. Then there is a rather classless, $\aleph_0$-saturated $\mathcal{M} \succ \mathcal{N}$ of cardinality κ.*

Our aim is to generalize Theorem 1 by pushing up the amount of saturation. We do this in Theorem 2. It is well known that for singular λ, a model of PA is λ-saturated iff it is λ^+-saturated. Thus, we need consider λ-saturation only for regular λ. It is also well known that if $\mathcal{N}$ has cardinality κ and is λ-saturated, then $\kappa^{<\lambda} = \kappa$. Thus, the following theorem is easily seen to be an optimal result. Notice that Shelah's previously mentioned theorem handles the instance of Theorem 2 in which κ is the successor of a regular cardinal.

Theorem 2 *Suppose $\mathcal{N} \models \mathsf{PA}$, and suppose λ is regular, $\aleph_1 \leq \lambda < \kappa = \kappa^{<\lambda}$ and $\kappa \geq |N|$. Then there is a rather classless, λ-saturated $\mathcal{M} \succ \mathcal{N}$ of cardinality κ.*

To prove Theorem 2 we will obtain a rather classless, λ-saturated model $\mathcal{M}$ as the union of a continuous chain $\langle \mathcal{M}_\alpha : \alpha < \lambda^+ \rangle$ of models of T, where each model in the chain is an elementary end extension of each of the previous models. For an ordinal $\alpha < \lambda^+$, if α is not a limit ordinal or if $\mathrm{cf}(\alpha) \geq \lambda$, then $\mathcal{M}_\alpha$ will be λ-saturated. We will start with a λ-saturated model $\mathcal{M}_0 \succ \mathcal{N}$ having cardinality κ and descending cofinality λ^+. Two types of constructions of elementary end extensions will be used, resulting in model $\mathcal{M}_{\alpha+1}$ which is λ-generated over $\mathcal{M}_\alpha$. The type of construction which is used when $\mathcal{M}_\alpha$ is not λ-saturated (that is, when α is a limit ordinal and $\mathrm{cf}(\alpha) < \lambda$) is discussed in §3. When $\mathcal{M}_\alpha$ is λ-saturated, we will use a construction involving compatible sequences of satisfaction classes as described in §2. The proof of Theorem 2 will be completed in §4. Some preliminaries will be given in §1. Some open questions appear in §5.

1 Preliminaries

Consult Kaye [4] as a background reference to models of Peano Arithmetic.

We let $\mathcal{L}_{\mathsf{PA}} = \{+, \cdot, 0, 1, \leq\}$ be the language of Peano Arithmetic. We assume that the logic has term-building capabilities. For any language $\mathcal{L} \supseteq \mathcal{L}_{PA}$, we let $\mathsf{PA}^*(\mathcal{L})$ (or just PA^* if $\mathcal{L}$ is understood) be the extension of PA by all instances of the induction scheme in the language $\mathcal{L}$. In this paper it will always be understood that $\mathcal{L}$ is finite.

Consider a model $\mathcal{M} \models \mathsf{PA}^*$. For $b \in M$, the set $[0, b] = \{x \in M : \mathcal{M} \models x \leq b\}$ is an initial segment of $\mathcal{M}$. A *cut* is an initial segment not of the form $[0, b]$. If I is a cut, then its *cofinality* $\mathrm{cf}(I)$ is the least cardinal κ such that some cofinal $X \subseteq I$ has cardinality κ. Similarly, $\mathrm{cf}(\mathcal{M})$ is the least cardinal κ such that some cofinal $X \subseteq M$ has cardinality κ. The *descending cofinality* of $\mathcal{M}$, denoted by $\mathrm{dcf}(\mathcal{M})$, is the least cardinality κ of some set $X \subseteq M$ of nonstandard elements such that for each nonstandard $a \in M$ there is $b \in X$ for which $\mathcal{M} \models b < a$. Alternatively, if $c \in M$ is nonstandard, then $\mathrm{dcf}(\mathcal{M}) = \mathrm{cf}(\{x \in M : \mathcal{M} \models x + n \leq c \text{ for each } n \in \omega\})$.

The following easy lemma can be proved by a union of chains argument.

