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Survey on the Fundamental Lemma

Ngô Bao Châu

This is a survey on the recent proof of the fundamental lemma. The fundamental lemma and the related transfer conjecture were formulated by R. Langlands in the context of endoscopy theory for automorphic representations in [**26**]. Important arithmetic applications follow from the endoscopy theory, including the transfer of automorphic representations from classical groups to linear groups and the construction of Galois representations attached to automorphic forms via Shimura varieties. Independent of applications, endoscopy theory is instrumental in building a stable trace formula that seems necessary to any decisive progress toward Langlands' conjecture on functoriality of automorphic representations.

There are already several expository texts on endoscopy theory and in particular on the fundamental lemma. The original text [**26**] and articles of Kottwitz [**19**], [**20**] are always the best places to learn the theory. The two introductory articles to endoscopy, one by Labesse [**24**], the other [**14**] written by Harris for the Book project are highly recommended. So are the reports on the proof of the fundamental lemma in the unitary case written by Dat for Bourbaki [**7**] and in general written by Dat and Ngo Dac for the Book project [**8**]. I have also written three expository notes on Hitchin fibration and the fundamental lemma : [**34**] reports on endoscopic structure of the cohomology of the Hitchin fibration, [**36**] is a more gentle introduction to the fundamental lemma, and [**37**] reports on the support theorem, a key point in the proof of the fundamental lemma written for the Book project. The survey follows the same plan as [**36**] but more details have been added.

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1. Orbital integrals over non-archimedean local fields

1.1. First example. Let V be a n-dimensional vector space over a non-archimedean local field F, for instant the field of p-adic numbers. Let $\gamma : V \to V$ be a linear endomorphism with distinct eigenvalues in an algebraic closure of F. The centralizer I_γ of γ is of the form

$$I_\gamma = E_1^\times \times \cdots \times E_r^\times$$

where $E_1, \ldots, E_r$ are finite separable extensions of F. This is a commutative locally compact topological group.

Let $\mathcal{O}_F$ denote the ring of integers in F. We consider the set of lattices of V that are sub-$\mathcal{O}_F$-modules $\mathcal{V} \subset V$ of finite type with maximal rank. We are interested in the subset $\mathcal{M}_\gamma$ of lattices $\mathcal{V}$ of V such that $\gamma(\mathcal{V}) \subset \mathcal{V}$. The group I_γ acts the set $\mathcal{M}_\gamma$. This set is infinite in general but the set of orbits under the action of I_γ is finite. We fix a Haar measure dt on the locally compact group I_γ. We consider a set of representatives of orbits of I_γ on $\mathcal{M}_\gamma$ and for each x in this set, let denote $I_{\gamma,x}$ the compact open subgroup of I_γ of elements stabilizing x. The finite sum

$$\sum_{x \in \mathcal{M}_\gamma / I_\gamma} \frac{1}{\mathrm{vol}(I_{\gamma,x}, dt)} \tag{1}$$

is a typical example of orbital integrals.

1.2. Another example. A basic problem in arithmetic geometry is the determination of the number of abelian varieties equipped with a principal polarization defined over a finite field $\mathbb{F}_q$. The set of isogeny classes of abelian varieties over finite fields is described by Honda and Tate. As usual, we first describe the set the principally polarized abelian varieties that are equipped with an isogeny to a fixed one in requiring that the isogeny be compatible with the polarization. We will be concerned only with ℓ-polarization for some fixed prime number ℓ which is different from the characteristic of $\mathbb{F}_q$.

Let A be a n-dimensional abelian variety over a finite field $\mathbb{F}_p$ equipped with a principal polarization. The $\mathbb{Q}_\ell$-Tate module of A

$$T_{\mathbb{Q}_\ell}(A) = \mathrm{H}_1(A \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$$

is a $2n$-dimensional $\mathbb{Q}_\ell$-vector space equipped with

- a non-degenerate alternating form that is induced by the polarization,
- a Frobenius operator σ_p induced from the $\mathbb{F}_p$-structure of A,
- a self-dual lattice $T_{\mathbb{Z}_\ell}(A) = \mathrm{H}_1(A \otimes \bar{\mathbb{F}}_p, \mathbb{Z}_\ell)$ which is stable under σ_p.

Let A' be a principally polarized abelian variety equipped with a ℓ-isogeny to A compatible with polarizations and defined over k. This isogeny induces an isomorphism between $\mathbb{Q}_\ell$-vector spaces $T_{\mathbb{Q}_\ell}(A)$ and $T_{\mathbb{Q}_\ell}(A')$ that is compatible with symplectic forms and Frobenius operators. Defining this ℓ-isogeny is thus equivalent to defining a self-dual lattice $\mathrm{H}_1(A', \mathbb{Z}_\ell)$ of $\mathrm{H}_1(A, \mathbb{Q}_\ell)$ stable under σ_p. Orbital integral for symplectic group enters in this way in the solution of the problem of counting the number of principally polarized abelian varieties over finite field within a fixed isogeny class.

The description of the set of p-isogenies where p is the characteristic of $\mathbb{F}_q$ is more complicated. The solution is based on the crystalline cohomology of the abelian variety instead of etale ℓ-adic cohomology, and can be translated into semi-linear algebra instead of linear algebra. Instead of orbital integral, the answer is expressed naturally in terms of twisted orbital integrals. Moreover, the test function is no longer the unit of the Hecke algebra but the characteristic function of the double class indexed a the minuscule coweight of the group of symplectic similitudes.

As isogeny is required to be compatible with polarization, the classification of principally polarized abelian varieties can't be immediately reduced to the classification of Honda and Tate. There is indeed a subtle difference between requiring A and A' to be isogenous or A and A' equipped with polarization to be isogenous. In [**23**], Kottwitz observed that this difficulty is of endoscopic nature. He expressed the number of points with values in a finite field on Siegel's moduli space of polarized abelian varieties in terms of orbital integral and twisted orbital integrals in taking into account the endoscopic phenomenon. He proved in fact the same result for a larger class of Shimura varieties classifying abelian varieties with polarization, endomorphisms and level structures.

1.3. General orbital integrals. Let G be a reductive group over F, $\mathfrak{g}$ its Lie algebra. Let γ be an element of $G(F)$ or $\mathfrak{g}(F)$ which is strongly regular semisimple in the sense that its centralizer I_γ if a F-torus. Choose a Haar measure dg on $G(F)$ and a Haar measure dt on $I_\gamma(F)$.

For $\gamma \in G(F)$ and for any compactly supported and locally constant function $f \in C_c^\infty(G(F))$, we set

$$\mathbf{O}_\gamma(f, dg/dt) = \int_{I_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\frac{dg}{dt}.$$

We have the same formula for an element of the Lie algebra $\gamma \in \mathfrak{g}(F)$ and for $f \in C_c^\infty(\mathfrak{g}(F))$. By definition, the orbital integral $\mathbf{O}_\gamma$ does not depend on γ but only on its conjugacy class. It also depends on the choice of Haar measures dg and dt.

We are mostly interested in the unramified case. We assume that G has a reductive model over $\mathcal{O}_F$ i.e. there exists a reductive group scheme over $\mathcal{O}_F$ whose generic fiber is G. This is the case for instant for Chevalley groups. We use a slight abuse of notation in assigning also the letter G to the reductive group scheme over $\mathcal{O}_F$. The group $K = G(\mathcal{O}_F)$ of integral points is a maximal compact subgroup of $G(F)$. We choose the Haar measure dg such that K has volume one. Consider the set

$$\mathcal{M}_\gamma = \{x \in G(F)/K \mid gx = x\}, \tag{2}$$

equipped with an action of $I_\gamma(F)$. The orbital integral of the characteristic function 1_K of K admits a concrete description

$$\mathbf{O}_\gamma(1_K, dg/dt) = \sum_{x\in I_\gamma(F)\backslash\mathcal{M}_\gamma} \frac{1}{\mathrm{vol}(I_\gamma(F)_x, dt)} \tag{3}$$

where x runs over a set of representatives of orbits of $I_\gamma(F)$ in $\mathcal{M}_\gamma$ and $I_\gamma(F)_x$ is the compact open subgroup of $I_\gamma(F)$ stabilizer of x. The function 1_K also called the unit of the Hecke algebra plays a very special role in the global setting.

