Kernels and cohomology groups for some finite covers

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ABSTRACT We extend work of G. Ahlbrandt and M. Ziegler to give a classification of the finite covers with fibre group of prime order p for the projective space over the field with p elements, and for the Grassmannian of k-sets from a disintegrated set (for $k \in \mathbb{N}$).
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This paper is a contribution to the study of the fine detail of the class of (countable) totally categorical structures, in particular the almost strongly minimal ones. The approach we adopt is the one initiated by G. Ahlbrandt and M. Ziegler in [1] and [2] and is purely algebraic. The results we obtain are explicit classification results (under restrictive hypotheses) and are phrased in the terminology of finite covers. It may be helpful if we give a brief impression of them without using this terminology.

Corollary 2.13 represents a classification of certain strongly minimal $\aleph_0$-categorical structures where the associated strictly minimal set is a projective geometry over a prime field. Theorems 3.6 and 3.12 classify certain almost strongly minimal structures in which the associated strictly minimal set is disintegrated. In all these cases it is assumed that the relative automorphism group of the structure over the strictly minimal set is abelian of prime exponent. A key question is whether there exists an expansion of the structure which is biinterpretable with the strictly minimal set (*splitting*). One corollary of our results is that this happens in all cases we consider.

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1 Introduction

1.1 Finite covers

If W is any set then the symmetric group $\mathrm{Sym}(W)$ on W can be considered as a topological group by taking as open sets arbitrary unions of cosets of pointwise stabilisers of finite subsets of W. In this topology, closed subgroups are precisely automorphism groups of first-order structures with domain W. In fact, if H is a subgroup of $\mathrm{Sym}(W)$ then the closure of H in $\mathrm{Sym}(W)$ is the set of elements of $\mathrm{Sym}(W)$ which, for each $n \in \mathbb{N}$, preserve each H-orbit on W^n. Thus we employ the following notation and terminology.

Definition 1.1 A *permutation structure* is a pair $\langle W; G\rangle$ where W is a non-empty set (the *domain*), and G is a closed subgroup of $\mathrm{Sym}(W)$ (the group of *automorphisms*). We shall usually write $G = \mathrm{Aut}(W)$ and refer simply to 'the permutation structure W.' If A is a subset of W and B a subset of W (or more generally of some set on which $\mathrm{Aut}(W)$ is acting in an obvious way), then $\mathrm{Aut}(A/B)$ denotes the permutations of A which extend to elements of $\mathrm{Aut}(W)$ fixing every element of B. We regard $\mathrm{Aut}(W)$ as a topological group with the subspace topology from $\mathrm{Sym}(W)$: a base of open neighbourhoods of the identity consists of subgroups $\mathrm{Aut}(W/X)$ for finite $X \subseteq W$. We shall write permutations on the left of the elements of W.

Permutation structures are obtained by taking automorphism groups of first-order structures on W, and we often regard a first-order structure as a permutation structure without explicitly saying so (by taking for the group of automorphisms of the permutation structure the automorphism group of the first-order structure). In this paper we will be primarily be concerned with the following permutation structures.

Definition 1.2 Let $\mathbb{F}$ be a finite field and V a vector space over $\mathbb{F}$. So V is a permutation structure with automorphism group $\mathrm{GL}(V)$, the group of invertible linear transformations of V. Let $k \in \mathbb{N}$, and let $[[V]]^k$ denote the set of k-dimensional subspaces of V. The group of permutations induced on $[[V]]^k$ by $\mathrm{GL}(V)$ is closed and the kernel of this action consists of the scalar linear transformations. Thus we may regard $[[V]]^k$ as a permutation structure with automorphism group $\mathrm{PGL}(V)$ (the quotient of $\mathrm{GL}(V)$ by the scalar transformations). This is the *Grassmannian* of k-subspaces of V. In the case $k = 1$, we also refer to this as the *projective space* of V.

Definition 1.3 Let D be any set and $k \in \mathbb{N}$. Let $[D]^k$ denote the set of subsets from D of size k. Then the group of permutations induced on $[D]^k$ by $\mathrm{Sym}(D)$ is closed and we refer to the permutation structure $\langle [D]^k; \mathrm{Sym}(D)\rangle$ as the *Grassmannian* of k-sets from (the disintegrated set) D.

We now give the group-theoretic definition of *finite cover.*

Definition 1.4 If C, W are permutation structures, then a finite-to-one surjection $\pi : C \to W$ is a *finite cover* if its fibres form an $\mathrm{Aut}(C)$-invariant partition of C, and the induced map $\mu : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ given by $\mu(g)w = \pi(g\pi^{-1}(w))$ for $g \in \mathrm{Aut}(C)$ and $w \in W$ has image $\mathrm{Aut}(W)$. We refer to μ as the *restriction* map. The *kernel* of the finite cover is $\ker \mu = \mathrm{Aut}(C/W)$. If this is the trivial group we say that the cover is *trivial.* We say that the cover is *split* if there is a closed complement to $\mathrm{Aut}(C/W)$ in $\mathrm{Aut}(C)$.

Remark. To provide a reference-point for model theorists, we give a model-theoretic version.

Definition 1.5 Let C and W be first-order structures. A finite-to-one surjection $\pi : C \to W$ is a *finite cover* of W if there is a 0-definable equivalence relation E on C whose classes are the fibres of π, and any relation on W^n (respectively, C^n) which is 0-definable in the 2-sorted structure (C, W, π) is already 0-definable in W (respectively, C).

A finite cover (in the sense of 1.5) $\pi : C \to W$ induces a homomorphism

$$\mu : \mathrm{Aut}(C) \to \mathrm{Aut}(W),$$

given by putting $\mu(g)(w) = \pi(g\pi^{-1}(w))$ for all $g \in \mathrm{Aut}(C)$ and $w \in W$. In fact, if W is countable $\aleph_0$-categorical, then 1.5 is equivalent to saying that the fibres of π are the classes of an $\mathrm{Aut}(C)$-invariant equivalence relation on C, and the map $\mathrm{Aut}(C) \to \mathrm{Aut}(W)$ induced by π has image $\mathrm{Aut}(W)$ (Lemma 1.1 of [7] ensures that Definition 1.5 implies the surjectivity). The cover is split if there is an expansion of (C, W, π) which is a trivial cover.

If $\pi : C \to W$ is a finite cover than the associated restriction map $\mu : \mathrm{Aut}(C) \to \mathrm{Aut}(W)$ is a continuous homomorphism and so the kernel of the cover $K = \mathrm{Aut}(C/W)$ is a closed normal subgroup of $\mathrm{Aut}(C)$. As all K-orbits on C are finite, it follows that K is compact (and in fact, profinite). By Lemma 1.1 of [7], μ maps open sets to open sets and closed subgroups to closed subgroups. In particular, the induced isomorphism $\mathrm{Aut}(C)/K \to \mathrm{Aut}(W)$ is a homeomorphism.

Definition 1.6 If C, C' are permutation structures with the same domain and $\pi : C \to W$ and $\pi' : C' \to W$ are finite covers with $\pi(c) = \pi'(c)$ for all $c \in C = C'$ then we say that π' is a *covering expansion* of π if $\mathrm{Aut}(C') \leq \mathrm{Aut}(C)$.