Lemma 1.1 *Let $\mathcal{N}$ be a model of* PA^*. *Let μ, κ, λ be cardinals such that μ and λ are regular, $\kappa^{<\lambda} = \kappa \geq \mu \geq \lambda > \aleph_0$, and $|N| \leq \kappa$. Then there is a λ-saturated $\mathcal{M} \succ \mathcal{N}$ such that $|M| = \kappa$ and $dcf(\mathcal{M}) = \mu$.* □

We let $\mathrm{Def}(\mathcal{M})$ be the set of all subsets of M which are definable in $\mathcal{M}$ allowing parameters. A subset $X \subseteq M$ is a *class* of $\mathcal{M}$ iff $X \cap [0, b] \in \mathrm{Def}(\mathcal{M})$ for each $b \in M$. If each class of $\mathcal{M}$ is in $\mathrm{Def}(\mathcal{M})$, then we say that $\mathcal{M}$ is *rather classless.*

Let $\mathcal{M} \models \mathsf{PA}^*(\mathcal{L})$ and let $b \in M$. We will refer to a (partial) satisfaction class $S \subseteq M$ as a $\sum_b$-satisfaction class if $(\mathcal{M}, S) \models \mathsf{PA}^*(\mathcal{L} \cup \{S\})$ and S decides satisfaction for just the $\sum_b$-formulas (from the point of view of $\mathcal{M}$). If S is a $\sum_b$-satisfaction class and $a \leq b$, then $S|a$ is the unique subset of S which is a $\sum_a$-satisfaction class. The following proposition contains some well known facts about satisfaction classes.

Proposition 1.2 *Let $\mathcal{M} \models$ PA^*, $n < \omega$ and $a, b \in M$.*
(1) If S is a $\sum_{b+n}$-satisfaction class for $\mathcal{M}$, then $S \in Def((\mathcal{M}, S|b))$.
(2) If S is a $\sum_{a+b}$ satisfaction class for $\mathcal{M}$, then there is $S_0 \in Def((\mathcal{M}, S))$ which is a $\sum_b$-satisfaction class for $(\mathcal{M}, S|a)$. □

A *satisfaction class* is just a $\sum_b$-satisfaction class for some nonstandard b. It is well known that if $\mathcal{M} \models \mathsf{PA}^*$ is countable, then $\mathcal{M}$ has a satisfaction class iff it is recursively saturated. The next proposition shows that certain uncountable models also have satisfaction classes.

Proposition 1.3 *Let λ be a regular cardinal. Suppose that $\mathcal{M} \models \mathsf{PA}^*$ is λ-saturated and that $cf(\mathcal{M}) = \lambda$. Then $\mathcal{M}$ has a satisfaction class.*

Proof: Let $\mathcal{N}_0 \equiv \mathcal{M}$ be countable and recursively saturated, and let S_0 be a satisfaction class for $\mathcal{N}_0$. Let $(\mathcal{N}, S) \equiv (\mathcal{N}_0, S_0)$ be such that $\mathrm{cf}(\mathcal{N}) = |N| = \lambda$. Then S is a satisfaction class for $\mathcal{N}$. Using the λ-saturation of $\mathcal{M}$, we can easily get a cofinal, elementary embedding of $\mathcal{N}$ into $\mathcal{M}$. Thus, we can assume that $\mathcal{N} \prec^{\mathrm{cf}} \mathcal{M}$. Then, according to Theorem 1.2 of [7], there is $S' \subseteq M$ such that $(\mathcal{N}, S) \prec (\mathcal{M}, S')$, and therefore S' is a satisfaction class for $\mathcal{M}$. □

If $\mathcal{M} \models \mathsf{PA}^*$ and $X \subseteq M$, then $\mathcal{M}$ is *generated* by X if $\mathcal{M}$ has no proper elementary substructures containing X. If $\mathcal{M}$ is generated by a set of cardinality at most λ, then $\mathcal{M}$ is λ-generated. If $A \subseteq M$, then $\mathcal{M}$ is

λ-generated over A if for some $X \subseteq M$, $|X| \leq \lambda$ and $\mathcal{M}$ is generated by $A \cup X$. If $\mathcal{M} \prec \mathcal{N}$ then $\mathcal{N}$ is a λ-generated elementary extension of $\mathcal{M}$ if $\mathcal{N}$ is λ-generated over M.

Lemma 1.4 *Suppose $\mathcal{N} \prec^{end} \mathcal{M} \models \mathsf{PA}^*$, $b \in N$, $cf(\mathcal{N}) \leq \lambda$, and $\mathcal{M}$ is λ-generated over $[0,b]$. Then $\mathcal{N}$ is λ-generated over $[0,b]$.*

Proof: Let $A \cup [0,b]$ generate $\mathcal{M}$, where $|A| \leq \lambda$; and let $C \subseteq N$ be cofinal, where $|C| \leq \lambda$. For each term $t(x,y)$ and elements $a \in A$ and $c \in C$ there is $d \leq c^c$ such that

$$\mathcal{M} \models \forall y < c\,(t(a,y) < c \rightarrow (d)_y = t(a,y)).$$

Each such d is in N, and there are at most λ of them. Clearly, the set of all such d generates $\mathcal{N}$ over $[0,b]$. □