If $G = \mathrm{GL}(n)$, the space of cosets $G(F)/K$ can be identified with the set of lattices in F^n so that we recover the lattice counting problem in the first example. For classical groups, orbital integrals for the unit function can also be expressed as the number of selfdual lattices fixed by an automorphism.

1.4. The Arthur-Selberg trace formula. We consider now a semi-simple group G defined over a global field F that can be either a number field or the field of rational functions on a curve defined over a finite field. It is of interest to understand the trace of Hecke operator on automorphic representations of G. The Arthur-Selberg trace formula is a powerful tool for this quest. It has the following form

$$\sum_{\gamma\in G(F)/\sim} \mathbf{O}_\gamma(f) + \cdots = \sum_{\pi} \mathrm{tr}_\pi(f) + \cdots \tag{4}$$

where γ runs over the set of anisotropic conjugacy classes of $G(F)$ and π over the set of discrete automorphic representations. The trace formula contains also more complicated terms related to hyperbolic conjugacy classes on one side and the continuous spectrum on the other side.

The test functions f are of the form $f = \otimes f_v$ with f_v being the unit function in Hecke algebra of $G(F_v)$ for almost all finite places v of F. The global orbital integral

$$\mathbf{O}_\gamma(f) = \int_{I_\gamma(F)\backslash G(\mathbb{A})} f(g^{-1}\gamma g)dg$$

is convergent for isotropic conjugacy classes $\gamma \in G(F)/\sim$. After choosing a Haar measure $dt = \bigotimes dt_v$ on $I_\gamma(\mathbb{A})$, we can express the above global integral as follows

$$\mathbf{O}_\gamma(f) = \mathrm{vol}(I_\gamma(F)\backslash I_\gamma(\mathbb{A}), dt) \prod_v \mathbf{O}_\gamma(f_v, dg_v/dt_v).$$

Local orbital integral of semisimple elements are convergent for every v and are equal to one for almost all v if the measure dt is chosen so that $I_\gamma(\mathcal{O}_v)$ has volume one for almost all v. The torus I_γ has an integral form that is well defined up to finitely many places. The volume term is finite when the global class γ is anisotropic.

Arthur introduced truncation operator to deal with the continuous spectrum and with non isotropic conjugacy classes. In his geometric expansion, Arthur has more complicated local integral that he calls weighted orbital integrals, see [**1**].

1.5. Shimura varieties. Similar strategy has been used for the calculation of the Hasse-Weil zeta function attached to Shimura varieties. In the special case of Shimura varieties classifying polarized abelian varieties with endomorphisms and level structure, Kottwitz established a formula for the number of points with values in a finite field $\mathbb{F}_q$. The formula he obtained is closed to the orbital side of (4) for the reductive group G entering in the definition of $\mathcal{S}$. Again, certain local identities of orbital integrals are needed to establish the equality of number $\sharp\mathcal{S}(\mathbb{F}_q)$ with a combination the orbital sides of (4) for G and a collection of smaller groups called endoscopic groups of G. Eventually, this strategy allows one to attach Galois representation to auto-dual automorphic representations of $\mathrm{GL}(n)$. For the most recent and complete results, see [**31**] and [**38**].

2. Stable trace formula

2.1. Stable conjugacy. In studying orbital integrals for other groups for $\mathrm{GL}(n)$, we encounter with an a priori annoying problem. For $\mathrm{GL}(n)$, two regular semisimple elements in $\mathrm{GL}(n, F)$ are conjugate if and only if they are conjugate in the larger group $\mathrm{GL}(n, \bar{F})$ where $\bar{F}$ is an algebraic closure of F and this latter condition is tantamount to request that γ and γ' have the same characteristic polynomial. For a general reductive group G, we have a characteristic polynomial map $\chi : G \to T/W$ where T is a maximal torus and W is its Weyl group. An element is said strongly regular semisimple if its centralizer is a torus. Strongly regular semisimple elements $\gamma, \gamma' \in G(\bar{F})$ have the same characteristic polynomial if and only if they are $G(\bar{F})$-conjugate. However, there are possibly more than one $G(F)$-conjugacy classes within the set of strongly regular semisimple elements having the same characteristic polynomial in $G(F)$. These conjugacy classes are said stably conjugate.

For a fixed $\gamma \in G(F)$, assumed strongly regular semisimple, the set of $G(F)$-conjugacy classes in the stable conjugacy of γ can be identified with the subset of elements $\mathrm{H}^1(F, I_\gamma)$ whose image in $\mathrm{H}^1(F, G)$ is trivial. For local fields, the group $\mathrm{H}^1(F, I_\gamma)$ is finite but for global field, it can be infinite.

2.2. Stable orbital integral and its κ-sisters. For a local non-archimedean field F, A_γ is a subgroup of the finite abelian group $\mathrm{H}^1(F, I_\gamma)$. One can form linear combinations of orbital integrals within a stable conjugacy class using characters of A_γ. In particular, the stable orbital integral

$$\mathbf{SO}_\gamma(f) = \sum_{\gamma'} \mathbf{O}_{\gamma'}(f)$$

is the sum over a set of representatives γ' of conjugacy classes within the stable conjugacy class of γ. One needs to choose in a consistent way Haar measures on different centralizers $I_{\gamma'}(F)$. For strongly regular semisimple γ, the tori $I_{\gamma'}$ for γ' in the stable conjugacy class of γ, are in fact canonically isomorphic so that we can transfer a Haar measure from $I_\gamma(F)$ to $I_{\gamma'}(F)$.

Obviously, the stable orbital integral $\mathbf{SO}_\gamma$ depends only on the characteristic polynomial of γ. If a is the characteristic polynomial of a strongly regular semisimple element γ, we set $\mathbf{SO}_a = \mathbf{SO}_\gamma$. A stable distribution is an element in the closure of the vector space generated by the distributions of the form $\mathbf{SO}_a$ with respect to the weak topology.

For each character $\kappa : A_\gamma \to \mathbb{C}^\times$, the κ-orbital integral is the linear combination

$$\mathbf{O}_\gamma^\kappa(f) = \sum_{\gamma'} \kappa(\mathrm{cl}(\gamma'))\mathbf{O}_{\gamma'}(f)$$

over a set of representatives γ' of conjugacy classes within the stable conjugacy class of γ, $\mathrm{cl}(\gamma')$ being the class of γ' in A_γ. For any γ' in the stable conjugacy class of γ, A_γ and $A_{\gamma'}$ are canonical isomorphic so that the character κ on A_γ defines a character of A'_γ. Now, $\mathbf{O}_\gamma^\kappa$ and $\mathbf{O}_{\gamma'}^\kappa$ are not equal but differ by the scalar $\kappa(\mathrm{cl}(\gamma'))$ where $\mathrm{cl}(\gamma')$ is the class of γ' in A_γ. Even though this transformation rule is simple enough, we can't a priori define κ-orbital $\mathbf{O}_a^\kappa$ for a characteristic polynomial a as in the case of stable orbital integral. This is a source of an important technical difficulty in the theory of endoscopy that is known as the transfer factor.

At least in the case of Lie algebra, there exists a section $\iota : \mathfrak{t}/W \to \mathfrak{g}$ due to Kostant of the characteristic polynomial map $\chi : \mathfrak{g} \to \mathfrak{t}/W$ and we set

$$\mathbf{O}_a^\kappa = \mathbf{O}_{\iota(a)}^\kappa.$$

Thanks to Kottwitz' calculation of transfer factor, we know that this naively looking definition turns out to be correct. This simplifies significantly the statement of the fundamental lemma and the transfer conjecture for Lie algebra [**22**].

