

Properly Embedded Minimal Planar Domains with Infinite Topology are Riemann Minimal Examples

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ABSTRACT. These notes outline recent developments in classical minimal surface theory that are essential in classifying the properly embedded minimal planar domains $M \subset \mathbb{R}^3$ with infinite topology (equivalently, with an infinite number of ends). This final classification result by Meeks, Pérez, and Ros [64] states that such an M must be congruent to a homothetic scaling of one of the classical examples found by Riemann [87] in 1860. These examples $\mathcal{R}_s, 0 < s < \infty$, are defined in terms of the Weierstrass \mathcal{P} -functions \mathcal{P}_t on the rectangular elliptic curve $\frac{\mathbb{C}}{\langle 1, t\sqrt{-1} \rangle}$, are singly-periodic and intersect each horizontal plane in \mathbb{R}^3 in a circle or a line parallel to the x -axis. Earlier work by Collin [22], López and Ros [49] and Meeks and Rosenberg [71] demonstrate that the plane, the catenoid and the helicoid are the only properly embedded minimal surfaces of genus zero with finite topology (equivalently, with a finite number of ends). Since the surfaces \mathcal{R}_s converge to a catenoid as $s \rightarrow 0$ and to a helicoid as $s \rightarrow \infty$, then the moduli space \mathcal{M} of all properly embedded, non-planar, minimal planar domains in \mathbb{R}^3 is homeomorphic to the closed unit interval $[0, 1]$.

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CONTENTS

1.	Introduction	282
2.	Background	285
2.1.	Weierstrass representation and the definition of flux	286

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2.2.	Maximum principles	287
2.3.	Monotonicity formula	288
2.4.	Stability, Plateau problem and barrier constructions	288
2.5.	The examples that appear in Theorem 1.1	290
3.	The case with r ends, $2 \leq r < \infty$	292
4.	The one-ended case	295
5.	Infinitely many ends I: one limit end is not possible	299
6.	Infinitely many ends II: two limit ends	309
6.1.	Curvature estimates and quasiperiodicity	310
6.2.	The Shiffman function	314
7.	Infinitely many ends III: The KdV equation	327
7.1.	Relationship between the KdV equation and the Shiffman function	327
7.2.	Algebro-geometric potentials for the KdV equation	328
7.3.	Proof of Theorem 6.11 provided that u is algebro-geometric	330
7.4.	Why u is algebro-geometric if $g \in \mathcal{M}_{\text{imm}}$	332
8.	The asymptotics of the ends of finite genus surfaces	335
	References	342

1. Introduction

In the last decade spectacular progress has been made in various aspects of classical minimal surface theory. Some of the successes obtained are the solutions of open problems which have been pursued since the birth of this subject in the 19-th century, while others have opened vast new horizons for future research. Among the first such successes, we would like to highlight the achievement of a deep understanding of topological aspects of proper minimal embeddings in three-space including their complete topological classification [23, 31, 32, 33, 34, 35, 36]. Equally important in this progress has been a comprehensive analysis of the behavior of limits of sequences of embedded minimal surfaces without *a priori* area or curvature bounds [9, 15, 16, 17, 18], with outstanding applications such as the classification of all simply-connected, properly embedded minimal surfaces [71]. Also, many deep results have been obtained on the subtle relationship between completeness and properness for complete immersed minimal surfaces [2, 29, 50, 51, 52, 53, 79], and how embeddedness introduces a strong dichotomy in this relationship [21, 61]. While all of these results are extremely interesting, they will not be treated in these notes (at least, not in depth) but they do give an idea of the enormous activity within this field; instead, we will explain the recent solution to the following long standing problem in classical minimal surface theory:

Classify all possible properly embedded minimal surfaces of genus zero in \mathbb{R}^3 .

Research by various authors help to understand this problem, the more relevant work being by Colding and Minicozzi [18, 21], Collin [22], López and Ros [49], Meeks, Pérez and Ros [64] and Meeks and Rosenberg [70, 71]. In fact, this problem has been one of main goals of the two authors of these notes (in collaboration with A. Ros) for over the past 15 years, and the long path towards its solution has been marked by the discovery of powerful techniques which have proved useful in other applications. Putting together all of these efforts, we now state the final solution to the above problem, whose proof appears in [64].

THEOREM 1.1. *Up to scaling and rigid motion, any connected, properly embedded, minimal planar domain in \mathbb{R}^3 is a plane, a helicoid, a catenoid or one of the Riemann minimal examples.¹ In particular, for every such surface there exists a foliation of \mathbb{R}^3 by parallel planes, where each plane intersects the surface transversely in a circle or a straight line.*

In these notes we will try to pass on to the reader a glimpse of the beauty of the arguments and different theories that intervene in the proof of Theorem 1.1. Among these auxiliary theories, we highlight the theory of integrable systems, whose applications to minimal and constant mean curvature surface theory have gone far beyond the existence results in the late eighties (Abresch [1], Bobenko [5], Pinkall and Sterling [85], based on the sinh-Gordon equation) to recent uniqueness theorems like the one that gives the title of these notes (based on the KdV equation), and the even more recent tentative solution to the Lawson conjecture by Kilian and Schmidt [45].

Before proceeding, we make a few general comments about the organization of this article, which relate to the proof of Theorem 1.1. In Section 2 we briefly introduce the main definitions and background material, including short discussions of the classical examples that appear in the statement of the above theorem. Since a complete, immersed minimal surface M without boundary in \mathbb{R}^3 cannot be compact, M must have *ends*.² As we are interested in planar domains, the allowed topology for our surfaces is that of the two-sphere minus a compact totally disconnected set $\mathcal{E} \neq \emptyset$ corresponding to the space of ends of the surface. A crucial result proved by Collin [22] in 1997 (see Conjecture 3.3 below) implies that when the cardinality $\#(\mathcal{E})$ of \mathcal{E} satisfies $2 \leq \#(\mathcal{E}) < \infty$, then the total Gaussian curvature of M is finite. Complete embedded minimal surfaces with finite total curvature comprise the best understood minimal surfaces in \mathbb{R}^3 ; the main reason for this is the fact discovered by Osserman that the underlying complex structure for every such minimal surface M is that of a compact Riemann \overline{M} surface minus

¹See Section 2.5 for further discussion of these surfaces.

²An *end* of a non-compact connected topological manifold M is an equivalence class in the set $\mathcal{A} = \{\alpha: [0, \infty) \rightarrow M \mid \alpha \text{ is a proper arc}\}$, under the equivalence relation: $\alpha_1 \sim \alpha_2$ if for every compact set $C \subset M$, α_1, α_2 lie eventually in the same component of $M - C$. If $\alpha \in \mathcal{A}$ is a representative proper arc of an end of M and $\Omega \subset M$ is a proper subdomain with compact boundary such that $\alpha \subset \Omega$, then we say that Ω *represents* the end e .

a finite number of points, and the classical analytic Weierstrass representation data on M extends across the punctures to meromorphic data on \overline{M} . Using the maximum principle together with their result that every complete, embedded minimal surface with genus zero and finite total curvature can be minimally deformed, López and Ros characterized the plane and the catenoid as being the unique embedded examples of finite total curvature and genus zero, see Theorem 3.2. We also explain Collin's and López-Ros' theorems in Section 3.

Section 4 covers the one-ended case of Theorem 1.1, which was solved by Meeks and Rosenberg (Theorem 4.2). To understand the proof of this result we need the local results of Colding and Minicozzi which describe the structure of compact, embedded minimal disks as essentially being modeled by either a plane or a helicoid, and their one-sided curvature estimate together with other results of a global nature such as their limit lamination theorem for disks, see Theorem 4.1 below.

The remainder of the article, except for the last section, focuses on the case in Theorem 1.1 where the surface has infinitely many ends. Crucial in this discussion is the Ordering Theorem by Frohman and Meeks (Theorem 5.1) as well as two topological results, the first on the non-existence of middle limit ends due to Collin, Kusner, Meeks and Rosenberg (Theorem 5.3) and the second on the non-existence of properly embedded minimal planar domains with just one limit end by Meeks, Pérez and Ros (Theorem 5.11). It follows from these two non-existence results that a properly embedded minimal planar domain must have exactly two limit ends. These key ingredients represent the content of Section 5.

As a consequence of the results in Section 5, in Section 6 we obtain strong control on the conformal structure and height differential of a properly embedded, minimal planar domain M with infinitely many ends. In this section we also explain how Colding-Minicozzi theory can be applied to obtain a curvature bound for such an M , after it has been normalized by a homothety so that its vertical flux is 1, which only depends on the length of the horizontal component of its flux vector (Theorem 6.3). This bound leads to a quasi-periodicity property of M , which is a cornerstone to finishing the classification problem. Also in Section 6 we introduce the Shiffman Jacobi function S_M and explain how the existence of a related holomorphic deformation of M preserving its flux is sufficient to reduce the proof of Theorem 1.1 in the case with infinitely many ends to the singly-periodic case, which was solved earlier by Meeks, Pérez and Ros [66].

In Section 7 we explain how the existence of the desired holomorphic deformation of M follows from the integration of an evolution equation for its Gauss map. The Shiffman function S_M will be crucial at this point by enabling us to reduce this evolution equation to an equation of type Korteweg de Vries (KdV). A classical condition that implies global integrability of the KdV equation, i.e. existence of (globally defined) meromorphic solutions $u(z, t)$ of the Cauchy problem associated to the KdV equation, is

that the initial condition $u(z)$ is an *algebraic-geometric* meromorphic function, a concept related to the hierarchy of the KdV equation. In our setting, the final step of the classification of the properly embedded minimal planar domains consists of proving that the initial condition $u(z)$ for the Cauchy problem of the KdV equation naturally associated to any quasiperiodic, possibly immersed, minimal planar domain M with two limit ends is algebraic-geometric (Section 7.4), which in turn is a consequence of the fact that the space of bounded Jacobi functions on such a surface M is finite dimensional (Theorem 7.3).

An important consequence of the proof of the classification of properly embedded minimal planar domains is the characterization of the asymptotic behavior of the ends of any properly embedded minimal surface $M \subset \mathbb{R}^3$ with finite genus. If $e \in \mathcal{E}(M)$ is an end, then there exists properly embedded domain $E(e) \subset M$ with compact boundary which represents e and such that in a natural sense $E(e)$ converges to the end of a plane, a catenoid, a helicoid or to one of the limit ends of a Riemann minimal example. This asymptotic characterization is due to Schoen [91] and Collin [22] when M has a finite number of ends greater than one, to Meeks, Pérez and Ros in the case M has an infinite number of ends, and to Meeks and Rosenberg [71], Bernstein and Breiner [3], and Meeks and Pérez [58, 59] in the case of just one end. These asymptotic characterization results will be explained in Section 8 of these notes.

2. Background

An isometric immersion $X = (x_1, x_2, x_3): M \rightarrow \mathbb{R}^3$ of a Riemannian surface into Euclidean space is said to be *minimal* if x_i is a harmonic function on M for each i (in the sequel, we will identify the Riemannian surface M with its image under the isometric embedding). Since harmonicity is a local property, we can extend the notion of minimality to immersed surfaces $M \subset \mathbb{R}^3$. We will always assume all surfaces under consideration are orientable. If $M \subset \mathbb{R}^3$ is an immersed oriented surface, we will denote by H the *mean curvature* function of X (average normal curvature) and by $N: M \rightarrow \mathbb{S}^2(1) \subset \mathbb{R}^3$ its Gauss map. Since $\Delta X = 2HN$ (here Δ is the Laplace-Beltrami operator on M), we have that M is minimal if and only if $H = 0$ identically. Expressing locally M as the graph of a function $u = u(x, y)$ (after a rotation), the last equation can be written as the following quasilinear second order elliptic PDE:

$$(1) \quad \operatorname{div}_0 \left(\frac{\nabla_0 u}{\sqrt{1 + |\nabla_0 u|^2}} \right) = 0,$$

where the subscript \bullet_0 indicates that the corresponding object is computed with respect to the flat metric in the plane.

Let Ω be a subdomain with compact closure in a surface $M \subset \mathbb{R}^3$ and let $u \in C_0^\infty(\Omega)$ be a compactly supported smooth function. The first variation of the area functional $A(t) = \operatorname{Area}((X + tuN)(\Omega))$ for the normally perturbed

immersions $X + utN$ (with $|t|$ sufficiently small) gives

$$(2) \quad A'(0) = -2 \int_{\Omega} uH \, dA,$$

where dA stands for the area element of M . Therefore, M is minimal if and only if it is a critical point of the area functional for all compactly supported variations. The second variation of area implies that any point in a minimal surface has a neighborhood with least-area relative to its boundary,³ and thus $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighborhood with least-area relative to its boundary. If one exchanges the area functional A by the *Dirichlet energy* $E = \int_{\Omega} |\nabla X|^2 dA$, then the two functionals are related by $E \geq 2A$, with equality if and only if the immersion $X: M \rightarrow \mathbb{R}^3$ is conformal. This relation between area and energy together with the existence of isothermal coordinates on every Riemannian surface allow us to state that a conformal immersion $X: M \rightarrow \mathbb{R}^3$ is minimal if and only if it is a critical point of the Dirichlet energy for all compactly supported variations (or equivalently, every point $p \in M$ has a neighborhood with least energy relative to its boundary). Finally, the relation $A_p = -dN_p$ between the differential of the Gauss map and the *shape operator*, together with the Cauchy-Riemann equations give that M is *minimal* if and only if its stereographically projected Gauss map $g: M \rightarrow \mathbb{C} \cup \{\infty\}$ is a holomorphic function. All these equivalent definitions of minimality illustrate the wide variety of branches of mathematics that appear in its study: Differential Geometry, PDE, Calculus of Variations, Geometric Measure Theory, Complex Analysis, etc.

The Gaussian curvature function⁴ K of a surface $M \subset \mathbb{R}^3$ can be written as $K = \kappa_1 \kappa_2 = \det A$, where κ_i are the principal curvatures and A the shape operator. Thus $|K|$ is the absolute value of the Jacobian of the Gauss map N . If M is minimal, then $\kappa_1 = -\kappa_2$ and $K \leq 0$, hence the *total curvature* $C(M)$ of M is the negative of the spherical area of M through its Gauss map, counting multiplicities:

$$(3) \quad C(M) = \int_M K \, dA = -\text{Area}(N: M \rightarrow \mathbb{S}^2(1)) \in [-\infty, 0].$$

In the sequel, we will denote by $\mathbb{B}(p, R) = \{x \in \mathbb{R}^3 \mid |x - p| < R\}$ and $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$.

2.1. Weierstrass representation and the definition of flux. Let $M \subset \mathbb{R}^3$ be a possibly immersed minimal surface, with stereographically projected Gauss map $g: M \rightarrow \mathbb{C} \cup \{\infty\}$. Since the third coordinate function x_3 of M is harmonic, it admits a locally well-defined harmonic conjugate function x_3^* . The *height differential of M* is the holomorphic 1-form $dh = dx_3 + ix_3^*$ (note that dh is not necessarily exact on M). The minimal

³This property justifies the word “minimal” for these surfaces.

⁴If needed, we will use the notation K_M to highlight the surface M of which K is the Gaussian curvature function.

immersion $X: M \rightarrow \mathbb{R}^3$ can be written up to translation by the vector $X(p_0)$, $p_0 \in M$, solely in terms of the *Weierstrass data* (g, dh) as

$$(4) \quad X(p) = \Re \int_{p_0}^p \left(\frac{1}{2} \left(\frac{1}{g} - g \right), \frac{i}{2} \left(\frac{1}{g} + g \right), 1 \right) dh,$$

where \Re stands for real part. The positive-definiteness of the induced metric and the independence of (4) with respect to the integration path give rise to certain compatibility conditions on the meromorphic data (g, dh) for analytically defining a minimal surface (Osserman [80]); namely, if we start with a meromorphic function g and a holomorphic one-form dh on an abstract Riemann surface M , then the map $X: M \rightarrow \mathbb{R}^3$ given by (4) is a conformal minimal immersion with Weierstrass data (g, dh) provided that two conditions hold:

(5)

The zeros of dh coincide with the poles and zeros of g , with the same order.

(6)

$$\overline{\int_{\gamma} g dh} = \int_{\gamma} \frac{dh}{g}, \quad \Re \int_{\gamma} dh = 0 \quad \text{for any closed curve } \gamma \subset M \text{ (period problem).}$$

The *flux vector* of M along a closed curve $\gamma \subset M$ is defined as

$$(7) \quad F(\gamma) = \int_{\gamma} \text{Rot}_{90^\circ}(\gamma') = \Im \int_{\gamma} \left(\frac{1}{2} \left(\frac{1}{g} - g \right), \frac{i}{2} \left(\frac{1}{g} + g \right), 1 \right) dh \in \mathbb{R}^3,$$

where Rot_{90° denotes the counterclockwise rotation by angle $\pi/2$ in the tangent plane of M at any point, and \Im stands for imaginary part. Both the period condition (6) and the flux vector (7) only depend on the homology class of γ in M .

