Two-Dimensional Temporal Logic

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ABSTRACT Two-dimensional combinations of temporal and modal logics have been studied for some time for their logical properties and their applications to natural language semantics and computer science. In this survey, we briefly describe a variety of these logics, concentrating on the temporal-temporal combinations, their properties and uses. We also look at some more recent results using irreflexivity rules, tiling and mosaic techniques.

1 Introduction

We survey some recent results about various two-dimensional temporal logics and some similar modal-temporal logics. We look at their simple logical properties and applications in computer science and artificial intelligence. For a more general account of multi-dimensional modal logic see [MV97] and for broader surveys of temporal logic see [GHR95].

The logics we are most concerned with are defined over frames consisting of a cross product of simpler structures. Valuations of propositional atoms will be made at ordered pairs and so truth of formulas is also evaluated at ordered pairs in structures. The accessibility relations of the modalities will be restricted by keeping one of the coordinates of the two-dimensions constant.

Such logics may be of interest to those investigating natural language semantics, describing changes in temporal information contained in databases, using interval temporal logics to describe the relationships between processes or states of extended duration, combining temporal logic with logics of possibility, knowledge or belief, describing systems of parallel processes or trying to find modal approximates to the first-order logic of two variable symbols.

In this paper we will briefly look at axiomatizations for these logics using

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Gabbay's irreflexivity rules, undecidability proofs using the tiling technique and we also describe an application to decidability questions of a technique which was initially used by Istvan Németi in the area of algebraic logic (see [Nem86])–the mosaic method.

2 Compass Logic

The most straight forwardly two-dimensional temporal logic is the compass logic introduced by Venema in [Ven90] and [Ven92]. The language contains four interrelated modal diamonds: $\Diamond_{\uparrow}$, $\Diamond_{\downarrow}$, $\Diamond_{\rightarrow}$ and $\Diamond_{\leftarrow}$. Structures for this language consist of two linear orders $(T_1, <_1)$ and $(T_2, <_2)$: we shall call such a pair a *rectangular frame*. Two-dimensional valuations for atoms are made at ordered pairs from $T_1 \times T_2$. We can think of $(T_1, <_1)$ as lying horizontally and $(T_2, <_2)$ as lying vertically on a cartesian grid. The truth of formulas is also defined at ordered pairs in a natural way: for example,

- $(T_1, <_1, T_2, <_2, g), t_1, t_2 \models p$ iff $p \in g(t_1, t_2)$
- $(T_1, <_1, T_2, <_2, g), t_1, t_2 \models \Diamond_{\uparrow} A$ iff there is $s_1 \in T_1$ such that $t_1 <_1 s_1$ and $(T_1, <_1, T_2, <_2, g), s_1, t_2 \models A$.

It is useful to define some abbreviations including the corresponding universal modalities and some boolean combinations of the basic modalities: for example,

$$\Box_{\uparrow} A \equiv \neg \Diamond_{\uparrow} \neg A$$
$$\Box A \equiv A \wedge \Box_{\uparrow} A \wedge \Box_{\downarrow} A \wedge \Box_{\rightarrow} A \wedge \Box_{\leftarrow} A$$
$$\Diamond_{-} A \equiv \Diamond_{\leftarrow} A \vee A \vee \Diamond_{\rightarrow} A$$

Notice that this notation extends the appealingly intuitive geographical analogy suggested by the notation for a modal two-dimensional logic in [Seg73]. The intuition further suggests another possible application of this logic: to the field of spatial reasoning. Although there are modal logics for spatial reasoning (such as the logic of convex hulls in [Ben96]), we know of no investigation of the use of modalities for compass directions in this field.

An example of the kind of statement one can make in the logic is

$$\Diamond_{\uparrow} \Box_{\rightarrow} A \rightarrow \Box_{\rightarrow} \Diamond_{\uparrow} A$$

which is actually a validity where by a validity we here mean a formula which is true at every ordered pair in every rectangular structure. A *satisfiable* formula, on the other hand, is a formula ϕ for which there exists some rectangular structure $\mathcal{T} = (T_1, <_1, T_2, <_2, g)$ and some pair (t_1, t_2) such that

$$\mathcal{T}, t_1, t_2 \models \phi.$$

Just to be clear, we also define semantic consequence by

$$\Gamma \models A$$

iff $\mathcal{T}, t_1, t_2 \models A$ whenever we have $\mathcal{T}, t_1, t_2 \models \gamma$ for all $\gamma \in \Gamma$.