Suppose $\pi : C \to W$ is a finite cover. For each $a \in W$ let $C(a)$ denote the fibre above a, that is $\{x \in C : \pi(x) = a\}$. We also define, for any $a \in W$, the *fibre group* $F(a)$ of the cover at a as the permutation group induced by $\mathrm{Aut}(C)$ on $C(a)$. The *binding group* at a is a normal subgroup of the fibre group, and is the permutation group induced on the fibre $C(a)$ by the kernel K. Clearly, if $\mathrm{Aut}(W)$ acts transitively on W then all of the fibre groups are isomorphic as permutation groups, as are the binding groups.

A finite cover is *principal* if its kernel is the product of the fibre groups. We can regard any finite cover $\pi : C \to W$ as a covering expansion of a principal finite cover $\pi_0 : C_0 \to W$ in a canonical way: we take as $\mathrm{Aut}(C_0)$ the group $K_0 \,\mathrm{Aut}(C)$, where K_0 is $\prod_{w \in W} F(w)$, the product of the fibre groups of π. It is easy to see that a principal finite cover is split.

In this paper we shall be concerned with determining, for primes p, the finite covers of the projective space over the field with p elements, and the Grassmannian of k-sets from a disintegrated set, where the fibre groups are of order p (acting transitively on a set with p elements). So these can be considered as covering expansions of an appropriate principal cover $\pi_0 : C_0 \to W$, and our classification is up to conjugacy of the automorphism groups in $\mathrm{Aut}(C_0)$. The strategy we use is that of G. Ahlbrandt and M. Ziegler from [2]: we first determine the possible kernels of covering expansions of π_0 and then we use cohomological methods to determine the actual covering expansions which can give rise to each kernel. The results are summarised in Corollary 2.13 (for the projective space case) and Theorems 3.6 and 3.12 (for the case of Grassmannians of disintégrated sets), but in particular, we note that all these covers split. In the rest of this section, we summarise the machinery we use. This is mostly taken from [2] and [10].

1.2 Kernels

Suppose $\pi_0 : C_0 \to W$ is a finite cover with abelian kernel K_0. So K_0 is a closed normal subgroup of $\Gamma_0 = \mathrm{Aut}(C_0)$ and we have the short exact sequence

$$1 \to K_0 \to \Gamma_0 \xrightarrow{\mu} G \to 1$$

where μ is restriction to W, and $G = \mathrm{Aut}(W)$. Recall that $\Gamma_0/K_0 \cong G$ as topological groups. Now consider Γ_0 acting on K_0 by conjugation. As K_0 is abelian, K_0 is in the kernel of this action, and so we get an action of $G = \Gamma_0/K_0$ on K_0. From now on we shall write K_0 additively, with the G-action on the left. Thus $gk = hkh^{-1}$, for $g \in G$, $k \in K_0$ and any $h \in \mu^{-1}(g)$. We have the following basic fact (see Lemma 6.2.1 of [8] for a proof).

Lemma 1.7 *With this notation K_0 is a topological G-module.* □

If π_0 is a principal finite cover (so $K_0 = \prod_{w \in W} \mathrm{Aut}(C_0(w))$) then π_0 is split. Let T be a closed complement to K_0 in $\mathrm{Aut}(C_0)$ and suppose K is a closed submodule of K_0. Then KT is a closed subgroup of $\mathrm{Aut}(C_0)$ and so can be thought of as the automorphism group of a split covering expansion of π_0 with kernel K. This gives part of the following, which is a result of Ahlbrandt and Ziegler (Lemma 2.1 of [2]):

Theorem 1.8 *Suppose W is a permutation structure with automorphism group G and $\pi_0 : C_0 \to W$ is a principal finite cover of W with abelian kernel K_0. Regard K_0 as a topological G-module. Then a subgroup K of K_0 is the kernel of some covering expansion of π_0 if and only if it is a closed G-submodule of K_0.* □

We shall mainly use this in the case where the fibre and binding groups are cyclic of order p, for some prime p. In this case, we can identify K_0 with the G-module ${\mathbb{F}_p}^W$ of functions from W into $\mathbb{F}_p$, the field of integers modulo p (the G-action is given by $(gf)(w) = f(g^{-1}w)$, for $f \in {\mathbb{F}_p}^W$, $g \in G$, and $w \in W$). So we are interested in the closed G-invariant subspaces of ${\mathbb{F}_p}^W$. These can sometimes more easily be described by making use of a simple instance of Pontriagin duality (for full details see [8]).

Definition 1.9 Let $\mathbb{F}_pW$ be the vector space of formal linear combinations of elements of W, and regard this as a G-module in the natural way. Let X be a subspace of $\mathbb{F}_pW$ and define its *annihilator* in ${\mathbb{F}_p}^W$ to be

$$X^\perp = \{f \in {\mathbb{F}_p}^W : \Sigma_{w\in W} a_w f(w) = 0 \text{ for all } \Sigma_{w\in W} a_w w \in X\}.$$

Note that $X^\perp \leq Y^\perp$ if and only if $Y \leq X$.

Theorem 1.10 *The closed G-invariant subspaces of ${\mathbb{F}_p}^W$ are precisely the annihilators $X^\perp$ of G-invariant subspaces X of $\mathbb{F}_pW$.* □

In summary, to determine kernels of finite covers of W where the fibre and binding groups are of prime order p, it is enough to determine the G-submodules of the permutation module $\mathbb{F}_pW$.

1.3 Derivations

Here we follow rather closely the approach of [2] as modified by Hodges and Pillay in [10].

Recall that if G is a group and M is a G-module, then a *derivation* from G to M is a map $d : G \longrightarrow M$ which satisfies $d(gh) = d(g)+gd(h)$ for all $g, h \in G$. An *inner* derivation is a derivation of the form d_a (for $a \in M$) where $d_a(g) = ga-a$ for all $g \in G$. The set of all such derivations forms an abelian group (with pointwise addition of functions), and the inner derivations form a subgroup. The quotient group is denoted by $H^1(G, M)$, and is referred to as the first cohomology group of G on M. If M is a topological G-module then the continuous derivations form a subgroup of the group of all derivations, and this clearly contains all the inner derivations. We denote the quotient group of continuous derivations modulo inner derivations by $H^1_c(G, M)$.

Suppose that $\pi_0 : C_0 \to W$ is a finite cover of the countably infinite permutation structure W, and suppose from now on that the kernel $K_0 = \mathrm{Aut}(C_0/W)$ is abelian. Then conjugation in $\mathrm{Aut}(C)$ gives K_0 the structure of a topological $\mathrm{Aut}(W)$-module. Let $\mu : \mathrm{Aut}(C_0) \to \mathrm{Aut}(W)$ be the restriction homomorphism. Suppose K is a (closed) G-invariant subgroup of K_0 such that there exists a closed subgroup H_0 of $\mathrm{Aut}(C_0)$ with $H_0 \cap K_0 = K$ and $\mu(H_0) = G$. The following is from ([10], Corollary 18).

