On "star" schemata of Kossak and Paris

Vladimir Kanovei[2] [3]

ABSTRACT Kossak and Paris introduced the "star" versions of the Induction and Collection schemata for Peano arithmetic, in which one admits, as extra parameters, subsets of a given nonstandard Peano model coded in a fixed elementary end extension of the model. We prove that the "star" schemata are not finitelly axiomatizable over recursively saturated models. A partial solution of a conjecture of Kossak and Paris is obtained.

Introduction

Kossak and Paris [2] have suggested the study of properties of second–order **PA** structures of the form $\langle M; N/M \rangle$, where M and N are nonstandard models of the Peano arithmetic, **PA**, N being an end extension of M (so that M is an initial segment of N), and N/M is the collection of all sets $X \subseteq M$ of the form $X = X' \cap M$, where $X' \subseteq N$ is an N-finite set (*i. e.* X' is coded in N as a finite set by some $a \in N$).

Let $\Sigma_n[N/M]$ denote the extension of the class of Σ_n formulas of the **PA** language by elements of M occuring in the usual way and sets $X \in N/M$ used as extra second–order parameters (with no quantification over them allowed).

This enrichment of the language leads us to the question: are the `Induction` and `Collection` schemata, restricted to the class of $\Sigma_{n+1}[N/M]$ formulas, really stronger than those restricted to $\Sigma_n[N/M]$ formulas ? Kossak and Paris obtained (see [2]) positive answers for the case when $n = 1$ or 2, and formulated it as a conjecture that the result should be true for all n.

This note is written to present a partial answer. We prove that, at least

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in the case when M is countable and recursively saturated, there exists a countable elementary end extension N of M such that M models the schemata for $\Sigma_{n-1}[N/M]$ formulas but does not model those for $\Sigma_{n+1}[N/M]$ formulas.

The level n is still missing. The other open problem is to eliminate the requirement that M is recursively saturated.

The proof involves a coding technique for subsets of **PA** models. In particular, we prove that, given a model $M \models \mathbf{PA}$ and a set $X \subseteq M$, for any n there exists a set $A \subseteq M$ such that M still models both `Induction` and `Collection` for $\Sigma_n(A)$ formulas (where A can occur as an extra second–order parameter), but X is $\Delta_{n+1}(A)$ in M.

1 Preliminaries

We give Kaye [1] as a general reference in matters of notation, but take some space to introduce more special notation which reflects the scope of the paper.

Let M be a countable **PA** model, fixed for the remainder.

An *M-finite set* will mean: a set $X \subseteq M$ coded in M as a finite set. The notion of an *M-finite sequence* (of elements of M) is understood similarly.

A set $X \subseteq M$ is *M-piecewise definable*, M-p. df. in brief, iff $X \cap u$ is M-finite for every M-finite set u.

Σ_n and Π_n will denote the ordinary classes of formulas in the **PA** language.

By $\mathit{\Sigma}_n$ (slanted !) we shall denote the collection of all Σ_n-formulas of the **PA** language, with elements of M allowed as parameters.

Let $\mathcal{X} \subseteq \mathcal{P}(M)$. By $\mathit{\Sigma}_n[\mathcal{X}]$ we shall denote the collection of all formulas obtained from $\mathit{\Sigma}_n$ by the permission to use terms composed from characteristic functions of sets $X \in \mathcal{X}$ to substitute **PA** variables. We write $\mathit{\Sigma}_n(X)$ or $\mathit{\Sigma}_n(X,Y)$ instead of resp. $\mathit{\Sigma}_n[\{X\}]$ or $\mathit{\Sigma}_n[\{X,Y\}]$.

By $\mathit{\Sigma}_n$ we shall also denote the collections of all subsets of M, $M \times M$ *etc.* definable in M by $\mathit{\Sigma}_n$ formulas (where, by definition, elements of M may occur as parameters). We define $\mathit{\Sigma}_\infty = \bigcup_{1 \le n \in \omega} \mathit{\Sigma}_n$.

Other similar notation, like $\mathit{\Pi}_n(X)$, has the corresponding meaning.

Finally, $\mathit{\Delta}... = \mathit{\Sigma}... \cap \mathit{\Pi}...$, in all cases.

It will always be the case that the subsets X of M involved as extra set parameters are M-piecewise definable.

2 The main results

Let Γ be a definability class. We shall consider the following schemata of axioms, where Φ is assumed to be a formula in Γ :

Γ-`Collection`:
$\forall a_0\, [\forall a < a_0\, \exists b\, \Phi(a,b) \implies \exists b_0\, \forall a < a_0\, \exists b < b_0\, \Phi(a,b)]$,

Γ-`Induction`: $\Phi(0)\, \&\, \forall a\, [\Phi(a) \implies \Phi(a+1)] \implies \forall a\, \Phi(a)$.

Let $N \models \mathbf{PA}$ be an end extension of M. Following Kossak and Paris [2], we consider the schemata for the classes $\Gamma = \Sigma_n[N/M]$. (Notation $B\Sigma_n^*$ and $I\Sigma_n^*$ was used in [2] to denote $\Sigma_n[N/M]$-`Collection` and $\Sigma_n[N/M]$ -`Induction`.)