If G is semisimple and simply connected, Steinberg constructed a section $\iota : T/W \to G$ of the characteristic polynomial map $\chi : G \to T/W$. It is tempting to define $\mathbf{O}_a^\kappa$ by Steinberg's section but we don't know whether this is the right definition as in the case of Lie algebra.

2.3. Stabilization. Let F be a global field and $\mathbb{A}$ denote its ring of adeles. Test functions for the trace formula are finite combination of functions f on $G(\mathbb{A})$ of the form $f = \bigotimes_{v\in|F|} f_v$ where for all v, f_v is a smooth function with compact support on $G(F_v)$ and for almost all finite place v, f_v is the characteristic function of $G(\mathcal{O}_v)$ with respect to an integral form of G which is well defined almost everywhere.

The trace formula defines a linear form in f. For each v, it induces an invariant linear form in f_v. There exists a Galois theoretical cohomological obstruction that prevents this linear form from being stably invariant. Let $\gamma \in G(F)$ be a strongly regular semisimple element. Let $(\gamma'_v) \in G(\mathbb{A})$ be an adelic element with γ'_v stably conjugate to γ for all v and conjugate for almost all v. There exists a cohomological obstruction that prevents the adelic conjugacy class (γ'_v) from being rational. In fact the map

$$\mathrm{H}^1(F, I_\gamma) \to \bigoplus_v \mathrm{H}^1(F_v, I_\gamma)$$

is not surjective in general. Let denote $\hat{I}_\gamma$ the dual complex torus of I_γ equipped with a finite action of the Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$. For each place v, the Galois group $\Gamma_v = \mathrm{Gal}(\bar{F}_v/F_v)$ of the local field also acts on $\hat{I}_\gamma$. By local Tate-Nakayama duality as reformulated by Kottwitz, $\mathrm{H}^1(F_v, I_\gamma)$ can be identified with the group of characters of $\pi_0(\hat{I}_\gamma^{\Gamma_v})$. By global Tate-Nakayama duality, an adelic class in $\bigoplus_v \mathrm{H}^1(F_v, I_\gamma)$ comes from a rational class in $\mathrm{H}^1(F, I_\gamma)$ if and only if the corresponding characters on $\pi_0(\hat{I}_\gamma^{\Gamma_v})$ restricted to $\pi_0(\hat{I}_\gamma^{\Gamma})$, sum up to the trivial character. The original problem with conjugacy classes within a stable conjugacy class, complicated by the presence of the strict subset A_γ of $\mathrm{H}^1(F, I_\gamma)$, was solved in Langlands [**26**] and in a more general setting by Kottwitz [**20**].

In [**26**], Langlands outlined a program to derive from the usual trace formula a stable trace formula. He proposed to remove first the above Galois cohomological obstruction so that the formula becomes a stable distribution and to introduce the correction terms appearing after a Fourier transform on the obstruction group that similar to the component group $\pi_0(\hat{I}_\gamma^{\Gamma})$. Those correction terms turn out to be κ-orbital integrals. Langlands conjectured that these κ-orbital integrals can also expressed in terms of stable orbital integrals of endoscopic groups. We shall formulate his conjecture with more details later.

Admitting these conjecture on local orbital integrals, Langlands and Kottwitz succeeded to stabilize the elliptic part of the trace formula. In particular, they showed how the different κ-terms for different γ fit in the stable trace formula for endoscopic groups. One of the difficulty is to keep track of the variation of the component group $\pi_0(\hat{I}_\gamma^{\Gamma})$ with γ. The whole trace formula was eventually stabilized by Arthur under the assumption of the weighted fundamental lemma.

In the course of the construction of the stable trace formula, special cases of the functoriality principle between a reductive groups and its endoscopic groups are also established.

2.4. Endoscopic groups. Assume for simplicity that G is a quasi-split group over F that splits over a finite Galois extension K/F. The finite group $\mathrm{Gal}(K/F)$ acts on the root datum of G. Let $\hat{G}$ denote the connected complex reductive group whose root system is related to the root system of G by exchanging roots and coroots. Following [**26**], we set ${}^L G = \hat{G} \rtimes \mathrm{Gal}(K/F)$ where the action of $\mathrm{Gal}(K/F)$ on $\hat{G}$ derives from its action on the root datum. For instant, $G = \mathrm{Sp}(2n)$ and $\hat{G} = \mathrm{SO}(2n+1)$ are dual groups and $\mathrm{SO}(2n)$ is selfdual.

By the Tate-Nakayama duality, a character κ of $\mathrm{H}^1(F, I_\gamma)$ corresponds to a semisimple element $\hat{G}$ well defined up to conjugacy. Let $\hat{H}$ be the neutral component of the centralizer of κ in ${}^LG$. For a given torus I_γ, we can define an action of the Galois group of F on $\hat{H}$ through the component group of the centralizer of κ in ${}^LG$. By duality, we obtain a quasi-split reductive group over F.

More agreeable is the case where the group G is split and has connected centre. In this case, the derived group of $\hat{G}$ is simply connected. This implies that the centralizer $\hat{G}_\kappa$ of the semisimple element κ is connected and therefore the endoscopic group H is split.

2.5. Transfer of stable conjugacy classes. The endoscopic group H is not a subgroup of G in general. Nevertheless, it is possible to transfer stable conjugacy classes from H to G. If G is split and has connected centre, in the dual side $\hat{H} = \hat{G}_\kappa \subset \hat{G}$ induces an inclusion of Weyl groups $W_H \subset W$. It follows that there exists a canonical map $T/W_H \to T/W$ that realizes the transfer of stable conjugacy classes from H to G. If $\gamma_H \in H(F)$ has characteristic polynomial a_H mapping to the characteristic polynomial a of $\gamma \in G(F)$, we will loosely say that γ and γ_H have the same characteristic polynomial.

Similar construction exists for Lie algebras as well. One can transfer stable conjugacy classes in the Lie algebra of H to the Lie algebra of Lie. Moreover, transfer of stable conjugacy classes is not limited to endoscopic relationship. For instant, one can transfer stable conjugacy classes in Lie algebras of groups with isogenous root systems. In particular, this transfer is possible between the Lie algebras of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$.

2.6. Applications of the endoscopy theory. Many known cases of functoriality principle fit in the endoscopic framework. In particular, the transfer known as general Jacquet-Langlands from a group to its quasi-split inner form. The transfer from classical group to $\mathrm{GL}(n)$ expected to follow from Arthur's work on stable trace formula is a particular case of theory of twisted endoscopy.

Endoscopy is however far from exhausting the functoriality principle. It is concerned mainly with "small" homomorphism of L-groups. However, the stable trace formula seems to be an indispensable tool to any serious progress toward understanding functoriality.

Endoscopy is also instrumental in the study of Shimura varieties and the proof of many cases of the global Langlands correspondence [**31**], [**38**].

3. Conjectures on orbital integrals

3.1. Transfer conjecture. The first conjecture is concerned with the possibility of transfer of smooth functions :

CONJECTURE 1. *For every $f \in C_c^\infty(G(F))$ there exists $f^H \in C_c^\infty(H(F))$ such that*

$$\mathbf{SO}_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma)\mathbf{O}_\gamma^\kappa(f) \tag{5}$$

for all strongly regular semisimple elements γ_H and γ having the same characteristic polynomial, $\Delta(\gamma_H, \gamma)$ being a factor which is independent of f.

Under the assumption γ_H and γ strongly regular semisimple with the same characteristic polynomial, their centralizers in H and G respectively are canonically isomorphic. We can therefore transfer Haar measures between those locally compact groups.