Corollary 1.11 *There is a one-to-one correspondence between the set of conjugacy classes of closed subgroups H of $\mathrm{Aut}(C_0)$ which satisfy $\mu(H) = G$ and $H \cap K_0 = K$, and $H^1_c(G, K_0/K)$.* □

In applications here, $\pi_0 : C_0 \to W$ will be a principal finite cover and we will be interested in classifying covering expansions of this which have as kernel some particular G-invariant closed subgroup K of K_0. Corollary 1.11 indicates that to do this we should compute the cohomology group $H^1_c(G, K_0/K)$. If this is trivial, then we can use the following.

Corollary 1.12 *Suppose $\pi_0 : C_0 \to W$ is a principal finite cover with abelian kernel K_0 and K is an $\mathrm{Aut}(W)$-invariant closed subgroup of K_0. If $H^1_c(\mathrm{Aut}(W), K_0/K) = \{0\}$, then there is a covering expansion of π_0 with kernel K. It is unique (up to conjugacy in $\mathrm{Aut}(C_0)$) and split.*

Proof. Existence of a split covering expansion follows from Theorem 1.8 and the remarks preceding it. The uniqueness follows from (1.11): the automorphism groups of any two covering expansions of π_0 with kernel K are conjugate in $\mathrm{Aut}(C_0)$. □

The following curious lemma will replace the use of envelopes in [2]. It allows us to deduce triviality of the cohomology groups we are concerned with from known results about 1-cohomology of finite general linear and symmetric groups.

Lemma 1.13 *Let Γ be a Hausdorff topological group and M a compact topological Γ-module. Suppose there exists $(G_i : i < \omega)$, an increasing chain of subgroups of Γ such that $G = \bigcup_{i<\omega} G_i$ is dense in Γ. Suppose also that for each i we have an open, G_i-invariant subgroup M_i of M, and that $M_{i+1} \leq M_i$ for all $i < \omega$ and $\bigcap_{i<\omega} M_i = \{0\}$. Suppose further that for all i, any continuous derivation from G_i to M/M_i is inner. Then any continuous derivation $d : \Gamma \longrightarrow M$ is inner.*

Proof. Note first that if two continuous derivations $\Gamma \longrightarrow M$ agree on a dense subgroup, then they must be equal. So (as inner derivations are continuous) it will suffice to prove that $\delta = d|G$ is inner. The hypotheses imply that M is metrizable, with a metric θ such that the diameters of the M_i tend to zero (in fact, M as a topological group is isomorphic to the inverse limit of the finite groups M/M_i).

For every $i < \omega$ there exists $a_i \in M$ such that for all $g \in G_i$ we have

$$\delta(g) + M_i = ga_i - a_i + M_i.$$

By compactness of M we may assume that the a_i converge to some $a \in M$. Let d_a denote the inner derivation obtained from a. Thus, for $g \in G_i$, for every $j > i$ there exists $m_j \in M_j$ such that

$$\theta(\delta(g), d_a(g)) = \theta(ga_j - a_j + m_j, ga - a).$$

Now, the m_j tend to 0 as j tends to infinity, and so (by continuity of the Γ-action) $\theta(\delta(g), d_a(g))$ can be arbitrarily small. So $\delta(g) = d_a(g)$. But this holds for all i, and so we conclude that $d = d_a$, as required. □

The following is easy, but useful.

Lemma 1.14 *Let Γ be a topological group and M a continuous Γ-module. Let N be a closed submodule of M and suppose that $H_c^1(\Gamma, M/N)$ and $H_c^1(\Gamma, N)$ are trivial. Then $H_c^1(\Gamma, M)$ is trivial.* □

We shall also require the following version of the 'long exact sequence of cohomology'. A proof can be found in [8] (or see ([5], III.6.1) for the discrete case). If M is a G-module, then $H^0(G, M)$ is the submodule of G-fixed elements of M.

Lemma 1.15 *Suppose G is a group and*

$$0 \to K \to M \to N \to 0$$

is an exact sequence of G-modules. Then there is an exact sequence of abelian groups:

$$0 \to H^0(G,K) \to H^0(G,M) \to H^0(G,N) \to$$

$$\to H^1(G,K) \to H^1(G,M) \to H^1(G,N).$$

If, moreover, G is a topological group and the the short exact sequence is a sequence of compact topological G-modules in which the homomorphisms are continuous, then there is a long exact sequence as above in which the H^1 terms are replaced by H_c^1. □

2 Projective spaces

Throughout this section p will be a prime and $\mathbb{F}_p$ will denote the field with p elements. If n is a cardinal then $V(n,p)$ denotes the vector space of dimension n over $\mathbb{F}_p$.

2.1 Kernels

Let $V = V(\aleph_0, p)$, $W = [[V]]^1$ and $G = \mathrm{GL}(V)$. Let $\pi_0 : C_0 \longrightarrow W$ be a principal finite cover with fibre groups cyclic of order p. In this subsection we determine all possible kernels of covering expansions of π_0. According to Theorem 1.8 and the remarks following it we may identify the kernel K_0 of π_0 with the G-module of functions ${\mathbb{F}_p}^W$, and we want to know the closed G-submodules of K_0. By Theorem 1.10 this problem is equivalent to determining all the G-submodules of the permutation module $\mathbb{F}_pW$. To do this we first investigate the finite case. So let $V_n = V(n, \mathbb{F}_p)$ and $G_n = \mathrm{GL}(V_n)$. We will determine precisely the G_n-submodule structure of the G_n-module $\mathbb{F}_p[[V_n]]^1$. Crucial to this study are the following natural 'incidence maps.'

Definition 2.1 Let k and l be integers satisfying $0 \leq l < k \leq n$. Then we define the map $\beta^n_{k,l} : \mathbb{F}_p[[V_n]]^k \longrightarrow \mathbb{F}_p[[V_n]]^l$ by $\beta^n_{k,l}(w) = \sum\{w' : w' \in [[w]]^l\}$ for $w \in [[V_n]]^k$, and extend linearly to the whole of $\mathbb{F}_p[[V_n]]^k$. So $\beta^n_{k,l}$ maps a k-dimensional subspace w of V_n to a formal sum of all the l-dimensional subspaces of w. This map is clearly a homomorphism of G_n-modules.

The images of these maps (with $l = 1$) provide us with a stock of submodules of $\mathbb{F}_p[[V_n]]^1$. Also consider the map $\beta^n_{1,0} : \mathbb{F}_p[[V_n]]^1 \longrightarrow \mathbb{F}_p$, the so-called *augmentation map*. Obviously, the kernel of this is an $\mathbb{F}_pG_n$-submodule of $\mathbb{F}_p[[V_n]]^1$ (known as the *augmentation submodule*) and it is easy to see that $\ker \beta^n_{1,0}$ is of codimension one in $\mathbb{F}_p[[V_n]]^1$.

Suppose $0 \leq s \leq t$. Denote the number of s-dimensional subspaces of $V(t,p)$ by $\begin{bmatrix} t \\ s \end{bmatrix}_p$ (a *Gaussian coefficient*). It is not difficult to compute this in terms of s, t and p and show that it is coprime to p.