Working with a hierarchy, one naturally wants to figure out whether a given property on a level $n+1$ is strictly stronger than it is on level n. Regarding the `Induction` and `Collection` schemata, Kossak and Paris obtained the following results (see [2]). First, every countable model $M \models \mathbf{PA}$ has an elementary end extension N such that M models $\Sigma_1[N/M]$-`Induction` but does not model $\Sigma_2[N/M]$-`Collection`. Second, every countable model $M \models \mathbf{PA}$ has an elementary end extension N such that M models $\Sigma_2[N/M]$-`Induction` but does not model $\Sigma_3[N/M]$-`Induction`. They conjectured that the results generalize to higher levels.

We do not know how to prove this conjecture even in the case of recursively saturated models M. The following theorem gives a partial result.

Theorem 2.1 (Main theorem)
Let $n \geq 2$. Assume that M is a countable recursively saturated model of **PA**. *There exists a countable elementary end extension N of M such that M models $\Sigma_{n-1}[N/M]$-`Induction` but does not model $\Sigma_{n+1}[N/M]$ -`Collection`.*

`Induction` usually implies `Collection`; for instance Σ_n-`Induction` implies Σ_n-`Collection` for any particular n, see *e.g.* Proposition 4.1 in Sieg [3], so the theorem formally yields the result for either of the schemata separately. However, to make the exposition self-contained, we shall prove independently that M also models $\Sigma_{n-1}[N/M]$-`Collection` and does not model $\Sigma_{n+1}[N/M]$-`Induction`.

The level n is still missing. On the other hand, the theorem implies that for any n we have "essential" gap at least for one of the successive pairs, $n-1, n$ and $n, n+1$.

The proof is based on two ideas concerning how to code subsets of Peano models. The first idea appears in the following theorem, perhaps of separate interest.

Theorem 2.2 *Let $n \geq 1$. Suppose that M is a countable model of* **PA** *and $T \subseteq M$ is an inductive set for M. Let finally $X \subseteq M$ be an M-p. df. set. Then there exists an M-p. df. set $A \subseteq M$ such that M models both* `Induction` *and* `Collection` *for $\Sigma_n(T,A)$, but X is $\Delta_{n+1}(T,A)$ in M.*

(A set $T \subseteq M$ is *inductive* iff M models $\Sigma_m(T)$-`Induction` for all m.

The set T enters the result and the proof as a uniform parameter.) Thus any M-p. df. set $X \subseteq M$ (*e. g.* X may effectively code a cofinal map from some $M_{\le a}$ to M, violating the `Collection` schema) can be coded in $\langle M; T, A\rangle$ at level $n+1$ in such a way that the schemata still hold in M at level n and below.

To prove Theorem 2.2, we introduce the notion of a $\Sigma_n(T)$-generic matrix in Section 3. A *matrix* here is essentially a sequence $\mu = \langle \mu_a : a \in M\rangle$ of functions $\mu_a \in 2^M$. We show that $\Sigma_n(T)$-generic matrices do not violate $\Sigma_n(T)$-`Induction` and $\Sigma_n(T)$-`Collection` in M.

Then we use, in Section 4, a "double" $\Delta_{n+1}(T)$ matrix $\mathrm{\mu}$, which is essentially a double sequence $\langle \mathrm{\mu}_{ai} : a \in M,\ i \in \{0,1\}\rangle$ of functions $\mathrm{\mu}_{ai} \in 2^M$, satisfying the property that, for any M-p. df. set $X \subseteq M$, putting $\mu_a = \mathrm{\mu}_{a1}$ iff $a \in X$ and $\mu_a = \mathrm{\mu}_{a0}$ otherwise, one obtains a $\Sigma_n(T)$-generic matrix $\mu = \langle \mu_a : a \in M\rangle$ independently on the choice of X. On the other hand, μ codes X in such a way that X is $\Delta_{n+1}(T,\mu)$ in M.

To prove Theorem 2.2, we apply this construction for a given M-p. df. set $X \subseteq M$. This results in a $\Sigma_n(T)$-generic matrix μ (so M models the schemata for the class $\Sigma_n(T,\mu)$) such that X is $\Delta_{n+1}(T,\mu)$ in M. It remains to convert μ to a set $A \subseteq M$.

Let us describe how this theorem works in the proof of Theorem 2.1. We consider a countable recursively saturated model M of **PA**. There exists an inductive satisfaction class $T \subseteq M$. Note that M models $\Sigma_m(T)$-`Induction` for all $m \in \omega$, and T satisfies the Tarski rules for a class of true formulas provided elements of M are adequately treated as Gödel numbers of **PA** formulas.

We then fix a cofinal M-p. df. map $\beta : M_{<a_0} \longrightarrow M$, a_0 being an arbitrary nonstandard element of M. Applying Theorem 2.2, we obtain an M-p. df. set $A \subseteq M$ such that β is $\Delta_{n+1}(T,A)$ in M and M satisfies the `Induction` and `Collection` schemata for $\Sigma_n(T,A)$.

As the second part of the proof of Theorem 2.1, we define in Section 5 a countable elementary end extension N of M (an ultrapower of M) such that

(1) Both T and A belong to N/M, therefore β is $\Delta_{n+1}[N/M]$ in M.