2 Compatible Sequences

In this section we will discuss compatible sequences of satisfaction classes and of definable types. Let $\mathcal{M} \models \mathsf{PA}^*$ and let $I \subseteq M$ be a cut. A sequence $\langle S_k : k \in I\rangle$ is a *compatible sequence of satisfaction classes* (or CSSC) *for* $\mathcal{M}$ if for each $k \in I$, S_k is a $\sum_k$-satisfaction class and, whenever $j < k \in I$, then $S_j \subseteq S_k$. It is just as easy to get CSSC's as it is to get just satisfaction classes. For, let S be a $\sum_b$-satisfaction class for $\mathcal{M}$, let $S_k = S|k$ for each $k \leq b$, and let $I \subseteq [0,b]$ be a cut of $\mathcal{M}$. Then $\langle S_k : k \in I\rangle$ is a CSSC.

Recall the notion of definable types and their properties as developed by Gaifman [1]. Let T be a completion of $\mathsf{PA}^*(\mathcal{L})$. A type $p(x)$ of T is definable if it is unbounded (that is, for each $\theta(x)$ in $p(x)$, the sentence $\forall w \exists x > w\theta(x)$ is in T) and for each $\mathcal{L}$-formula $\varphi(x,u)$ there is an $\mathcal{L}$-formula $\sigma_\varphi(u)$ such that whenever t is a constant $\mathcal{L}$-term, then $\varphi(x,t) \in p(x)$ iff $\sigma_\varphi(t) \in T$. If $\mathcal{M} \models T$ and $p(x)$ is a definable type of T, then $p(x)$ can be extended to a type $p^M(x) = \{\varphi(x,a) : \varphi(x,u)$ is an $\mathcal{L}$-formula, $a \in M$, and $\mathcal{M} \models \sigma_\varphi(a)\}$. If $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{N}$ is generated by c over $\mathcal{M}$ and c realizes $p^M(x)$, then we say that $\mathcal{N}$ is a $p(x)$-extension of $\mathcal{M}$ generated by c. Each $p(x)$-extension $\mathcal{N}$ of $\mathcal{M}$ is conservative; that is, whenever $X \in \mathrm{Def}(\mathcal{N})$, then $X \cap M \in \mathrm{Def}(\mathcal{M})$.

Now let $\mathcal{M} \models \mathsf{PA}^*(\mathcal{L})$ and let $\langle S_k : k \in I\rangle$ be a CSSC for $\mathcal{M}$. For each $k \in I$, let $T_k = \mathrm{Th}((\mathcal{M}, S_k))$, which is a theory in the language $\mathcal{L}_k = \mathcal{L} \cup \{S_k\}$. We say that $\langle p_k(x) : k \in I\rangle$ is a *compatible sequence of definable types* (or CSDT) (*relative to $\langle S_k : k \in I\rangle$ and $\mathcal{M}$*) if the following two conditions hold:

(1) For each $k \in I$, $p_k(x)$ is a definable type of T_k. (Thus, for each $\mathcal{L}_k$-formula $\varphi(x,u)$ there is an $\mathcal{L}_k$-formula $\sigma^k_\varphi(u)$ such that whenever t is a constant $\mathcal{L}_k$-term, then $\varphi(x,t) \in p_k(x)$ iff $\sigma^k_\varphi(t) \in T_k$.)

(2) For $j < k \in I$, if $\psi(x)$ is an $\mathcal{L}_j$-formula and $\varphi(x, u)$ is the $\mathcal{L}_k$-formula derived from $\psi(x)$ by replacing each occurrence of S_j by $S_k|u$, then $\psi(x) \in p_j(x)$ iff $(\mathcal{M}, S_k) \models \sigma_\varphi^k(j)$.

The formula $\varphi(x, u)$, which is syntactically derived from $\psi(x)$, is related to $\psi(x)$ in the following semantic way: for any $a \in M$, $(\mathcal{M}, S_j) \models \psi(a)$ iff $(\mathcal{M}, S_k) \models \varphi(a, j)$.

Here is the point of the preceding definition of a CSDT. Suppose $\langle S_k : k \in I\rangle$ is a CSSC for $\mathcal{M}$ and $\langle p_k(x) : k \in I\rangle$ is a CSDT. Suppose $j < k \in I$. Let $(\mathcal{N}, S'_k)$ be an elementary extension of $(\mathcal{M}, S_k)$ and let $c \in N$ realize $p_k^M(x)$ over $(\mathcal{M}, S_k)$. Then c (as an element of $(\mathcal{N}, S'_k|j)$) realizes $p_j^M(x)$ over $(\mathcal{M}, S_j)$.