Below we will consider the expressive power of this logic and give an axiomatization for it. We will also see that validity is undecidable.

3 Variations

Variations on this logic arise in the usual ways: semantically, we can restrict our attention to certain subsets of the set of rectangular structures; syntactically, we can consider other temporal operators; or we can combine such variations.

Examples of restrictions are to only consider frames $(T_1, <_1, T_2, <_2)$ (i) with each $(T_i, <_i)$ being dense, (ii) with $(T_1, <_1) = (T_2, <_2)$ or (iii) with each $(T_i, <_i)$ being the natural numbers order. The logics resulting from such restrictions will, of course, usually have more validities.

Considering the language, then a more expressive two-dimensional temporal logic can be obtained by using Kamp's *until* U and *since* S operators. In one dimensional temporal logic, *until* (technically, the strict version of *until*) is defined as follows:

- $\mathcal{T}, t \models U(A, B)$ iff there is some time s such that $t < s$ and $\mathcal{T}, s \models A$ and for all times r such that $t < r < s$ we have $\mathcal{T}, r \models B$.

Since is defined dually, i.e. with $>$ instead of $<$. For two dimensions we end up with four operators: U^h, S^h, U^v and S^v, with a pair for each of the horizontal and vertical ordering. The compass operators can easily be defined in this language: for example $\Diamond_{\uparrow}\, A \equiv U^v(A, \top)$.

Note that there are also less expressive versions of *until* and *since* (called *non-strict*). In two-dimensions the horizontal (easterly) non-strict *until* has defining clause:

- $(T_1, <_1, T_2, <_2, g), t_1, t_2 \models U^h_{ns}(A, B)$ iff there is some $s_1 \in T_1$ such that $t_1 \leq_1 s_1$ and $(T_1, <_1, T_2, <_2, g), s_1, t_2 \models A$ and for all $r_1 \in T_1$ such that $t_1 \leq_1 r_1 <_1 s_1$ we have $(T_1, <_1, T_2, <_2, g), r_1, t_2 \models B$.

The difference between strict and non-strict versions of *until* and *since* is sometimes important in applications where time is taken to be the integers or natural numbers. In such a situation we also have the "next-time" operators $\textcircled{\rightarrow}$, $\textcircled{\uparrow}$, $\textcircled{\leftarrow}$ and $\textcircled{\downarrow}$. The semantic clause for $\textcircled{\rightarrow}$, for example, is

$$(\mathbb{N}, <, \mathbb{N}, <, g), n, m \models \textcircled{\rightarrow}\, \alpha \text{ iff } (\mathbb{N}, <, \mathbb{N}, <, g), n+1, m \models \alpha$$

while that for $\textcircled{\downarrow}$ is

$$(\mathbb{N}, <, \mathbb{N}, <, g), n, m \models \textcircled{\downarrow}\, \alpha \text{ iff } n > 0 \text{ and } (\mathbb{N}, <, \mathbb{N}, <, g), n, m-1 \models \alpha.$$

It is not hard to show that the next-time operators are definable from the strict *until-since* operators but not from the non-strict ones.

When the two dimensions of time are based on the same linear order $(T, <)$ (we can call such structures *square*) we have a diagonal in our structures: $\{(t,t) | t \in T\}$. Then we can make the language even more expressive by including modal constants which represent being on the diagonal or on one side of it as opposed to the other. For example we can introduce δ as a constant which is only true on the diagonal. Being in the north-west half-plane is determined by the truth of the formula $\Diamond_{\rightarrow} \delta$.

Harel in [Har83] has considered a two-dimensional logic with just $\Diamond_{\rightarrow}$, $\Diamond_{\uparrow}$, $\bigcirc_{\uparrow}$ and $\bigcirc_{\rightarrow}$ over natural numbers squares.

In square structures we can also follow Vlach and Åqvist(see [GHR94]) and introduce a *converse* modality J (here written $\otimes$) and *projection* modalities J_1 and J_2. The definitions are:

$$(T,<,T,<,g),t,u \models \otimes\alpha \text{ iff } (T,<,T,<,g),u,t \models \alpha$$
$$(T,<,T,<,g),t_1,t_2 \models J_i\alpha \text{ iff } (T,<,T,<,g),t_i,t_i \models \alpha$$

There are many other variations on these logics: we can also relax some of our semantical assumptions instead of restricting them (e.g. consider structures where each $(T_i, <_i)$ is not necessarily a linear irreflexive order), or reduce the expressiveness of the language (e.g. do not use the "past" operators such as $\Diamond_{\leftarrow}$ and $\Diamond_{\downarrow}$) instead of increasing it.