2.2. Maximum principles. Since minimal surfaces can be written locally as solutions of the PDE (1), they satisfy certain maximum principles.

THEOREM 2.1 (Interior and Boundary Maximum Principles [91]). *Let M_1, M_2 be connected minimal surfaces in \mathbb{R}^3 and p an interior point to both surfaces, such that $T_p M_1 = T_p M_2 = \{x_3 = 0\}$. If M_1, M_2 are locally expressed as the graphs of functions u_1, u_2 around p and $u_1 \leq u_2$ in a neighborhood of p , then $M_1 = M_2$ in a neighborhood of p . The same conclusion holds if p is a boundary point of both surfaces and additionally, $T_p \partial M_1 = T_p \partial M_2$.*

We also dispose of more sophisticated versions of the maximum principle, where a first contact point of two minimal surfaces does not occur at a finite point but at infinity, amongst which we state two. The first one (whose proof we sketch for later purposes) was proved by Hoffman and Meeks, and the second one is due to Meeks and Rosenberg.

THEOREM 2.2 (Half-space Theorem [41]). *A properly immersed, non-planar minimal surface without boundary cannot be contained in a half-space.*

SKETCH OF PROOF. Arguing by contradiction, suppose that a surface $M \subset \mathbb{R}^3$ as in the hypotheses is contained in $\{x_3 \geq 0\}$ (and so, $M \subset \{x_3 > 0\}$ by Theorem 2.1) but is not contained in $\{x_3 > c\}$ for any $c > 0$. Since M is proper, we can find a ball $\mathbb{B}(p, r)$ centered at a point $p \in \{x_3 = 0\}$ such that $M \cap \mathbb{B}(p, r) = \emptyset$. Consider a vertical half-catenoid C with negative logarithmic growth, completely contained in $\{x_3 \leq 0\} \cup \mathbb{B}(p, r)$, whose waist circle Γ is centered at $q = p + \varepsilon(0, 0, 1)$, $\varepsilon > 0$ small. Then $M \cap C = \emptyset$. Now deform $C = C(1)$ by a one-parameter family of non-compact annular pieces of vertical catenoids $\{C(r)\}_{r \in (0, 1]}$, all having the same boundary Γ as C , with negative logarithmic growths converging to zero as $r \rightarrow 0$ and whose Gaussian curvatures blow up at the limit point of the waist circles of $C(r)$, which is the point q . Then, the surfaces $C(r)$ converge on compact subsets of $\mathbb{R}^3 - \{q\}$ to the plane $\{x_3 = \varepsilon\}$, and so, M achieves a first contact point with one of the catenoids, say $C(r_0)$, in this family; the usual maximum principle for M and $C(r_0)$ gives a contradiction. \square

THEOREM 2.3 (Maximum Principle at Infinity [73]). *Let $M_1, M_2 \subset \mathbb{R}^3$ be disjoint, connected, properly immersed minimal surfaces with (possibly empty) boundary.*

i) *If $\partial M_1 \neq \emptyset$ or $\partial M_2 \neq \emptyset$, then after possibly re-indexing,*

$$\text{dist}(M_1, M_2) = \inf\{\text{dist}(p, q) \mid p \in \partial M_1, q \in M_2\}.$$

ii) *If $\partial M_1 = \partial M_2 = \emptyset$, then M_1 and M_2 are flat.*

2.3. Monotonicity formula. Monotonicity formulas, as well as maximum principles, play a crucial role in many areas of Geometric Analysis. For instance, we will see in Theorem 5.3 how the monotonicity formula can be used to discard middle limit ends for a properly embedded minimal surface. We will state this basic result without proof here; it is a consequence of the classical coarea formula applied to the distance function to a point $p \in \mathbb{R}^3$, see for instance Corollary 4.2 in [14] for a detailed proof.

THEOREM 2.4 (Monotonicity Formula [10, 46]). *Let $X: M \rightarrow \mathbb{R}^3$ be a connected, properly immersed minimal surface. Given $p \in \mathbb{R}^3$, let $A(R) = \text{Area}(X(M) \cap \mathbb{B}(p, R))$. Then, $A(R)R^{-2}$ is non-decreasing. In particular, $\lim_{R \rightarrow \infty} A(R)R^{-2} \geq \pi$ with equality if and only if M is a plane.*

2.4. Stability, Plateau problem and barrier constructions. Recall that every (orientable) minimal surface $M \subset \mathbb{R}^3$ is a critical point of the area functional for compactly supported normal variations. M is said to be *stable* if it is a local minimum for such a variational problem. The following well-known result indicates how restrictive is stability for complete minimal surfaces. It was proved independently by Fischer-Colbrie and Schoen [30], do Carmo and Peng [25], and Pogorelov [86] for orientable surfaces and more recently by Ros [89] in the case of non-orientable surfaces.

THEOREM 2.5. *If $M \subset \mathbb{R}^3$ is a complete, immersed, stable minimal surface, then M is a plane.*

The *Plateau Problem* consists of finding a *compact* surface of least area spanning a given boundary. This problem can be solved under certain circumstances; this existence of compact solutions together with a taking limits procedure leads to construct non-compact *stable* minimal surfaces in \mathbb{R}^3 that are constrained to lie in regions of space whose boundaries have non-negative mean curvature. A huge amount of literature is devoted to this procedure, but we will state here only a particular version.

Let W be a compact Riemannian three-manifold with boundary which embeds in the interior of another Riemannian three-manifold. W is said to have *piecewise smooth, mean convex boundary* if ∂W is a two-dimensional complex consisting of a finite number of smooth, two-dimensional compact simplices with interior angles less than or equal to π , each one with non-negative mean curvature with respect to the inward pointing normal. In this situation, the boundary of W is a good barrier for solving Plateau problems in the following sense:

THEOREM 2.6 ([54, 75, 76, 94]). *Let W be a compact Riemannian three-manifold with piecewise smooth mean convex boundary. Let Γ be a smooth collection of pairwise disjoint closed curves in ∂W , which bounds a compact orientable surface in W . Then, there exists an embedded orientable surface $\Sigma \subset W$ with $\partial\Sigma = \Gamma$ that minimizes area among all orientable surfaces with the same boundary.*

Instead of giving an idea of the proof of Theorem 2.6, we will illustrate it together with a taking limits procedure to obtain a particular result in the non-compact setting. Consider two disjoint, connected, properly embedded minimal surfaces M_1, M_2 in \mathbb{R}^3 and let W be the closed complement of $M_1 \cup M_2$ in \mathbb{R}^3 that has both M_1 and M_2 on its boundary.

- (1) We first show how to produce compact, least-area surfaces in W with prescribed boundary lying in the boundary ∂W . Note that W is a complete flat three-manifold with boundary, and ∂W has mean curvature zero. Meeks and Yau [76] proved that W embeds isometrically in a homogeneously regular⁵ Riemannian three-manifold \widetilde{W} diffeomorphic to the interior of W and with metric \widetilde{g} . Morrey [78] proved that in a homogeneously regular manifold, one can solve the classical Plateau problem. In particular, if Γ is an embedded 1-cycle in \widetilde{W} which bounds an orientable finite sum of differentiable simplices in \widetilde{W} , then by standard results in geometric measure theory [28], Γ is the boundary of a compact, least-area embedded surface $\Sigma_\Gamma(\widetilde{g}) \subset \widetilde{W}$. Meeks and Yau proved that the metric \widetilde{g} on \widetilde{W} can be approximated by a family of homogeneously regular metrics $\{g_n\}_{n \in \mathbb{N}}$ on \widetilde{W} , which converges smoothly on compact subsets

⁵A Riemannian three-manifold N is *homogeneously regular* if given $\varepsilon > 0$ there exists $\delta > 0$ such that δ -balls in N are ε -uniformly close to δ -balls in \mathbb{R}^3 in the C^2 -norm. In particular, if N is compact, then N is homogeneously regular.

of \widetilde{W} to \widetilde{g} , and each g_n satisfies a convexity condition outside of $W \subset \widetilde{W}$, which forces the least-area surface $\Sigma_\Gamma(g_n)$ to lie in W if Γ lies in W . A subsequence of the $\Sigma_\Gamma(g_n)$ converges to a smooth minimal surface Σ_Γ of least-area in W with respect to the original flat metric, thereby finishing our description of how to solve (compact) Plateau-type problems in W .

- (2) We now describe the limit procedure to construct a non-compact, stable minimal surface with prescribed boundary lying in M_1 . Let $M_1(1) \subset \dots \subset M_1(n) \subset \dots$ be a compact exhaustion of M_1 , and let $\Sigma_1(n)$ be a least-area surface in W with boundary $\partial M_1(n)$, constructed as in the last paragraph. Let α be a compact arc in W which joins a point in $M_1(1)$ to a point in $\partial W \cap M_2$. By elementary intersection theory, α intersects every least-area surface $\Sigma_1(n)$. By compactness of least-area surfaces, a subsequence of the surfaces $\Sigma_1(n)$ converges to a properly embedded area-minimizing surface Σ in W with a component Σ_0 which intersects α ; this proof of the existence of Σ is due to Meeks, Simon and Yau [74].
- (3) We finish this application of the barrier construction, as follows. Since Σ_0 separates \mathbb{R}^3 , Σ_0 is orientable and so, Theorem 2.5 insures that Σ_0 is a plane. Hence, M_1 and M_2 lie in closed half-spaces of \mathbb{R}^3 , and the Half-space Theorem (Theorem 2.2) implies that both M_1, M_2 are planes.

The above items 1–3 give the following generalization of Theorem 2.2, also due to Hoffman and Meeks.

THEOREM 2.7 (Strong Half-space Theorem [41]). *If M_1 and M_2 are two disjoint, properly immersed minimal surfaces in \mathbb{R}^3 , then M_1 and M_2 are parallel planes.*

2.5. The examples that appear in Theorem 1.1. We will now use the Weierstrass representation for introducing the minimal planar domains characterized in Theorem 1.1.

The plane. $M = \mathbb{C}$, $g(z) = 1$, $dh = dz$. It is the only complete, flat minimal surface in \mathbb{R}^3 .

The catenoid. $M = \mathbb{C} - \{0\}$, $g(z) = z$, $dh = \frac{dz}{z}$ (Figure 1 left). This surface has genus zero, two ends and total curvature -4π . Together with the plane, the catenoid is the only minimal surface of revolution (Bonnet [6]). As we will see in Theorem 3.2, the catenoid and the plane are the unique complete, embedded minimal surfaces with genus zero and finite total curvature. Also, the catenoid was characterized by Schoen [91] as being the unique complete, immersed minimal surface with finite total curvature and two embedded ends.

The helicoid. $M = \mathbb{C}$, $g(z) = e^z$, $dh = i dz$ (Figure 1 center). When viewed in \mathbb{R}^3 , the helicoid has genus zero, one end and infinite total curvature. Together with the plane, the helicoid is the only ruled minimal surface

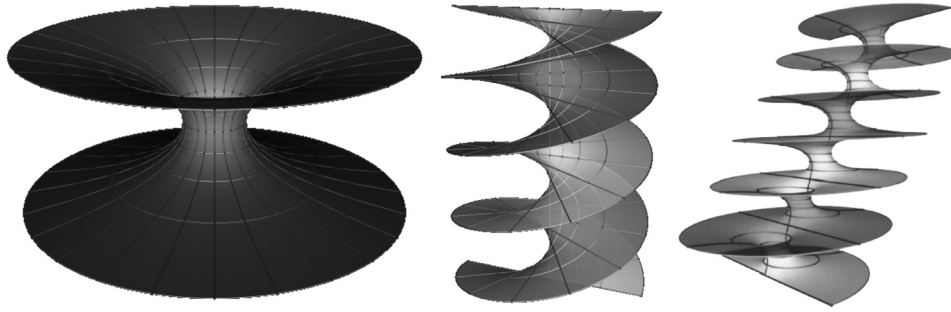


FIGURE 1. Left: Catenoid. Center: Helicoid. Right: One of the Riemann minimal examples. Figures courtesy of Matthias Weber.

(Catalan [8]), and we will see in Theorem 4.2 below that it is the unique properly embedded, simply-connected minimal surface. The vertical helicoid is invariant by a vertical translation T and by a 1-parameter family of screw motions⁶ S_θ , $\theta > 0$. Viewed in \mathbb{R}^3/T or in \mathbb{R}^3/S_θ , the helicoid is a properly embedded minimal surface with genus zero, two ends and finite total curvature. The catenoid and the helicoid are *conjugate* minimal surfaces, in the sense that the coordinate functions of one of these surfaces are the harmonic conjugates of the coordinate functions of the other one; in this case, we consider the catenoid to be defined on its universal cover $e^z: \mathbb{C} \rightarrow \mathbb{C} - \{0\}$ in order for the harmonic conjugate of x_3 to be well-defined.

The Riemann minimal examples. They form a one-parameter family, with Weierstrass data $M_\lambda = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z(z - \lambda)(\lambda z + 1)\} - \{(0, 0), (\infty, \infty)\}$, $g(z, w) = z$, $dh = A_\lambda \frac{dz}{w}$, for each $\lambda > 0$, where A_λ is a non-zero complex number satisfying $A_\lambda^2 \in \mathbb{R}$ (one of these surfaces is represented in Figure 1 right). Together with the plane, catenoid and helicoid, these examples were characterized by Riemann [88] as the unique minimal surfaces which are foliated by circles and lines in parallel planes. Each Riemann minimal example M_λ is topologically a cylinder minus an infinite set of points which accumulates at infinity to the top or bottom ends of the cylinder; hence, M_λ is topologically the unique planar domain with two limit ends. Furthermore, M_λ is invariant under reflection in the (x_1, x_3) -plane and by a translation T_λ ; the quotient surface $M_\lambda/T_\lambda \subset \mathbb{R}^3/T_\lambda$ has genus one and two planar ends, provided that T_λ is the generator of the orientation preserving translations of M_λ . The conjugate minimal surface of M_λ is $M_{1/\lambda}$ (the case $\lambda = 1$ gives the only self-conjugate surface in the family). See [64] for a more precise description of these surfaces.

⁶A *screw motion* S_θ is the composition of a rotation of angle θ around the x_3 -axis with a translation in the direction of this axis.

3. The case with r ends, $2 \leq r < \infty$

Complete minimal surfaces with *finite total curvature* can be naturally thought of as compact algebraic objects, which explains why these surfaces form the most extensively studied family among complete minimal surfaces.

THEOREM 3.1 (Huber [42], Osserman [81]). *Let $M \subset \mathbb{R}^3$ be a complete (oriented), immersed minimal surface with finite total curvature. Then, M is conformally a compact Riemann surface \overline{M} minus a finite number of points, and the Weierstrass representation (g, dh) of M extends meromorphically to \overline{M} . In particular, the total curvature of M is a multiple of -4π .*

Under the hypotheses of the last theorem, the Gauss map g has a well-defined finite degree on \overline{M} , and equation (3) implies that the total curvature of M is -4π times the degree of g . The Gauss-Bonnet formula relates the degree of g with the genus of \overline{M} and the number of ends (Jorge and Meeks [43]); although this formula can be stated in the more general immersed case, we will only consider it when all the ends of M are embedded:

$$(8) \quad \deg(g) = \text{genus}(\overline{M}) + \#(\text{ends}) - 1.$$

The asymptotics of a complete, embedded minimal surface in \mathbb{R}^3 with finite total curvature are also well-understood: after a rotation, each embedded end of such a surface is a graph over the exterior of a disk in the (x_1, x_2) -plane with height function

$$(9) \quad x_3(x_1, x_2) = a \log r + b + \frac{c_1 x_1 + c_2 x_2}{r^2} + \mathcal{O}(r^{-2}),$$

where $r = \sqrt{x_1^2 + x_2^2}$, $a, b, c_1, c_2 \in \mathbb{R}$ and $\mathcal{O}(r^{-2})$ denotes a function such that $r^2 \mathcal{O}(r^{-2})$ is bounded as $r \rightarrow \infty$ (Schoen [91]). When the *logarithmic growth* a in (9) is not zero, the end is called a *catenoidal end* (and the surface is asymptotic to a half-catenoid); if $a = 0$, we have a *planar end* (and the surface is asymptotic to a plane). A consequence of the asymptotics (9) is that for minimal surfaces with finite total curvature, completeness is equivalent to properness (this is also true for immersed surfaces).

The classification of the complete embedded minimal surfaces with genus zero and finite total curvature in \mathbb{R}^3 was solved in 1991 by López and Ros [49]. Their result is based on the fact that every surface in this family can be deformed through minimal surfaces of the same type; this is a strong property which we will encounter in more general situations, as in Theorem 6.11 below.