So, now let $(\mathcal{N}_k, S'_k)$ be a $p_k(x)$-extension of $(\mathcal{M}, S_k)$ generated by c_k, for each $k \in I$. If $j < k \in I$, then there is a unique elementary embedding $f_{jk} : (\mathcal{N}_j, S'_j|i)_{i\leq j} \rightarrow (\mathcal{N}_k, S'_k|i)_{i\leq j}$ which is the identity on M and for which $f(c_j) = c_k$. It follows from the uniqueness that these embeddings are compatible; that is, $f_{jk} \circ f_{ij} = f_{ik}$. Therefore, we can assume, without loss of generality, that $(\mathcal{N}_j, S'_j|i)_{i\leq j} \prec (\mathcal{N}_k, S'_k|i)_{i\leq j}$ and $c_j = c_k$ when $j < k \in I$. Let $(\mathcal{N}, R_k)_{k\in I} = \bigcup\{(\mathcal{N}_k, S'_k|i)_{i\leq k} : k \in I\}$ and let $c = c_k$ (for any $k \in I$). We will refer to $(\mathcal{N}, R_k)_{k\in I}$ as the *canonical* $\langle p_k(x) : k \in I\rangle$*-extension* of $(\mathcal{M}, S_k)_{k\in I}$ generated by c. If the specific CSDT is not important, we will refer to $(\mathcal{N}, R_k)_{k\in I}$ as a canonical extension of $(\mathcal{M}, S_k)_{k\in I}$.

The following easily proved lemma contains the main properties of canonical extensions.

Lemma 2.1 *Suppose $(\mathcal{N}, R_k)_{k\in I}$ is a canonical extension of $(\mathcal{M}, S_k)_{k\in I}$. Then $(\mathcal{N}, R_k)_{k\in I}$ is a conservative extension of $(\mathcal{M}, S_k)_{k\in I}$. If $cf(I) = \lambda$, then $\mathcal{N}$ is a λ-generated extension of $\mathcal{M}$.*

Proof: That the extension is conservative follows from the fact that $(\mathcal{N}, R_k)_{k\in I}$ is the union of an elementary chain each member of which is a conservative extension of some reduct of $(\mathcal{M}, S_k)_{k\in I}$.

Let $A \subseteq I$ be cofinal such that $|A| = \lambda$, and let $c \in N$ generate the canonical extension. Now let $C \subseteq N$ be such that $|C| = \lambda$, $c \in C$ and for each term $t(x, u)$ in the language of $(\mathcal{M}, S_k)_{k\in A}$ there is $d \in C$ such that $(\mathcal{N}, R_k)_{k\in I} \models \forall x \leq c((d)_x = t(x, c))$. Clearly, $(\mathcal{N}, R_k)_{k\in I}$ is generated by $C \cup M$. □

It still needs to be demonstrated that CSDT's exist. That is the content of the next lemma, which, in fact, will show something stronger, namely that compatible sequences of minimal types exist. For each $n < \omega$ we say that a type $p(x)$ of T is n-indiscernible if for any n-ary formula $\psi(x_0, x_1, \ldots, x_{n-1})$ there is a formula $\varphi(x)$ in $p(x)$ such that either the sentence

$$\forall \bar{x}[(\varphi(x_0) \wedge \varphi(x_1) \wedge \cdots \wedge \varphi(x_{n-1}) \wedge x_0 < x_1 < \cdots < x_{n-1}) \rightarrow \psi(\bar{x})]$$

or the sentence

$$\forall \bar{x}[(\varphi(x_0) \wedge \varphi(x_1) \wedge \cdots \wedge \varphi(x_{n-1}) \wedge x_0 < x_1 < \cdots < x_{n-1}) \rightarrow \neg\psi(\bar{x})]$$

is in T. Recall (see [6]) that the type $p(x)$ is minimal iff it is unbounded and n-indiscernible for each $n < \omega$. Also, $p(x)$ is minimal iff it is unbounded and 2-indiscernible. Minimal types are definable.

Lemma 2.2 *Let $\mathcal{M} \models \mathsf{PA}^*$ and let $I \subseteq M$ be a cut. Suppose $\langle S_k : k \in I \rangle$ is a CSSC. Then there exists a CSDT $\langle p_k(x) : k \in I \rangle$.*