The "transfer" factor $\Delta(\gamma_H, \gamma)$, defined by Langlands and Shelstad in [**27**], is a power of the cardinal of the residue field and a root unity which is a sign in most cases. This sign takes into account the fact that $\mathbf{O}_\gamma^\kappa$ depends on the choice of γ in its stable conjugacy class. In the case of Lie algebra, if we pick $\gamma = \iota(a)$ where ι is the Kostant section to the characteristic polynomial map, this sign equals one, according to Kottwitz in [**22**]. According to Kottwitz again, if the derived group of G is simply connected, Steinberg's section would play the same role for Lie group as Kostant's section for Lie algebra.

3.2. Fundamental lemma. Assume that we are in unramified situation i.e. both G and H have reductive models over $\mathcal{O}_F$. Let $1_{G(\mathcal{O}_F)}$ be the characteristic function of $G(\mathcal{O}_F)$ and $1_{H(\mathcal{O}_F)}$ the characteristic function of $H(\mathcal{O}_F)$.

CONJECTURE 2. *The equality (5) holds for $f = 1_{G(\mathcal{O}_F)}$ and $f^H = 1_{H(\mathcal{O}_F)}$.*

There is a more general version of the fundamental lemma. Let $\mathcal{H}_G$ be the algebra of $G(\mathcal{O}_F)$-biinvariant functions with compact support of $G(F)$ and $\mathcal{H}_H$ the similar algebra for G. Using Satake isomorphism we have a canonical homomorphism $b : \mathcal{H}_G \to \mathcal{H}_H$. Here is the more general version of the fundamental lemma.

CONJECTURE 3. *The equality (5) holds for any $f \in \mathcal{H}_G$ and for $f^H = b(f)$.*

3.3. Lie algebras. There are similar conjectures for Lie algebras. The transfer conjecture can be stated in the same way with $f \in C_c^\infty(\mathfrak{g}(F))$ and $f^H \in C_c^\infty(\mathfrak{h}(F))$. Idem for the fundamental lemma with $f = 1_{\mathfrak{g}(\mathcal{O}_F)}$ and $f^H = 1_{\mathfrak{h}(\mathcal{O}_F)}$.

Waldspurger stated a conjecture called the non standard fundamental lemma. Let G_1 and G_2 be two semisimple groups with isogenous root systems i.e. there exists an isogeny between their maximal tori which maps a root of G_1 on a scalar multiple of a root of G_2 and conversely. In this case, there is an isomorphism $\mathfrak{t}_1/W_1 \simeq \mathfrak{t}_2/W_2$. We can therefore transfer regular semisimple stable conjugacy classes from $\mathfrak{g}_1(F)$ to $\mathfrak{g}_2(F)$ and back.

CONJECTURE 4. *Let $\gamma_1 \in \mathfrak{g}_1(F)$ and $\gamma_2 \in \mathfrak{g}_2(F)$ be regular semisimple elements having the same characteristic polynomial. Then we have*

$$\mathbf{SO}_{\gamma_1}(1_{\mathfrak{g}_1(\mathcal{O}_F)}) = \mathbf{SO}_{\gamma_2}(1_{\mathfrak{g}_2(\mathcal{O}_F)}). \tag{6}$$

The absence of transfer conjecture makes this conjecture particularly agreeable.

3.4. History of the proof. All the above conjectures are now theorems. Let me sketch the contribution of different peoples coming into its proof.

The theory of endoscopy for real groups is almost entirely due to Shelstad.

First case of twisted fundamental lemma was proved by Saito, Shintani and Langlands in the case of base change for GL(2). Kottwitz had a general proof for the fundamental lemma for unit element in the case of base change.

Particular cases of the fundamental lemma were proved by different peoples : Labesse-Langlands for SL(2) [**25**], Kottwitz for SL(3) [**18**], Kazhdan and Waldspurger for SL(n) [**16**], [**39**], Rogawski for U(3) [**4**], Laumon-Ngô for U(n) [**30**], Hales, Schroder and Weissauer for Sp(4). Whitehouse also proved the weighted fundamental lemma for Sp(4).

In a landmark paper, Waldspurger proved that the fundamental lemma implies the transfer conjectures. Due to his and Hales' works, the case of Lie group follows from the case of Lie algera. Waldspurger also proved that the twisted fundamental lemma follows from the combination of the fundamental lemma with his non standard variant [**42**]. In [**13**], Hales proved that if we know the fundamental lemma for the unit for almost all places, we know it for the entire Hecke algebra for all places. In particular, if we know the fundamental lemma for the unit element at all but finitely many places, we also know it at the remaining places.

Following Waldspurger and independently Cluckers, Hales and Loeser, it is enough to prove the fundamental lemma for a local field in characteristic p, see [**41**] and [**6**].

For local fields of Laurent series, the approach using algebraic geometry was eventually successful. The local method was first introduced by Goresky, Kottwitz and MacPherson [**11**] based on the affine Springer fibers constructed by Kazhdan and Lusztig [**17**]. The Hitchin fibration was introduced in this context in [**33**]. Laumon and I used this approach, combined with previous work of Laumon [**29**] in order to prove the fundamental lemma for unitary group in [**30**]. The general case was proved in [**35**] with essentially the same strategy as in [**30**] with one major exception. The equivariant cohomology is no longer used the determination of the support of simple perverse sheaves occurring in the cohomology of Hitchin fibration.

4. Geometric method : local picture

4.1. Affine Springer fibers. Let $k = \mathbb{F}_q$ be a finite field with q elements. Let G be a reductive group over k and $\mathfrak{g}$ its Lie algebra. Let denote

$F = k((\pi))$ and $\mathcal{O}_F = k[[\pi]]$. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. According to Kazhdan and Lusztig [**17**], there exists a k-scheme $\mathcal{M}_\gamma$ whose set of k points is

$$\mathcal{M}_\gamma(k) = \{g \in G(F)/G(\mathcal{O}_F) \mid \mathrm{ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_F)\}.$$

They proved that the affine Springer fiber $\mathcal{M}_\gamma$ is finite dimensional and locally of finite type.

There exists a finite dimensional k-group scheme $\mathcal{P}_\gamma$ acting on $\mathcal{M}_\gamma$. We know that $\mathcal{M}_\gamma$ admits a dense open subset $\mathcal{M}_\gamma^{\mathrm{reg}}$ which is a principal homogenous space of $\mathcal{P}_\gamma$. The group connected components $\pi_0(\mathcal{P}_\gamma)$ of $\mathcal{P}_\gamma$ is possibly infinite and $\mathcal{M}_\gamma$ not of finite type. The group $\mathcal{P}_\gamma$ is a quotient of $I_\gamma(F)$ viewed as infinite dimensional group over k. The action of $\mathcal{P}_\gamma$ on $\mathcal{M}_\gamma$ is induced from $I_\gamma(F)$.

Let us consider a simple but important example. Let $G = \mathrm{SL}_2$ and let γ be the diagonal matrix

$$\gamma = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}.$$

In this case $\mathcal{M}_\gamma$ is an infinite chain of projective lines with the point ∞ in each copy being identified with the point 0 of the next one. The group $\mathcal{P}_\gamma$ is $\mathbb{G}_m \times \mathbb{Z}$ with $\mathbb{G}_m$ acts on each copy of $\mathbb{P}^1$ by rescaling and the generator of $\mathbb{Z}$ acts by translation from each copy to the next one. The dense open orbit is obtained by removing from $\mathcal{M}_\gamma$ its double points. The group $\mathcal{P}_\gamma$ over k is closely related to the centralizer of γ is over F which is just the multiplicative group $\mathbb{G}_m$ in this case. The surjective homomorphism

$$I_\gamma(F) = F^\times \to k^\times \times \mathbb{Z} = \mathcal{P}_\gamma(k)$$

attaches to a nonzero Laurent series the first no zero coefficient and the valuation.