(2) Every element of N/M belongs to $\Delta_2(T,A)$ in M.

Now (1) implies that $\Sigma_{n+1}[N/M]$-`Collection` and $\Sigma_{n+1}[N/M]$-`Induction` fail in M. On the other hand, it follows from (2) that `Collection` and `Induction` for the class $\Sigma_{n-1}[N/M]$ hold in M by the choice of A.

We do not know how to reduce $\Delta_2(T,\mu)$ to $\Delta_1(T,\mu)$ in (2), that would improve Σ_{n-1} to Σ_n in Theorem 2.1. The other open problem is to eliminate the assumption that the given model M is recursively saturated. (This property is used in the ultrapower construction of N.)

3 Generic matrices

This section starts the proof of Theorem 2.2. Thus let us suppose that M is a nonstandard countable model of **PA**, and T is an inductive set for M (so that M models $\Sigma_n(T)$-`Induction` for all n), but not necessarily a satisfaction class. For instance, this includes the case when T is the empty set; then the classes $\Sigma_n(T)$ *etc.* below become equal to Σ_n *etc.*

Generic matrices

We wish to consider generic sequences of maps from M into $2 = \{0, 1\}$. Technically, this can be realized in the form of generic matrices.

A *matrix* is an arbitrary function $\mu : M \times M \longrightarrow \{0, 1\}$. Alternatively, a matrix μ can be seen as the indexed family $\langle \mu_a : a \in M \rangle$, where every $\mu_a \in 2^M$ is defined by $\mu_a(l) = \mu(a, l)$ for all l.

Let COND denote the set of all M-finite functions p such that

i) the domain $\mathtt{dom}\, p$ is an M-finite subset of $M \times M$;

ii) all values of p are among 0 and 1.

Elements of COND, called *conditions* below, are identified with their codes in M, so that COND is understood as a definable class in M.

The set COND is ordered by inclusion: $p \le p'$ iff p' extends p as a function. In this case, we say that p' is *stronger than* p. A set $D \subseteq$ COND is *dense* iff every $p \in$ COND is extended by some $p' \in D$.

Let $C \subseteq$ COND. A condition p *decides* C iff either $p \in C$ or there is no stronger condition $p' \in C$. We observe that the set $\{p : p \text{ decides } C\}$ is dense; and if a set C is dense then *deciding* C is equivalent to *belonging to* C.

A matrix μ *extends* a condition $p \in$ COND iff $p \subseteq \mu$, *i.e.* $p(a, l) = \mu(a, l)$ for all $\langle a, l \rangle \in \mathtt{dom}\, p$. A matrix μ *decides* a set $C \subseteq$ COND iff μ extends a condition which decides C. As above if C is dense then μ decides C iff μ extends a condition in C.

Now we introduce the notion of a $\Sigma_n(T)$-generic matrix. The definition intends to meet the following two requirements of opposite character:

1. Any $\Sigma_n(T)$-generic matrix μ has to decide $\Delta_{n+1}(T)$ sets, *and*:

2. One would be able to define a $\Sigma_n(T)$-generic matrix μ of class $\Delta_{n+1}(T)$.

The latter requirement implies that *some*, and even some *dense*, $\Delta_{n+1}(T)$ sets cannot be decided by μ. However we can decide a reasonably large subfamily of dense $\Delta_{n+1}(T)$ sets. For example this subfamily will contain all sets of the form $\{p : p \text{ decides } C\}$ where C is an arbitrary $\Sigma_n(T)$ subset of COND.

Suppose that $E \subseteq M \times \textsc{Cond}$. We put $E^b = \{m : \langle b, m\rangle \in E\}$ for all elements $b \in M$, and

$$\mathcal{D}_{cE} = \{p \in \textsc{Cond} : p \text{ decides every } E^b,\ b < c\}.$$

Evidently $\mathcal{D}_{cE}$ is dense in $\textsc{Cond}$ provided E is $\Sigma_n(T)$ in M for some n.

Definition 3.1 Let Γ be a definability type. A matrix μ is Γ-*generic* iff for every Γ-set E and every $c \in M$, μ extends a condition $p \in \mathcal{D}_{cE}$. □

Proposition 3.2 *Let $n \geq 1$. There exists a $\Sigma_n(T)$-generic matrix μ of class $\Delta_{n+1}(T)$ in M.*

Proof Using a $\Sigma_n(T)$ set universal for all $\Sigma_n(T)$ sets in M get an appropriate $\Delta_{n+1}(T)$ enumeration of all relevant sets $\mathcal{D}_{cE}$, and define μ as the limit of a certain increasing $\Delta_{n+1}(T)$ sequence of conditions. □

Forcing

Let us consider the extension of the language of **PA** by the set $T \subseteq M$ as an extra second order parameter, as above, and a one more constant, $\breve{\mu}$, for a generic matrix. In other words, now "terms" of the form $\breve{\mu}(a,k)$ are admitted to substitute **PA** variables. In particular, $\Sigma_n(\breve{\mu}, T)$ will denote the collection of all $\Sigma_n(T)$ formulas where in addition $\breve{\mu}$ may occur in the mentioned way. The notation $\Pi_n(\breve{\mu}, T)$ is treated similarly.