Lemma 2.2 *Let l and k be integers satisfying $0 \leq k < l < r \leq n$. Then we have that $\beta^n_{l,k}\beta^n_{r,l} = \begin{bmatrix} r-k \\ l-k \end{bmatrix}_p \beta^n_{r,k}$.*

Proof. The homomorphism $\beta^n_{r,l}$ maps an r-dimensional subspace x of V_n to the formal sum of the l-dimensional subspaces of x. Let $a = \beta^n_{r,l}(x)$. Now the coeffcient of a k-dimensional subspace y of V_n in $\beta^n_{l,k}(a)$ is equal to the number of l-dimensional subspaces of x which contain y, which is $\begin{bmatrix} r-k \\ l-k \end{bmatrix}_p$. □

Lemma 2.3 *Let r, l and k be integers satisfying $0 \leq k < l < r \leq n$. The we have that* $\mathrm{im}\ \beta^n_{r,k} \leq \mathrm{im}\ \beta^n_{l,k}$.

Proof. This follows from the above because $\begin{bmatrix} r-k \\ l-k \end{bmatrix}_p$, considered as an element of $\mathbb{F}_p$, is non-zero. □

So we have the following chain of submodules:

$$0 \leq \text{im } \beta^n_{n,1} \leq \text{im } \beta^n_{n-1,1} \leq \cdots \leq \text{im } \beta^n_{3,1} \leq \text{im } \beta^n_{2,1} \leq \mathbb{F}_p[[V_n]]^1.$$

Futher, we have:

Lemma 2.4 *The two submodules* im $\beta^n_{n,1}$ *and* ker $\beta^n_{1,0}$ *of* $\mathbb{F}_p[[V_n]]^1$ *are incomparable.*

Proof. This follows from the fact that im $\beta^n_{n,1}$ is one-dimensional, and $\left[{n \atop 1}\right]_p$ is coprime to p. □

We now give a result due to P. Delsarte which is the the most important (and difficult) ingredient needed for calculating the submodule structure of $\mathbb{F}_p[[V_n]]^1$. In its original form, the result involves so-called (non-primitive) generalised Reed-Muller codes, but the version presented here has been 'translated' into a directly applicable form. For the orginal statement of the result, see Theorem 8 in [6] (and see Chapter 5 of [3] for a nice treatment of the coding-theoretic background).

Theorem 2.5 *Let U be a G_n-submodule of $\mathbb{F}_p[[V_n]]^1$ such that U is not contained in* ker $\beta^n_{1,0}$. *Then $U =$* im $\beta^n_{k,1}$ *for some k satisfying $2 \leq k \leq n$, and moreover these are all distinct.* □

We now have enough to describe the submodule structure of $\mathbb{F}_p[[V_n]]^1$, so we collect our results together as:

Theorem 2.6 *Let $V_n = V(n, \mathbb{F}_p)$ and $G_n = GL(V_n)$. Then the proper G_n-submodules of $\mathbb{F}_p[[V_n]]^1$ are:*

- ker $\beta^n_{1,0}$
- im $\beta^n_{k,1}$ *for $k = 2, 3, \ldots, n$*
- im $\beta^n_{k,1} \cap$ ker $\beta^n_{1,0}$ *for $k = 2, 3, \ldots, n$,*

and these are all distinct. Moreover, we have that

$$0 < \text{im } \beta^n_{n,1} < \text{im } \beta^n_{n-1,1} < \cdots < \text{im } \beta^n_{2,1} < \mathbb{F}_p[[V]]^1$$

is a composition series for $\mathbb{F}_p[[V_n]]^1$.

Proof. Delsarte's theorem gives the G_n-submodules of $\mathbb{F}_p[[V_n]]^1$ not contained in ker $\beta^n_{1,0}$. If U is a submodule contained in ker $\beta^n_{1,0}$ then (by Lemma 2.4) $U +$ im $\beta^n_{n,1}$ must therefore equal im $\beta^n_{k,1}$ for some k and (as ker $\beta^n_{1,0} \cap$ im $\beta^n_{k,1}$ contains U and is of codimension 1 in im $\beta^n_{k,1}$) we have $U =$ ker $\beta^n_{1,0} \cap$ im $\beta^n_{k,1}$. □

We can now give the corresponding result for $\mathbb{F}_p W$.

Theorem 2.7 *Let $V = V(\aleph_0, \mathbb{F}_p)$ and $G = GL(V)$. Let $W = [[V]]^1$. Then the G-submodules of $\mathbb{F}_p W$ are:*

- $\ker \beta_{1,0}$
- $\mathrm{im}\ \beta_{k,1}$ *for* $k = 2, 3, \ldots$
- $\mathrm{im}\ \beta_{k,1} \cap \ker \beta_{1,0}$ *for* $k = 2, 3, \ldots,$

and these are all distinct. Moreover, we have that

$$0 < \ldots < \mathrm{im}\ \beta_{n,1} < \cdots < \mathrm{im}\ \beta_{2,1} < \mathbb{F}_p W$$

is a composition series for $\mathbb{F}_p W$.

Proof. Let $x \in \mathbb{F}_p W$. Then for large enough n we have that $x \in \mathbb{F}_p[[V_n]]^1$. So by Theorem 2.6, the G_n-submodule generated by x equals $\mathrm{im}\ \beta^n_{k,1}$ or $\mathrm{im}\ \beta^n_{k,1} \cap \ker \beta^n_{1,0}$, for some k. In the former case we may assume that $x = \beta_{k,1}(w)$ for some $w \in [[V_n]]^k$, and then clearly the G-submodule of $\mathbb{F}_p W$ generated x is $\mathrm{im}\ \beta_{k,1}$. In the latter case we may assume that $x = \beta_{k,1}(w - w')$ for some distinct $w, w' \in [[V_n]]^k$ (as this is a generator for $\mathrm{im}\ \beta^n_{k,1} \cap \ker \beta^n_{1,0}$) and then the G-submodule generated by x is seen to be $\mathrm{im}\ \beta_{k,1} \cap \ker \beta_{1,0}$. □

So we have described all the G-submodules of $\mathbb{F}_p W$. Now we use Theorem 1.10 to describe all the closed G-invariant subgroups of $K_0 = {\mathbb{F}_p}^W$, by taking annihilators. Corresponding to the submodule $\mathrm{im}\ \beta_{k+1,1}$ of $\mathbb{F}_p W$ we have the folllowing submodule of ${\mathbb{F}_p}^W$:

$$\begin{aligned} Pol_k &= \left\{ f \in {\mathbb{F}_p}^W \ : \ \Sigma_{x \in W} a_x f(x) = 0 \text{ for all } \Sigma_{x \in W} a_x x \in \mathrm{im}\ \beta_{k+1,1} \right\} \\ &= \left\{ f \in {\mathbb{F}_p}^W \ : \ \textstyle\sum_{x \in [[w]]^1} f(x) = 0 \text{ for all } w \in [[V]]^{k+1} \right\}, \end{aligned}$$

and the annihilator of $\ker \beta_{1,0}$ is Con, which is the submodule of constant functions. Thus we have:

Corollary 2.8 *The closed, G-invariant submodules of ${\mathbb{F}_p}^W$ are*

- Con
- Pol_k *for* $k = 1, 2, \ldots$
- $Pol_k + Con$ *for* $k = 1, 2, \ldots,$

and these are all distinct. □

Remark 2.1 The above is proved in ([1], Theorem 1.11) for the case $p = 2$. We have used the same notation for the submodules as in [2].