In general there is no such an explicit description of the affine Springer fiber. The group $\mathcal{P}_\gamma$ is nevertheless rather explicit. In fact, it can be quite helpful to keep in mind that $\mathcal{M}_\gamma$ is a kind of equivariant compactification of the group $\mathcal{P}_\gamma$.

4.2. Counting points over finite fields. The stabilization of the trace formula suggests that we count the number of points of the quotient $[\mathcal{M}_\gamma/\mathcal{P}_\gamma]$ as an algebraic stack. $[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k)$ is not a set but a groupoid. The cardinal of a groupoid $\mathcal{C}$ is the number

$$\sharp\mathcal{C} = \sum_x \frac{1}{\sharp\mathrm{Aut}(x)}$$

for x in a set of representative of its isomorphism classes and $\sharp\mathrm{Aut}(x)$ being the order of the group of automorphisms of x. In our case, it can be proved that

$$\sharp[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k) = \mathbf{SO}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}, dg/dt) \tag{7}$$

for an appropriate choice of Haar measure on the centralizer. Roughly speaking, the Haar measure is related to the kernel of the homomorphism $I_\gamma(F) \to \mathcal{P}_\gamma(k)$.

The group $\pi_0(\mathcal{P}_\gamma)$ of geometric connected components of $\mathcal{P}_\gamma$ is an abelian group of finite type equipped with an action of Frobenius σ_q. For every character of finite order $\kappa : \pi_0(\mathcal{P}_\kappa) \to \mathbb{C}^\times$ fixed by σ_ℓ, we consider the finite sum

$$\sharp[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k)_\kappa = \sum_x \frac{\kappa(\mathrm{cl}(x))}{\sharp\mathrm{Aut}(x)}$$

where $\mathrm{cl}(x) \in \mathrm{H}^1(k, \mathcal{P}_\gamma)$ is the class of the $\mathcal{P}_\gamma$-torsor $\pi^{-1}(x)$ where $\pi : \mathcal{M}_\gamma \to [\mathcal{M}_\gamma/\mathcal{P}_\gamma]$ is the quotient map. By a counting argument similar to the stable case, we have

$$\sharp[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k)_\kappa = \mathbf{O}_\gamma^\kappa(1_{\mathfrak{g}(\mathcal{O}_F)}, dg/dt)$$

This provides a cohomological interpretation for κ-orbital integrals. Let fix an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$ so that κ can be seen as taking values in $\bar{\mathbb{Q}}_\ell$. Then we have the formula

$$\mathbf{O}_\gamma^\kappa(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sharp\mathcal{P}_\gamma^0(k)^{-1}\mathrm{tr}(\sigma_q, \mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa).$$

For simplicity, assume that the component group $\pi_0(\mathcal{P}_\gamma)$ is finite. Then $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa$ is the biggest direct summand of $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)$ on which $\mathcal{P}_\gamma$ acts through the character κ. When $\pi_0(\mathcal{O}_\gamma)$ is infinite, the definition of $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa$ is more complicated.

By taking $\kappa = 1$, we obtained a cohomological interpretation of the stable orbital integral

$$\mathbf{SO}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sharp\mathcal{P}_\gamma^0(k)^{-1}\mathrm{tr}(\sigma_q, \mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_{st})$$

where the index st means the direct summand where $\mathcal{P}_\gamma$ acts trivially under the assumption $\pi_0(\mathcal{P}_\gamma)$ be finite.

This cohomological interpretation is essentially the same as the one given by Goresky, Kottwitz and MacPherson [**11**]. It allows us to shift focus from a combinatorial problem of counting lattices to a geometric problem of computing ℓ-adic cohomology. However, the calculation of ℓ-adic cohomology of the affine Springer fiber is no easier than the calculation of the orbital integrals themselves.

4.3. More about $\mathcal{P}_\gamma$. We don't know much about $\mathcal{M}_\gamma$. The only information which is available in general is that $\mathcal{M}_\gamma$ is in a loose sense an equivariant compactification of a group $\mathcal{P}_\gamma$ that we know better.

There are two simple but useful facts about the group $\mathcal{P}_\gamma$. A formula for its dimension was conjectured by Kazhdan and Lusztig and proved by Bezrukavnikov [**3**]. The component group $\pi_0(\mathcal{P}_\gamma)$ can also be described precisely. The centralizer I_γ is a torus over F. If G is split, the monodromy of I_γ determines a subgroup $\rho(\Gamma)$ of the Weyl group W well determined up to conjugation. Assume that the center of G is connected. Then $\pi_0(\mathcal{P}_\gamma)$

is the group of $\rho(\Gamma)$-coinvariants of the group of cocharacters $X_*(T)$ of the maximal torus of G. In general, the formula is slightly more complicated.

Let denote $a \in (\mathfrak{t}/W)(F)$ the image of $\gamma \in \mathfrak{g}(F)$. If the affine Springer fiber $\mathcal{M}_\gamma$ is non empty, then a can be extended to a $\mathcal{O}$-point of $\mathfrak{t}/W$. By construction, the group $\mathcal{P}_\gamma$ depends only on $a \in (\mathfrak{t}/W)(\mathcal{O})$ and is denoted by $\mathcal{P}_a$. In general $\mathcal{M}_\gamma$ does not depend only on a. We restrict ourselves to the Kostant section and have an affine Springer fiber $\mathcal{M}_a$ that depends only on a. This choice is consistent with Kottwitz's construction of the transfer factor. This is also helpful for connecting with the global picture.

5. Geometric method : global picture

5.1. The case of SL(2)**.** The description of the Hitchin system in the case of $G = \mathrm{SL}(2)$ is simple and instructive.

Let X be a smooth projective curve over a field k. We assume that X is geometrically connected and its genus is at least 2. A Higgs bundle for SL(2) over X consists in a vector bundle V of rank two with trivialized determinant $\bigwedge^2 V = \mathcal{O}_X$ and equipped with a Higgs field $\phi : V \to V \otimes K$ satisfying the equation $\mathrm{tr}(\phi) = 0$. Here K denotes the canonical bundle and $\mathrm{tr}(\phi) \in \mathrm{H}^0(X, K)$ is a 1-form. The moduli stack of Higgs bundle $\mathcal{M}$ is Artin algebraic and locally of finite type. Let Bun_G denote the moduli stack of principal G-bundles on X. Over the stable locus of Bun_G, $\mathcal{M}$ can be identified with the cotangent of Bun_G by Serre's duality. As a cotangent, $\mathcal{M}$ is naturally equipped with a symplectic structure. Hitchin constructed explicitly a family of d Poisson commuting algebraically independent functions on $\mathcal{M}$ where d is half the dimension of $\mathcal{M}$. In other words, $\mathcal{M}$ is an algebraic completely integrable system.

In SL(2) case, we can associate with a Higgs bundle (V, ϕ) the quadratic differential $a = \det(\phi) \in \mathrm{H}^0(X, K^{\otimes 2})$. By Riemann-Roch, the dimension of $\mathrm{H}^0(X, K^{\otimes 2})$ is also equal to half the dimension of $\mathcal{M}$. According to Hitchin, the mapping $(V, \phi) \mapsto \det(\phi)$ defines a family of d Poisson commuting algebraically independent functions on $\mathcal{M}$.

Following Hitchin, the fibers of the map $f : \mathcal{M} \to \mathcal{A} = \mathrm{H}^0(X, K^{\otimes 2})$ can be described by the recipe of spectral curve. A section $a \in \mathrm{H}^0(X, K^{\otimes 2})$ determines a curve Y_a on the total space $|K|$ of K by pulling back the section $-a$ by the ramified 2-covering $|K| \to |K^{\otimes 2}|$. For any a, $p_a : Y_a \to X$ is a covering of degree 2 of X. If $a \neq 0$, the curve Y_a is reduced. For generic a, the curve Y_a is smooth but in general, it can be singular. It can be even reducible if $a = b^{\otimes 2}$ for certain $b \in \mathrm{H}^0(X, K)$.