Proof: We begin with a well known observation (see [2]) concerning the effectiveness of Ramsey's Theorem: If $k < \omega$ and $R_0, R_1, \cdots, R_k \subseteq \omega \times \omega$ are recursive binary relations, then there is an infinite Δ_3-set $H \subseteq \omega$ such that H is homogeneous for each R_i (that is, if $i < k$ then either whenever $x, y \in H$ and $x < y$ then $\langle x, y \rangle \in R_i$ or whenever $x, y \in H$ and $x < y$ then $\langle x, y \rangle \notin R_i$). This observation easily relativizes, yielding: If $k, n < \omega$, $X \subseteq \omega$ is an infinite $\sum_n$-set and $R_0, R_1, \cdots, R_k \subseteq \omega \times \omega$ are binary $\sum_n$-relations, then there is an infinite $\sum_{n+4}$-set $H \subseteq X$ which is homogeneous for each R_i. This statement is formalizable and provable as a scheme in PA^*. Moreover, if $k \in I$, then the following sentence can be formalized and shown to hold in $(\mathcal{M}, S_k)$: if $j+4 \leq k$ and $X \subseteq M$ is an unbounded $\sum_j$-set, then there is an unbounded $\sum_{j+4}$-set $Y \subseteq X$ which is homogeneous for the first j binary $\sum_j$-relations $R \subseteq M^2$. Therefore, in $(\mathcal{M}, S_k)$ we can formally define the sequence $\langle H_j : j \leq k \text{ and } j \equiv 1 (\operatorname{mod} 4) \rangle$, where $H_1 = M$ and H_{4i+5} is the first unbounded $\sum_{4i+5}$-set $Y \subseteq H_{4i+1}$ which is homogeneous for each of the first $4i+1$ binary $\sum_{4i+1}$-relations.

Notice that H_{4i+1} is independent of k (as long as $4i+1 \leq k \in I$). Now "fill in" the sequence $\langle H_k : k \in I \rangle$ by setting $H_k = H_{4i+1}$, where $4i \leq k \leq 4i+3$.

Clearly, for any $k \in I$ and $n < \omega$, H_{k+n} is definable in $(\mathcal{M}, S_k)$ without using parameters. Let $p_k(x)$ be the type consisting of all formulas $\varphi(x)$ in the language $\mathcal{L}_k$ such that for some $n < \omega$, $(\mathcal{M}, S_k) \models \forall x\, (x \in H_{k+n} \rightarrow \varphi(x))$. It is now an easy matter to check that $\langle p_k(x) : k \in I \rangle$ is a CSDT for $\langle S_k : k \in I \rangle$, completing the proof of the lemma. □

The next lemma shows how to construct λ-saturated, proper elementary end extensions of some λ-saturated models.

Lemma 2.3 *Let $\lambda \geq \aleph_1$ be a regular cardinal, let $\mathcal{M} \models \mathsf{PA}^*$ be λ-saturated, let $I \subseteq M$ be a cut for which $cf(I) \geq \lambda$, and let $\langle S_k : k \in I \rangle$ be a CSSC. If $(\mathcal{N}, R_k)_{k \in I}$ is a canonical extension of $(\mathcal{M}, S_k)_{k \in I}$, then $\mathcal{N}$ is λ-saturated.*

Proof: First, we show that $\mathrm{cf}(\mathcal{N}) = \mathrm{cf}(I) \geq \lambda$. This is an immediate consequence of Proposition 1.2(2) since whenever $m \in I$ there is $k \in I$ such that $m + n < k$, for each $n < \omega$, and for any such k, there is a function

$g \in \mathrm{Def}((\mathcal{M}, S_k))$ such that for any function $f \in \mathrm{Def}((\mathcal{M}, S_m)), (\mathcal{M}, S_k) \models \exists w \forall x > w(f(x) < g(x))$.

Next, we show that $\mathcal{N}$ is λ-saturated. Let $\mu < \lambda$ and let $\{A_\alpha : \alpha < \mu\}$ be a collection of nonempty, definable subsets of $\mathcal{N}$ which is closed under finite intersections. We need to show $\bigcap\{A_\alpha : \alpha < \mu\} \neq \emptyset$. Since $\mathrm{cf}(\mathcal{N}) \geq \lambda$, we can assume that each A_α is bounded. Since $\mathrm{cf}(I) \geq \lambda$, we can get $k \in I$ large enough so that $A_\alpha \in N_k$ for each $\alpha < \mu$. (Refer to the notation in the definition of canonical extension.) Then, for each $\alpha < \mu$, there is an $\mathcal{L}_k$-term $t_\alpha(u, v)$ and an element $b_\alpha \in M$ such that $(\mathcal{N}_k, S'_k) \models A_\alpha = t_\alpha(b_\alpha, c)$. Since $\mathcal{M}$ is λ-saturated, there is $d \in M$ such that $(d)_{b_\alpha} = t_\alpha$ for each $\alpha < \mu$. Let $m \in I$ be such that $k + n < m$ for each $n < \omega$. By Proposition 1.2(2), there is a $\sum_k$-satisfaction class for $(\mathcal{N}_k, S'_k)$ in $\mathrm{Def}((\mathcal{N}_m, S'_m))$, so that there is an $\mathcal{L}_m$-term $t(u, v, w)$ such that for each $\alpha < \mu$, $(\mathcal{N}_m, S'_m) \models A_\alpha = t(b_\alpha, c, d)$. Hence, there is $e \in N$ such that $A_\alpha = (e)_{b_\alpha}$ for each $\alpha < \mu$.