There are also less neatly two-dimensional combinations of temporal logics in the literature. For example, there are the logics arising from general *Temporalizing* [FG92] and combining [FG96] techniques. Temporalizing allows the adding of a temporal logic on top of any other logic. Truth is evaluated in two-dimensional structures but only a restricted language is available– formulas with a horizontal modality nested inside a vertical one, say, are outlawed. Combining or Fibring techniques, on the other hand, allow the full two-dimensional language but also allow very complex models without commutativity of the two accessibility relations $<_1$ and $<_2$. Such structures are sometimes known as independent combinations of modal logics [Tho80]. These logics are used to investigate the preservation of various logical properties under combination logics. They can also sometimes be the only way of keeping combinations of logic decidable. For more recent results in this area see, for example, [KW91], [Spa93] and [GS97].

4 Expressive Completeness

With all the variations on temporal language even for a fixed semantic domain, questions of relative expressiveness and expressive completeness are bound to be raised. Many of these have been answered by Venema and de Rijke in [Ven90], [Ven92], [Ven94] and [dR93].

Expressive completeness for temporal languages (see [GHR94]) concerns the well-known translation ([vB84]) from temporal or modal formulas into formulas in a monadic first-order logic. A temporal logic is expressively complete if and only if the translation is reversible. Without going into details we just summarize that in the one-dimensional case there do exist expressively complete temporal logics (e.g. the logic with U and S was proved to be so by [Kam68] for Dedekind complete flows of time) while for two-dimensional structures, Venema (in [Ven90]) has shown that no finite set of operators is complete for the class of all square frames built from a dense linear order.

However, [Ven94] presents an expressively complete two-dimensional logic when we restrict our attention to flat structures, i.e. structures in which the valuation g is such that $p \in g(t,u)$ iff $p \in g(t,v)$ for any t, v, u.

5 An Axiomatization

Here we adapt the systems in [Ven90] (for an interval logic), [GHR94] (for Vlach and Åqvist's logic) and [Fin94] (for a two-dimensional until-since logic) to give a complete finite axiom system for the compass logic. There is a slightly different axiomatization for the compass logic in [MV97]. Recall that the semantics are defined over frames with any pair of linear orders.

The inference rules are modus ponens and temporal generalization:

$$\frac{A, A \to B}{B} \qquad \frac{A}{[N]\, A} \qquad \frac{A}{[S]\, A} \qquad \frac{A}{[E]\, A} \qquad \frac{A}{[W]\, A}$$

along with two versions of the Gabbay IRR rule ([Gab81]):

$$\frac{\langle E\rangle\, q \wedge [S]\, \neg \langle E\rangle\, q \to A}{A} \quad \text{provided the atom } q \text{ does not appear in } A$$

and the version with $\langle N\rangle$ instead of $\langle E\rangle$ and $[W]$ instead of $[S]$.

The axiom schemas include the usual ones for linear temporal logic:

A1: all classical tautologies,

A2: $[N]\,(A \to B) \to ([N]\, A \to [N]\, B)$

A3: $A \to [N]\, \langle S\rangle\, A$

A4: $[N]\, A \to [N]\, [N]\, A$

A5: $\langle N\rangle\, ([N]\, A \wedge B \wedge C) \to [N]\, (A \vee \langle N\rangle\, B \vee C)$

with south facing versions of A2, A3 and A4 and east-west versions of all these and schemas C(SE), C(SW), C(NW), C(NE) to describe the interaction of the two dimensions:

C(SE): $\Diamond_{\downarrow} \Diamond_{\rightarrow} A \leftrightarrow \Diamond_{\rightarrow} \Diamond_{\downarrow} A$
e.t.c.