For a condition p, let p^+ be the matrix which extends p by zeros, that is,

$$p^+(a,l) = \begin{cases} p(a,l) & \text{whenewer} \quad \langle a,l\rangle \in \mathtt{dom}\, p \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.3 The forcing relation $p\,\mathtt{forc}\,\varphi$ is introduced; here $p \in \textsc{Cond}$ while φ is a closed formula of one of the classes $\Sigma_n(\breve{\mu}, T)$, $\Pi_n(\breve{\mu}, T)$; $n \geq 1$

1. Let $\varphi(\breve{\mu}, T)$ be a closed $\Sigma_1(\breve{\mu}, T)$ formula. We set $p\,\mathtt{forc}\,\varphi(\breve{\mu}, T)$ iff the computation of the truth value of $\varphi(p^+, T)$ in M gives the result true after an M-finite number of steps, in such a way that every value $p^+(a,l)$ which factually occurs in the computation satisfy $\langle a,l\rangle \in \mathtt{dom}\, p$.

2. $p\,\mathtt{forc}\,\exists a\, \varphi(a)$ iff there exists $a \in M$ such that $p\,\mathtt{forc}\,\varphi(a)$.

3. Let Φ be a closed $\Pi_m(\breve{\mu}, T)$ formula, $m \geq 1$. Then $p\,\mathtt{forc}\,\Phi$ iff none among the conditions p' extending p forces Φ^-. (Here Φ^- denotes the result of straightforward transformation of $\neg\Phi$ to $\Sigma_n(\breve{\mu}, T)$ form.) □

Proposition 3.4 *Let $\Phi(a_1, ..., a_m)$ be a $\Sigma_n(\breve{\mu}, T)$ formula, $n \geq 1$. Then the set $\{\langle a_1, ..., a_m, p\rangle \in M^m \times \text{COND} : p \, \mathtt{forc} \, \Phi(a_1, ..., a_m)\}$ is $\Sigma_n(T)$ in M.*

Proof The statement in the case $n = 1$ follows from item 1 of Definition 3.3; then the result extends to the general case by induction. □

In particular the set $\{p \in \text{COND} : p \, \mathtt{forc} \, \Phi\}$ is $\Sigma_n(T)$ in M for any closed $\Sigma_n(T)$-formula Φ.

Corollary 3.5 *Assume that $n \geq 1$. Let μ be an $\Sigma_n(T)$-generic matrix. Then for any $m \leq n$ and any closed $\Sigma_m(T)$ formula Φ there exists a condition p, extended by μ, which decides Φ (i. e.* forces Φ or forces Φ^-.) □

The following lemma connects the truth of **PA** formulas, having T and a generic matrix μ as extra parameters, with the forcing.

Lemma 3.6 *Assume that $n \geq 1$. Let μ be an $\Sigma_n(T)$-generic matrix. Let $\varphi(\breve{\mu})$ be a $\Sigma_m(\breve{\mu}, T)$ formula, $1 \leq m \leq n+1$. Then $\varphi(\mu)$ is true in M iff some condition p extended by μ forces $\varphi(\breve{\mu})$.*

Proof The proof goes on by induction on m. The case $m = 1$ is easy.

To carry out the step, suppose that $m \leq n$. Consider a $\Sigma_{m+1}(\breve{\mu}, T)$ formula $\varphi(\breve{\mu})$ of the form $\exists a \, \psi(a, \breve{\mu})$, where ψ is a $\Pi_m(\breve{\mu}, T)$ formula.

Assume that $\varphi(\mu)$ is true. Then $\psi(a, \mu)$ holds in M for some $a \in M$, so that the $\Sigma_m(T)$ formula $\psi^-(a, \mu)$ is false and, by the induction hypothesis, none among conditions p extended by μ forces $\psi^-(a, \breve{\mu})$. By Corollary 3.5, there exists a condition $p \subset \mu$ which forces $\psi(a, \breve{\mu})$. Therefore $p \, \mathtt{forc} \, \varphi(\breve{\mu})$.

Conversely suppose that a condition $p \subset \mu$ forces $\varphi(\breve{\mu})$, that is, forces $\psi(a, \breve{\mu})$ for some a. We prove that $\psi(a, \mu)$ is true in M. Assume on the contrary that $\psi(a, \mu)$ is false, that is, $\psi^-(a, \mu)$ is true in M. Applying the induction hypothesis, we obtain a condition $p' \subset \mu$ which forces $\psi^-(a, \breve{\mu})$. One may assume that $p \subseteq p'$ since p also is extended by μ. This is a contradiction because $p \, \mathtt{forc} \, \psi(a, \breve{\mu})$. □

Lemma 3.7 *Let $n \geq 1$. Suppose that μ is a $\Sigma_n(T)$-generic matrix. Then M satisfies both* `Induction` *and* `Collection` *for formulas in $\Sigma_n(T, \mu)$.*

Proof `Induction`. Consider a $\Sigma_n(\breve{\mu}, T)$ formula $\Phi(\breve{\mu}, a)$. It suffices to prove that if the set $A = \{a \in M : \neg \, \Phi(\mu, a) \text{ in } M\}$ is nonempty then it contains a least element in M. Consider an arbitrary $a' \in A$. By Proposition 3.4 and the genericity, μ extends a condition p which decides every sentence $\Phi(\breve{\mu}, a)$, $a \leq a'$, in M. We pick the M-least $a \leq a'$ such that p forces $\neg \, \Phi(\breve{\mu}, a)$, and use Lemma 3.6 having in mind that Π_n is convertable to Σ_{n+1}.