In the finite total curvature setting, the deformation is explicitly given in terms of the Weierstrass representation: If (g, dh) is the Weierstrass pair of a minimal surface M , then for each $\lambda > 0$ the pair $(\lambda g, dh)$ satisfies condition (5) and the second equation in (6). The first equation in (6) holds for $(\lambda g, dh)$ provided that the flux vector $F(\gamma)$ of M along every closed curve $\gamma \subset M$ is vertical. If M is assumed to have genus zero and finite total curvature, then the homology classes of M are generated by loops around

its planar and/or catenoidal ends. It is easy to check that the flux vector of a catenoidal (resp. planar) end along a non-trivial loop is $(0, 0, \pm 2\pi a)$ after assuming that the limiting normal vector at the end is $(0, 0, \pm 1)$. Since embeddedness implies that all the ends are parallel, then all the flux vectors of our complete embedded minimal surface M with genus zero and finite total curvature are vertical; hence $(\lambda g, dh)$ defines a minimal immersion X_λ by the formula (4) for each $\lambda > 0$; note that for $\lambda = 1$ we obtain the starting surface M .

A direct consequence of the maximum principle is that smooth deformations of compact minimal surfaces remain embedded away from their boundaries. The strong control on the asymptotics for complete embedded minimal surfaces of finite total curvature implies embeddedness throughout the entire deformation $\{X_\lambda\}_{\lambda>0}$. This last property excludes both points in M with horizontal tangent plane and planar ends (a local analysis of the deformation around such points and ends produce self-intersections in X_λ for values of the parameter λ close to zero or infinity), which in turn implies that the height function x_3 of M is proper without critical points. A simple application of Morse theory gives that M has just two ends, in which case the characterization of the catenoid was previously solved by Schoen [91]. This is a sketch of the proof of the following result.

THEOREM 3.2 (López, Ros [49]). *The plane and the catenoid are the only complete, embedded minimal surfaces in \mathbb{R}^3 with genus zero and finite total curvature.*

The next step in our classification of all properly embedded minimal planar domains is to understand the case when the number of ends is finite and at least two. In 1993, Meeks and Rosenberg [70] showed that if a properly embedded minimal surface $M \subset \mathbb{R}^3$ has at least two ends, then every annular end $E \subset M$ either has finite total curvature or it satisfies the hypotheses of the following conjecture, which was solved by Collin in 1997.

CONJECTURE 3.3 (Generalized Nitsche Conjecture, Collin's Theorem [22]). *Let $E \subset \{x_3 \geq 0\}$ be a properly embedded minimal annulus with $\partial E \subset \{x_3 = 0\}$, such that E intersects each plane $\{x_3 = t\}$, $t > 0$, in a simple closed curve. Then, E has finite total curvature.*

As a direct consequence of the last result and Theorem 3.2, we have that the plane and the catenoid are the unique properly embedded minimal surfaces in \mathbb{R}^3 with genus zero and r ends, $2 \leq r < \infty$.

Collin's original proof of Conjecture 3.3 is a beautiful and long argument based on the construction of auxiliary minimal graphs which serve as guide posts to elucidate the shape of E in space. For later purposes, it will be more useful for us to briefly explain a later proof due to Colding and Minicozzi, which is based on the following scale invariant bound for the Gaussian curvature of any embedded minimal disk in a half-space.

THEOREM 3.4 (One-sided Curvature Estimates, Colding, Minicozzi [18]). *There exists $\varepsilon > 0$ such that the following holds. Given $r > 0$ and an embedded minimal disk $M \subset \mathbb{B}(2r) \cap \{x_3 > 0\}$ with $\partial M \subset \partial \mathbb{B}(2r)$, then for any component M' of $M \cap \mathbb{B}(r)$ which intersects $\mathbb{B}(\varepsilon r)$,*

$$\sup_{M'} |K_M| \leq r^{-2}.$$

Before sketching the proof of the Nitsche Conjecture, we will make a few comments about the one-side curvature estimates. The catenoid shows that the hypothesis in Theorem 3.4 on M to be simply-connected is necessary. Theorem 3.4 implies that if an embedded minimal disk is close enough to (and lies at one side of) a plane, then reasonably large components of it are graphs over this plane. The proof of Theorem 3.4 is long and delicate, see [15, 16, 18].

Returning to the Nitsche Conjecture, we see that it follows directly from the next result. Given $\varepsilon \in \mathbb{R}$, we denote by \mathcal{C}_ε the conical region $\{x_3 > \varepsilon \sqrt{x_1^2 + x_2^2}\}$.

THEOREM 3.5 (Colding, Minicozzi [11]). *There exists $\delta > 0$ such that any properly embedded minimal annular end $E \subset \mathcal{C}_{-\delta}$ has finite total curvature.*

SKETCH OF PROOF OF THEOREM 3.5. The argument starts by showing, for each $\delta > 0$, the existence of a sequence $\{y_j\}_j \subset E - \mathcal{C}_\delta$ with $\|y_j\| \rightarrow \infty$ (this is done by contradiction: if for a given $\delta > 0$ this property fails, then one use E together with the boundary of \mathcal{C}_δ as barriers to construct an end of finite total curvature contained in \mathcal{C}_δ , which is clearly impossible by the controlled asymptotics of catenoidal and planar ends). The next step consists of choosing suitable radii $r_j > 0$ such that the connected component M_j of $E \cap \mathbb{B}(y_j, 2r_j)$ which contains y_j is a disk. Now if $\delta > 0$ is sufficiently small in terms of the ε appearing in the one-sided curvature estimates, we can apply Theorem 3.4 and conclude a bound for the supremum of the absolute Gaussian curvature of the component M_j^1 of $M_j \cap \mathbb{B}(y_j, r_j)$ which contains y_j . A Harnack type inequality together with this curvature bound gives a bound for the length of the intrinsic gradient of x_3 in the intrinsic ball \mathcal{B}_j in M_j^1 centered at y_j with radius $5r_j/8$, which in turn implies (by choosing ε sufficiently small) that \mathcal{B}_j is a graph with small gradient over $x_3 = 0$, and one can control a bound by below of the diameter of this graph. This allows to repeat the above argument exchanging y_j by a point y_j^1 in \mathcal{B}_j^1 at certain distance from y_j , and the estimates are carefully done so that the procedure can be iterated to go entirely around a curve $\gamma_j \subset E$ whose projection to the (x_1, x_2) -plane links once around the x_3 -axis. The graphical property of γ_j implies that either γ_j can be continued inside E to spiral indefinitely or it closes up with linking number one with the x_3 -axis. The first possibility contradicts that E is properly embedded, and in the second case the topology of E implies that $\partial E \cup \gamma_j$ bounds an annulus E_j .

The above gradient estimate gives a linear growth estimate for the length of γ_j in terms of $\|y_j\|$, from where the isoperimetric inequality for doubly connected minimal surfaces by Osserman and Schiffer [82] gives a quadratic growth estimate for the area of E_j . Finally, this quadratic area growth property together with the finiteness of the topology of E imply that E has finite total curvature by the Gauss-Bonnet formula, finishing the outline of proof.

4. The one-ended case

In our goal of classifying the properly embedded minimal surfaces with genus zero, the simplest topology occurs when the number of ends is one, and the surface is simply-connected. In spite of this apparent simplicity, this problem remained open until 2005, when Meeks and Rosenberg gave a complete solution by using the one-side curvature estimates (Theorem 3.4) and other aspects of Colding-Minicozzi theory that we comment on in this section.

Classical minimal surface theory allows us to understand the structure of limits of sequences of embedded minimal surfaces with fixed genus, when the sequence has uniform local area and curvature bounds, see for instance the survey by Pérez and Ros [84]. Colding and Minicozzi faced the same problem in the absence of such uniform local bounds in a series of papers starting in 2004 [9, 15, 16, 17, 18]. Their most important structure theorem deals with the case in which all minimal surfaces in the sequence are disks whose Gaussian curvature blows up near the origin. To understand this phenomenon, one should think of a sequence of rescaled helicoids $M_n = \lambda_n H = \{\lambda_n x \mid x \in H\}$, where H is a fixed vertical helicoid with axis the x_3 -axis and $\lambda_n \in \mathbb{R}^+$, $\lambda_n \searrow 0$. The curvature of the sequence $\{M_n\}_n$ blows up along the x_3 -axis and the M_n converge away from the axis to the foliation \mathcal{L} of \mathbb{R}^3 by horizontal planes. The x_3 -axis is the singular set of C^1 -convergence $S(\mathcal{L})$ of M_n to \mathcal{L} , and each leaf L of \mathcal{L} extends smoothly across $L \cap S(\mathcal{L})$ (i.e. $S(\mathcal{L})$ consists of removable singularities of \mathcal{L}). The same behavior is mimicked by any sequence of embedded minimal disks in balls centered at the origin with radii tending to infinity:

THEOREM 4.1 (Limit Lamination Theorem for Disks, Colding, Minicozzi [18]).

Let $M_n \subset \mathbb{B}(R_n)$ be a sequence of embedded minimal disks with $\partial M_n \subset \partial \mathbb{B}(R_n)$ and $R_n \rightarrow \infty$. If $\sup |K_{M_n \cap \mathbb{B}(1)}| \rightarrow \infty$, then there exists a subsequence of the M_n (denoted in the same way) and a Lipschitz curve $S: \mathbb{R} \rightarrow \mathbb{R}^3$ such that up to a rotation of \mathbb{R}^3 ,

- (1) $x_3(S(t)) = t$ for all $t \in \mathbb{R}$.
- (2) Each M_n consists of exactly two multigraphs⁷ away from $S(\mathbb{R})$ which spiral together.

- (3) For each $\alpha \in (0, 1)$, the surfaces $M_n - S(\mathbb{R})$ converge in the C^α -topology to the foliation $\mathcal{L} = \{x_3 = t\}_{t \in \mathbb{R}}$ by horizontal planes.
- (4) $\sup |K_{M_n \cap \mathbb{B}(S(t), r)}| \rightarrow \infty$ as $n \rightarrow \infty$, for any $t \in \mathbb{R}$ and $r > 0$.

SKETCH OF PROOF. Similar as in the proof of the one-sided curvature estimates (Theorem 3.4), the proof of this theorem is involved and runs through various papers [15, 16, 18] (references [12, 13, 14, 19, 20] by Colding and Minicozzi are reading guides for the complete proofs of these results). We will content ourselves with a rough idea of the argument. The first step consists of showing that the embedded minimal disk M_n with large curvature at some interior point can be divided into multivalued graphical building blocks $u_n(\rho, \theta)$ defined on annuli,⁷ and that these basic pieces fit together properly, in the sense that the number of sheets of $u_n(\rho, \theta)$ rapidly grows as the curvature blows up and at the same time, the sheets do not accumulate in a half-space. This is obtained by means of sublinear and logarithmic bounds for the *separation*⁷ $w_n(\rho, \theta)$ as a function of $\rho \rightarrow \infty$. Another consequence of these bounds is that by allowing the *inner radius*⁷ of the annulus where the multigraph is defined to go to zero, the sheets of this multigraph collapse (i.e. $|w_n(\rho, \theta)| \rightarrow 0$ as $n \rightarrow \infty$ for ρ, θ fixed); thus a subsequence of the u_n converges to a smooth minimal graph through $\rho = 0$. The fact that the R_n go to ∞ then implies this limit graph is entire and, by the classical Bernstein's Theorem [4], it is a plane.

The second step in the proof uses the one-sided curvature estimates in the following manner: once it has been proven that an embedded minimal disk M contains a highly sheeted double multigraph \widetilde{M} , then \widetilde{M} plays the role of the plane in the one-sided curvature estimate, which implies that reasonably large pieces of M consist of multigraphs away from a cone with axis "orthogonal" to the double multigraph. The fact that the singular set of convergence is a Lipschitz curve follows because the aperture of this cone is universal (another consequence of Theorem 3.4). \square

With the above discussion in mind, we can now state the main result of this section.

THEOREM 4.2 (Meeks, Rosenberg [71]). *If $M \subset \mathbb{R}^3$ is a properly embedded, simply-connected minimal surface, then M is a plane or a helicoid.*

SKETCH OF PROOF. Take a sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ with $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, and consider the rescaled surface $\lambda_n M$. Since M is simply-connected, Theorem 4.1 gives that a subsequence of $\lambda_n M$ converges on compact subsets of \mathbb{R}^3 to a minimal foliation \mathcal{L} of \mathbb{R}^3 by parallel planes, with singular set of convergence $S(\mathcal{L})$ being a Lipschitz curve that can be parametrized by the height over the planes in \mathcal{L} . Furthermore, a consequence

⁷In polar coordinates (ρ, θ) with $\rho > 0$ and $\theta \in \mathbb{R}$, a k -valued graph on an annulus of inner radius r and outer radius R , is a single-valued graph of a function $u(\rho, \theta)$ defined over $\{(\rho, \theta) \mid r \leq \rho \leq R, |\theta| \leq k\pi\}$, k being a positive integer. The *separation* between consecutive sheets is $w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta) \in \mathbb{R}$.

of the proof of Theorem 4.1 in our case is that for n large, the almost flat multigraph which starts to form on $\lambda_n M$ near the origin extends all the way to infinity. From here it can be shown that the limit foliation \mathcal{L} is independent of the sequence $\{\lambda_n\}_n$. After a rotation of M and replacement of the $\lambda_n M$ by a subsequence, we can suppose that the $\lambda_n M$ converge to the foliation \mathcal{L} of \mathbb{R}^3 by horizontal planes, on compact subsets outside of the singular set of convergence given by a Lipschitz curve $S(\mathcal{L})$ parametrized by its x_3 -coordinate. In particular, $S(\mathcal{L})$ intersects each horizontal plane exactly once.

The next step consists of proving that M intersects transversely each of the planes in \mathcal{L} . The idea now is to consider the solid vertical cylinder $E = \{x_1^2 + x_2^2 \leq 1, -1 \leq x_3 \leq 1\}$. After a homothety and translation, we can assume that $S(\mathcal{L}) \cap \{x_3 = 0\} = \{\vec{0}\}$ and $S(\mathcal{L}) \cap \{-1 \leq x_3 \leq 1\}$ is contained in the convex component of the solid cone whose boundary has the origin as vertex and that passes through the circles in ∂E . The Colding-Minicozzi picture of Theorem 4.1 implies that for n large, $\lambda_n M$ intersects $\partial E \cap \{-1 < x_3 < 1\}$ in a finite number of spiraling curves. For simplicity, we will suppose additionally that the foliation of $\partial E \cap \{-1 < x_3 < 1\}$ by horizontal circles is transversal to $(\lambda_n M) \cap [\partial E \cap \{-1 < x_3 < 1\}]$ (in general, one needs to deform slightly these horizontal circles to almost horizontal Jordan curves to have this transversality property), and consider the foliation of E given by the flat disks $D(t)$ bounded by these circles (here $t \in [-1, 1]$ denotes height; in general, $D(t)$ is a minimal almost flat disk constructed by Rado's theorem). Since at a point of tangency, the minimal surfaces $\lambda_n M$ and $D(t)$ intersect hyperbolically (negative index), Morse theory implies that each minimal disk $D(t)$ intersects $\lambda_n M$ transversely in a simple arc for all n large. This property together with the openness of the Gauss map of the original surface M , implies that M is transverse to \mathcal{L} , as desired. In terms of the Weierstrass representation, we now know that the stereographical projection of the Gauss map $g: M \rightarrow \mathbb{C} \cup \{\infty\}$ can be expressed as $g(z) = e^{H(z)}$ for some holomorphic function $H: M \rightarrow \mathbb{C}$.