To see that the IRR rules are sound, just suppose that the atom q does not appear in the formula A but that $\psi = ((\Diamond_{\rightarrow} q \wedge \Box_{\downarrow} \neg \Diamond_{\rightarrow} \neg q) \rightarrow A)$ is valid. Now consider any rectangular structure $\mathcal{T} = (T_1, <_1, T_2, <_2, g)$ and any $(t_1, t_2) \in T_1 \times T_2$. Define a new structure $\mathcal{T}' = (T_1, <_1, T_2, <_2, h)$ with valuation $h : (T_1 \times T_2) \rightarrow 2^L$ via

$$q \in h(u_1, u_2) \text{ iff } u_2 = t_2$$
$$p \in h(u_1, u_2) \text{ iff } p \in g(u_1, u_2) \quad \text{for } p \neq q.$$

It is clear that $\mathcal{T}', t_1, t_2 \models \Diamond_{\rightarrow} q \wedge \Box_{\downarrow} \neg \Diamond_{\rightarrow} \neg q$ and so, as ψ is valid, we have $\mathcal{T}', t_1, t_2 \models A$. A simple induction shows that for formulas like A which do not contain the atom q we have $\mathcal{T}, t_1, t_2 \models A$. Thus the first IRR rule is valid and it is straight forward to show that the whole axiom system is sound.

The completeness part of the proof is made easy by the presence of the IRR rules. Instead of considering the set of maximally consistent sets of formulas as in many traditional completeness proofs we can confine our attention to a subset of such sets called the IRR sets. For each of these sets there is some atom which indicates its "latitude" relative to other comparable IRR sets and there is some atom which indicates its "longitude". It is then straight forward to arrange the sets of formulas in a two-dimensional grid and prove we have built a model of a given consistent set of formulas.

It is an open question whether there are axiomatizations for compass logic without using IRR-style rules (as there are in the case of some logics of historical necessity [Zan90]) but it is thought likely that there are not. Note that, as we will see below, it is not possible to axiomatize compass logic at all if we restrict the linear orders to be the natural numbers time.

6 Applications to Natural Language Semantics

Much of the initial motivation for studying multi-dimensional temporal logics came from the study of tenses in natural language. In giving formal semantics to various constructs it was realised that evaluating the truths of complex expressions was not adequately done with reference to a single time of evaluation. Often, for example, evaluating the truth of a past tense expression is not a simple matter of finding a time in the past at which some subexpression is true. The subexpression may depend on that past point and the overall time of utterance for its truth. Extra dimensions –often just 1 extra dimension– are used for points of reference in the semantics of various perfect tenses (see [GHR94], [Gue78]) and in special temporal constructs such as "now" (see [Kam71]) and "then" (see [Vla73]).

For a simple example consider the statement

> when he was gaoled he didn't know that he would be released before now.

Although, the statement contains an epistemic modality to confuse the matter slightly, it is clear that its truth at time s could be adequately established by finding a time t in the past at which the following statement is true:

> he is being gaoled and he doesn't know that he will be released before now

provided that we are careful about the use of the word "now" and suppose that it refers to time s. Thus the truth of the latter statement obviously can only be evaluated at a pair of time points: one t specifying when the evaluation takes place and the other, s, specifying what time corresponds to that word "now".

7 Applications to Databases

Another very important use of temporal logic is in dealing with databases which make use of time. We call these *temporal databases.* Time can be relevant to a database in one or both of two different ways. Each change to the contents of the database will be made at some time: we refer to this as the *transaction* time of the database update. Databases often also store information about the time of events: we refer to the actual time of occurrence of an event as its *valid* time. Depending on which of these uses is made of time or on whether both approaches have a role to play, we can identify several different types of temporal databases but what is common to all, as with all systems which change over time, is that describing or reasoning about their evolution is very conveniently done with temporal logic. With both the forms of temporal information involved, it was thus suggested in [Fin92], that describing the evolution of a temporal database is best done with two-dimensional temporal logic. This is because, for example, at a certain transaction time today, say, we might realize that our database has not been kept up to date and we may add some data about an event which occurred (at a valid time) last week. Thus a one-dimensional model which represents this-morning's view of the history of the recorded world, is changed, by the afternoon, into a new one-dimensional model by having the state of its view about last week altered. A series of one-dimensional models arranged from one day to the next is clearly a structure for a two-dimensional temporal logic. This application has recently been developed into a logical language for controlling temporal databases (see [FR97]). Furthermore, it has recently been shown ([FG92]) that the restricted kind of logic needed for database applications is much more amenable to reasoning with than the usually undecidable general two-dimensional logics.