By Cayley-Hamilton theorem, if $a \neq 0$, the fiber $\mathcal{M}_a$ can be identified with the moduli space of torsion-free sheaf $\mathcal{F}$ on Y_a such that $\det(p_{a,*}\mathcal{F}) = \mathcal{O}_X$. If Y_a is smooth, $\mathcal{M}_a$ is identified with a translation of a subabelian variety $\mathcal{P}_a$ of the Jacobian of Y_a. This subabelian variety consists in line bundle $\mathcal{L}$ on Y_a such that $\mathrm{Nm}_{Y_a/X}\mathcal{L} = \mathcal{O}_X$.

Hitchin used similar construction of spectral curve to prove that the generic fiber of f is an abelian variety.

5.2. Picard stack of symmetry. Let us observe that the above definition of $\mathcal{P}_a$ is valid for all a. For any a, the group $\mathcal{P}_a$ acts on $\mathcal{M}_a$ because of the formula

$$\det(p_{a,*}(\mathcal{F} \otimes \mathcal{L})) = \det(p_{a,*}\mathcal{F}) \otimes \mathrm{Nm}_{Y_a/X}\mathcal{L}.$$

In [**33**], we construct $\mathcal{P}_a$ and its action on $\mathcal{M}_a$ for any reductive group. Instead of the canonical bundle, K can be any line bundle of large degree. We defined a canonical Picard stack $g : \mathcal{P} \to \mathcal{A}$ acting on the Hitchin fibration $f : \mathcal{M} \to \mathcal{A}$ relatively to the base $\mathcal{A}$. In general, $\mathcal{P}_a$ does not act simply transitively on $\mathcal{M}_a$. It does however on a dense open subset of $\mathcal{M}_a$. This is why we can think about the Hitchin fibration $\mathcal{M} \to \mathcal{A}$ as an equivariant compactification of the Picard stack $\mathcal{P} \to \mathcal{A}$.

Consider the quotient $[\mathcal{M}_a/\mathcal{P}_a]$ of the Hitchin fiber $\mathcal{M}_a$ by its natural group of symmetries. In [**33**], we observed a product formula

$$[\mathcal{M}_a/\mathcal{P}_a] \simeq \prod_v [\mathcal{M}_{v,a}/\mathcal{P}_{v,a}] \tag{8}$$

where for all $v \in X$, $\mathcal{M}_{v,a}$ is the affine Springer fiber at the place v attached to a and $\mathcal{P}_a$ is its symmetry group that appeared in 4.3. For all but finitely many v, $\mathcal{P}_{a,v}$ acts simply transitively on $\mathcal{M}_{v,a}$. The sign $\simeq$ means homeomorphism. It does not seem to be an isomorphism in general. However, for the purpose of ℓ-adic cohomology, it does not make any difference with an isomorphism.

Even though the Hitchin fibers $\mathcal{M}_a$ are organized in a family, individually, their structure depends significantly on a. For generic a, $\mathcal{P}_a$ acts simply transitively on $\mathcal{M}_a$ so that all quotients appearing in the product formula are trivial. In this case, all affine Springer fibers appearing on the right hand side are zero dimensional. For bad parameter a, affine Springer fibers have positive dimension. The Hitchin fibration allows us to have a control on the bad fibers from the good fibers. This is the basic idea of our global geometric method.

5.3. Counting points with values in a finite field. Let k be a finite field of characteristic p with q elements. In counting the numbers of points with values in k on a Hitchin fiber, we noticed a remarkable connection with the trace formula.

In choosing a global section of K, we identify K with the line bundle $\mathcal{O}_X(D)$ attached to an effective divisor D. It also follows an injective map $a \mapsto a_F$ from $\mathcal{A}(k)$ into $(\mathfrak{t}/W)(F)$. The image is a finite subset of $(\mathfrak{t}/W)(F)$ that can be described easily with help of the exponents of $\mathfrak{g}$ and the divisor D. Thus points on the Hitchin base correspond essential to rational stable conjugacy classes, see [**33**] and [**34**].

For simplicity, assume that the kernel $\ker^1(F,G)$ of the map

$$\mathrm{H}^1(F,G) \to \prod_v \mathrm{H}^1(F_v,G)$$

is trivial. Following Weil's adelic desription of vector bundle on a curve, we can express the number of points on $\mathcal{M}_a = f^{-1}(a)$ as a sum of global orbital integrals

$$\sharp\mathcal{M}_a(k) = \sum_\gamma \int_{I_\gamma(F)\backslash G(\mathbb{A}_F)} 1_D(\mathrm{ad}(g)^{-1}\gamma)dg \tag{9}$$

where γ runs over the set of conjugacy classes of $\mathfrak{g}(F)$ with a as the characteristic polynomial, F being the field of rational functions on X, $\mathbb{A}_F$ the ring of adèles of F, 1_D a very simple function on $\mathfrak{g}(\mathbb{A}_F)$ associated with a choice of divisor within the linear equivalence class D. In summing over $a \in \mathcal{A}(k)$, we get an expression very similar to the geometric side of the trace formula for Lie algebra.

Without the assumption on the triviality of $\ker^1(F,G)$, we obtain a sum of trace formula for inner form of G induced by elements of $\ker^1(F,G)$. This further complication turns out to be a simplification when we stabilize the formula, see [**34**]. In particular, instead of the subgroup A_γ of $\mathrm{H}^1(F,I_\gamma)$ as in 2.1, we deal with the group $\mathrm{H}^1(F,I_\gamma)$ it self.

At this point, it is a natural to seek a geometric interpretation of the stabilization process as explained in 2.3. Fix a rational point $a \in \mathcal{A}(k)$ and consider the quotient morphism

$$\mathcal{M}_a \to [\mathcal{M}_a/\mathcal{P}_a]$$

If $\mathcal{P}_a$ is connected then for every point $x \in [\mathcal{M}_a/\mathcal{P}_a](k)$, there is exactly $\sharp\mathcal{P}_a(k)$ points with values in k in the fiber over x. It follows that

$$\sharp\mathcal{M}_a(k) = \sharp\mathcal{P}_a(k)\sharp[\mathcal{M}_a/\mathcal{P}_a](k)$$

where $\sharp[\mathcal{M}_a/\mathcal{P}_a](k)$ can be expressed by stable orbital integrals by the product formula 8 and by 7. In general, the component group $\pi_0(\mathcal{P}_a)$ prevents the number $\sharp\mathcal{M}_a(k)$ from being expressed as stable orbital integrals.

5.4. Variation of the component groups $\pi_0(\mathcal{P}_a)$. The dependence of the component group $\pi_0(\mathcal{P}_a)$ on a makes the combinatorics of the stabilization of the trace formula rather intricate. Geometrically, this variation can be packaged in a sheaf of abelian group $\pi_0(\mathcal{P}/\mathcal{A})$ over $\mathcal{A}$ whose fibers are $\pi_0(\mathcal{P}_a)$.

If the center G is connected, it is not difficult to express $\pi_0(\mathcal{P}_a)$ from a in using a result of Kottwitz [**21**]. A point $a \in \mathcal{A}(\bar{k})$ defines a stable conjugacy class $a_F \in (\mathfrak{t}/W)(F \otimes_k \bar{k})$. We assume a_F is regular semisimple so that there exists $g \in \mathfrak{g}(F \otimes_k \bar{k})$ whose characteristic polynomial is a. The centralizer I_x is a torus which does not depend on the choice of x but only on a. Its monodromy can expressed as a homomorphism $\rho_a : \mathrm{Gal}(F \otimes_k \bar{k}) \to \mathrm{Aut}(\mathbb{X}_*)$ where $\mathbb{X}_*$ is the group of cocharacters of a maximal torus of G.