The next goal is to demonstrate that M is conformally \mathbb{C} , M intersects every horizontal plane in just one arc and its height function can be written $x_3 = \Re(z)$, $z \in \mathbb{C}$. In the original proof by Meeks and Rosenberg, all of these properties can be deduced from the non-existence of *asymptotic curves* in M , a concept that we now explain⁸. Note that the non-existence of points in M with vertical normal vector implies that the intrinsic gradient ∇x_3 of the third coordinate function does not vanish on M . An integral curve $\gamma: [0, \infty) \rightarrow M$ of ∇x_3 is called an *asymptotic curve* if γ limits to a finite height as its parameter goes to ∞ . Suppose for the moment that M does not admit asymptotic curves. Consider a component Γ of $M \cap \{x_3 = 0\}$, which we know it is smooth. The mapping $F: \Gamma \times \mathbb{R} \rightarrow M$ given by $F(p, t) = \gamma_p(t)$ where γ_p is the unique integral curve of ∇x_3 with $\gamma_p(0) = p$,

⁸ For an alternative short argument, see Remark 8.4 below.

is a local diffeomorphism. Using that M does not have asymptotic curves, it can be shown that $\partial F(\Gamma \times \mathbb{R})$ is empty, hence $F(\Gamma \times \mathbb{R}) = M$ since M is connected. Now consider the holomorphic function $h = x_3 + ix_3^* : M \rightarrow \mathbb{C}$, where x_3^* is the (globally well-defined) harmonic conjugate function of x_3 . Again the transversality of M to every horizontal plane implies that h is a local biholomorphism. Since $\Gamma \subset \{x_3 = 0\}$, h maps Γ diffeomorphically onto an interval $I \subset i\mathbb{R} \subset \mathbb{C}$. As M has no asymptotic curves, h maps any integral curve γ_p of ∇x_3 onto a complete horizontal line in \mathbb{C} . Thus $h(M) = \mathbb{R} \times I$ and h is a biholomorphism between these two surfaces. The first sentence of this paragraph will be proved provided that $I = i\mathbb{R}$. Otherwise, $[0, \infty) \times I$ is conformally the closed unit disk minus a closed interval in its boundary, which is not parabolic as a Riemann surface (i.e. bounded harmonic functions on it are not determined by their boundary values). Therefore $M \cap \{x_3 \geq 0\}$ is not parabolic, which contradicts Theorem 4.3 below. It remains to prove that M does not admit asymptotic curves. The argument is by contradiction: if $\gamma \subset M$ is an asymptotic curve, then one can find a piece of M which is a graph with infinitely many connected components above a certain horizontal plane, with zero boundary values. This contradicts the existence of an upper bound for the number of components of a minimal graph over a possibly disconnected, proper domain in \mathbb{R}^2 with zero boundary values (Meeks and Rosenberg proved their own version of this bound in [71] following previous arguments of Colding and Minicozzi for harmonic functions; later and sharper versions of this bound can be found in the papers by Li and Wang [47] and Tkachev [95]).

At this point, we know that the Weierstrass pair of M is $(g, dh) = (e^{H(z)}, dz)$, $z \in \mathbb{C}$, where H is an entire function. The last step in the proof is to show that $H(z)$ is a linear function of the form $az + b$ (because in that case M is an associate surface to a vertical helicoid; but such a surface is embedded only if it is actually a helicoid). Assuming that H is a polynomial, the explicit expression of the Gaussian curvature K in terms of the Weierstrass data implies that $H(z)$ is linear if and only if M has bounded curvature. This fact completes the proof of Theorem 4.2 provided that H is a polynomial and K is bounded. On the other hand, Theorem 4.1 and a clever blow-up argument on the scale of curvature allows us to argue in the bounded curvature setting, and we then are left with ruling out the case that H has an essential singularity at ∞ . This is done by analyzing the inverse image of a latitude by the Gauss map of the surface. This concludes our sketch of proof. \square

In the last proof we mentioned a result on parabolicity for minimal surfaces with boundary, which we next state. The proof of this auxiliary result uses the harmonic measure and universal harmonic functions (see for instance [60] for these concepts), and we will skip its proof here.

THEOREM 4.3 (Collin, Kusner, Meeks, Rosenberg [23]). *Let M be a connected, properly immersed minimal surface in \mathbb{R}^3 , possibly with boundary. Then, every component of the intersection of M with a closed half-space is a parabolic surface with boundary.*

5. Infinitely many ends I: one limit end is not possible

In the sequel, we will consider the case of M being a properly embedded minimal surface, whose topology is that of a sphere \mathbb{S}^2 minus an infinite, compact, totally disconnected subset $\mathcal{E}(M)$. Viewed as a subset $\mathcal{E}(M)$ of \mathbb{S}^2 , the set $\mathcal{E}(M)$ of ends of M must have accumulation points, which are called *limit ends*⁹ of M . The isolated points in $\mathcal{E}(M)$ are called *simple ends*.

Next we explain the first two ingredients needed to understand the geometry of properly embedded minimal surfaces with more than one end: the notion of limit tangent plane at infinity and the Ordering Theorem. Every properly embedded minimal surface $M \subset \mathbb{R}^3$ with more than one end admits in one of its two closed complements a properly embedded minimal surface Σ with finite total curvature and compact boundary (produced via the barrier construction method, see Section 2.4). By the discussion in Section 3, the ends of Σ are of catenoidal or planar type with parallel normal vectors at infinity since Σ is embedded. The plane passing through the origin which is orthogonal to the limiting normal vectors at the ends of Σ does not depend on Σ , and it is called the *limit tangent plane at infinity* of M (for details, see Callahan, Hoffman and Meeks [7]).

THEOREM 5.1 (Ordering Theorem, Frohman, Meeks [33]). *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then, the space $\mathcal{E}(M)$ of ends of M is linearly ordered geometrically by the relative heights of the ends over the (x_1, x_2) -plane, and $\mathcal{E}(M)$ embeds topologically as a compact totally disconnected subspace of $[0, 1]$ in an ordering preserving way.*

Proof that the subset $X \subset \mathcal{E}(M)$ of all the ends of M with proper annular representatives has a natural linear ordering. Suppose $M \subset \mathbb{R}^3$ is a properly embedded minimal surface and X is the set of ends which have proper annular representatives. By Collin’s Theorem (Conjecture 3.3), every proper annular representative of an end in X has finite total curvature and thus, it is asymptotic to a horizontal plane or to a half-catenoid (recall that M has horizontal limit tangent plane at infinity). Since these ends of M are all graphs over complements of compact subdomains in the (x_1, x_2) -plane as in equation (9), we have that the set of ends in X has a natural linear ordering by relative heights over the (x_1, x_2) -plane, and the Ordering Theorem is proved for this restricted collection of ends. □

REMARK 5.2. *The proof of Theorem 5.1 in the general case is more involved. For later purposes, we will only indicate what is done. One starts the proof by using the barrier construction to find ends of finite total curvature in one of the closed complements of M in space, adapted to each of the non-annular ends of M ; more precisely, one separates each non-annular end*

⁹See Section 2.7 of [60] for a generalization of the notion of limit end to a non-compact connected n -manifold.

representative E of M with compact boundary from the non-compact domain $M - E$ by a properly embedded, orientable least area surface $\Sigma_1 \subset \mathbb{R}^3 - M$ with $\partial\Sigma_1 = \partial E = \partial(M - E)$ constructed using M as a barrier against itself. The asymptotics of such a Σ_1 consists of a positive number of graphical ends of planar or catenoidal type, with vertical limiting normal vector. Then one uses these naturally ordered surfaces of the type Σ_1 to extend the linear ordering to the entire set of ends of M , see [33] for further details.

Since $\mathcal{E}(M) \subset [0, 1]$ is a compact subspace, the above linear ordering on $\mathcal{E}(M)$ lets us define the *top* (resp. *bottom*) end e_T (resp. e_B) of M as the unique maximal (resp. minimal) element in $\mathcal{E}(M)$. If $e \in \mathcal{E}(M)$ is neither the top nor the bottom end of M , then it is called a *middle* end of M . Another key result, related to conformal properties and area growth, is the following non-existence result for middle limit ends for a properly embedded minimal surface.

THEOREM 5.3 (Collin, Kusner, Meeks, Rosenberg [23]). *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then, any limit end of M must be a top or bottom end. In particular, M can have at most two limit ends, each middle end is simple and the number of ends of M is countable.*

SKETCH OF PROOF. The arguments in Remark 5.2 insure that every middle end of a surface M as described in Theorem 5.3 can be represented by a proper subdomain $E \subset M$ with compact boundary such that E “lies between two half-catenoids”. This means that E is contained in a neighborhood W of the (x_1, x_2) -plane, W being topologically a slab, whose width grows at most logarithmically with the distance from the origin. This constraint on a middle end representative can be used in the following way to deduce that the area of this end grows at most quadratically in terms of the distance to the origin.

For simplicity, we will assume that E is trapped between two horizontal planes, rather than between two half-catenoids,¹⁰ i.e. $E \subset W := \{(x_1, x_2, x_3) \mid r \geq 1, 0 \leq x_3 \leq 1\}$, where $r = \sqrt{x_1^2 + x_2^2}$. We claim that both $|\nabla x_3|^2$ and $\Delta \ln r$ are in $L^1(E)$, where ∇, Δ are the intrinsic gradient and laplacian on M . It is not hard to check using the inequality in equation (10) below that the function $\ln r - x_3^2$ restricts to every minimal surface lying in W as a superharmonic function.¹¹ In particular, the restriction $f: E \rightarrow \mathbb{R}$ is superharmonic and proper. Suppose $f(\partial E) \subset [-1, c]$ for some $c > 0$. Replacing E by $f^{-1}[c, \infty)$ and taking $t > c$, the Divergence Theorem gives

¹⁰The general case can be treated in a similar way, although the auxiliary function f is more complicated.

¹¹This is called a *universal superharmonic function* for the region W ; for instance, x_1 or $-x_1^2$ are universal superharmonic functions for all of \mathbb{R}^3 .

(we can also assume that both c, t are regular values of f):

$$\int_{f^{-1}[c,t]} \Delta f \, dA = - \int_{f^{-1}(c)} |\nabla f| \, ds + \int_{f^{-1}(t)} |\nabla f| \, ds,$$

where dA, ds are the corresponding area and length elements. As f is superharmonic, the function $t \mapsto \int_{f^{-1}[c,t]} \Delta f \, dA$ is monotonically decreasing and bounded from below by $-\int_{f^{-1}(c)} |\nabla f| \, ds$. In particular, Δf lies in $L^1(E)$. Furthermore, $|\Delta f| = |\Delta \ln r - 2|\nabla x_3|^2| \geq -|\Delta \ln r| + 2|\nabla x_3|^2$.

At this point we need a useful inequality also due to Collin, Kusner, Meeks and Rosenberg [23], valid for every immersed minimal surface in \mathbb{R}^3 :

$$(10) \quad |\Delta \ln r| \leq \frac{|\nabla x_3|^2}{r^2} \quad \text{in } M - (x_3\text{-axis}).$$

By the estimate (10), we have $|\Delta f| \geq (2 - \frac{1}{r^2}) |\nabla x_3|^2$. Since $r^2 \geq 1$ in W , it follows $|\Delta f| \geq |\nabla x_3|^2$ and thus, both $|\nabla x_3|^2$ and $\Delta \ln r$ are in $L^1(E)$, as desired. Geometrically, this means that outside of a subdomain of E of finite area, E can be assumed to be as close to being horizontal as one desires, and in particular, for the radial function r on this horizontal part of E , $|\nabla r|$ is almost equal to 1.

Let $r_0 = \max r|_{\partial E}$ and $E(t)$ be the subdomain of E that lies inside the region $\{r_0^2 \leq x_1^2 + x_2^2 \leq t^2\}$. Since

$$\int_{E(t)} \Delta \ln r \, dA = - \int_{r=r_0} \frac{|\nabla r|}{r} \, ds + \int_{r=t} \frac{|\nabla r|}{r} \, ds = \text{const.} + \frac{1}{t} \int_{r=t} |\nabla r| \, ds$$

and $\Delta \ln r \in L^1(E)$, then the following limit exists:

$$(11) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_{r=t} |\nabla r| \, ds = C$$

for some positive constant C . Thus, $t \mapsto \int_{r=t} |\nabla r| \, ds$ grows at most linearly as $t \rightarrow \infty$. By the coarea formula, for t_1 fixed and large,

$$(12) \quad \int_{E \cap \{t_1 \leq r \leq t\}} |\nabla r|^2 \, dA = \int_{t_1}^t \left(\int_{r=\tau} |\nabla r| \, ds \right) \, d\tau;$$

hence, $t \mapsto \int_{E \cap \{t_1 \leq r \leq t\}} |\nabla r|^2 \, dA$ grows at most quadratically as $t \rightarrow \infty$. Finally, since outside of a domain of finite area E is arbitrarily close to horizontal and $|\nabla r|$ is almost equal to one, we conclude that the area of $E \cap \{r \leq t\}$ grows at most quadratically as $t \rightarrow \infty$. In fact, from (11) and (12) it follows that

$$\int_{E \cap \{r \leq t\}} dA = \frac{C}{2} t^2 + o(t^2),$$

where $t^{-2}o(t^2) \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, it can be proved that the constant C must be an integer multiple of 2π (using the quadratic area growth property, the homothetically shrunk surfaces $\frac{1}{n}E$ converge as $n \rightarrow \infty$ in the sense of geometric measure theory to a locally finite, minimal integral varifold V with empty boundary, and V is supported on the limit of $\frac{1}{n}W$,

which is the (x_1, x_2) -plane; thus V is an integer multiple of the (x_1, x_2) -plane, which implies that C must be an integer multiple of 2π .

Finally, every end representative of a minimal surface must have asymptotic area growth at least equal to half of the area growth of a plane (as follows from the monotonicity formula, Theorem 2.4). Since we have checked that each middle end e of a properly embedded minimal surface has a representative with at most quadratic area growth, then e admits a representative which have exactly one end, and this means that e is never a limit end. \square

We remark that Theorem 5.3 does not make any assumption on the genus of the minimal surface. Our next goal is to discard the possibility of just one limit end for properly embedded minimal surfaces with *finite* genus (in particular, this result holds in our search of the examples with genus zero), which is the content of Theorem 5.11 below. In order to understand the proof of this theorem, we will need the following notions and results.

DEFINITION 5.4. Suppose that $\{M_n\}_n$ is a sequence of connected, properly embedded minimal surfaces in an open set $A \subset \mathbb{R}^3$. Given $p \in A$ and $n \in \mathbb{N}$, let $r_n(p) > 0$ be the largest radius of an extrinsic open ball $\mathbb{B} \subset A$ centered at p such that \mathbb{B} intersects M_n in simply-connected components. If for every $p \in A$ the sequence $\{r_n(p)\}_n$ is bounded away from zero, we say that $\{M_n\}_n$ is *locally simply-connected in A* . If $A = \mathbb{R}^3$ and for all $p \in \mathbb{R}^3$, the radius $r_n(p)$ is bounded from below by a fixed positive constant for all n large, we say that $\{M_n\}_n$ is *uniformly locally simply-connected*.

DEFINITION 5.5. A *lamination* of an open subset $U \subset \mathbb{R}^3$ is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ where \mathcal{L} is a closed subset of U and $\mathcal{A} = \{\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta\}_\beta$ is a collection of coordinate charts of \mathbb{R}^3 (here \mathbb{D} is the open unit disk, $(0, 1)$ the open unit interval and U_β an open subset of U); the local product structure is described by the property that for each β , there exists a closed subset C_β of $(0, 1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$. It is customary to denote a lamination only by \mathcal{L} , omitting the charts in \mathcal{A} . A lamination \mathcal{L} is a *foliation of U* if $\mathcal{L} = U$. Every lamination \mathcal{L} naturally decomposes into a union of disjoint connected surfaces, called the *leaves* of \mathcal{L} . A lamination is *minimal* if all its leaves are minimal surfaces.

The simplest examples of minimal laminations of \mathbb{R}^3 are a closed family of parallel planes, and $\mathcal{L} = \{M\}$, where M is a properly embedded minimal surface. A crucial ingredient of the proof of the key topological result described in Theorem 5.11 will be the following structure theorem of minimal laminations in \mathbb{R}^3 :

THEOREM 5.6 (Structure of Minimal Laminations in \mathbb{R}^3). *Let $\mathcal{L} \subset \mathbb{R}^3$ be a minimal lamination. Then, one of the following possibilities hold.*

- (1) $\mathcal{L} = \{L\}$ where L is a properly embedded minimal surface in \mathbb{R}^3 .

- (2) \mathcal{L} has more than one leaf. In this case, $\mathcal{L} = \mathcal{P} \cup \mathcal{L}_1$ where \mathcal{P} consists of the disjoint union of a non-empty closed set of parallel planes, and \mathcal{L}_1 is a (possibly empty) collection of pairwise disjoint, complete embedded minimal surfaces. Each $L \in \mathcal{L}_1$ has infinite genus, unbounded Gaussian curvature, and is properly embedded in one of the open slabs and half-spaces components of $\mathbb{R}^3 - \mathcal{P}$, which we will call $C(L)$. If $L, L' \in \mathcal{L}_1$ are distinct, then $C(L) \cap C(L') = \emptyset$. Finally, each plane Π contained in $C(L)$ divides L into exactly two components.

The proof of Theorem 5.6 can be found in two papers, due to Meeks and Rosenberg [71] and Meeks, Pérez and Ros [67]. Although the proof of the above theorem is a bit long, we next devote some paragraphs to comment some aspects of this proof since many of the arguments that follow will be used somewhere else in these notes.