Let $b \in M$ be such that $b_\alpha < b$ for each $\alpha < \mu$. Let $E \in \mathrm{Def}(\mathcal{N})$ be the set of $f \in N$ such that for each definable $F \subseteq [0, b]$, $\emptyset \neq \bigcap\{(e)_x : x \in F\}$ iff $\emptyset \neq \bigcap\{(f)_x : x \in F\}$. In particular, $a \in E$. Clearly $E \cap M \neq \emptyset$, so let $f \in E \cap M$. By the λ-saturation of $\mathcal{M}$, there is $r \in M$ such that $r \in \bigcap\{(f)_{b_\alpha} : \alpha < \mu\}$. For each $e' \in E$, there is r' such that for each $x < b$, $r' \in (e')_x$ iff $r \in (f)_x$. Thus, there is s such that for each $x < b$, $s \in (e)_x$ iff $r \in (f)_x$, so in particular, $s \in \bigcap\{(e)_{b_\alpha} : \alpha < \mu\}$. Thus, $\bigcap\{A_\alpha : \alpha < \mu\} \neq \emptyset$. □

The next corollary will not be explicitly used but is mentioned for its independent interest.

Corollary 2.4 *If $\mathcal{M}$ is λ-saturated and $cf(\mathcal{M}) = \lambda$, then $\mathcal{M}$ has a λ-saturated, λ-generated elementary end extension.*

Proof: By Lemma 1.3, $\mathcal{M}$ has a $\sum_b$-satisfaction class S for some nonstandard b. Since $\mathcal{M}$ is λ-saturated, it has a cut I such that $\mathrm{cf}(I) = \lambda$ and $b \notin I$. Then $\langle S|k : k \in I\rangle$ is a CSSC. By Lemma 2.2 there is a CSDT. Now apply Lemmas 2.1 and 2.3 to get the λ-saturated, λ-generated elementary end extension. □

3 Extending bdd λ-saturated models

In this section we present a simple lemma which shows that some bdd λ-saturated models have elementary end extensions which are λ-saturated.

First, we define bdd λ-saturation. Let λ be a regular cardinal and let $\mathcal{M} \models \mathsf{PA}^*(\mathcal{L})$. Then $\mathcal{M}$ is *bdd λ-saturated* if for each $b \in M$ there is a λ-saturated $\mathcal{N} \prec^{\mathrm{end}} \mathcal{M}$ such that $b \in N$. If $\mathcal{M}$ is λ-saturated, then $\mathcal{M}$ is bdd λ-saturated. Indeed, $\mathcal{M}$ is λ-saturated iff $\mathcal{M}$ is bdd λ-saturated and $\mathrm{cf}(\mathcal{M}) \geq \lambda$. If $\mathcal{M}$ is bdd λ-saturated and $\mathrm{cf}(\mathcal{M}) = \kappa$, then a *filtration* for $\mathcal{M}$ is a continuous sequence $\langle \mathcal{M}_\alpha : \alpha < \kappa\rangle$ whose union is $\mathcal{M}$ such that

$\mathcal{M}_\alpha \prec^{\text{end}} \mathcal{M}_\beta \prec^{\text{end}} \mathcal{M}$ whenever $\alpha < \beta < \kappa$ and $\mathcal{M}_\alpha$ is λ-saturated unless $\omega \leq \text{cf}(\alpha) < \lambda$. Every bdd λ-saturated $\mathcal{M}$ has a filtration.

Lemma 3.1 *Let $\lambda \geq \aleph_1$ be regular. Suppose $\mathcal{M}$ is a bdd λ-saturated model of* PA* *such that $cf(\mathcal{M}) < \lambda$ and $\mathcal{M}$ is λ-generated over an initial segment. Then $\mathcal{M}$ has a λ-saturated, λ-generated elementary end extension.*

Proof: Let $\text{cf}(\mathcal{M}) = \kappa < \lambda$, and let $b \in M$ be such that $\mathcal{M}$ is λ-generated over $[0, b]$. Because it is bdd λ-saturated, $\mathcal{M}$ has a filtration $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$ such that that $b \in M_0$. Then $\mathcal{M}_0$ has a filtration $\langle \mathcal{N}_\beta : \beta < \lambda \rangle$ with $b \in N_0$. By Lemma 1.4, $\mathcal{M}_0$ is λ-generated over $[0, b]$