`Collection`. Consider a $\Sigma_n(\check{\mu}, T)$ formula $\Phi(\check{\mu}, a, b)$. Let $a_0 \in M$. It suffices to find $b_0 \in M$ such that the following holds in M :

$$\forall a < a_0 \, [\exists b \, \Phi(\mu, a, b) \implies \exists b < b_0 \, \Phi(\mu, a, b)\,].$$

By the genericity there exists a condition p, extended by μ, which decides the formula $\exists b \, \Phi(\check{\mu}, a, b)$ for all $a < a_0$. By Proposition 3.4 the forcing relation is definable in $\langle M; T \rangle$; hence for any $a < a_0$ there exists the M-least element $b = b(a) \in M$ such that either p `forc` $\Phi(\check{\mu}, a, b)$ or $b = 0$ and p does not force $\exists b \, \Phi(\check{\mu}, a, b)$. Moreover there exists $b_0 = \max_{a<a_0} b(a) \in M$, as required. □

4 Coding sets by generic matrices

In this section, we complete the proof of Theorem 2.2; a **PA** model M and an inductive for M set $T \subseteq M$ remain fixed. Let us also fix a number $n \geq 1$.

Let a *double matrix* be any function $\upmu : M \times 2 \times M \longrightarrow 2 = \{0, 1\}$. A double matrix $\upmu$ can be seen as the indexed family $\langle \upmu_{ai} : a \in M \,\&\, i \in \{0, 1\} \rangle$ where every "row" $\upmu_{ai} \in 2^M$ is defined by $\upmu_{ai}(l) = \upmu(a, i, l)$ for all $l \in M$.

In this case for any set $X \subseteq M$ we define a matrix $\mu = \upmu * X = \langle \mu_a : a \in M \rangle$ by $\mu_a = \upmu_{a1}$ whenever $a \in X$ and $\mu_a = \upmu_{a0}$ otherwise. Matrices of the form $\upmu * X$ generated by M-p. df. sets $X \subseteq M$ will be called *M-flips* of $\upmu$.

Lemma 4.1 *There exists a double matrix $\upmu$ which is $\Delta_{n+1}(T)$ in M, $\upmu_{a0} \neq \upmu_{a1}$ for any $a \in M$, and all M-flips of $\upmu$ are $\Sigma_n(T)$-generic.*

Proof Let DCond denote the set of all M-finite functions $\mathbf{p}$ such that

1) The domain $\mathbf{dom}\,\mathbf{p}$ is an M-finite subset of $M \times 2 \times M$ satisfying the following requirement: $\langle a, 0, l \rangle \in \mathbf{dom}\,\mathbf{p} \iff \langle a, 1, l \rangle \in \mathbf{dom}\,\mathbf{p}$.

2) All values of $\mathbf{p}$ are among the numbers 0 and 1.

Elements of DCond, called *double conditions*, are identified with their codes in M so that DCond is understood as a subset of M. We put

$$|\mathbf{p}| = \mathbf{dom}\,\mathbf{dom}\,\mathbf{dom}\,\mathbf{p} = \{a : \exists i \, \exists b \, (\langle a, i, b \rangle \in \mathbf{dom}\,\mathbf{p})\}$$

for any double condition $\mathbf{p}$; this is an M-finite subset of M, of course. The set DCond is ordered by inclusion.

Now we introduce flips of double conditions. Let $\mathbf{p} \in$ DCond, $u = |\mathbf{p}|$, and $U \subseteq u$ is an M-finite set. We define $p = \mathbf{p} * U \in$ Cond, an *M-flip* of $\mathbf{p}$, as follows: $p(a, l) = \mathbf{p}(a, 1, l)$ whenever $a \in U$, and $p(a, l) = \mathbf{p}(a, 0, l)$ otherwise. Thus, an M-flip of a double condition is a condition in Cond.

Assertion *Let* $\mathbf{p} \in \mathrm{DC{\small OND}}$, $E \subseteq M \times \mathrm{C{\small OND}}$ *be a* $\Sigma_m(T)$ *set in* M *for some* m, *and* $c \in M$. *There exists a double condition* $\mathbf{p}'$ *extending* $\mathbf{p}$ *such that every* M*-flip of* $\mathbf{p}'$ *decides each of the sets* E^b, $b < c$.