- (1) First note that the local structure of a lamination \mathcal{L} of \mathbb{R}^3 implies that the Gaussian curvature function of the leaves of \mathcal{L} is locally bounded (in bounded extrinsic balls). Reciprocally, if a complete embedded minimal surface $M \subset \mathbb{R}^3$ has locally bounded Gaussian curvature (bounded in extrinsic balls), then its closure \overline{M} has the structure of a minimal lamination of \mathbb{R}^3 . This holds because if $\{p_n\}_n \subset M$ is a sequence of points that converges to some $p \in \overline{M}$, then the local boundedness property of K_M implies that there exists $\delta = \delta(p) > 0$ such that for n sufficiently large, M is a graph G_n over the disk $D(p_n, \delta) \subset T_{p_n}M$ of radius δ and center p_n , and we have C^k -bounds for these graphs for all k . Up to a subsequence, the planes $T_{p_n}M$ converge to some plane P with $p \in P$, and the $G_n|_{D(p_n, \delta/2)}$ converge to a minimal graph G_∞ over $D(p, \frac{\delta}{2}) \subset P$; the absence of self-intersections in M and the maximum principle insure that both P and G_∞ are independent of the sequence $\{p_n\}_n$, and that each G_n is disjoint from G_∞ ; thus we have constructed a local structure of lamination around $p \in \overline{M}$, which can be continued analytically along \overline{M} .
- (2) Given a minimal lamination \mathcal{L} of an open set $A \subset \mathbb{R}^3$ and a point $p \in \mathcal{L}$, we say that p is a *limit point* if for all $\varepsilon > 0$ small enough, the ball $\mathbb{B}(p, \varepsilon)$ intersects \mathcal{L} in an infinite number of (disk) components. If a leaf $L \in \mathcal{L}$ contains a limit point, then L consists entirely of limit points, and L is then called a *limit leaf* of \mathcal{L} .
- (3) Let \mathcal{L} be a minimal lamination of an open set $A \subset \mathbb{R}^3$ and L a limit leaf of \mathcal{L} . Then, the universal covering space \tilde{L} of L is stable: By lifting arguments, this property can be reduced to the case L is simply-connected, and this particular case follows by expressing every compact disk $\Omega \subset L$ as the uniform limit of disjoint compact disks $\Omega(n)$ in leaves $L(n) \in \mathcal{L}$, and then writing the $\Omega(n)$ as normal graphs over Ω of functions $u_n \in C^\infty(\Omega)$ such that $u_n \rightarrow 0$ as $n \rightarrow \infty$

in the C^k -topology for each k . The lamination structure of \mathcal{L} allows us assume that $0 < u_n < u_{n+1}$ in Ω for all n . After normalizing suitably $u_{n+1} - u_n$, we produce a positive limit $v \in C^\infty(\Omega)$, which satisfies the linearized version of the minimal surface equation (the so-called *Jacobi equation*), $\Delta v - 2K_L v = 0$ in Ω . Since $v > 0$ in Ω , it is a standard fact that the first eigenvalue of the Jacobi operator $\Delta - 2K_L$ in Ω is positive. Since Ω is an arbitrary compact disk in L , we deduce that L is stable (in fact, a recent general result of Meeks, Perez and Ros implies that any limit leaf L of \mathcal{L} is stable without assuming it is simply-connected, see [62, 69]).

- (4) A direct consequence of items 1 and 3 above together with Theorem 2.5 is that if $M \subset \mathbb{R}^3$ is a connected, complete embedded minimal surface with locally bounded curvature, then:
- (a) M is proper in \mathbb{R}^3 , or
 - (b) M is proper in an open half-space (resp. slab) of \mathbb{R}^3 with limit set the boundary plane (resp. planes) of this half-space (resp. slab).

In particular, in order to understand the structure of minimal laminations of \mathbb{R}^3 , we only need to analyze the behavior in a neighborhood of a limit plane.

- (5) In both this item and item 6 below, $M \subset \mathbb{R}^3$ will denote a connected, embedded minimal surface with K_M locally bounded, such that M is not proper in \mathbb{R}^3 and P is a limit plane of M ; after a rotation, we can assume $P = \{x_3 = 0\}$ and M limits to P from above. We claim that *for any $\varepsilon > 0$, the surface $M \cap \{0 < x_3 \leq \varepsilon\}$ has unbounded curvature*. This follows because otherwise, we can choose a smaller $\varepsilon > 0$ so that the vertical projection $\pi: M \cap \{0 \leq x_3 \leq \varepsilon\} \rightarrow P$ is a submersion. Let Δ be a component of $M \cap \{0 \leq x_3 \leq \varepsilon\}$. Since $\Delta - \partial\Delta$ is properly embedded in the simply-connected slab $\{0 < x_3 < \varepsilon\}$, it separates this slab. It follows that each vertical line in \mathbb{R}^3 intersects Δ transversally in at most one point. This means that Δ is a graph over its orthogonal projection in $\{x_3 = 0\}$. In particular, Δ is proper in $\{0 \leq x_3 \leq \varepsilon\}$. A straightforward application of the proof of the Halfspace Theorem 2.2 now gives a contradiction.
- (6) *For every $\varepsilon > 0$, the surface $M \cap \{0 < x_3 \leq \varepsilon\}$ is connected.* The failure of this property produces, using two components $M(1), M(2)$ of $M \cap \{0 < x_3 \leq \varepsilon\}$ as barriers, a properly embedded minimal surface $\Sigma \subset \{0 < x_3 \leq \varepsilon\}$ which is stable, orientable and has the same boundary as, say, $M(1)$. Thus Σ satisfies curvature estimates away from its boundary (Schoen [90]), which allows us to replace M by Σ in the arguments of item 5 to obtain a contradiction.

By items 1–6 above, Theorem 5.6 will be proved provided that we check that every $L \in \mathcal{L}_1$ has infinite genus, which in turn is a consequence of the next result.

PROPOSITION 5.7 (Meeks, Pérez and Ros [67]). *If $M \subset \mathbb{R}^3$ is a complete embedded minimal surface with finite genus and locally bounded Gaussian curvature, then M is proper.*

SKETCH OF PROOF. One argues by contradiction assuming that M is not proper in \mathbb{R}^3 . By item 4 above, M is proper in an open region $W \subset \mathbb{R}^3$ which is (up to a rotation and finite rescaling) the slab $\{0 < x_3 < 1\}$ or the halfspace $\{x_3 > 0\}$. Since M is not proper in \mathbb{R}^3 , it cannot have finite topology (by Theorem 5.8 below). As M has finite genus, then it has infinitely many ends. Using similar arguments as in the proof of the Ordering Theorem (Theorem 5.1), every pair of ends of M can be separated by a stable, properly embedded minimal surface with compact boundary. This gives a linear ordering of the ends of M by relative heights over the (x_1, x_2) -plane, in spite of M not being proper in \mathbb{R}^3 but only being proper in W . In this new setting, the arguments in the proof of Theorem 5.3 apply to give that the middle ends of M are simple, hence topologically annuli and asymptotically planar or catenoidal.

If some middle end E of M is planar, then all of the middle ends of M below E are clearly planar. Furthermore, M has unbounded curvature in every slab $\{0 < x_3 \leq \varepsilon\}$, $\varepsilon > 0$ small (by item 5 before this proposition). The contradiction now follows from the proof of the curvature estimates of the two-limit-ended case (Theorem 6.3 below), which can be extended to this situation.¹²

Hence all the annular ends of M are catenoidal (in particular, $W = \{x_3 > 0\}$). By flux arguments one can show that M cannot have a top limit end, so it has exactly one limit end which is its bottom end, that limits to $\{x_3 = 0\}$, with all the annular catenoidal ends of positive logarithmic growth. The rest of the proof only uses the (connected) portion $M(\varepsilon)$ of M in a slab of the form $\{0 < x_3 \leq \varepsilon\}$ with $\varepsilon > 0$ small enough so that $M(\varepsilon)$ is a planar domain. Since $K_{M(\varepsilon)}$ is not bounded but is locally bounded, we can find a divergent sequence $p_n \in M(\varepsilon)$ with $x_3(p_n) \rightarrow 0$ and $|K_{M(\varepsilon)}|(p_n) \rightarrow \infty$. The one-sided curvature estimate (Theorem 3.4) implies that there exists a sequence $r_n \searrow 0$ such that $M \cap \mathbb{B}(p_n, r_n)$ contains some component which is not a disk. Then the desired contradiction will follow after analyzing a new sequence of minimal surfaces $\widehat{M}(n)$, obtained by *blowing-up* $M(\varepsilon)$ around p_n on the scale of topology. This new notion deserves some brief explanation, since it will be useful in other applications (for instance, in the proof of Theorem 5.11 below). It consists of homothetically expanding $M(\varepsilon)$ so that in the new scale, p_n becomes the origin and the surface $\widehat{M}(n)$ obtained from expansion of $M(\varepsilon)$ in the n -th step, intersects every ball in \mathbb{R}^3 of radius less than $1/2$ in simply-connected components for n large, but $\widehat{M}(n) \cap \overline{\mathbb{B}}(1)$ contains a component which is not a disk (roughly speaking, this is achieved by using the inverse of the injectivity radius function of

¹²We remark that Theorem 6.3 assumes that M is proper in \mathbb{R}^3 ; nevertheless, here we need to apply the arguments in its proof of this situation in which M is proper in W .

$M(\varepsilon)$ as the ratio of expansion). In the new scale, the contradiction will appear after consideration of two cases, depending on whether the sequence $\{K_{\widehat{M}(n)}\}_n$ is locally bounded or unbounded in \mathbb{R}^3 .

If the sequence $\{K_{\widehat{M}(n)}\}_n$ is not locally bounded, then one applies to $\{\widehat{M}(n)\}_n$ the generalization by Colding-Minicozzi of Theorem 4.1 to planar domains (see Theorem 5.9 below), and concludes that after extracting a subsequence, the $\widehat{M}(n)$ converge to a minimal foliation \mathcal{F} of \mathbb{R}^3 by parallel planes, with singular set of convergence $S(\mathcal{F})$ consisting of two Lipschitz curves that intersect the planes in \mathcal{F} exactly once each. By Meeks' regularity theorem for the singular set (see Theorem 5.10 below), $S(\mathcal{F})$ consists of two straight lines orthogonal to the planes in \mathcal{F} . Furthermore, the distance between these two lines is at most 2 (this comes from the non-simply-connected property of some component of $\widehat{M}(n) \cap \overline{\mathbb{B}}(1)$), and this limit picture allows to find a closed curve $\delta_n \subset \widehat{M}(n)$ arbitrarily close to a doubly covered straight line segment l contained in one of the planes of \mathcal{F} , such that each of the extrema of l lies in one of the singular lines in $S(\mathcal{F})$. Since $\widehat{M}(n)$ has genus zero, δ_n separates $\widehat{M}(n)$. Therefore, δ_n is the boundary of a non-compact subdomain $\Omega_n \subset \widehat{M}(n)$ with a finite number of vertical catenoidal ends, all with positive logarithmic growth. By the Divergence Theorem, the flux of $\widehat{M}(n)$ along δ_n is vertical. But this flux converges to a vector parallel to the planes of \mathcal{F} , of length at most 4. It follows that the planes of \mathcal{F} are vertical. This fact leads to a contradiction with the maximum principle applied to the function x_3 on the domain $\Delta(n) \subset \widehat{M}(n)$ with finite topology and boundary δ_n (note that $\Delta(n)$ must lie above the height $\min x_3|_{\delta_n}$ because the ends of $\Delta(n)$ are catenoidal with positive logarithmic growth). Therefore, $\{K_{\widehat{M}(n)}\}_n$ must be locally bounded.

Finally if $\{K_{\widehat{M}(n)}\}_n$ is locally bounded, then the arguments in item 1 before Proposition 5.7 (extended to a sequence of embedded surfaces instead of a single surface) imply that after extracting a subsequence, the $\widehat{M}(n)$ converge to a minimal lamination $\widehat{\mathcal{L}}$ of \mathbb{R}^3 . The non-simply-connected property of some component of $\widehat{M}(n) \cap \overline{\mathbb{B}}(1)$ is then used to prove that $\widehat{\mathcal{L}}$ contains a non-flat leaf \widehat{L} . Since \widehat{L} is not a plane, it is not stable, which in turn implies that the multiplicity of the convergence of portions of $\widehat{M}(n)$ to \widehat{L} is 1 (the arguments for this property are similar to those applied in item 3 before Proposition 5.7). This fact and the verticality of the flux of the $\widehat{M}(n)$ imply that \widehat{L} has vertical flux as well. Then one finishes this case by discarding all possibilities for such an \widehat{L} as a limit of the $\widehat{M}(n)$; the verticality of the flux of \widehat{L} together with the López-Ros deformation argument explained in Section 3 are crucial here (for instance, one of the possibilities for \widehat{L} is being a properly embedded minimal planar domain with two limit ends; this case is discarded by using Theorem 6.4 below). See Theorem 7 in [67] for further details. \square

In the last proof we used three auxiliary results, which we next state for future reference.

THEOREM 5.8 (Colding-Minicozzi [21]). *If $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface with finite topology, then M is proper.*

THEOREM 5.9 (Limit Lamination Theorem for Planar Domains, Colding, Minicozzi [9]). *Let $M_n \subset \mathbb{B}(R_n)$ be a locally simply-connected sequence of embedded minimal planar domains with $\partial M_n \subset \partial \mathbb{B}(R_n)$, $R_n \rightarrow \infty$, such that $M_n \cap \mathbb{B}(2)$ contains a component which is not a disk for any n . If $\sup |K_{M_n \cap \mathbb{B}(1)}| \rightarrow \infty$, then there exists a subsequence of the M_n (denoted in the same way) and two vertical lines S_1, S_2 , such that*

- (a) M_n converges away from $S_1 \cup S_2$ to the foliation \mathcal{F} of \mathbb{R}^3 by horizontal planes.
- (b) Away from $S_1 \cup S_2$, each M_n consists of exactly two multivalued graphs spiraling together. Near S_1 and S_2 , the pair of multivalued graphs form double spiral staircases with opposite handedness at S_1 and S_2 . Thus, circling only S_1 or only S_2 results in going either up or down, while a path circling both S_1 and S_2 closes up.

THEOREM 5.10 (Regularity of $S(\mathcal{L})$, Meeks [56]). *Suppose $\{M_n\}_n$ is a locally simply-connected sequence of properly embedded minimal surfaces in a three-manifold, that converges C^α to a minimal foliation \mathcal{L} outside a locally finite collection of Lipschitz curves $S(\mathcal{L})$ transverse to \mathcal{L} . Then, $S(\mathcal{L})$ consists of a locally finite collection of integral curves of the unit Lipschitz normal vector field to \mathcal{L} . In particular, the curves in $S(\mathcal{L})$ are $C^{1,1}$ and orthogonal to the leaves of \mathcal{L} .*

After all these preliminaries, we are ready to discard the one-limit-ended case in our search of all properly embedded minimal surfaces with finite genus.

THEOREM 5.11 (Meeks, Pérez, Ros [68]). *If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with finite genus, then M cannot have exactly one limit end.*

SKETCH OF PROOF. After a rotation, we can suppose that M has a horizontal limit tangent plane at infinity and by Theorems 5.1 and 5.3, its ends are linearly ordered by relative increasing heights, $\mathcal{E}(M) = \{e_1, e_2, \dots, e_\infty\}$ with e_∞ being the limit end of M and its top end. Since for each finite n the end e_n has an annular representative, Collin’s Theorem implies that e_n has a representative with finite total curvature, which is therefore asymptotic to a graphical annular end of a vertical catenoid or plane. By embeddedness, this catenoidal or planar end has vertical limit normal vector, and its logarithmic growth a_n satisfies $a_n \leq a_{n+1}$ for all n . Note that $a_1 < 0$ (otherwise we contradict Theorem 2.2). By flux reasons, $a_n < 0$ for all n (one a_k being positive implies that $a_n \geq a_k > 0$ for all $k \geq n$, which cannot be balanced

by finitely many negative logarithmic growths; a similar argument works if one a_k is zero).

The next step consists of analyzing the limits of M under homothetic shrinkings:

CLAIM 5.12. *Given a sequence $\lambda_n \searrow 0$, the sequence of surfaces $\{\lambda_n M\}_n$ is locally simply-connected in $\mathbb{R}^3 - \{\vec{0}\}$.*

The proof of this property is by contradiction: if it fails around a point $p \in \mathbb{R}^3 - \{\vec{0}\}$, then one blows-up $\lambda_n M$ on the scale of topology around p , as we did in the proof of Proposition 5.7. Thus, we produce expanded versions $\widehat{M}(n)$ of $\lambda_n M$ so that p becomes the origin and $\widehat{M}(n)$ intersects any ball in \mathbb{R}^3 of radius less than $1/2$ in simply-connected components for n large, but $\widehat{M}(n) \cap \mathbb{B}(1)$ contains a component which is not simply-connected. We will now try to mimic the arguments in the last two paragraphs of the proof of Proposition 5.7, commenting only the differences between the two situations.