We now provide a different description of the composition factors of the finite-dimensional G_n-module $\mathbb{F}_p[[V_n]]^1$.

Definition 2.9 Suppose E is any field of characteristic p and consider the ring $E[x_1, \ldots, x_n]$ of polynomials in variables $x_1, \ldots, x_n$ as a module for G_n (with G_n acting by substitutions in the usual way). The G_n-action preserves the total degree of an element of the ring, and leaves invariant the ideal I generated by $x_1^p, \ldots, x_n^p$. We let $X_E(i)$ be the image in $E[x_1, \ldots, x_n]/I$ of the homogeneous polynomials of degree i, and regard this as a G_n-module. This is called the module of *truncated polynomials* of degree i.

Remarks 2.2 (i) When $p = 2$, the module $X_{\mathbb{F}_p}(i)$ is just the i-th exterior power of V_n.

(ii) If $i \leq n(p-1)$ there is a natural basis of $X_E(i)$ consisting of $x_1^{a_1} \ldots x_n^{a_n} + I$ where the $a_j \leq p-1$ and $\Sigma_j a_j = i$. It is easy to see that $X_E(i)$ is naturally isomorphic with $X_{\mathbb{F}_p}(i) \otimes_{\mathbb{F}_p} E$.

(iii) If i is divisible by $p-1$ then the scalar transformations in G_n act trivially on $X_{\mathbb{F}_p}(i)$.

Some of the results about these modules which we wish to quote are stated in the literature for the case where E is algebraically closed. The following trivial lemma allows us to deduce the corresponding results for the prime field.

Lemma 2.10 *Let $E \subseteq F$ be a field extension and G a group. Suppose M is a finite dimensional EG-module. Extend this to an FG-module $M' = F \otimes_E M \geq M$.*

(i) If M' is an irreducible FG-module, then M is an irreducible EG-module.

(ii) If M has a series of submodules with m non-zero factors $M_1, \ldots,$ M_m and M' has a composition series with m composition factors, then M_1 $, \ldots, M_m$ are irreducible EG-modules and the composition factors of M' are $F \otimes_E M_1, \ldots, F \otimes_E M_m$.

(iii) If M is irreducible and N is an EG-module such that M' and $F \otimes_E N$ are isomorphic as FG-modules, then N and M are isomorphic EG-modules.

Proof. (i) If H is an E-subspace of M of E-dimension k, then the F-subspace H' of M' spanned by H has F-dimension k. Moreover, if H is G-invariant, then so is H'.

(ii) As in (i), taking the F-subspaces of M' generated by the terms of the series of submodules of M gives a series of submodules of M' with m non-zero factors isomorphic to $F \otimes_E M_1, \ldots, F \otimes_E M_m$. The statement now follows from the Jordan-Hölder theorem and (i).

(iii) Without loss of generality, we may assume that N is an EG-submodule of M' of E-dimension n. Let $x_1, \ldots, x_n$ be an E-basis for M and $y_1, \ldots, y_n$ an E-basis for N. For each $g \in G$ the matrix representing g with respect to each of these two bases has entries in E. Moreover there is an invertible matrix A with entries in F which for each $g \in G$ conjugates the one matrix to the other. The entries of A therefore give a non-zero solution to a system of linear equations with coefficients in E, and so we can actually find such a solution in E. This matrix of elements of E can then be used to set up a non-zero EG-homomorphism from M to N. The rest follows from irreducibility of M, and comparison of dimensions. □

Now we let S_n denote the elements of G_n of determinant 1. Clearly we can regard any G_n-module as an S_n-module, just by restriction. Let F denote an algebraically closed field of characteristic p.

Theorem 2.11 *Suppose $n \geq 3$. Then the composition factors of $\mathbb{F}_p[[V_n]]^1$, viewed as an S_n-module, are $\left\{X_{\mathbb{F}_p}(i(p-1)) \,:\, i = 0, 1, \ldots, n-1\right\}$.*

Proof. By a result of I. Suprunenko and A. Zalesskii (Theorem 1.8(a) of [14], together with the description of the highest weights of the modules $X_F(i(p-1))$ in, for example, the fourth paragraph of [13]) the composition factors of $F[[V_n]]^1$, considered as FS_n-modules, are $X_F(i(p-1))$ for $i = 0, \ldots, n-1$. In particular, there are n composition factors. Also Theorem 2.6 gives a series of submodules of the $\mathbb{F}_pS_n$-module $\mathbb{F}_p[[V_n]]^1$ with n non-zero factors.The result now follows from Lemma 2.10. □

Remark 2.3 The case $p = 2$ (where the composition factors are exterior powers of V_n) is proved directly in [1].

2.2 Cohomology groups

As above, we let $V = V(\aleph_0, p)$ be a countably infinite dimensional vector space over the field with p elements, and $G = \mathrm{GL}(\aleph_0, p)$ its automorphism group. Let $W = [[V]]^1$, considered as a permutation structure with automorphisms those permutations induced by G. Let $\pi_0 : C_0 \to W$ be the principal finite cover of W with fibre groups cyclic of order p (and each fibre of size p). Let K_0 be the kernel of this. In Theorem 2.8 we described the closed G-invariant subgroups K of K_0: a result which was deduced from the parallel situation of finite-dimensional V.

We now show:

Theorem 2.12 *For each possible kernel K we have $H^1_c(G, K_0/K) = \{0\}$.*

Applying (1.12) we get:

Corollary 2.13 *All covering expansions of π_0 split. Any such covering expansion is determined (up to conjugacy in $\mathrm{Aut}(C_0)$) by its kernel, and the possibilities for the kernels are given in Theorem 2.8.* □

Remarks 2.4 For the case $p = 2$ Ahlbrandt and Ziegler ([2]) deduce Theorem 2.12 from results of G. Bell ([4]) about the vanishing of the first cohomology groups of the finite general linear groups $GL(n, 2)$ acting on exterior powers of $V(n,2)$ (if $n \geq 4$), together with results on envelopes in totally categorical structures. Instead of using envelopes, we use Lemma 1.13. In place of Bell's results, we shall use the following. The cases not covered by Bell's work are due to A. Kleschev. The notation is as in Theorem 2.11.