8 Intervals

As described in [vB95], there are many and various motivations for using an interval temporal logic: these include philosophical considerations of time and events, natural language, processes in computations and planning problems. For constraint problems in planning it may be sufficient to just consider networks of intervals with each pair related by some subset of the 13 possible basic relations (see [All81] for details). However, for more sophisticated reasoning about intervals an interval temporal logic such as that in [HS86] is better suited.

The modalities in [HS86] include:

- $\langle A \rangle$: at some interval beginning immediately after the end of the current one,

- $\langle B \rangle$: at some interval during the current one, beginning when the current one begins, and

- $\langle E \rangle$: at some interval during the current one, ending when the current one ends

As suggested in [Ven90], it is rewarding to notice that interval temporal logics are closely related to two-dimensional temporal logics. We can use the compass language to describe interval structures. Then the interval logic is much the same as the compass logic with a diagonal but with half the points missing. These logics are intertranslatable.

Suppose that $(T, <)$ is a linear order. An interval structure over $(T, <)$ is obtained by adding a valuation for the atoms on intervals i.e. on pairs (s, t) where $s \leq t$ in T. Truth of formulas is also evaluated at intervals.

We turn this into (half a) 2-dimensional structure $(T, <, T, <, g)$ for the compass language with the diagonal constant δ by the following moves:

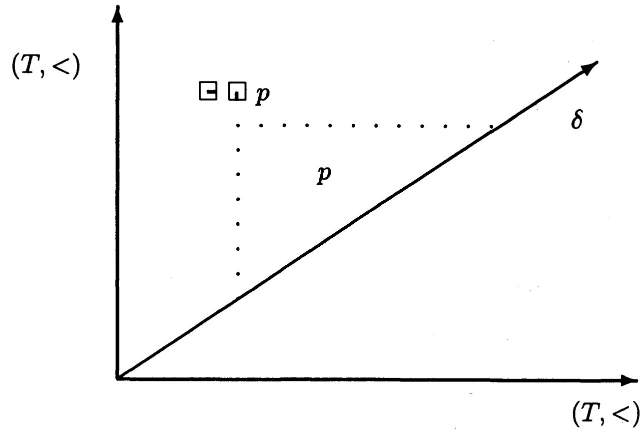
- truth of formulas and the valuation of atoms only takes place at pairs (s, t) with $s \leq t$,

- $p \in g(t_1, t_2)$ iff $t_1 \leq t_2$ and $p \in h(t_1, t_2)$

- $(T, <, T, <, g), s, t \models \delta$ iff $s = t$

Then interval properties can be expressed in compass logic:

- "all subintervals satisfy p"
  is $\boxminus \, \boxbar \, p$



- "any interval strictly in the future satisfies p"
  is $\Diamond\,(\delta \wedge \boxdot \boxminus (\neg\Diamond\, \delta \rightarrow p))$

In fact Venema shows that all the formulas of Halpern and Shoham's logic can be easily translated: for example, $\langle B\rangle p$ simply translates to $\Diamond\ p$.

We know from a preceding section that the compass logic is not expressively complete. In fact, in [Ven91], Venema describes a useful construct which is not able to be expressed in the above interval logics. This is the two-place *chop* operator C which is defined by

$$\mathcal{T}, t, u \models \phi C \psi \text{ iff } \exists v \text{ such that } t \leq v \wedge v \leq u \text{ and}$$
$$\mathcal{T}, t, v \models \phi \text{ and } \mathcal{T}, v, u \models \psi$$

This operator would seem to have application in natural language semantics (as "and-then"), program semantics (as sequential composition), planning and be a generally useful composition construct in two-dimensional logics.

9 Undecidability via Tiling

The truly two-dimensional temporal logics we have met above are all undecidable. This is usually quite straight forward to show using the domino or tiling technique. The technique was first applied to two-dimensional logic and other logics of programs in [Har83] where many different tiling techniques are used to establish various levels of undecidability of the logics. Other techniques such as coding of Turing machines runs (see [HS86] or [HV89]) can be used but the tiling approach is very natural in this context. In [MR96] it is shown that the compass logic itself is undecidable. The authors use tiling but the proof is not completely straight forward. A much better demonstration of tiling in action can be gained from considering the two-dimensional temporal logic X^2 with "next" operators $\ominus$, $\circledcirc$, $\ominus$

and $\obar$ as well as the compass operators used to describe structures where both dimensions of time are the natural numbers. This is very similar to the two-dimensional logic actually considered in [Har83].