If $\{K_{\widehat{M}(n)}\}_n$ is not locally bounded in \mathbb{R}^3 , then Theorems 5.9 and 5.10 produce a limit picture for the $\widehat{M}(n)$ which is a foliation \mathcal{F} of \mathbb{R}^3 by parallel planes, with two parallel straight lines as singular set of convergence. Then one finds connection loops $\delta_n \subset \widehat{M}(n)$ as in the proof of Proposition 5.7, which together with a flux argument, imply that the foliation \mathcal{F} is by vertical planes. Now the contradiction comes from the maximum principle applied to the function x_3 on the domain $\Delta(n) \subset \widehat{M}(n)$ with finite topology and boundary δ_n (note that $\Delta(n)$ now lies below the height $\max x_3|_{\delta_n}$ because the ends of $\Delta(n)$ are catenoidal with negative logarithmic growth, compare with the situation in the next to the last paragraph of the proof of Proposition 5.7).

If $\{K_{\widehat{M}(n)}\}_n$ is locally bounded in \mathbb{R}^3 , then after replacing by a subsequence, the $\widehat{M}(n)$ converge to a minimal lamination $\widehat{\mathcal{L}}$ of \mathbb{R}^3 with at least one non-flat leaf \widehat{L} . The difference now with the last paragraph in the proof of Proposition 5.7) is that we know that \widehat{L} is proper in \mathbb{R}^3 (since it has genus zero and by Proposition 5.7). Therefore, this current situation is even simpler than the one in the last paragraph of the proof of Proposition 5.7, where we discarded all the possibilities for \widehat{L} by using that its flux is vertical (for instance, the case of \widehat{L} being a two-limit ended planar domain is discarded using Theorem 6.4 below). This finishes the sketch of the proof of Claim 5.12.

Once we know that $\{\lambda_n M\}_n$ is locally simply-connected in $\mathbb{R}^3 - \{\vec{0}\}$, it can be proved that the limits of certain subsequences of $\{\lambda_n M\}_n$ consist of (possibly singular) minimal laminations \mathcal{L} of $H(*) = \{x_3 \geq 0\} - \{0\} \subset \mathbb{R}^3$ containing $\partial H(*)$ as a leaf. Subsequently, one checks that every such a limit lamination \mathcal{L} has no singular points and that the singular set of convergence $S(\mathcal{L})$ of $\lambda_n M$ to \mathcal{L} is empty. In particular, taking $\lambda_n = \|p_n\|^{-1}$ where p_n is

any divergent sequence on M , the fact that $S(\mathcal{L}) = \emptyset$ for the corresponding limit minimal lamination \mathcal{L} insures that the Gaussian curvature of M decays at least quadratically in terms of the distance function to the origin. In this situation, the Quadratic Curvature Decay Theorem (Theorem 5.13 below), insures that M has finite total curvature, which is impossible since M has an infinite number of ends. \square

We finish this section by stating an auxiliary result that was used in the last paragraph of the sketch of proof of Theorem 5.11.

THEOREM 5.13 (Quadratic Curvature Decay Theorem, Meeks, Pérez, Ros [63]). *Let $M \subset \mathbb{R}^3 - \{\vec{0}\}$ be an embedded minimal surface with compact boundary (possibly empty), which is complete outside the origin $\vec{0}$; i.e. all divergent paths of finite length on M limit to $\vec{0}$. Then, M has quadratic decay of curvature if and only if its closure in \mathbb{R}^3 has finite total curvature.*

6. Infinitely many ends II: two limit ends

Consider a properly embedded minimal planar domain $M \subset \mathbb{R}^3$ with two limit ends and horizontal limit tangent plane at infinity. Our goal in Sections 6 and 7 is to prove that M is one of the Riemann minimal examples introduced in Section 2.5.

From the preceding sections we know that the limit ends of M are its top and bottom ends and each middle end of M has a representative which is either planar or catenoidal. Indeed, all middle ends of M are planar: otherwise M contains a catenoidal end e with, say, logarithmic growth $a > 0$. Consider an embedded closed curve $\gamma \subset M$ separating the two limit ends of M (for this is useful to think of M topologically as a sphere minus a closed set \mathcal{E} which consists of an infinite sequence of points that accumulates at two different points). Let M_1 be the closure of the component of $M - \gamma$ which contains e . By embeddedness, all the annular ends of M_1 above e must have logarithmic growth at least a , which contradicts that the flux of M along $\gamma \subset M$ is finite.

Once we know that the middle ends of M are all planar, we can separate each pair of consecutive planar ends of M by a horizontal plane which intersects M in a compact set. An elementary analysis of the third coordinate function restricted to the subdomain $M(-) = M \cap \{-\infty < x_3 \leq 0\}$ (which is a parabolic Riemann surface with boundary by Theorem 4.3) shows that $M(-)$ is conformally equivalent to the closed unit disk $\overline{\mathbb{D}}$ minus a closed subset $\{\xi_k\}_k \cup \{0\}$ where $\xi_k \in \overline{\mathbb{D}} - \{0\}$ converges to zero as $k \rightarrow \infty$, and that the third coordinate function of M can be written as $x_3(\xi) = \alpha \log |\xi|$, $\xi \in \overline{\mathbb{D}}$, where $\alpha > 0$ (in particular, there are no points in M where the tangent plane is horizontal, or equivalently M intersects every horizontal plane of \mathbb{R}^3 in a Jordan curve or an open arc). After a suitable homothety so that M has vertical component of its flux vector along a compact horizontal

section equal to 1 (i.e. $\alpha = \frac{1}{2\pi}$), we deduce that the following properties hold:

- (1) M can be conformally parametrized by the cylinder $\mathbb{C}/\langle i \rangle$ (here $i = \sqrt{-1}$) punctured in an infinite discrete set of points $\{p_j, q_j\}_{j \in \mathbb{Z}}$ which correspond to the planar ends of M .
- (2) The stereographically projected Gauss map of M extends through the planar ends to a meromorphic function g on $\mathbb{C}/\langle i \rangle$ which has double zeros at the points p_j and double poles at the q_j (otherwise we contradict that M intersects every horizontal plane asymptotic to a planar end in an open arc).
- (3) The height differential of M is $dh = dz$ with z being the usual conformal coordinate on \mathbb{C} (equivalently, the third coordinate function of M is $x_3(z) = \Re(z)$).
- (4) The planar ends are ordered by their heights so that $\Re(p_j) < \Re(q_j) < \Re(p_{j+1})$ for all j with $\Re(p_j) \rightarrow \infty$ (resp. $\Re(p_j) \rightarrow -\infty$) when $j \rightarrow \infty$ (resp. $j \rightarrow -\infty$).

6.1. Curvature estimates and quasiperiodicity. The next step consists of proving that every surface M as before admits an estimate for its absolute Gaussian curvature that depends solely on an upper bound for the horizontal component of the flux vector $F(M)$ of M along a compact horizontal section. Note that $F(M)$ does not depend on the height of the horizontal plane which produces the compact section since the flux of a planar end is zero; for this reason, we will simply call $F(M)$ the *flux vector* of M .

THEOREM 6.1 (Meeks, Pérez, Ros [67]). *Let $\{M(i)\}_i \subset \mathcal{M}$ be a sequence of properly embedded minimal surfaces with genus zero, two limit ends, horizontal limit tangent plane at infinity and flux vector $F(M(i)) = (H(i), 1) \in \mathbb{R}^2 \times \mathbb{R}$. If $\{H(i)\}_i$ is bounded from above in \mathbb{R}^2 , then the Gaussian curvature function $K_{M(i)}$ of the $M(i)$ is uniformly bounded.*

SKETCH OF PROOF. Arguing by contradiction, assume $\{|K_{M(i)}|\}_i$ is not uniformly bounded. Then, we blow up $M(i)$ on the scale of curvature, which means that we choose suitable points $p(i) \in M(i)$ such that after translating $M(i)$ by $-p(i)$ and expanding by $\lambda(i) = \sqrt{|K_{M(i)}(p(i))|} \rightarrow +\infty$, we produce new properly embedded minimal surfaces $M'(i) \subset \mathbb{R}^3$ which, after passing to a subsequence, converge uniformly on compact subsets of \mathbb{R}^3 with multiplicity 1 to a properly embedded minimal surface $H \subset \mathbb{R}^3$ with $\vec{0} \in H$, $|K_H(\vec{0})| = 1$ and $|K_H| \leq 1$ in H . Since H is a limit with multiplicity 1 of surfaces of genus zero, then H has genus zero.

If H is not simply-connected, then the discussion in previous sections shows that one can find an embedded closed curve $\gamma \subset H$ such that the flux of H along γ is finite and non-zero. This is a contradiction, since γ produces related non-trivial loops $\gamma'(i) \subset M'(i)$ converging to γ as $i \rightarrow \infty$;

if we call $\gamma(i)$ to the loop in $M(i)$ which corresponds to $\gamma'(i)$ in the original scale, then the third component of the flux of $M(i)$ along $\gamma(i)$ times $\lambda(i)$ converges as $i \rightarrow \infty$ to the third component of the flux of H along γ , which is finite. Since $\lambda(i) \rightarrow \infty$, we deduce that the third component of the flux of $M(i)$ along $\gamma(i)$ tends to zero, which is impossible by our normalization on $F(i)$. Therefore, H is simply-connected.

Since H is a non-flat, properly embedded minimal surface which is simply-connected, Theorem 4.2 implies that H is a helicoid. Furthermore, H is a vertical helicoid, since its Gauss map does not take vertical directions (because the Gauss maps of the surfaces $M'(i)$ share the same property). Once this first helicoidal limit H of rescalings of the $M(i)$ has been found, one rescales and rotates again the $M(i)$ in a rather delicate way:

CLAIM 6.2. *There exist a universal $\tau_0 > 0$ and angles $\theta(i) \in [0, 2\pi)$ such that for any $\tau > \tau_0$, one can find numbers $\mu(\tau, i) > 0$ and embedded closed curves $\delta(\tau, i) \subset \mu(\tau, i)\text{Rot}_{\theta(i)}(M(i) - p(i))$ (here Rot_θ denotes the rotation of angle θ around the x_3 -axis) so that the flux of the rotated and rescaled surface $\mu(\tau, i)\text{Rot}_{\theta(i)}(M(i) - p(i))$ along $\delta(\tau, i)$ decomposes as*

$$(13) \quad \text{Flux}(\mu(\tau, i)\text{Rot}_{\theta(i)}(M(i) - p(i)), \delta(\tau, i)) = V(\tau, i) + W(\tau, i)$$

where $V(\tau, i), W(\tau, i) \in \mathbb{R}^3$ are vectors such that $\lim_{i \rightarrow \infty} V(\tau, i) = (12\tau, 0, 0)$ and $\{\|W(\tau, i)\|\}_i$ is bounded by a constant independent of τ .

Assuming this technical property, the proof of Theorem 6.1 finishes as follows. First note that for any properly embedded minimal surface $M \subset \mathbb{R}^3$ with genus zero, two limit ends and horizontal limit tangent plane at infinity, the angle between the flux vector $F(M)$ and its horizontal component $(H(M), 0)$ is invariant under translations, homotheties and rotations around the x_3 -axis. By (13), the corresponding angles for the flux vectors of the surfaces $\mu(\tau, i)\text{Rot}_{\theta(i)}(M(i) - p(i))$ tend to zero as $i \rightarrow \infty$ and $\tau \rightarrow \infty$. But those angles are nothing but the angles for $M(i)$, which are bounded away from zero because of the hypothesis of $\{H(M(i))\}_i$ being bounded above. This contradiction proves the theorem, modulo the above claim.

The proof of Claim 6.2 is delicate, and we will only mention that it uses some results of Colding-Minicozzi (Theorems 3.4 and 5.9) and the rescaling on the scale of topology that we explained in the proof of Proposition 5.7. For details, see [67]. □

With the curvature bound given in Theorem 6.1 in hand, one can use standard arguments based on the maximum principle to find an embedded regular neighborhood of constant positive radius bounded from below by a constant which only depends on the curvature bound of the minimal surface (see for instance Lemma 1 in [66]). The existence of such a regular neighborhood implies uniform local area bounds for a sequence of surfaces $\{M(i)\}_i$ under the hypotheses of Theorem 6.1. Finally, these curvature and

area bounds allow one to apply classical results for taking limits (after extracting a subsequence) of suitable translations of the $M(i)$. Summarizing, we have the next statement.

THEOREM 6.3. *Suppose M is a properly embedded minimal surface in \mathbb{R}^3 with genus zero and two limit ends. Assume that M is normalized by a rotation and homothety so that it has horizontal limit tangent plane at infinity and the vertical component of its flux equals 1. Then:*

- (1) *The middle ends $\{e_n \mid n \in \mathbb{Z}\}$ of M are planar and have heights $\mathcal{H} = \{x_3(e_n) \mid n \in \mathbb{Z}\}$ such that $x_3(e_n) < x_3(e_{n+1})$ for all $n \in \mathbb{Z}$;*
- (2) *$\lim_{n \rightarrow \infty} x_3(e_n) = \infty$ and $\lim_{n \rightarrow -\infty} x_3(e_n) = -\infty$;*
- (3) *Every horizontal plane intersects M in a simple closed curve when its height is not in \mathcal{H} and in a single properly embedded arc when its height is in \mathcal{H} ;*
- (4) *M has bounded Gaussian curvature, with the bound of its curvature depending only on an upper bound of the horizontal component of the flux of M .*
- (5) *If the Gaussian curvature of M is bounded from below in absolute value by ε^2 , then M has a regular neighborhood of radius $1/\varepsilon$ and so, the spacings $S(n) = x_3(e_{n+1}) - x_3(e_n)$ between consecutive ends are bounded from below by $2/\varepsilon$. Furthermore, these spacings are also bounded by above.*
- (6) *M is quasiperiodic in the following sense. There exists a divergent sequence $V(n) \in \mathbb{R}^3$ such that the translated surfaces $M + V(n)$ converge to a properly embedded minimal surface $M(\infty)$ of genus zero, two limit ends, horizontal limit tangent plane at infinity and with the same flux as M .*

SKETCH OF PROOF. Everything has been proved (at least, outlined with some details) except for item 6 and the last sentence of item 5. To see that item 6 holds, one starts by noticing that for any two consecutive planar ends e_n, e_{n+1} of M there is a point $p(n)$ with $x_3(e_n) < x_3(p(n)) < x_3(e_{n+1})$ where the tangent plane of M is vertical (by continuity, since the Gauss map alternates from the north pole to the south pole or vice versa at e_n, e_{n+1}). Extrinsicly close to $p(n)$ we can find a point $q(n) \in M$ such that $|K_M(q(n))| \geq \varepsilon > 0$, for ε fixed and small (otherwise we produce relatively large flat vertical regions in M around $p(n)$, which contradicts that the third component of the flux vector of M is one). Now one considers the translated surfaces $M - q(n)$, which have local area and curvature estimates. The arguments in the paragraph just before the statement of Theorem 6.3 give that a subsequence of the $M - q(n)$ converges with multiplicity 1 to a connected, non-flat, properly embedded minimal surface $M(\infty)$ of genus zero, with length of the vertical component of its flux vector at most 1. Since $M(\infty)$ is not flat, the open mapping property for the Gauss map of minimal surfaces implies that the Gauss map of $M(\infty)$ omits the vertical directions. Since the vertical component of the flux of $M(\infty)$ is finite, $M(\infty)$ cannot

be a helicoid. By Theorem 4.2, $M(\infty)$ is not simply-connected. Since it has genus zero, $M(\infty)$ has at least two ends, and thus it must have a planar or a catenoid type end (which must be horizontal). In particular, $M(\infty)$ has a horizontal limit tangent plane at infinity.

Next we check that $M(\infty)$ cannot have catenoidal ends. Otherwise, $M(\infty)$ contains a horizontal, strictly convex Jordan curve $\Gamma(\infty)$, along which this limit surface has vertical flux. Since $M(\infty)$ is the limit of (a subsequence of) $M - q(n)$, the curve $\Gamma(\infty)$ produces horizontal, strictly convex Jordan curves $\Gamma(n) \subset M$, with $\Gamma(n) - q(n) \rightarrow \Gamma(\infty)$ as $n \rightarrow \infty$. As the ends of M are planar and the flux of $M(\infty)$ along $\Gamma(\infty)$ is vertical, we deduce that the flux of M along $\Gamma(n)$ is also vertical for n large enough. Now one applies a variant of the López-Ros deformation argument to the portion of M bounded by $\Gamma(n) \cup \Gamma(n+k)$, with n, k large, to find a contradiction. Therefore, $M(\infty)$ cannot have catenoidal ends.

Finally, the non-existence of catenoidal ends on $M(\infty)$ implies that $M(\infty)$ neither has finitely many ends (recall it is not simply-connected) nor does it have exactly one limit end.¹³ Therefore, $M(\infty)$ has two limit ends and item 6 of Theorem 6.3 is proved. The last sentence in item 5 is also proved, since if the spacing between the ends of M were unbounded, then a variation of the above arguments would yield a limit surface of translations of M with a catenoidal end, which is impossible. \square

For the sake of completeness, we next state a technical result which was used at the end of the proofs of Proposition 5.7 and Claim 5.12. Although this proposition and this claim were stated in earlier sections of this article, chronologically they were obtained later than the next property.