Theorem 2.14 *For $n \geq 4$ and $i = 0, \ldots, n-1$ we have*

$$H^1(S_n, X_{\mathbb{F}_p}(i(p-1))) = 0.$$

Proof. For $p = 2$ this is in [4]. For $p = 3$ it is ([12], Theorem 4.8). For $p > 3$ it is ([12], Theorem 4.6). □

Corollary 2.15 *If $n \geq 4$ and M is a submodule or quotient module of the $\mathbb{F}_p S_n$-module $\mathbb{F}_p[[V_n]]^1$, then $H^1(S_n, M) = \{0\}$.*

Proof. This follows from Theorem 2.11, Lemma 1.14 (for discrete groups), and Theorem 2.14. □

Proof of 2.12. We use Lemma 1.13 with $\Gamma = \mathrm{GL}(V)$ and $M = K_0/K$. Remember that $K_0 = \mathbb{F}_p{}^W$ and K is a closed, Γ-invariant subgroup of K_0. Let $(V_i : 5 \leq i < \omega)$ be an increasing chain of finite dimensional subspaces of V (with V_i of dimension i) with union the whole of V. Let T_i be a complement to V_i in V, and choose these so that $T_i \geq T_{i+1}$ for all i. Let

$$G_i = \{g \in \Gamma : gV_i = V_i,\ g|V_i \text{ has determinant } 1 \text{ and } gx = x\ \forall x \in T_i\}.$$

Then the G_i form an increasing chain whose union is dense in Γ, and G_i is naturally isomorphic to the special linear group $\mathrm{SL}(V_i)$ (called S_i in the above). Let K_i be those functions in K_0 which are zero on $[[V_i]]^1$. Thus, K_0/K_i is isomorphic to $\mathbb{F}_p{}^{[[V_i]]^1}$. Let $M_i = (K + K_i)/K$. Then

$$M/M_i = (K_0/K)/(K + K_i/K) \cong K_0/(K + K_i) \cong (K_0/K_i)/(K + K_i/K_i)$$

and all these isomorphisms hold as isomorphisms of G_i-modules. So M/M_i is isomorphic to a quotient module of $\mathbb{F}_p{}^{[[V_i]]^1}$. But the latter is isomorphic to $\mathbb{F}_p[[V_i]]^1$ (this module is self-dual) and so by Corollary 2.15 we get $H^1(G_i, M/M_i) = \{0\}$. Lemma 1.13 is now applicable, and this finishes the proof of 2.12. □

3 Grassmannians of a disintegrated set

Throughout this section p will be a prime number. Let $D = \mathbb{N}$, $G = \mathrm{Sym}(D)$, let $k \in \mathbb{N}$ and let $W = [D]^k$, the Grassmannian of k-sets from the disintegrated set D. We shall describe the finite covers of W with fibre group of order p (and fibres of size p). So we wish to find all the closed G-submodules K of $K_0 = \mathbb{F}_p{}^W$, and compute the cohomology groups $H^1_c(G, K_0/K)$ in each case. We shall use facts about the representation theory of the finite symmetric groups for which we refer to [11]. We use the following notation throughout.

We denote by M^k the G-module $\mathbb{F}_p[D]^k$. If $n \in \mathbb{N}$ let $[n]^k$ be the set of k-subsets from $\{1, \dots, n\}$, and let $M^k[n] = \mathbb{F}_p[n]^k$, which we regard as a $\mathrm{Sym}(n)$-module. The *Specht submodule* S^k of M^k is the G-submodule generated by the element:

$$e = \sum_{r=0}^{k} \sum_{A \in [k]^r} (-1)^r \{x \leq k : x \notin A\} \cup \{x + k : x \in A\}.$$

Also, if $n \geq 2k$ then the $\mathrm{Sym}(n)$-submodule of $M^k[n]$ generated by this is called the Specht submodule of $M^k[n]$ and is denoted by $S^k[n]$. There is a natural $\mathrm{Sym}(n)$-invariant inner product on $M^l[n]$ (with $[n]^l$ as an orthonormal basis), and any submodule of $M^l[n]$ either contains $S^l[n]$ or is orthogonal to it (see [11], Theorem 4.8). The Specht modules can also be characterised as the intersections of the kernels of the module homomorphisms $\beta_{k,l}$ (and $\beta^n_{k,l}$) definied below (cf. [11], Corollary 17.18). In particular, this shows that $S^k[n] = M^k[n] \cap S^k$. It can be shown that S^k is irreducible ([9], Corollary 3.3) (although this is not necessarily true of $S^k[n]$ for arbitrary n).

3.1 Kernels

As in the previous section, if $0 \leq l < k$ we define a map $\beta_{k,l} : M^k \to M^l$ by setting, for $w \in W$,

$$\beta_{k,l}(w) = \sum_{x \in [w]^l} x$$

and extending linearly. This is clearly a G-homomorphism. Similarly, we define maps $\beta^n_{k,l} : M^k[n] \to M^l[n]$ for $0 \leq l \leq k \leq n$. The following in proved in [9]:

Theorem 3.1 *Any proper, non-zero G-submodule of M^k is an intersection of kernels of homomorphisms $\beta_{k,l}$ for $0 \leq l < k$. The composition factors of M^k are Specht modules $S^0, \dots, S^k$ (where S^0 is the one-dimensional trivial module $\mathbb{F}_p$).* □

Remark 3.1 Note that there are only finitely many submodules of M^k. In fact, there is an algorithm which enables one to write down the full submodule lattice of M^k (and the only computation involved in this is checking divisibility by p of a finite number of binomial coefficients). See [9] for further details.

We can now use Theorem 1.10 to describe the closed G-submodules of ${\mathbb{F}_p}^W$. Consider the maps $\alpha_{l,k} : {\mathbb{F}_p}^{[D]^l} \to {\mathbb{F}_p}^W$ given by

$$(\alpha_{l,k}(f))(w) = \sum_{x \in [w]^l} f(x).$$

It is easy to see that these are continuous G-module homomorphisms and that $f \in {\mathbb{F}_p}^W$ annihilates $\ker(\beta_{k,l})$ if and only if f is in the image of $\alpha_{l,k}$. Thus we have:

Corollary 3.2 *Any closed G-submodule of ${\mathbb{F}_p}^W$ is a sum of images of homomorphisms $\alpha_{l,k}$, for some $0 \leq l \leq k$. There is a (topological) composition series of ${\mathbb{F}_p}^W$ by closed G-submodules, where the composition factors are duals of Specht modules $(S^l)^*$ for $0 \leq l \leq k$, each appearing with multiplicity one.* □

Remark 3.2 The dual $(S^l)^*$ consists of all linear functions $f : S^l \to \mathbb{F}_p$. We can regard this as the quotient module of ${\mathbb{F}_p}^{[D]^l}$ by the annihilator in ${\mathbb{F}_p}^{[D]^l}$ of S^l.