Let $C \subseteq M$ be a cofinal subset of M such that $|C| = \kappa < \lambda$. More specifically, let $C = \{c_\alpha : \alpha < \kappa\}$ where $c_\alpha \in M_{\alpha+1} \backslash M_\alpha$. By the λ-saturation of $\mathcal{M}_0$ there is a function $f_0 : C \to M_0$ which is elementary over $[0, b]$ and $f_0(c_\alpha) \in N_{\alpha+1} \backslash N_\alpha$ for each $\alpha < \kappa$. By Lemma 1.4, $\mathcal{N}_\kappa$ is λ-generated over $[0, b]$. Let $X \subseteq M$ be such that $|X| \leq \lambda$ and $X \cup [0, b]$ generates $\mathcal{M}$, and let $Y \subseteq N_\kappa$ be such that $Y \cup [0, b]$ generates $\mathcal{N}_\kappa$ and $|Y| \leq \lambda$. By a back-and-forth argument, we can extend f_0 to f, which is elementary over $[0, b]$ such that $X \subseteq D = \text{dom} f$ and $Y \subseteq f``D$. Since X and Y generate $\mathcal{M}$ and $\mathcal{N}_\kappa$, respectively, over $[0, b]$, we can extend f to an isomorphism of $\mathcal{M}$ and $\mathcal{N}_\kappa$. Since $\mathcal{M}_0$ is λ-generated over $[0, b]$, it certainly is a λ-generated extension of $\mathcal{N}_\kappa$. Thus, $\mathcal{N}_\kappa$ has a λ-saturated, λ-generated elementary end extension (namely $\mathcal{M}_0$), then so does $\mathcal{M}$. □

4 Proving Theorem 2

The proof of Theorem 2 will be completed in this section.

Let $\mathcal{M}_0 \succ \mathcal{N}$ be λ-saturated such that $\text{dcf}(\mathcal{M}_0) = \lambda^+$ and $|M_0| = \kappa$. Moreover, we want $\mathcal{M}_0$ to have a $\sum_b$-satisfaction class S for some nonstandard $b \in M_0$. (To get such a model $\mathcal{M}_0$, apply Lemma 1.1 to the theory T together with the sentences asserting that S is a satisfaction class.) Let $\langle I_\alpha : \alpha < \lambda^+ \rangle$ be a sequence of cuts of $\mathcal{M}_0$ such that:

(1) if $\alpha < \lambda^+$, then $I_\alpha \subseteq [0, b]$ and $\text{cf}(I_\alpha) = \lambda$;

(2) if $\alpha < \beta < \lambda^+$, then $I_\beta \subseteq I_\alpha$ and $I_\beta \neq I_\alpha$;

(3) $\omega = \bigcap\{I_\alpha : \alpha < \lambda^+\}$.

Let $S_k^0 = S|k$ for each $k \in I_0$, and then let $\mathcal{M}_0^* = (\mathcal{M}_0, S_k^0)_{k \in I_0}$. Notice that $\langle S_k^0 : k \in I_0 \rangle$ is a CSSC.

Construct a sequence $\langle \mathcal{M}_\alpha^* : \alpha < \lambda^+ \rangle$ of structures $\mathcal{M}_\alpha^* = (\mathcal{M}_\alpha, S_k^\alpha)_{k \in I_\alpha}$ such that for each $\alpha < \lambda^+$, the following hold:

(4) if $\alpha = 0$, α is a successor ordinal or $\text{cf}(\alpha) = \lambda$, then $\mathcal{M}_{\alpha+1}^*$ is a canonical extension of $(\mathcal{M}_\alpha, S_k^\alpha)_{k \in I_{\alpha+1}}$;

(5) if $\aleph_0 \leq \mathrm{cf}(\alpha) < \lambda$, then $\mathcal{M}^*_{\alpha+1}$ is an elementary end extension of $(\mathcal{M}_\alpha, S^\alpha_k)_{k \in I_{\alpha+1}}$ and $\mathcal{M}_{\alpha+1}$ is a λ-saturated, λ-generated extension of $\mathcal{M}_\alpha$;

(6) if α is a limit ordinal, then $\mathcal{M}^*_\alpha = \bigcup\{(\mathcal{M}_\beta, S^\beta_k)_{k \in I_\alpha} : \beta < \alpha\}$.