THEOREM 6.4 (Meeks, Pérez, Ros [67]). *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with genus zero, two limit ends and horizontal limit tangent plane at infinity. Then, M has non-zero horizontal flux along any compact horizontal section.*

SKETCH OF PROOF. Arguing by contradiction, suppose that M has vertical flux. The main tool used here is again a variant of the López-Ros deformation (see Section 3). Since M has vertical flux, the López-Ros deformation produces a one-parameter family $\{X_\lambda: M \rightarrow \mathbb{R}^3\}_\lambda$ of complete minimal immersions with $X_1(M) = M$. Since the middle ends of M are planar, for λ large X_λ has self-intersections. As X_1 is an embedding, it makes sense to consider the largest $\lambda_0 \in [1, \infty)$ such that X_λ is injective for all $\lambda \in [1, \lambda_0)$. The usual maximum principle for minimal surfaces implies that limits of embedded minimal surfaces are embedded, from where we conclude that X_{λ_0} is also an embedding. The contradiction will follow by

¹³Although $M(\infty)$ is known to have genus zero, we cannot apply Theorem 5.11 here to avoid the possibility that $M(\infty)$ has just one limit end, since the end of the proof of Claim 5.12 uses this Theorem 6.3 to discard the existence of a certain minimal surface \hat{L} with two limit ends.

proving that, whenever X_λ is injective, then for nearby parameter values λ' , $X_{\lambda'}$ is also injective (just apply this property to $\lambda = \lambda_0$). First note that the height differential is preserved along the López-Ros deformation, and thus the heights of the planar ends of X_λ remain constant in λ . From here is not difficult to prove the desired openness property of the injective parameter values, provided that the surface M is periodic (invariant by a translation in \mathbb{R}^3), since one can then argue with the quotient surface under the translation, which has finitely many ends. The key observation here is that the quasiperiodicity property in item 6 of Theorem 6.3 applied to $X_{\lambda_0}(M)$ is enough to preserve the embeddedness of $X_\lambda(M)$ for nearby values of λ . \square

6.2. The Shiffman function. Next we explain the last tool necessary to finish our classification of properly embedded minimal planar domains with two limit ends: the *Shiffman function*. This is a particular example of a *Jacobi function*, which can be considered to be an infinitesimal deformation for a minimal surface M of the kind we are considering.

In the sequel we will denote by \mathcal{M} the space of properly embedded, minimal planar domains $M \subset \mathbb{R}^3$ with two limit ends, horizontal limit tangent plane at infinity and flux vector $F(M) = (h, 0, 1)$ for some $h = h(M) > 0$, identified up to translations. Recall that we proved that for every $M \in \mathcal{M}$, properties 1–4 stated before Theorem 6.1 hold. In particular, the planar sections $M \cap \{x_3 = c\}$, $c \in \mathbb{R}$, are either Jordan curves or Jordan arcs. In the natural conformal coordinate $z = x + iy$ such that the height differential of M is dz , the level curve $M \cap x_3^{-1}(c)$ corresponds to $y \mapsto z_c(y) = c + iy$ in the z -plane. The planar curvature of this level curve is

$$(14) \quad \kappa_c(y) = \left[\frac{|g|}{1 + |g|^2} \Re \left(\frac{g'}{g} \right) \right] \Big|_{z=z_c(y)},$$

where g is the stereographically projected Gauss map of M and the prime stands for derivative with respect to z .

DEFINITION 6.5. In the above setting, the *Shiffman function* of M is

$$(15) \quad S_M = \Lambda \frac{\partial \kappa_c}{\partial y} = \Im \left[\frac{3}{2} \left(\frac{g'}{g} \right)^2 - \frac{g''}{g} - \frac{1}{1 + |g|^2} \left(\frac{g'}{g} \right)^2 \right],$$

where the induced metric ds^2 on M by the inner product of \mathbb{R}^3 is $ds^2 = \Lambda^2 |dz|^2$ (i.e. $\Lambda = \frac{1}{2}(|g| + |g|^{-1})$) and \Im stands for imaginary part.

REMARK 6.6. *The Shiffman function can also be defined for surfaces not in \mathcal{M} : we only need $(g(z), dh = dz)$ to be the Weierstrass pair of a minimal surface with respect to a local conformal coordinate. Every minimal surface admits such a local representation around a point with non-vertical normal vector.*

Perhaps the most remarkable property of S_M is that, from a variational point of view, it is an infinitesimal deformation of M by minimal surfaces, i.e. S_M lies in the kernel of the *Jacobi operator* $L = \Delta - 2K$ of M (here K stands for the Gaussian curvature function of M). Functions in the kernel of L are called *Jacobi functions*. In his pioneering paper [93], Shiffman himself exploited this property when he proved that if a minimal annulus M is bounded by two circles in parallel planes, then M is foliated by circles in the intermediate planes. For a surface $M \in \mathcal{M}$ and with the notation of items 1–4 before the statement of Theorem 6.1, its Shiffman function S_M can be defined globally on $M = (\mathbb{C}/\langle i \rangle) - \{p_j, q_j\}_j$. By writing the local expression of g around a zero or pole, it is not difficult to check that S_M is bounded around the middle ends of M . Hence S_M has a continuous extension to the cylinder $\mathbb{C}/\langle i \rangle$ obtained after attaching to M its planar ends. Since S_M solves the Jacobi equation $LS_M = 0$ and L is of the form $\Delta + q$ for q smooth on $\mathbb{C}/\langle i \rangle$ (now Δ refers to the laplacian in the flat metric on $\mathbb{C}/\langle i \rangle$), then elliptic regularity implies that S_M extends smoothly to $\mathbb{C}/\langle i \rangle$. Another key property of S_M for every $M \in \mathcal{M}$ is that S_M is bounded on $\mathbb{C}/\langle i \rangle$, as we next explain.

DEFINITION 6.7. A meromorphic function $g: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ is said to be *quasiperiodic* if it satisfies the following two conditions:

- (1) There exists a constant $C > 0$ such that the distance between any two distinct points in $g^{-1}(\{0, \infty\}) \subset \mathbb{C}/\langle i \rangle$ is at least C and given any $p \in g^{-1}(\{0, \infty\}) - \{p\}$ of distance less than $1/C$ from p .
- (2) For any divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of the meromorphic functions $g_k(z) = g(z + z_k)$ which converges uniformly on compact subsets of $\mathbb{C}/\langle i \rangle$ to a non-constant meromorphic function $g_\infty: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ (thus g_∞ is also quasiperiodic).

Note that items 5, 6 of Theorem 6.3 imply that the (stereographically projected) Gauss map of a surface $M \in \mathcal{M}$ extends across the planar ends to a quasiperiodic meromorphic function. This quasiperiodicity of g together with equation (15) clearly imply that S_M is bounded on $\mathbb{C}/\langle i \rangle$.

Let us return to the definition of the Shiffman function. Since the conformal factor Λ in (15) is a positive function, we deduce that $S_M = 0$ on M if and only if M is foliated by pieces of circles and straight lines in horizontal planes. In the nineteenth century, Riemann [87, 88] classified all minimal surfaces with such a foliation property: they reduce to the examples appearing in Theorem 1.1. Therefore, a possible approach to proving Theorem 1.1 is to verify that $S_M = 0$. Instead of doing this directly, we will show that given $M \in \mathcal{M}$, the Shiffman function S_M can be integrated, in the sense that there exists a one-parameter family $\{M_t\}_t \subset \mathcal{M}$ such that $M_0 = M$ and the normal component of the variational vector of this variation, when restricted to each M_t , is exactly the Shiffman Jacobi function S_{M_t} . In fact,

in our arguments below we will need to complexify our framework by considering complex-valued Jacobi functions of the type $v + iv^*$, where v^* is the Jacobi conjugate function of a Jacobi function v (see the definition of $\mathcal{J}_{\mathbb{C}}(g)$ below) and *holomorphic* deformations $t \mapsto (g_t, dz)$ of the Weierstrass data (g, dz) of $M \in \mathcal{M}$; this holomorphicity of the deformation will be crucial in the proof of the linearity of the complex-valued Shiffman function $S_M + iS_M^* \in \mathcal{J}_{\mathbb{C}}(g)$ of $M \in \mathcal{M}$ (Proposition 6.17); in turn, the linearity of S_M will imply that M is periodic (Lemma 6.15) and subsequently, that M is a Riemann minimal example (Proposition 6.16). In order for this framework to make sense, we need discuss some technical issues.

First note that the Weierstrass pair of every $M \in \mathcal{M}$ depends solely on the Gauss map g (not on the height differential $dh = dz$), hence one can think of a deformation of M inside \mathcal{M} as a certain kind of deformation of g in an appropriate space of *allowed* Gauss maps; namely, we consider the space of meromorphic functions

$$\mathcal{W} = \left\{ g: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\} \text{ quasiperiodic} \mid (g) = \prod_{j \in \mathbb{Z}} p_j^2 q_j^{-2} \right\},$$

where (g) denotes the divisor of zeros and poles of g on $\mathbb{C}/\langle i \rangle$. By Theorem 6.3, the Gauss map of every $M \in \mathcal{M}$ lies in \mathcal{W} . Note that by Definition 6.7, any limit g_{∞} of a convergent subsequence of $g_k(z) = g(z + z_k)$, with $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$ being a divergent sequence, satisfies that g_{∞} lies in \mathcal{W} . The divisor of zeros $Z = \prod_j p_j^2$ of a function $g \in \mathcal{W}$ is quasiperiodic, in the sense that for every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of $\{Z + z_k\}_k$ which converges in the Hausdorff distance on compact subsets of $\mathbb{C}/\langle i \rangle$ to a divisor Z_{∞} in $\mathbb{C}/\langle i \rangle$, and a similar property holds for poles. Reciprocally, two disjoint quasiperiodic divisors $Z = \prod_j p_j^2$, $P = \prod_j q_j^2$ in $\mathbb{C}/\langle i \rangle$ define a unique quasiperiodic meromorphic function g (up to multiplicative non-zero constants) whose principal divisor is $(g) = Z/P$. Existence is given in Douady and Douady [26], while uniqueness follows from the Liouville theorem together with fact that the function $f = g_1/g_2$, where $g_1, g_2 \in \mathcal{W}$ have $(g_1) = (g_2)$, is bounded on $\mathbb{C}/\langle i \rangle$ (if f were unbounded on $\mathbb{C}/\langle i \rangle$, then we could find a divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$ such that $f(z_k)$ diverges; but the quasiperiodicity of g_1 and g_2 implies that, after extracting a subsequence, $f_k(z) = f(z + z_k)$ converges uniformly on compact subsets of \mathbb{C} to a meromorphic function $f_{\infty}: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ which is not constant infinity, thus f_{∞} has no poles by the Hurwitz theorem, which contradicts that $f_k(0) = f(z_k) \rightarrow \infty$ as $k \rightarrow \infty$). This association, to each $g \in \mathcal{W}$ of the quasiperiodic set of its zeros and poles in $\mathbb{C}/\langle i \rangle$ together with the value of g at a prescribed point $z_0 \in (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$, leads us to define the notion of holomorphic dependence of a curve $t \mapsto g_t \in \mathcal{W}$:

DEFINITION 6.8. Given $\varepsilon > 0$, we say that a curve $t \in \mathbb{D}(\varepsilon) := \{t \in \mathbb{C} \mid |t| < \varepsilon\} \rightarrow g_t \in \mathcal{W}$ is *holomorphic* if the corresponding functions

$p_j(t), q_j(t), g_t(z_0)$ depend holomorphically on t . In this case, the function $\dot{g}: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ given by $z \in \mathbb{C}/\langle i \rangle \mapsto \frac{d}{dt}|_{t=0} g_t(z)$ is meromorphic on $\mathbb{C}/\langle i \rangle$. We will call \dot{g} the *infinitesimal deformation of g associated to the curve $t \mapsto g_t$* .

If \dot{g} is the infinitesimal deformation of $g = g_0 \in \mathcal{W}$ associated to the curve $t \mapsto g_t$ and g has principal divisor $(g) = \prod_j p_j^2 q_j^{-2}$, then the principal divisor of \dot{g} clearly satisfies $(\dot{g}) \geq \prod_j p_j q_j^{-3}$. The converse also holds: if $g \in \mathcal{W}$ and f is a meromorphic function on $\mathbb{C}/\langle i \rangle$ whose principal divisor verifies $(f) \geq \prod_j p_j q_j^{-3}$, then f can be proved to be the infinitesimal deformation of g associated to a holomorphic curve $t \mapsto g_t \in \mathcal{W}$ with $g_0 = g$. This leads to define a sort of *tangent space at g to \mathcal{W}* , which is formally the complex linear space of infinitesimal deformations of g associated to holomorphic curves:

$$(16) \quad T_g \mathcal{W} = \left\{ f: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\} \text{ meromorphic} \mid (f) \geq \prod_j p_j q_j^{-3} \right\}.$$

REMARK 6.9. Given $g \in \mathcal{W}$, the functions $g, g' \in T_g \mathcal{W}$ are respectively the *infinitesimal deformations at $t = 0$ associated to the holomorphic curves $t \mapsto (t + 1)g(z), t \mapsto g(z + t)$* .

The usual period problem can be reformulated in \mathcal{W} as follows. Let $\gamma = \{it \mid t \in [0, 1]\}$ be the generator of the homology of the cylinder $\mathbb{C}/\langle i \rangle$. Given $g \in \mathcal{W}$, the list $((\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\}), g, dh = dz)$ is the Weierstrass data of a complete, immersed minimal surface in \mathbb{R}^3 with individually embedded horizontal planar ends at the zeros and poles of g if and only if the corresponding period problem (6) can be solved (note that condition (5) holds automatically). Equation (6) can be encoded in the *period map* $\text{Per}: \mathcal{W} \rightarrow \mathbb{C}^2 \times \mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}}$, defined as

$$(17) \quad \text{Per}(g) = \left(\int_{\gamma} \frac{dz}{g}, \int_{\gamma} g dz, \left\{ \text{Res}_{p_j} \left(\frac{dz}{g} \right) \right\}_j, \left\{ \text{Res}_{q_j} (g dz) \right\}_j \right).$$

The subset of $g \in \mathcal{W}$ such that (g, dz) solves the period problem can be written as

$$(18) \quad \mathcal{M}_{\text{imm}} = \text{Per}^{-1}\{(a, \bar{a}, 0, 0) \mid a \in \mathbb{C}\}.$$

DEFINITION 6.10. A *quasiperiodic, immersed minimal surface of Riemann type* is a minimal surface $M \subset \mathbb{R}^3$ which admits a Weierstrass pair of the form (g, dz) on $(\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$ where g lies in \mathcal{M}_{imm} .

Given $g \in \mathcal{W}$ (not necessarily in \mathcal{M}_{imm}), we denote by $N = \left(\frac{2g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1} \right) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ its related spherical ‘‘Gauss map’’. The Jacobi equation $Lv = \Delta v + |\nabla N|^2 v = 0$ can be formally written as

$$(19) \quad v_{z\bar{z}} + 2 \frac{|g'|^2}{(1 + |g|^2)^2} v = 0.$$

A function $v: (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\}) \rightarrow \mathbb{R}$ which satisfies equation (19) is called a *Jacobi function associated to g* . We will denote by $\mathcal{J}(g)$ to the space of Jacobi functions associated to g . Since $N: (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\}) \rightarrow \mathbb{S}^2$ is harmonic, then $\Delta N + |\nabla N|^2 N = 0$, which implies that the functions $v = \langle N, a \rangle$ where $a \in \mathbb{R}^3$, are always Jacobi functions associated to g . Such functions will be called *linear Jacobi functions*.

If $M \subset \mathbb{R}^3$ is a quasiperiodic, immersed minimal surface of Riemann type (i.e. $g \in \mathcal{M}_{\text{imm}}$), then we define its *Shiffman function* S_M by equation (15). Note that the formula (15) makes sense for every $g \in \mathcal{W}$, but in order S_M to be bounded at each point in $g^{-1}(\{0, \infty\})$ we need g to lie in \mathcal{M}_{imm} .

Given a minimal surface M , let $B(N) \subset M$ denote the set of branch points of the Gauss map N of M . Given a Jacobi function $v: M \rightarrow \mathbb{R}$, the formula

$$(20) \quad X_v := vN + \frac{1}{|N_z|^2} \{v_z N_{\bar{z}} + v_{\bar{z}} N_z\}: M - B(N) \rightarrow \mathbb{R}^3$$

defines a branched minimal immersion (possibly constant) with the same Gauss map N as M , where z is any local conformal coordinate on M . The *support function*¹⁴ of X_v is $\langle X_v, N \rangle = v$. The correspondence $v \mapsto X_v$ was studied by Montiel and Ros [77] (see also Ejiri and Kotani [27]). Linear Jacobi functions $v = \langle N, a \rangle$, $a \in \mathbb{R}^3$, produce constant maps X_v (and vice versa), and the law $v \mapsto X_v$ can be considered to be a linear isomorphism from the linear space of Jacobi functions on M modulo the subspace of linear Jacobi functions, onto the linear space of all branched minimal immersions $X: M - B(N) \rightarrow \mathbb{R}^3$ with Gauss map N modulo the constant maps.