Proof Let us fix an enumeration $\{q_k : 1 \le k \le k_0\}$ (where $k_0 \in M$) of all M-flips of $\mathbf{p}$ in M. We need one more definition. Let $q \subseteq p$ be a pair of conditions in $\mathrm{C{\small OND}}$, q' a condition satisfying $\mathtt{dom}\, q' = \mathtt{dom}\, q$. We define the substitution $p' = p[q/q']$ as follows:

$$p'(a,l) = \begin{cases} q'(a,l) & \text{iff} \quad \langle a,l\rangle \in \mathtt{dom}\, q = \mathtt{dom}\, q' \\ p(a,l) & \text{iff} \quad \langle a,l\rangle \in \mathtt{dom}\, p \setminus \mathtt{dom}\, q \end{cases}$$

Let now q be one of the conditions q_k. One can construct an increasing M-finite sequence of conditions, $q = p_0 \subseteq p_1 \subseteq p_2 \subseteq \ldots \subseteq p_{k_0} = p'$ such that every condition $p'_k = p_k[q/q_k]$ $(1 \le k \le k_0)$ decides each of the sets E^b, $b < c$. In particular every $p'[q/q_k]$ decides each E^b. We define $\mathbf{p}' \in \mathrm{DC{\small OND}}$ as follows:

$$\mathbf{p}'(a,i,l) = \begin{cases} \mathbf{p}(a,i,l) & \text{if} \quad \langle a,i,l\rangle \in \mathtt{dom}\, \mathbf{p} \\ p'(a,l) & \text{if} \quad \langle a,l\rangle \in \mathtt{dom}\, p' \text{ and } \langle a,i,l\rangle \notin \mathtt{dom}\, \mathbf{p} \end{cases}$$

Then every M-flip of $\mathbf{p}'$ is equal to some $p'[q/q_k]$. $\square$ (the assertion)

Now, using the assertion, one ends the proof of Lemma 4.1 in the way outlined above, for the proof of Proposition 3.2, in addition taking care of requirement $\upmu_{a0} \ne \upmu_{a1}$. (The latter would easily follow from a very moderate amount of genericity of $\upmu$ itself, which indeed we shall not exploit.)

Note that the assertion can capture only the flips generated by M-finite sets (the reasoning essentially proceeds in M). This is why X was required to be an M-p. df. set in the lemma. $\square$ (Lemma 4.1)

Let us complete the proof of Theorem 2.2. Suppose that X is an M-p. df. set. Let $\upmu$ be the double matrix given by Lemma 4.1. Then $\mu = \upmu * X$ is a $\Sigma_n(T)$-generic matrix, so that M models both `Induction` and `Collection` for $\Sigma_n(T,\mu)$ by Lemma 3.7. On the other hand, since $\upmu_{a0} \ne \upmu_{a1}$ for all a, we have

$$a \in X \iff \forall l\,[\mu(a,l) = \upmu(a,1,l)] \iff \exists l\,[\mu(a,l) \ne \upmu(a,0,l)],$$

so X is $\Delta_{n+1}(T,\mu)$ in M because $\upmu$ is chosen to be $\Delta_{n+1}(T)$ in M. It remains to convert the matrix $\mu \in 2^{M\times M}$ to a set $A \subseteq M$. $\square$ (Theorem 2.2)

5 The extension

This section proves Theorem 2.1. Thus we suppose that $n \ge 2$ and M is a countable recursively saturated model of **PA**. Then there exists an inductive satisfaction class $T \subseteq M$ for M. In particular, M models $\Sigma_m(T)$-`Induction` for all m.

Since M is countable, there exists a cofinal increasing sequence $\langle b_k : k \in \omega\rangle$ in M. We put $d_k = 2^k 3^{b_k}$. Let us fix a number $a_0 \in M \setminus \omega$ and a $1-1$ map $\beta : M_{<a_0}$ onto the set $\Delta = \{d_k : k \in \omega\}$.

Note that β is M-p. df. as Δ is a cofinal subset of M of order type ω. Hence by Theorem 2.2 there exists an M-p. df. set $A \subseteq M$ such that M models both `Induction` and `Collection` for $\Sigma_n(T, A)$, and β is $\Delta_{n+1}(T, A)$ in M.

The continuation of the proof involves the following lemma.

Lemma 5.1 *There is a countable elementary end extension N of M such that*

(a) *Both A and T belong to N/M.*

(b) *Every element of N/M is $\Delta_2(T, A)$ in M.*

Let us demonstrate that the lemma implies Theorem 2.1. Requirement (b) guarantees that every $\Sigma_{n-1}[N/M]$ subset of M belongs to $\Sigma_n(T, A)$ in M. So M models $\Sigma_{n-1}[N/M]$-`Collection` and $\Sigma_{n-1}[N/M]$-`Induction` by Lemma 3.7.

Requirement (a) implies $\beta \in \Delta_{n+1}[N/M]$ in M by the choice of A. It immediately follows that $\Sigma_{n+1}[N/M]$-`Collection` fails in M by the choice of β. To see that $\Sigma_{n+1}[N/M]$-`Induction` fails as well, consider a $\Sigma_{n+1}[N/M]$ formula $\Phi(k)$ which says that there exist numbers $a < a_0$ and b satisfying $\beta(a) = 2^k 3^b$. It is clear that $\Phi(k)$ is true in M iff $k \in \omega$. □ (Theorem 2.1)

Proof of the lemma. The required extension N will be defined as an ultrapower of M of the form $N = \mathtt{Ult}_{\mathcal{U}}\,\mathcal{F}$, where

$$\mathcal{F} = \{f \in M^M : f \text{ is definable in } M \text{ by a } \mathbf{PA} \text{ formula with parameters in } M\}$$

and $\mathcal{U}$ is an ultrafilter in the algebra $\mathcal{A}$ of all subsets of M definable in M by a **PA** formula with parameters in M.