3.2 Cohomology groups

Lemma 3.3 *If $a \in \mathbb{N}$ is such that $p^a \geq l$ and $n = p^a + 2l - 1$ then the Specht module $S^l[n]$ is self-dual and irreducible.*

Proof. This follows from ([11], Theorem 23.13). □

Lemma 3.4 *If $S^l[n]$ is irreducible, then the map obtained by considering a derivation into $S^l[n]$ as a derivation into $M^l[n]$ gives an embedding of $H^1(\mathrm{Sym}(n), S^l[n])$ into $H^1(\mathrm{Sym}(n), M^l[n])$*

Proof. Applying the long exact sequence of cohomology (Lemma 1.15) to the exact sequence

$$0 \to S^l[n] \to M^l[n] \to M^l[n]/S^l[n] \to 0$$

we get the exact sequence

$$H^0(\mathrm{Sym}(n), M^l[n]) \xrightarrow{\phi} H^0(\mathrm{Sym}(n), M^l[n]/S^l[n]) \to H^1(\mathrm{Sym}(n), S^l[n]) \xrightarrow{\psi} H^1(\mathrm{Sym}(n), M^l[n])$$

where ψ is as described in the statement of the lemma. So it is only necessary to show that ϕ is surjective. Let T be the subspace of vectors of $M^l[n]$ orthogonal to $S^l[n]$ under the natural form on $M^l[n]$. As $S^l[n]$ is irreducible, we have $M^l[n] = S^l[n] \oplus T$. Now suppose $x + S^l[n] \in M^l[n]/S^l[n]$ is fixed by all elements of $\mathrm{Sym}(n)$. Then there exists a unique $y \in T$ such that $y - x \in S^l[n]$. It follows that y is fixed by $\mathrm{Sym}(n)$ and $\phi(y) = x + S^l[n]$, as required. □

Corollary 3.5 *If $p \neq 2$, $n = p^a + 2l - 1$ and $p^a \geq l$ then* $H^1(\mathrm{Sym}(n), S^l[n]) = \{0\}$.

Proof. By Shapiro's lemma ([5], III.6.2) we have

$$H^1(\mathrm{Sym}(n), M^l[n]) \cong H^1(\mathrm{Sym}(l) \times \mathrm{Sym}(n-l), \mathbb{F}_p)$$

which is trivial, as any derivation into a trivial module is a homomorphism, and $\mathrm{Sym}(l) \times \mathrm{Sym}(n-l)$ has no homomorphic image of order p. The result now follows from the above two lemmas. □

Theorem 3.6 *If $p \neq 2$ then for all closed G-submodules K of K_0 we have $H^1_c(G, K_0/K) = \{0\}$. All finite covers of W with fibre group of order p split and (assuming the fibres have order p) each such cover is determined by its kernel. The possibilities for the kernels are described in Corollary 3.2.*

Proof. By Corollary 3.2 all of this follows exactly as in Corollary 2.13 once we show that $H^1_c(G, (S^l)^*) = \{0\}$ for $l = 0, \ldots, k$. We use Lemma 1.13 (with $\Gamma = G$ and $M = (S^l)^*$), as in the proof of Theorem 2.12. Let $n_1 < n_2 < \cdots$ be such that $S^l[n_i]$ is irreducible and self dual for all $i \in \mathbb{N}$. Let G_i consist of permutations in G fixing each $n > n_i$, and let M_i consist of linear functions $S^l \to \mathbb{F}_p$ which are zero on $S^l[n_i]$. This is an open G_i-invariant subgroup of M and M/M_i is isomorphic (as a G_i-module) to $(S^l[n_i])^*$, which by choice of n_i is isomorphic (as G_i-module) to $S^l[n_i]$. The result now follows from Lemma 1.13 and Corollary 3.5. □

3.3 The case $p = 2$

Throughout we assume that $p = 2$ and $n = 2^a + 2l - 1$ where $2^a > l$. Thus $S^l[n]$ is self-dual and irreducible. We denote by Z_2 the cyclic group of order 2.

As before, $\pi_0 : C_0 \to W$ is a principal finite cover with fibres of size 2 and fibre groups Z_2. The difficulty with this case is that the cohomology groups $H^1_c(G, (S^l)^*)$ are not all zero.

Lemma 3.7 *If $k \geq 2$ there are covering expansions $\pi_1 : C_1 \to W$ and $\pi_2 : C_2 \to W$ of π_0 with trivial kernels, whose automorphism groups T_1 and T_2 are not conjugate in* $\mathrm{Aut}(C_0)$. *Thus, $H^1_c(G, K_0) \neq \{0\}$.*

Proof. The connection between the two parts of the claim is given by Corollary 1.11. It follows from a version of Shapiro's lemma ([8]) that $H^1_c(G, K_0) = Z_2$, but we shall describe explicitly π_1 and π_2.

We take π_1 to be a covering expansion with trivial fibre group: pick a transversal of the fibres and let π_1 be the expansion of π_0 by this (as a unary relation). Now let $w = \{1, \ldots, k\} \in W$ and let $H \leq G$ be those elements of the stabiliser of w which induce an even permutation on w. This is of index 2 in the stabiliser of w. Let C_2 be the set of left cosets of H in G (regarded as a permutation structure with automorphisms those permutations induced by left multiplication by elements of G) and $\pi_2 : C_2 \to W$ be given by $\pi_2(gH) = gw$. This is a finite cover with fibres of size 2 and fibre group Z_2. It is clear that the only element of G fixing each element of W is the identity. So π_2 may be regarded as a covering expansion of π_0 with trivial kernel and $\mathrm{Aut}(C_1)$ and $\mathrm{Aut}(C_2)$ are not conjugate in $\mathrm{Aut}(C_0)$ (as they have different fibre groups). □

By Shapiro's lemma ([5], III.6.2) we have the following.

Lemma 3.8

$$H^1(\mathrm{Sym}(n), M^l[n]) = \begin{cases} Z_2 & \text{if } l = 0, 1 \\ Z_2 \times Z_2 & \text{if } l \geq 2. \end{cases} \quad \square$$

Shapiro's lemma can be used to calculate the derivations $\mathrm{Sym}(n) \to M^l[n]$. If $l \geq 2$ then (modulo inner derivations) these are as follows. Note that to specify such a derivation, it is enough to give its value on each of the transpositions $(i, i+1) \in \mathrm{Sym}(n)$ for $i = 1, \ldots, n-1$ (as these generate $\mathrm{Sym}(n)$).

- 0
- $\delta_0 : \mathrm{Sym}(n) \to M^l[n]$ given by

$$\delta_0(g) = \begin{cases} 0 & \text{if } g \text{ is even} \\ \mathrm{j} = \sum_{w \in [n]^l} w & \text{if } g \text{ is odd.} \end{cases}$$

- $\delta_1 : \mathrm{Sym}(n) \to M^l[n]$ given by

$$\delta_1((i, i+1)) = \sum \{w \in [n]^l \ : \ i, i+1 \in w\}.$$

- $\delta_2 : \mathrm{Sym}(n) \to M^l[n]$ given by

$$\delta_2((i, i+1)) = \sum \{w \in [n]^l \ : \ i, i+1 \notin w\}.$$

If $l = 0$ or 1 then δ_0 gives the non-zero element of $H^1(\mathrm{Sym}(n), M^l[n])$.