We fix some denumerable set C of *colours*. *Tiles* are 4-tuples of colours and we define four projection maps so that each tile $\tau = (\mathrm{left}(\tau), \mathrm{right}(\tau), \mathrm{up}(\tau), \mathrm{down}(\tau))$. We say of a finite set T of tiles that T tiles $\mathbb{N} \times \mathbb{N}$ iff there is a map $\rho : \mathbb{N} \times \mathbb{N} \to T$ such that for each $i, j \in \mathbb{N}$,

- $\mathrm{up}(\rho(i,j)) = \mathrm{down}(\rho(i, j+1))$
- and $\mathrm{right}(\rho(i,j)) = \mathrm{left}(\rho(i+1, j))$.

The tiling problem for $\mathbb{N} \times \mathbb{N}$ is:

- given a finite set T of tiles, does T tile $\mathbb{N} \times \mathbb{N}$?

In [Rob71] it is shown that the tiling problem is co-r.e.-complete: and hence undecidable.

It is now straight forward to use this result to show the undecidability of the logic X^2.

Given a finite set T of tiles we define (recursively) a formula ϕ_T of the logic such that the satisfiability of ϕ_T is equivalent to the tiling of $\mathbb{N} \times \mathbb{N}$ by T. This will be clear. Since deciding validity is just deciding satisfiability of negations, this shows that validity in X^2 is r.e.-hard.

The formula ϕ_T, which uses the elements of T as propositional atoms is simply the conjunction of the following:

$$
\begin{array}{l}
\neg \ominus \top \wedge \neg \obar \top \\
\Box \bigvee_{\tau \in T} \tau \\
\Box \bigwedge_{\tau \neq \tau'} \neg(\tau \wedge \tau') \\
\Box \bigwedge_{\mathrm{up}(\tau) \neq \mathrm{down}(\tau')} \neg(\tau \wedge \obar\ \tau') \\
\Box \bigwedge_{\mathrm{right}(\tau) \neq \mathrm{left}(\tau')} \neg(\tau \wedge \ominus\ \tau')
\end{array}
$$

Such tiling proofs tend to be more complicated for other two-dimensional temporal logics. When the underlying flows are not necessarily the natural numbers then we must use the temporal logic to code in a discrete sub-flow. When the logic does not have next-time or *until* operators then the coding gets more complicated again. See [MR96] for details.

Note that we could have easily followed Harel and modified this proof using a different tiling problem to show that the logic X^2 actually has a Π_1^1-hard validity problem. This implies that the logic is not recursively axiomatisable. In fact, when the underlying linear orders are restricted to be natural numbers, integers or reals then many of the two-dimensional and interval logics we have seen are non-axiomatisable. When the full class of linear orders are available then this modified tiling problem, involving the infinite repetition of a certain tile, is not able to be encoded in the two-dimensional logic and we can only prove undecidability.

10 Combinations of Temporal and Modal Logic

Since they are similar we consider a few combined temporal-modal logics which exhibit a two-dimensional character. There is a survey of many of these logics in [Tho84].

The simplest of these is perhaps the logic which we will call $FP\Diamond$ (following [Rey98]). Models consist of a two-dimensional valuation on a cross product of a linear order and a set. The temporal operators F and P operate along the linear dimension perpendicularly to a modal $S5$ $\Diamond$ operator on the set.

$(U, T, <, g), u, t \models FA$ iff $\exists s \in T$ such that $t < s$ and $(U, T, <, g), u, s \models A$
$(U, T, <, g), u, t \models \Diamond A$ iff $\exists v \in U$ such that $(U, T, <, g), v, t \models A$
etc.

The logic was briefly mentioned in [Tho84] as being not very interesting. However, we will look at some of its niceties in the next section. It is also a logic which appears as a special case of many and various other combined temporal logics.