We will construct this sequence by recursion. Suppose that $\gamma < \lambda^+$ and that we already have $\langle \mathcal{M}^*_\alpha : \alpha < \gamma \rangle$ such that for each $\alpha < \gamma$, (4)–(6) hold. If γ is a limit ordinal, then let $\mathcal{M}^*_\gamma = \bigcup\{(\mathcal{M}_\alpha, S^\alpha_k)_{k \in I_\gamma} : \alpha < \gamma\}$. If $\gamma = \alpha + 1$ and either $\alpha = 0$ or α is a successor ordinal or $\mathrm{cf}(\alpha) = \lambda$, then apply Lemmas 2.1, 2.2 and 2.3 to get $\mathcal{M}^*_\gamma$. If $\gamma = \alpha + 1$ and $\aleph_0 \leq \mathrm{cf}(\alpha) < \lambda$, let $j \in I_\alpha \backslash I_\gamma$ and apply Lemma 3.1 to the structure $(\mathcal{M}_\alpha, S^\alpha_j)$, getting $(\mathcal{M}_\gamma, S^\gamma_j)$, and then letting $\mathcal{M}^*_\gamma = (\mathcal{M}_\gamma, S^\gamma_j | k)_{k \in I_\gamma}$.

Now let $\mathcal{M} = \bigcup\{\mathcal{M}_\alpha : \alpha < \lambda^+\}$. Clearly, $\mathcal{M}$ is λ-saturated and $|M| = \kappa$. Suppose $X \subseteq M$ is a class and that $X \notin \mathrm{Def}(\mathcal{M})$. By an argument like the one in the proof of Lemma 3.1 of [7], there is some $\alpha < \lambda^+$ such that $X \cap M_\alpha \notin \mathrm{Def}(\mathcal{M}_\alpha)$. Then, by Lemma 2.4 of [7], there is $k \in I_\alpha$ such that $X \cap M_\alpha \notin \mathrm{Def}((\mathcal{M}_\alpha, S^\alpha_k))$. Let $\beta < \lambda^+$ be such that $\alpha < \beta$, $\mathrm{cf}(\beta) = \lambda$ and $k \notin I_\beta$. Then $X \cap M_\alpha \in \mathrm{Def}(\mathcal{M}^*_\beta)$, so for some $j \in I_\beta$, $X \cap M_\beta \in \mathrm{Def}((\mathcal{M}_\beta, S^\beta_j))$. Since $\mathcal{M}^*_\alpha \prec \mathcal{M}^*_\beta$, then $X \cap M_\alpha \in \mathrm{Def}((\mathcal{M}_\alpha, S^\alpha_j)) \subseteq \mathrm{Def}((\mathcal{M}_\alpha, S^\alpha_k))$, entailing a contradiction.

This completes the proof of Theorem 2. □

5 Questions

The model $\mathcal{M}$ which was constructed in the proof of Theorem 2 has cofinality λ^+. With a little more effort, we can get $\mathrm{cf}(\mathcal{M}) = \lambda^{++}$ provided, of course, that $\lambda^{++} \leq \kappa$. This suggests the following question.

Question 5.1 For which cardinals κ, λ, μ do there exist rather classless, λ-saturated models $\mathcal{M} \models \mathsf{PA}$ such that $|M| = \kappa$ and $\mathrm{cf}(\mathcal{M}) = \mu$?

In [7], assuming $V = L$, for each $\kappa \geq \aleph_2$, such that $\mathrm{cf}(\kappa) > \aleph_0$ and κ is not weakly compact, we constructed a rather classless, κ-like $\aleph_0$-saturated model of PA. By the theorem of Keisler [5], if $\kappa = \lambda^+$ and λ is regular, then we can get a rather classless, κ-like λ-saturated model of PA. This suggests the following question.

Question 5.2 (Assume $V = L$.) For which cardinals κ, λ do there exist rather classless, κ-like, λ-saturated models of PA?

6 References

[1] Gaifman, H., Models and types of Peano's arithmetic, Ann. Math. Logic **9** (1976), 223-306.

[2] Jockusch, C.G., Ramsey's theorem and recursion theory, J. Symb. Logic **37** (1972), 268-280.

[3] Kaufmann, M., A rather classless model, Proc. Amer. Math. Soc. **62** (1977), 330-333.

[4] Kaye, Richard, *Models of Peano Arithmetic,* Oxford University Press, Oxford, 1991.

[5] Keisler, H.J., Models with tree structures, in: *Proceedings of the Tarski Symposium* (eds. L. Henkin et al.), American Math. Soc., 1974, Providence, RI, 331-348.

[6] Kossak, R., Kotlarski, H., Schmerl, J.H., On maximal subgroups of the automorphism group of a countable recursively saturated model of PA, Ann. Pure Appl. Logic **65** (1993), 125-148.

[7] Schmerl, J.H., Recursively saturated, rather classless models of Peano Arithmetic, in: *Logic Year 1979-80* (eds. M. Lerman et al.), Lecture Notes in Mathematics **859**, Springer-Verlag, Berlin, 1981, pp. 268-282.

[8] Shelah, S., Models with second order properties II. Trees with no undefined branches, Annals Math. Logic **14** (1978), 73-87.

Author address

Department of Mathematics,
University of Connecticut
Storrs, CT 06269, USA
email: `schmerl@math.uconn.edu`