The component group $\pi_0(\mathcal{P}_a)$ is isomorphic to the group of coinvariants of $\mathbb{X}_*$ under the action of $\rho_a(\mathrm{Gal}(F \otimes_k \bar{k}))$.

This isomorphism can be made canonical after choosing a rigidification. Let's fix a point $\infty \in X$ and choose a section of the line bundle K non vanishing on a neighborhood of ∞. Consider the covering $\tilde{\mathcal{A}}$ of $\mathcal{A}$ consisting of a pair $\tilde{a} = (a, \tilde{\infty})$ tale where $a \in \mathcal{A}$ regular semisimple at ∞ i.e. $a(\infty) \in (\mathfrak{t}/W)^{rs}$ and $\tilde{\infty} \in \mathfrak{t}^{rs}$ mapping to $a(\infty)$. The map $\tilde{\mathcal{A}} \to \mathcal{A}$ is etale, more precisely, finite etale over a Zariski open subset of $\mathcal{A}$. Over $\tilde{\mathcal{A}}$, there exists a surjective homomorphism from the constant sheaf $\mathbb{X}_*$ to $\pi_0(\mathcal{P})$ whose fiber admits now a canonical description as coinvariants of $\mathbb{X}_*$ under certain subgroup of the Weyl group depending on a.

When the center of G isn't connected, the answer is somehow subtler. In the SL_2 case, there are three possibilities. We say that a is generic, or stable if the spectral curve Y_a has at least one unibranched ramification point over X. In particular, if Y_a is smooth, all ramification points are unibranched. In this case $\pi_0(\mathcal{P}_a) = 0$. We say that a is hyperbolic if the spectral curve Y_a is reducible. In this case on can express $a = b^{\otimes 2}$ for some $b \in \mathrm{H}^0(X, K)$. If a is hyperbolic, we have $\pi_0(\mathcal{P}_a) = \mathbb{Z}$. The most interesting case is the case where a is neither stable nor hyperbolic i.e. the spectral curve Y_a is irreducible but all ramification points have two branches. In this case $\pi_0(\mathcal{P}_a) = \mathbb{Z}/2\mathbb{Z}$ and we say that a is endoscopic. We observe that a is endoscopic if and only if the normalization of Y_a is an unramified double covering of X. Such a covering corresponds to a nontrivial line bundle $\mathcal{E}$ on X such that $\mathcal{E}^{\otimes 2} = \mathcal{O}_X$. Moreover we can express $a = b^{\otimes 2}$ where $b \in \mathrm{H}^0(X, K \otimes \mathcal{E})$.

The upshot of this calculation can be summarized as follows. The free rank of $\pi_0(\mathcal{P}_a)$ jumps exactly when a is hyperbolic i.e. when a comes from a Levi subgroup of G. The torsion group of $\pi_0(\mathcal{P}_a)$ jumps exactly when a is endoscopic i.e. when a comes from an endoscopic group of G. These statement are in fact valid in general.

5.5. Stable part. We can construct an open subset $\mathcal{A}^{ani}$ of $\mathcal{A}$ over which $\mathcal{M} \to \mathcal{A}$ is proper and $\mathcal{P} \to \mathcal{A}$ is of finite type. In particular for every $a \in \mathcal{A}^{ani}(\bar{k})$, the component group $\pi_0(\mathcal{P}_a)$ is a finite group. In fact the converse assertion is also true : $\mathcal{A}^{ani}$ is precisely the open subset of $\mathcal{A}$ where the sheaf $\pi_0(\mathcal{P}/\mathcal{A})$ is an finite.

By construction, $\mathcal{P}$ acts on direct image $f_*\mathbb{Q}_\ell$ as an oject of the derived category of ℓ-adic sheaves on $\mathcal{A}$. The homotopy lemma implies that the induced action on the perverse sheaves of cohomology ${}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell)$ factors through the sheaf of components $\pi_0(\mathcal{P}/\mathcal{A})$ which is finite over $\mathcal{A}^{ani}$. Over this open subset, Deligne's theorem assures the purity of the above perverse sheaves. The finite action of $\pi_0(\mathcal{P}/\mathcal{A}^{ani})$ decomposes ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)$ into a direct sum.

This decomposition is at least as complicated as the sheaf $\pi_0(\mathcal{P}/\mathcal{A})$. In fact, this reflects exactly the combinatoric complexity of the stabilization of the trace formula as we have seen in 2.3.

We define the stable part ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)_{st}$ as the largest direct factor acted on trivially by $\pi_0(\mathcal{P}/\mathcal{A}^{ani})$. For every $a \in \mathcal{A}^{ani}(k)$, it can be showed by using the argument of 5.3 that the alternating sum of the traces of the Frobenius operator σ_a on ${}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell)_{st,a}$ can be expressed as stable orbital integrals.

Theorem 1. *Assume $k = \mathbb{C}$. For every integer n the perverse sheaf ${}^p\mathrm{H}^i(f_*^{ani}\mathbb{Q}_\ell)_{st}$ is completely determined by its restriction to any non empty open subset of $\mathcal{A}$. More preceisely, it can be recovered from its restriction by the functor of intermediate extension.*

When k is a finite field, we proved a weaker variant of this theorem which is strong enough the proof of the fundamental lemma. For simplicity, let's pretend here that the above theorem is also proved in positive characteristic.

Let G_1 and G_2 be two semisimple groups with isogenous root systems like $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$. The corresponding Hitchin fibration $f_\alpha : \mathcal{M}_\alpha \to \mathcal{A}$ for $\alpha \in \{1, 2\}$ map to the same base. For a generic a, $\mathcal{P}_{1,a}$, and $\mathcal{P}_{2,a}$ are essentially isogenous abelian varieties. It follows that ${}^p\mathrm{H}^i(f_{1,*}\mathbb{Q}_\ell)_{st}$ and ${}^p\mathrm{H}^i(f_{2,*}\mathbb{Q}_\ell)_{st}$ restricted to a non empty open subset of $\mathcal{A}$ are isomorphic local systems. With the intermediate extension, we obtain an isomorphism between perverse sheaves ${}^p\mathrm{H}^i(f_{1,*}\mathbb{Q}_\ell)_{st}$ and ${}^p\mathrm{H}^i(f_{2,*}\mathbb{Q}_\ell)_{st}$. We derive from this isomorphism the Waldspurger conjecture 6.

5.6. Support. By decomposition theorem, the pure perverse sheaves ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)$ are geometrically direct sum of simple perverse sheaves. Following Goresky and MacPherson, for a simple perverse sheaf K over base S, there exists an irreducible closed subscheme $i : Z \hookrightarrow S$ of S, an open subscheme $j : U \hookrightarrow Z$ of Z and a local system $\mathcal{K}$ on Z such that $K = i_* j_{!*}\mathcal{K}[\dim(Z)]$. In particular, the support $Z = \mathrm{supp}(K)$ is well defined.

The theorem 1 can be reformulated as follows. Let K be a simple perverse sheaf geometric direct factor of ${}^p\mathrm{H}^i(f_*^{ani}\mathbb{Q}_\ell)_{st}$. Then the support of K is the whole base $\mathcal{A}$.

In general, the determination of the support of constituents of a direct image is a rather difficult problem. This problem is solved to a large extent for Hitchin fibration and for more general abelian fibration. The complete answer involves endoscopic parts as well as the stable part.

5.7. Endoscopic part. Consider again the SL_2 case. In this case $\mathcal{A} - \{0\}$ is the union of closed strata $\mathcal{A}^{hyp}$ and $\mathcal{A}^{endo}$ that are the hyperbolic and endoscopic loci and the open stratum $\mathcal{A}^{st}$. The anisotropic open subset is $\mathcal{A}^{endo} \cup \mathcal{A}^{st}$. Over $\mathcal{A}^{ani}$, the sheaf $\pi_0(\mathcal{P})$ is the unique quotient of the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ that is trivial on the open subset $\mathcal{A}^{st}$ and non trivial on the closed subset $\mathcal{A}^{endo}$.