$FP\Diamond$ logic is a restriction of the Synchronized Ockhamist branching-time logic of [DZ94]. The semantical structures (called $T \times W$ frames by [Tho84]) for this logic involve the cross product of a linear order $(T, <)$ and a set W along with equivalence relations $\sim_t$ on W for each $t \in T$. The equivalence relations must satisfy the property that $w \sim_t w'$ and $t' < t$ implies $w \sim_{t'} w'$. The order $(T, <)$ represents time and the elements of the set W represents alternative histories. The $\sim_t$-class containing w can be used to represent the histories which are possible from the point of view of the world (t, w). Thus the modality $\Diamond_2$ defined by

$$(T, <, W, \{\sim_t\}, g), t, w \models \Diamond_2 \alpha \text{ iff } \quad \exists w' \in W \text{ such that } w \sim_t w' \text{ and } (T, <, W, \{\sim_t\}, g), t, w' \models \alpha$$

represents the idea of "at this time in some history which is currently considered possible". The modality $\Diamond_1$ defined by

$$(T, <, W, \{\sim_t\}, g), t, w \models \Diamond_1 \alpha \text{ iff } \exists w' \in W \text{ such that } (T, <, W, \{\sim_t\}, g), t, w' \models \alpha$$

represents the idea of "at this time in some history". It is this latter modality which extends the non-temporal modality in the logic $FP\Diamond$. Logics very similar to the synchronized logic form bases for logics of agency [BP90] and causation [vK93]. There are axiomatizations of such logic in [vKar] and [DZ96]. It seems to be an open problem whether this logic is decidable.

Logics of historical necessity or Ockhamist logics are closely related examples of a combined logic. They are not a neatly two-dimensional logics but we do have a modal logic of possibility in some sense orthogonal to a linear temporal logic. These logics are obtained by removing the $\Diamond_1$ modality from the Synchronized logic above. They are described in [Bur79] while there are axiomatizations in [Zan85], [Zan96] and [GHR94]. A special case of this logic is proved decidable in [GS85].

Many combinations of time and other modalities arise from formal investigations into how knowledge (or belief) changes over time. These logics are usually designed for reasoning about systems of multiple agents. See [FHMV95] for a comprehensive survey. A temporal-epistemic logic for n agents will use n knowledge modalities. Thus the versions which are of relevance to us here are simple ones, formalizing the changes in knowledge of one lone agent who knows about the world and her or his own knowledge. $S5$ is commonly taken to be the non-temporal logic of knowledge appropriate for one agent. So we can formalize the semantics of the temporal-epistemic logic using a two-dimensional frame very similar in general form to those for Synchronized historical necessity. However, the accessibility relation for the knowledge modality does not have to be restricted to being between worlds (t, w) and (t', w') with $t = t'$. In the case with time being the natural numbers these logics are well studied. In [FHMV95] there is an EXPSPACE-complete decision procedure and complete axiomatization for a logic like $FP\Diamond$ but with natural numbers time: this is a temporal-epsitemic logic of one agent who doesn't forget, doesn't learn and who knows the time.

We have mentioned that $FP\Diamond$ logic has applications to systems of parallel processes. There has been some work in developing two-dimensional logic for such applications. In [RS85], for example, we find a logic combining temporal and spatial modalities. Once again the temporal dimension is the natural numbers and we have the other dimension based on a set of processes. However, there is a set of names for links which may or may not connect one process to another. The language uses *until* in the temporal direction but has a spatial modality for each link as well as one for the transitive closure of all links. This leads to a highly undecidable, unaxiomatisable logic. In [SG87] on the other hand, we have a similar logic but without the linking modalities. There is just the one existential spatial modality as in $FP\Diamond$. With *until* as the temporal connective and the natural numbers as time, this logic is stated to be EXPSPACE-complete.

There are a wealth of two-dimensional non-temporal modal logics which have been investigated. As we will see in the next section some of the results and techniques have also some application to temporal logics. One of the most fruitful areas here has been the investigation of modal versions of first-order logic and their cylindric algebra counterparts. If we look at first-order logic with no function symbols, relations of arity only 1 or 2 and only two variable symbols then we can regard the existential quantifier as a modal operator and come with a two-dimensional modal logic which is the same as that in [Seg73]. There is a very recent account of this area in [AvBN97]. In [VM95], a similar modal logic is studied but its modal semantics is generalized from the first-order motivation. This logic is proved decidable and the proof is an example of the mosaic method which we now turn to.

11 Decidability via The Mosaic Method

The logic $FP\Diamond$ is worth a closer look. This is not just because it may have some application to systems of unbounded numbers of processes computing in parallel. It is also interesting that the mosaic method can be used to show the decidability of validity in this logic.

¿From its beginnings in [Nem86], the mosaic method has been increasingly used in proving decidability and completeness for various multi-modal logics. It is well explained in [VM95] where it is used to prove completeness and decidability of the logic LC_n.