Since T and A may not be **PA** definable in M, we have to use an ultrafilter as the principal coding tool in order to fulfill requirement (a). The ultrafilter $\mathcal{U}$ will be defined in two steps.

Coding T and A

We put $P_a = \{x \in M : a\text{-th prime divides } x \text{ in } M\}$ and $R_a = M \setminus P_a$. Let $Z = \{2b : b \in T\} \cup \{2b + 1 : b \in A\}$; we now define

$$Q_a = \begin{cases} P_a & \text{iff } a \in Z \\ R_a & \text{otherwise} \end{cases}$$

We set finally $\mathcal{U}_0 = \{Q_a : a \in M\}$.

Notice that all M-finite intersections of sets Q_a are unbounded in M. Furthermore, if $\mathcal{U} \subseteq \mathcal{A}$ is an ultrafilter such that $\mathcal{U}_0 \subseteq \mathcal{U}$, and $N = \mathtt{Ult}_{\mathcal{U}}\,\mathcal{F}$, then both T and A belong to N/M, which yields requirement (a) of the lemma.

Trying to expand $\mathcal{U}_0$ *to an ultrafilter*

For any set $B \in \mathcal{A}$, it must be determined which among the sets B, $M \setminus B$, belongs to $\mathcal{U}$. We have two restrictions on this expansion, of somewhat opposite direction. First we must satisfy requirement (b) of the lemma; second we have to guarantee that $N = \mathtt{Ult}_{\mathcal{U}}\,\mathcal{F}$ is an *end* extension.

Let $\{\phi_p(x)\}_p$ be a formal recursive enumeration in **PA** of all formulas of the language of **PA** having x as the only free variable. We are willing to set $A_p = \{x \in M : M \models \phi_p(x)\}$ for all $p \in M$. This is inconsistent, generally speaking, since many "formulas" ϕ_p may not have a definite external meaning. The satisfaction class T converts the definition to legitimate form. We put

$$A_p = \{x \in M : \ulcorner \phi_p(x) \urcorner \in T\} \quad \text{and} \quad C_p = M \setminus A_p\,;$$

where $\ulcorner \cdot \urcorner$ denotes the Gödel number of $\cdot$.[4] We define an auxiliary ultrafilter $\mathcal{U}' = \mathcal{U}_0 \cup \{B_p : p \in M\}$ where each B_p is either A_p or C_p. The definition of B_p goes on in $\langle M; T, A\rangle$ by the (internal) induction on p as follows:

$$B_p = \begin{cases} A_p & \text{iff} \quad A_p \text{ is compatible with } \mathcal{U}_0 \cup \{B_q : q < p\}\,; \\ C_p & \qquad \text{otherwise} \end{cases}$$

We say that a set $B \subseteq M$ is *compatible* with a family $\mathcal{X}$ of subsets of M iff $B \cap \bigcap \mathcal{X}'$ is unbounded in M for any M-finite $\mathcal{X}' \subseteq \mathcal{X}$. We say that $\mathcal{X}$ is *compatible* iff M is compatible with $\mathcal{X}$. (In the way how the proof goes on, the notion of an M-finite family of subsets of M is well defined.)

Justification

The inductive definition of B_p is carried out in $\langle M; T, A\rangle$ (meaning M with T and A as extra second-order parameters). Therefore to see that the definition is legitimate we have to justify it in the frameworks of our assumptions.

We recall that M satisfies both `Induction` and `Collection` for $\Sigma_n(T, A)$, where $n \geq 2$ is a fixed number (from Theorem 2.1).

[4] Notice that every subset of M definable in M by a **PA** formula with parameters in M is equal to some A_p, $p \in M$, *but not vice versa*, because the family of all sets A_p contains sets of nonstandard M-finite levels of the arithmetical hierarchy in M, available via T.

Let a *good sequence* mean an M-finite binary sequense $s = \langle i_0, ..., i_r \rangle \in M$ corresponding to the construction of A_p and C_p in the sense that, for all $p \le r$,

$$i_p = \begin{cases} 1 & \text{iff} \quad (*) \quad A_p \text{ is compatible with } \mathcal{U}_0 \cup \{B_q : q < p\} \\ 0 & \qquad\qquad \text{otherwise} \end{cases}$$

In other words $i_p = 1$ iff $B_p = A_p$

We now explore the "complexity" of the requirement $(*)$ (saying that the set $B_{ps} = A_p \cap \bigcap_{q<p,\, s(q)=1} A_q \cap \bigcap_{q<p,\, s(q)=0} C_q$ has unbounded, in M, intersection with any among sets $Q_{<b} = \bigcap_{a<b} Q_a$).