Lemma 3.9 *Let $l \neq 0$. There exists an inner derivation d_x* : $\mathrm{Sym}(n) \to M^l[n]$ *with* $\mathrm{im}(\delta_j + d_x) \subseteq S^l[n]$ *if and only if $l = 2$ and $j = 2$.*

Proof. **Case $j = 1$.** Suppose $l > 2$. Let $x \in M^l[n]$. Recall that

$$S^l[n] = \bigcap_{r=0}^{l-1} \ker \beta^n_{l,r}.$$

So it is enough to find $0 \leq r < l$ such that $\beta^n_{l,r} \circ (\delta_1 + d_x) \neq 0$. Now,

$$\beta^n_{l,l-2}(\delta_1((i,i+1)) + x + (i,i+1)x) = \beta^n_{l,l-2}(\sum\{w \in [n]^l : i, i+1 \in w\}) \\ +\beta^n_{l,l-2}(x + (i,i+1)x).$$

Moreover, $x+(i,i+1)x$ is a sum of terms of the form $\{i\}\cup w+\{i+1\}\cup w$ for some $w \in [n\backslash\{i,i+1\}]^{l-1}$. Thus no element of the form $w' \in [n\backslash\{i,i+1\}]^{l-2}$ appears in the support of $\beta^n_{l,l-2}(x+(i,i+1)x)$. However any such w' appears in $\beta^n_{l,l-2}(\sum\{w \in [n]^l : i,i+1 \in w\})$ with coefficient 1. Thus $\delta_1 + d_x \not\leq \ker \beta^n_{l,l-2}$.

Case $j = 0$. The image of δ_0 is a one-dimensional submodule of $M^l[n]$, so is not contained in $S^l[n]$ (which is irreducible). If $\mathrm{im}(\delta_0 + d_x) \subseteq S^l[n]$ for some $x \in M^l[n]$ then as in the previous case $\beta^n_{l,r}(\mathbf{j} + (i,i+1)x + x) = 0$ for $0 \leq r < l$. It is now easy to argue that this implies $\beta^n_{l,r}(\mathbf{j}) = 0$, and so $\mathbf{j} \in S^l[n]$, a contradiction.

Case $j = 2$. If $0 \leq r < l$ then for $x \in M^l[n]$:

$$\beta^n_{l,r}(\delta_2((i,i+1)) + x + (i,i+1)x) =$$

$$\binom{n-2-r}{l-r} \sum_{w \in [n\backslash\{i,i+1\}]^r} w + \sum_{y \in Y}(\{i\} \cup y + \{i+1\} \cup y)$$

for some $Y \subseteq [n \setminus \{i,i+1\}]^{r-1}$.

So, this is zero for all r only if each of the binomial coefficients

$$\binom{n-2}{l}, \binom{n-2-1}{l-1}, \ldots, \binom{n-2-(l-1)}{1}$$

is divisible by 2. By ([11], Corollary 22.5) this happens if and only if

$$n - 2 - l \equiv -1 \bmod 2^s$$

where 2^s is the smallest power of 2 greater than l. Recalling that $n = 2^a + 2l - 1$ and $2^a > l$, this says that $l = 2$. Note that if $l = 2$ then the above shows that $\mathrm{im}(\delta_2) \subseteq S^2[n]$.

It remains to show that (in the case $l = 2$) there is no inner derivation d_x such that $\mathrm{im}(\delta_1 + d_x) \subseteq S^2[n]$. But in this case $\delta_1 = \delta_0 + \delta_2$ and $\mathrm{im}(\delta_2) \subseteq S^2[n]$, which implies $\mathrm{im}(\delta_0 + d_x) \subseteq S^2[n]$. This contradicts the result already established for $j = 0$. □

Corollary 3.10 *For $l \geq 1$ we have*

$$H^1(\mathrm{Sym}(n), S^l[n]) = \begin{cases} 0 & \text{if } l \neq 2 \\ Z_2 & \text{if } l = 2. \end{cases}$$

Proof. This follows directly from the above lemma and Lemma 3.4. □

Corollary 3.11 *(i) We have*

$$H^1_c(G,(S^l)^*) = \begin{cases} 0 & \text{if } l \neq 2 \\ Z_2 & \text{if } l = 2. \end{cases}$$

(ii) If K is a closed G-submodule of $K_0 = \mathbb{F}_2^W$ then $H^1_c(G, K_0/K)$ is trivial if $(S^2)^*$ is not a composition factor of K_0/K, otherwise, it is of order 2.

Proof. (i) The case $l = 0$ follows from the fact that S^0 is the trivial module $\mathbb{F}_2$, and G has no subgroup of index 2. In the other cases where $l \neq 2$ the result follows from Corollary 3.10 and Lemma 1.13 exactly as in Corollary 3.6.

Now suppose that $l = 2$. First note that there is a non-inner derivation $d : G \to (S^2)^*$ (otherwise $H^1_c(G, \mathbb{F}_2^{[D]^2})$ is trivial, by Corollary 3.2 and Lemma 1.14, contradicting Lemma 3.7). Now suppose that d' is another non-inner derivation. Then by Lemma 1.13 (or rather, its proof) for infinitely many choices of n, the derivations induced by d and d' on $(S^2[n])^*$ are non-inner. Thus by Corollary 3.10, and the fact that these $S^2[n]$ are self-dual, we deduce that $d - d'$ induces an inner derivation on $(S^2[n])^*$ for infinitely many n. It follows from the proof of Lemma 1.13 that $d - d'$ is inner.

(ii) If $(S^2)^*$ is not a composition factor of K_0/K this follows from Corollary 3.2 and Lemma 1.14. More generally, K_0 has the property that the trivial G-module appears as a (topological) composition factor with multiplicity 1, and this appears as a submodule of K_0 (the constant functions). Thus if M is a closed submodule of a continuous homomorphic image of K_0 and N is a closed submodule of this then any fixed point of G on M/N comes from a fixed point of G on M. So then the long exact sequence (Lemma 1.15) shows that the sequence

$$0 \to H^1_c(G,N) \to H^1_c(G,M) \to H^1_c(G,M/N)$$

is exact. It is now easy to deduce (ii) from (i) (and Corollary 3.2). □

Remark 3.3 By the dual version of Corollary 3.16 of [9] we have that $(S^2)^*$ is a composition factor of K_0/K if and only if $\mathrm{im}(\alpha_{2,k})$ is not contained in K (cf. Corollary 3.2).

We use the notation of Lemma 3.7.

Theorem 3.12 *Let $\pi : C \to W$ be a covering expansion of π_0 with kernel K. Then* $\mathrm{Aut}(C)$ *is conjugate in* $\mathrm{Aut}(C_0)$ *to KT_1 or to KT_2 (and only the first of these if $k = 1$). These are non-conjugate in* $\mathrm{Aut}(C_0)$ *if and only if* $\mathrm{im}(\alpha_{2,k}) \not\leq K$*. In any case, π is split.*

Proof. Let $d : G \to K_0$ be the derivation corresponding to π_2 of Lemma 3.7. Then the derivation corresponding to the split cover with automorphism group KT_2 and kernel K is $\bar{d}$, the result of composing d with the natural map $K_0 \to K_0/K$. By Corollary 3.11 and Corollary 1.11 the theorem follows once we show that if $\mathrm{im}(\alpha_{2,k}) \not\leq K$ then $\bar{d}$ is not inner. But if $\bar{d}$ is inner, there exists $a \in K_0$ such that $d - d_a$ has image in K, so is a derivation into K. Now, if $\mathrm{im}(\alpha_{2,k}) \not\leq K$ then $(S^2)^*$ is a composition factor of K_0/K, so not a composition factor of K and thus all continuous derivations into K are inner (by Corollary 3.11). So d is inner, contradicting Lemma 3.7. □

4 References

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