Often, completeness and decidability proofs proceed in a step-by-step manner adding one point at a time to eventually build a model of a satisfiable formula. In the mosaic method we instead try to find a set of small pieces of a model which satisfies a certain closure property. This will be enough to guarantee that the small pieces can be put together to form a model. The actual putting together can either be done by a very simple step-by-step operation (as in [VM95]) or (as shown recently in [HHM+96]), we might be able to use new techniques (of Herwig and Hrushovski) to immediately find the model.

In using the mosaic method to give a decidability proof we need to define mosaics appropriate for the logic and define closure properties (dependent on a given formula) for a finite set of mosaics so that the existence of such a set of a certain size will be equivalent to the existence of a model for the formula.

The logic $FP\Diamond$ is an interesting candidate for a decidability proof via the mosaic method because, as shown in [Rey98], the logic does not have the finite model property. This shows that the finite set of mosaics with the closure property is not just a finite model in disguise.

Here is a brief summary of the decidability proof presented in [Rey98]. Suppose that we are interested in the satisfiability of the formula ϕ. Recall that the structures consist of a set U, a linearly ordered set $(T, <)$ and a valuation $g : (U \times T) \rightarrow 2^L$ for the atoms in L. The accessibility relation for F is $\{((u,t),(u,s)) | t < s\}$ while that for $\Diamond$ is $\{((u,t),(v,t))\}$. We use the usual symbols G, H and $\Box$ for the universal modalities defined from F, P and $\Diamond$ respectively.

The object playing the part of a mosaic for this logic is called a *segment* and consists of a finite set X with a pair $\mu(x)$ and $\nu(x)$ of sets of subformulas (or negations of subformulas) of ϕ for each $x \in X$. We also impose some conditions on the sets. Let us try to motivate these. We want the segment to represent the small piece of a model $\mathcal{T} = (U, T, <, g)$ consisting of the set of pairs of sets of subformulas of ϕ which one obtains by choosing some s and t from T with $s < t$ and looking at

$$\{(A,B)| \quad \text{there is some } u \in U \text{ such that } \alpha \in A \text{ iff } \mathcal{T},u,s \models \alpha \text{ and } \beta \in B \text{ iff } \mathcal{T},u,t \models \beta\}$$

In other words, slice the structure at two different times and look at the sets of formulas which are true on the slices on corresponding U lines.

Note that because we are interested only in subformulas of ϕ there will be only a finite number of pairs in any segment. The conditions that we impose on a segment are 10 in number but quite straight forward given the motivation. For example,

- we require the sets $\mu(x)$ and $\nu(x)$ to be maximally boolean consistent
- if $G\alpha \in \mu(x)$ we require $\alpha \in \nu(x)$ and $G\alpha \in \nu(x)$
- if $\Box\alpha \in \mu(x)$ we require α to be in each $\mu(y)$
- and, importantly, if $\Diamond\alpha \in \mu(x)$ we require there to be some $y \in X$ such that $\alpha \in \mu(y)$.

This is (part of) the definition of just a single segment. We then have to find some closure conditions on a finite set of such segments which will guarantee that they can be (copied as many times as necessary and) put together to build a model of ϕ. These closure properties are such as to allow us to show how to construct a model of ϕ in a step-by-step manner reminiscent of the construction in [Bur82] where defects are cured successively, thus gradually building the model. For example, if (X,μ,ν) is in our set of mosaics and $F\alpha \in \nu(x)$ then we would want there to be a mosaic (X',μ',ν') also in the set which contains some $x' \in X'$ with $\alpha \in \nu'(x')$ and which is such that it could be glued consistently after the first mosaic. Note that the gluing process sometimes involves multiplying copies of various U lines and it is in this way that we can end up with U being an infinite set.

At the eventual (possibly infinite) end of this building process we have a structure in which the labels of formulas on points are equal to the subformulas of ϕ which are true at that point in the model. Provided that we started the process with a segment with ϕ itself in a label then we will end with a model of ϕ.

The decision procedure is thus to check through the finite (but large) number of possible sets of segments for any that satisfy the closure properties. Provided we show also that any satisfiable formula has such a set of segments then the construction described above guarantees that we have a correct procedure.

It is worth mentioning that the complexity of the decision problem for this logic is an open problem: it is nexptime hard but the procedure described above is double exponential.

12 References

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