Since A, T are M-p. df. sets, we can associate with any $b \in M$ a particular **PA** formula $\gamma_b(x)$ with parameters in M such that $Q_{<b} = \{x : \gamma_b(x)\}$ in M and the map $b \longmapsto \ulcorner \gamma_b \urcorner$ is $\Delta_1(T, A)$, *i. e.* recursive w. r. t. T and A, in M. Let Φ_{pbs} denote the following perhaps infinite but M-finite sequence of symbols in M, which looks like a **PA** formula from the M-th point of view:

$$\forall x \, \exists y \ge x \, [\, \phi_p(y) \;\&\; \gamma_b(y) \;\&\; \forall q < p \, (s(q) = 1 \iff \phi_q(y))\,] .$$

(By the way we cannot add the quantifier $\forall b$ because this would involve T and A as parameters.) Since T is a satisfaction class for M, we have, for all p, $b \in M$ and an M-finite sequence s,

$$\ulcorner \Phi_{pbs} \urcorner \in T \iff \text{the intersection } B_{ps} \cap Q_{<b} \text{ is unbounded} .$$

Then $(*) \iff \forall b \, (\ulcorner \Phi_{pbs} \urcorner \in T)$, so that the property of "being a good sequence" can be expressed by a formula of the type

$$\text{(bounded quantifier) } (\, \Sigma_1(T, A)\text{-formula } \;\&\; \Pi_1(T, A)\text{-formula}\,) ,$$

(because the function $p, b, s \longmapsto \ulcorner \Phi_{pbs} \urcorner$ is $\Delta_1(T, A)$ in M), which is within both $\Sigma_2(T, A)$ and $\Pi_2(T, A)$. It follows that the formula "there exists a good sequence of length k" is $\Sigma_2(T, A)$ as well.

Therefore we can apply $\Sigma_2(T, A)$-`Induction` (a good sequence obviously cannot be maximal) getting a good sequence s_k in M of length k for any $k \in M$. (The uniqueness of a good sequence for any fixed length is easily verified.) This conclusion justifies the construction of sets B_p, $p \in M$.

One more important consequence from our consideration is that the set $S = \{p \in M : B_p = A_p\}$ is $\Delta_2(T, A)$ in M; this will be used below.

The ultrapower

Thus the sets B_p are well defined, and so is $\mathcal{U}' = \mathcal{U}_0 \cup \{B_p : p \in M\}$, the auxiliary ultrafilter, therefore $\mathcal{U} = \mathcal{U}' \cap \mathcal{A}$ is an ultrafilter in $\mathcal{A}$, and $\mathcal{U}$ is compatible in the sense above. We shall prove that $N = \mathtt{Ult}_{\mathcal{U}} \mathcal{F}$ is the required extension of M. There are just two points which we have to check: first, N is an end extension of M, second, requirement (b) of Lemma 5.1.

End extension

By the choice of $\mathcal{F}$, to guarantee that N is an end extension of M, it suffices to prove that if $W \subseteq M \times M$ is definable in M by a **PA** formula (parameters in M, but not T or A, allowed), $c_0 \in M$, $W_k = \{a : \langle k, a\rangle \in W\}$ for all k, and $X = \bigcup_{c<c_0} W_c \in \mathcal{U}$ then there exists $c < c_0$ such that $W_c \in \mathcal{U}$.

To prove this fact assume on the contrary that $W_c \notin \mathcal{U}$ for all $c < c_0$. Let us verify that $W_{<k} = \bigcup_{c<k} W_c \notin \mathcal{U}$ for all $k \le c_0$ by induction on k; this immediately leads to contradiction. Since we have $\Sigma_2(T, A)$-`Induction` in M, it suffices to check that the property $W_{<k} = \bigcup_{c<k} W_c \notin \mathcal{U}$ can be expressed in M by a $\Sigma_2(T, A)$ formula. Such a formula can be defined as follows:

$$\exists p\,[\,(p \notin S \,\&\, W_{<k} = A_p) \bigvee (p \in S \,\&\, W_{<k} = C_p)\,],$$

where, we recall, $S = \{p \in M : B_p = A_p\}$ is a $\Delta_2(T, A)$ set in M.

It remains to replace the equality $W_{<k} = A_p$ by something like the formula $\ulcorner \forall x\,(x \in W_{<k} \iff \phi_p(x))\urcorner \in T$ and accordingly replace $W_{<k} = C_p$.

Requirement (b) of Lemma 5.1

The following is sufficient: if $W \subseteq M \times M$ and $W_k = \{a : \langle k, a\rangle \in W\}$ for all $k \in M$ are as above then $Y = \{k : W_k \in \mathcal{U}\} \in \Delta_2(T, A)$. We observe that

$$k \in Y \iff \exists p\,[\,p \in S \,\&\, W_k = A_p\,] \iff \forall p\,[\,p \in S \implies W_k \ne C_p\,].$$

Both the equality $W_k = A_p$ and the inequality $W_k \ne C_p$ can be reduced to $\Delta_1(T, A)$ as above, therefore $Y \in \Delta_2(T, A)$, as required. $\Box$
(Lemma 5.1)

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Added in proof: It was in September 1997, after the final version of this paper had been submitted, that Richard Kaye let me know a nice improvement of the reasonning in Section 5, which seems to close the gap between $n-1$ and $n+1$ in Theorem 2.1.

Author address

Moscow Transport Engineering Institute
e-mail: kanovei@mech.math.msu.su
kanovei@math.uni-wuppertal.de