The group $\mathbb{Z}/2\mathbb{Z}$ acts on ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)$ and decomposes it into a direct sum

$$ {}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell) = {}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)_+ \oplus {}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)_-. $$

By its very construction, the restriction of the odd part ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)_-$ to the open subset $\mathcal{A}^{st}$ is trivial.

For every simple perverse sheaf K direct factor of ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)_-$, the support of K is contained in one of the irreducible components of the endoscopic locus $\mathcal{A}^{endo}$. In reality, we prove that the support of a simple perverse sheaf K direct factor of ${}^p\mathrm{H}^n(f_*^{ani}\mathbb{Q}_\ell)_-$ is one of the irreducible components of the endoscopic locus.

In general case, the monodromy of $\pi_0(\mathcal{P}/\mathcal{A})$ prevents the result from being formulated in an agreeable way. We encounter again with the complicated combinatoric in the stabilization of the trace formula. In geometry, it is possible to avoid this unpleasant combinatoric by passing to the etale covering $\tilde{\mathcal{A}}$ of $\mathcal{A}$ defined in 5.4. Over $\tilde{\mathcal{A}}$, we have a surjective homomorphism from the constant sheaf $\mathbb{X}_*$ onto the sheaf of component group $\pi_0(\mathcal{P}/Ac)$ which is finite over $\mathcal{A}^{ani}$. Over $\mathcal{A}^{ani}$, there is a decomposition in direct sum

$$ {}^p\mathrm{H}^n(\tilde{f}_*^{ani}\mathbb{Q}_\ell) = \bigoplus_\kappa {}^p\mathrm{H}^n(\tilde{f}_*^{ani}\mathbb{Q}_\ell)_\kappa $$

where $\tilde{f}^{ani}$ is the base change of f to $\tilde{\mathcal{A}}^{ani}$ and κ are characters of finite order $\mathbb{X}_* \to \mathbb{Q}_\ell{}^\times$.

For any κ as above, the set of geometric points $\tilde{a} \in \tilde{\mathcal{A}}^{ani}$ such that κ factors through $\pi_0(\mathcal{P}_a)$, forms a closed subscheme $\tilde{\mathcal{A}}_\kappa^{ani}$ of $\tilde{\mathcal{A}}^{ani}$. One can check that the connected components of $\tilde{\mathcal{A}}_\kappa^{ani}$ are exactly of the form $\tilde{\mathcal{A}}_H^{ani}$ for endoscopic groups H that are certain quasi-split groups with $\hat{H} = \hat{G}_\kappa^0$.

Theorem 2. *Let $k = \mathbb{C}$. Let K be a simple perverse sheaf geometric direct factor of $\tilde{\mathcal{A}}_\kappa^{ani}$. Then the support of K is one of the $\tilde{\mathcal{A}}_H$ as above.*

Again, in characteristic p, we prove a weaker form which is strong enough to imply the fundamental lemma. Let's pretend here that the above theorem is proved in positive characteristic.

The geometric version of the fundamental lemma states that the restriction of ${}^p\mathrm{H}^n(\tilde{f}_*^{ani}\mathbb{Q}_\ell)_\kappa$ to $\tilde{\mathcal{A}}_H$ is isomorphic with ${}^p\mathrm{H}^{n+2r}(\tilde{f}_{H,*}^{ani}\mathbb{Q}_\ell)_{st}(-r)$ for certain shifting integer r. Here f_H is the Hitchin fibration for H and $\tilde{f}_H^{ani}$ is its base change to $\tilde{\mathcal{A}}_H^{ani}$. The support theorems 1 and 2 allow us to reduce the problem to an arbitrarily small open subset of $\tilde{\mathcal{A}}_H^{ani}$. On a small open subset of $\tilde{\mathcal{A}}_H^{ani}$, this isomorphism can be constructed by direct calculation, mainly based on the example of the infinite chain of projective lines as in the case of SL(2).

We refer to [**37**] for an account of the proof of the support theorem.

6. Weighted fundamental lemma

In order to stabilize the whole trace formula, Arthur needs more complicated local identities known as weighted fundamental lemma. These identities, conjectured by Arthur, are now theorems due to efforts of Chaudouard, Laumon and Waldspurger. As in the case of the fundamental lemma, Waldspurger proved that the weighted fundamental lemma for a p-adic field is equivalent to the same lemma for the Laurent formal series field $\mathbb{F}_p((\pi))$ as long as the residual characteristic is large with respect to the group G. Chaudouard, Laumon also used the Hitchin fibration and a support theorem to prove the weighted fundamental lemma in positive characteristic case.

The weighted fundamental lemma as stated by Arthur is rather intricate a combinatorial identity. It is in fact easier to explain the weighted fundamental lemma from the point of view of the Hitchin fibration than from the point of view of the trace formula.

We already observed that over the open subset $\mathcal{A}^{ani}$ of $\mathcal{A}$, the Hitchin fibration $f^{ani} : \mathcal{M}^{ani} \to \mathcal{A}^{ani}$ is a proper map. Chaudouard and Laumon made the important observation that an appropriate stability condition make it possible to extend f^{ani} to a proper map $f^{\chi-st} : \mathcal{M}^{\chi-st} \to \mathcal{A}^{\heartsuit}$ where $\mathcal{A}^{\heartsuit}$ is the open subset of $\mathcal{A}$ consisting in $a \in \mathcal{A}$ with regular semisimple generic fiber $a_F \in (\mathfrak{t}/W)(F \otimes_k \bar{k})$.

The stability condition depends on an arbitrary choice of $\chi \in \mathbb{X}_* \otimes \mathbb{R}$. For general χ, the condition χ-stability and χ-semistability become equivalent. For those χ, the morphism $f^{\chi-st} : \mathcal{M}^{\chi-st} \to \mathcal{A}^{\heartsuit}$ is proper. In counting number of points on the fibers of $f^{\chi-st}$, they obtained formula involving weighted orbital integrals. Remarkably, this formula shows that the number of points does not depends on the choice of χ. Chaudouard and Laumon were also able to extend the support theorems 1 and 2 and from this deduce the weighted fundamental lemma [**5**].

7. Perspective

The method used to prove the fundamental lemma should be useful to local identities issued from the comparison of trace formula and relative trace formulas. In fact the first instance of fundamental lemma proved by this geometric method is a relative fundamental lemma conjectured by Jacquet and Ye [**32**]. Recently, Z. Yun proved a fundamental lemma conjectured by Jacquet, Rallis [**43**]. We can expect that other fundamental lemmas can be proved following the same general pattern too. Technically, it can still be challenging. In fact, the support theorem was proved by three completely different method in each of the three cases Jacquet-Ye, Langlands-Shelstad or Jacquet-Rallis. In the unitary case, a weak version of the support theorem was proved by yet another method by Laumon and myself.

The general method is based so far on a geometric interpretation of the orbital side of the trace formula. It is legitimate to ask if it is possible to insert geometry to the spectral side as well. At least for a Riemann surface, the answer seems to be yes. In a joint work in progress with E. Frenkel and R. Langlands, we noticed a closed relationship between the trace formula and Beilinson-Drinfeld's conjecture in geometric Langlands program. We should mention the related work [**10**] of Frenkel and Witten on a manifestation of endoscopy in Kapustin-Witten's proposal for geometric Langlands conjecture.

The endoscopy theory has been essentially completed. We have at our disposal the stable trace formula. It seems now the great times to read "Beyond endoscopy" written by Langlands some years ago [**28**]. Though the difficulty is formidable, his proposal possibly leads us to the understanding of the functoriality of automorphic representations.

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School of mathematics, Institute for Advanced Study, Princeton NJ 08540 USA, and Département de mathématiques, Université Paris-Sud, 91405 Orsay France