The *conjugate Jacobi function* v^* of a Jacobi function $v: M \rightarrow \mathbb{R}$ is defined (locally) as the support function $\langle (X_v)^*, N \rangle$ of the conjugate minimal immersion $(X_v)^*$ (recall that such a conjugate minimal immersion is an isometric minimal immersion of the underlying Riemannian surface, whose coordinate functions are the harmonic conjugates to the ones of X_v and whose Gauss map is the same as for M). Since the conjugate minimal immersion is defined modulo additive vector-valued constants, then v^* is defined up to additive scalar constants. Furthermore, v^* is globally well-defined precisely when $(X_v)^*$ is globally well-defined.

We consider the complex linear space

$$\mathcal{J}_{\mathbb{C}}(g) = \{v + iv^* \mid v \in \mathcal{J}(g) \text{ and } v^* \text{ is globally defined}\}.$$

Thus, $\mathcal{J}_{\mathbb{C}}(g)$ is the space of support functions $\langle X, N \rangle$ of holomorphic maps $X: (\mathbb{C}/\langle i \rangle) - B(N) \rightarrow \mathbb{C}^3$ whose real and imaginary parts have differentials orthogonal to $N = \left(\frac{2g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1} \right) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$. The (complex) linear

¹⁴Since dilations of a minimal immersion $X: M \rightarrow \mathbb{R}^3$ are also minimal and have the same Gauss map N as X , the normal component of the variational part of the deformation $X_t = \langle tX, N \rangle$ is a Jacobi function. This function $\langle X, N \rangle$ is called the *support function* of X .

functions of g , $L_{\mathbb{C}}(g) := \{\langle N, a \rangle \mid a \in \mathbb{C}^3\}$, form a complex linear subspace of $\mathcal{J}_{\mathbb{C}}(g)$.

With the above framework, we are ready to state a precise formulation for the integration of the Shiffman function of every surface $M \in \mathcal{M}$, which will be proved in Section 7.

THEOREM 6.11. *Let M be a quasiperiodic, immersed minimal surface of Riemann type, with Gauss map $g : \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ and height differential dz . Let $(g) = \prod_{j \in \mathbb{Z}} p_j^2 q_j^{-2}$ be the principal divisor of g and let $z_0 \in (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$. Then, the Shiffman function S_M of M can be holomorphically integrated in the following sense: There exist $\varepsilon > 0$ and families $\{p_j(t)\}_j$, $\{q_j(t)\}_j \subset \mathbb{C}/\langle i \rangle$, $a(t) \in \mathbb{C} - \{0\}$ such that*

- i) For each $j \in \mathbb{Z}$, the functions $t \in \mathbb{D}(\varepsilon) \mapsto p_j(t)$, $t \mapsto q_j(t) \in \mathbb{C}/\langle i \rangle$ are holomorphic with $p_j(0) = p_j$, $q_j(0) = q_j$. Also, the function $t \mapsto a(t)$ is holomorphic as well.*
- ii) For any $t \in \mathbb{D}(\varepsilon)$, the divisor $\prod_j p_j(t)^2 q_j(t)^{-2}$ defines an element $g_t \in \mathcal{M}_{\text{imm}}$ with $g_0 = g$ and $g_t(z_0) = a(t)$. Let M_t be the quasiperiodic, immersed minimal surface of Riemann type with Weierstrass pair (g_t, dz) .*
- iii) For every $t \in \mathbb{D}(\varepsilon)$, the derivative of $t \mapsto g_t$ with respect to t equals*

$$\frac{d}{dt}g_t = \frac{i}{2} \left(g_t''' - 3 \frac{g_t' g_t''}{g_t} + \frac{3}{2} \frac{(g_t')^3}{g_t^2} \right) \quad \text{on } \mathbb{C}/\langle i \rangle.$$

Furthermore if M is embedded, then the surfaces M_t are also embedded for $|t|$ sufficiently small.

The condition *iii*) in Theorem 6.11 implies that if $\Psi_t = \left(\frac{1}{2}(\frac{1}{g_t} - g_t), \frac{i}{2}(\frac{1}{g_t} + g_t), 1\right) dz$ is the usual Weierstrass form associated to the pair (g_t, z) , then

$$(21) \quad \left\langle \frac{d}{dt} \Big|_t \int^z \Psi_t, N_t \right\rangle = -\frac{1}{2}(S_{M_t} + iS_{M_t}^*) + \langle a(t), N_t \rangle,$$

where $N_t = \left(\frac{2\Re(g_t)}{|g_t|^2+1}, \frac{2\Im(g_t)}{|g_t|^2+1}, \frac{|g_t|^2-1}{|g_t|^2+1}\right)$ is the spherical Gauss map of M_t and $a(t) \in \mathbb{C}^3$. In other words, the normal component of the complex-valued variational field of $t \mapsto (g_t, dz)$ is proportional to the (complex) Shiffman function $S_{M_t} + iS_{M_t}^*$ of M_t modulo an additive linear function of N_t , for all t . To understand why this is true, we need two auxiliary results which will be also used in Section 7.

PROPOSITION 6.12. *Given $g \in \mathcal{M}_{\text{imm}}$, we have:*

- (1) Let $h : \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic function which is a rational expression of g and its derivatives with respect to z up to some order, such that*

$$(22) \quad \dot{g}(h) = \left(\frac{g^3 h'}{2g'} \right)'$$

belongs to $T_g\mathcal{W}$. Then, the complex valued function

$$(23) \quad f(h) = \frac{g^2 h'}{g'} + \frac{2gh}{1 + |g|^2}$$

lies in $\mathcal{J}_{\mathbb{C}}(g)$, is quasiperiodic¹⁵ and bounded on $\mathbb{C}/\langle i \rangle$. Furthermore, for every closed curve $\Gamma \subset \mathbb{C}/\langle i \rangle$,

$$(24) \quad \int_{\Gamma} \frac{\dot{g}(h)}{g^2} dz = \int_{\Gamma} \dot{g}(h) dz = 0.$$

(2) Reciprocally, if $\dot{g} \in T_g\mathcal{W}$ satisfies (24), then there exists a meromorphic function h on $\mathbb{C}/\langle i \rangle$ such that (22) holds.

SKETCH OF PROOF. To prove item 1, assume $\dot{g}(h) \in T_g\mathcal{W}$. Then there exists a holomorphic curve $t \mapsto g_t \in \mathcal{W}$ such that $g_0 = g$ and $\frac{d}{dt}|_{t=0} g_t = \dot{g}(h)$. Therefore $\langle \frac{d}{dt}|_0 \int^z \Psi_t, N \rangle \in \mathcal{J}_{\mathbb{C}}(g)$, where $\Psi_t = \left(\frac{1}{2} \left(\frac{1}{g_t} - g_t \right), \frac{i}{2} \left(\frac{1}{g_t} + g_t \right), 1 \right) dz$ and $N = \left(\frac{2\Re(g)}{|g|^2+1}, \frac{2\Im(g)}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1} \right)$. A simple calculation gives

$$(25) \quad \int^z \frac{\dot{g}(h)}{g^2} dz = \frac{gh'}{2g'} + h, \quad \int^z \dot{g}(h) dz = \frac{g^3 h'}{2g'}$$

up to additive complex numbers, from where it is straightforward to get

$$(26) \quad \left\langle \frac{d}{dt} \Big|_0 \int^z \Psi_t, N \right\rangle = -\frac{1}{2} f(h) + \langle a, N \rangle,$$

for some $a \in \mathbb{C}^3$. Equation (26) implies that $f \in \mathcal{J}_{\mathbb{C}}(g)$. The quasiperiodicity of $f(h)$ follows directly from the quasiperiodicity of g (recall that h is a rational function of g and its derivatives). In order to prove that $f(h)$ is bounded on $\mathbb{C}/\langle i \rangle$, one first shows that $f(h)$ is bounded around every zero and pole of g and around every zero of g' (this is a direct computation using the behavior of $\dot{g}(h)$ at zeros and poles of g , see the definition of $T_g\mathcal{W}$). Then the boundedness of $f(h)$ in $\mathbb{C}/\langle i \rangle$ can be reduced to work away from the discrete set $g^{-1}(\{0, \infty\}) \cup (g')^{-1}(0)$, where it follows directly because g is quasiperiodic, h is a rational expression of g and its derivatives, and f is given in terms of g, h by the formula (23). Therefore $f(h)$ is bounded on $\mathbb{C}/\langle i \rangle$. Finally, (24) is a direct consequence of (25). This gives item 1 of the proposition. Concerning item 2, equation (25) together with the hypothesis (24) allow us to find a meromorphic function h on $\mathbb{C}/\langle i \rangle$ such that (22) holds. \square

¹⁵Given $g \in \mathcal{W}$, a Jacobi function $v \in \mathcal{J}(g)$ is said to be *quasiperiodic* if for every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of the functions $v_k(z) = v(z + z_k)$ which converges uniformly on compact subsets of $(\mathbb{C}/\langle i \rangle) - g_{\infty}^{-1}(\{0, \infty\})$ to a function v_{∞} , where $g_{\infty} \in \mathcal{W}$ is the limit of (a subsequence of) $\{g_k(z) = g(z + z_k)\}_k$, which exists since g is quasiperiodic. Note that $v_{\infty} \in \mathcal{J}(g_{\infty})$ and that if v_{∞} is constant, then $v_{\infty} = 0$. A similar definition can be made exchanging $\mathcal{J}(g)$ by $\mathcal{J}_{\mathbb{C}}(g)$.

REMARK 6.13. *An important interpretation of equation (24) is that $\dot{g}(h)$ lies in the kernel of the differential $d\text{Per}_g$ of the period map at $g \in \mathcal{M}_{\text{imm}}$, see equation (17).*

It is clarifying to analyze some examples of $\dot{g}(h)$ and $f(h)$ coming from particular choices of h in Proposition 6.12. If one takes $h = c_1 + \frac{c_2}{g^2}$ in (22) with $c_1, c_2 \in \mathbb{C}$, then $\dot{g}(h) = 0$ (and vice versa). In this case, $f(h)$ is a complex linear combination of $\frac{g}{1+|g|^2}, \frac{\bar{g}}{1+|g|^2}$, which can be viewed as a horizontal linear function of the ‘‘Gauss map’’ g . If one takes $h = \frac{1}{g}$ in (22), then $\dot{g}(h) = -\frac{1}{2}g'$ and $f(h) = \frac{1-|g|^2}{1+|g|^2}$, which is a vertical linear function of g . A slightly more subtle choice of h explains the nature of condition *iii*) in Theorem 6.11:

COROLLARY 6.14. *Let M be a quasiperiodic, immersed minimal surface of Riemann type. Then, its Shiffman function S_M given by (15) admits a globally defined conjugate Jacobi S_M^* , and $S_M + iS_M^* = f$ is given by equation (23) for*

$$(27) \quad h = h_S = \frac{i(g')^2}{2g^3}.$$

In particular:

- (1) *Both S_M, S_M^* are bounded on the cylinder $M \cup g^{-1}(\{0, \infty\})$.*
- (2) *The corresponding infinitesimal deformation $\dot{g}_S = \dot{g}(h_S) \in T_g\mathcal{W}$ is given by*

$$(28) \quad \dot{g}_S = \frac{i}{2} \left(g''' - 3\frac{g'g''}{g} + \frac{3(g')^3}{g^2} \right).$$

- (3) *If $\dot{g}_S = 0$ on M , then both S_M, S_M^* are linear.*

PROOF. Note that h defined by (27) is a rational expression of g and g' . A direct computation gives that plugging (27) into (22) we obtain (28), and that this last expression has the correct behavior at $g^{-1}(\{0, \infty\})$ expressed in (16), thus $\dot{g}_S \in T_g\mathcal{W}$. Now item 1 of this corollary follows from Proposition 6.12. Regarding item 3, if $\dot{g}(h_S) = 0$ then (22) gives $h_S = b - \frac{c}{g^2}$ for $b, c \in \mathbb{C}$. Substituting in (23) we have $S_M + iS_M^* = 2c\frac{\bar{g}}{1+|g|^2} + 2b\frac{g}{1+|g|^2}$. Hence, both S_M, S_M^* are linear. \square

We said above that a way of finishing the proof of Theorem 1.1 is to show that if $M \in \mathcal{M}$, then $S_M = 0$. Next we explain that it is enough to demonstrate a weaker condition, namely that S_M is linear. This is the goal of Lemma 6.15 and Proposition 6.16 below, which do not use Theorem 6.11 in their proofs.¹⁶

¹⁶Theorem 6.11 will be used when proving the linearity of S_M for every $M \in \mathcal{M}$, see Proposition 6.17.

LEMMA 6.15. *Suppose that the Shiffman function S_M of a quasiperiodic, immersed minimal surface of Riemann type $M \subset \mathbb{R}^3$, is linear. Then, M is singly-periodic (invariant by a translation), and its smallest orientable quotient surface $M/\langle v \rangle \subset \mathbb{R}^3/\langle v \rangle$ is a properly immersed minimal torus with two planar ends and total curvature -8π , where $v \in \mathbb{R}^3 - \{0\}$ is a translation vector of M .*

SKETCH OF PROOF. The argument uses the notion of conjugate Jacobi functions and the Montiel-Ros correspondence between Jacobi functions and branched minimal immersions explained before the statement of Theorem 6.11. Suppose that M is a minimal surface verifying the hypotheses in the lemma, and let N be its Gauss map. Since the Shiffman function S_M of M is assumed to be linear, then its related branched minimal immersion X_v is constant, where $v = S_M$. Thus the conjugate minimal immersion $(X_v)^*$ is also constant, which implies that the conjugate Jacobi function S_M^* of S_M exists globally and it is linear. The linearity of the complex valued Jacobi function $S_M + iS_M^*$ together with equation (15) imply that there exists $a \in \mathbb{C}^3$ such that

$$(29) \quad \frac{3}{2} \left(\frac{g'}{g} \right)^2 - \frac{g''}{g} - \frac{1}{1 + |g|^2} \left(\frac{g'}{g} \right)^2 = \langle N, a \rangle.$$

After writing the last right-hand-side in terms of g and manipulating algebraically the resulting equation, one arrives to the following ODE for g :

$$\bar{g} \left(\frac{3}{2} \frac{(g')^2}{g} - g'' - B - a_3 g \right) = \frac{g''}{g} - \frac{1}{2} \left(\frac{g'}{g} \right)^2 + Ag - a_3,$$

where $a = (a_1, a_2, a_3)$, $2a_1 = A + B$ and $2a_2 = i(A - B)$. Since g is holomorphic and not constant, we deduce that

$$\frac{3}{2} \frac{(g')^2}{g} - g'' - B - a_3 g = 0, \quad \frac{g''}{g} - \frac{1}{2} \left(\frac{g'}{g} \right)^2 + Ag - a_3 = 0.$$

After elimination of g'' in both equations, we have $(g')^2 = g(-Ag^2 + 2a_3g + B)$. This equation implies that the map $\pi = (g, g')$ is a possibly branched holomorphic covering from the cylinder $M \cup \{\text{planar ends}\} \equiv \mathbb{C}/\langle i \rangle$ onto the compact Riemann surface $\Sigma = \{(\xi, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = \xi(-A\xi^2 + 2a_3\xi + B)\}$. Clearly, Σ is either a sphere or a torus.

Suppose for the moment that Σ is a sphere. Then, consider the meromorphic differential $\frac{d\xi}{w}$ on Σ , whose pullback by π is $\pi^*\left(\frac{d\xi}{w}\right) = \frac{dg}{g'} = dz$. Given a pole $P \in \Sigma$ of $\frac{d\xi}{w}$, choose a point $z_0 \in \mathbb{C}/\langle i \rangle$ such that $\pi(z_0) = P$. The residue of $\frac{d\xi}{w}$ at P can be computed as the integral of $\frac{d\xi}{w}$ along a small closed curve $\Gamma_P \subset \Sigma$ which winds once around P . After lifting Γ_P through π locally around z_0 we obtain a closed curve $\tilde{\Gamma}_P \subset \mathbb{C}/\langle i \rangle$ which winds a positive integer number of times around z_0 , depending on the branching order of π at z_0 . Hence the residue of $\frac{d\xi}{w}$ at P equals a positive integer multiple of the residue of dz at z_0 , which is zero. Therefore, $\frac{d\xi}{w}$ has residue zero at

all its poles, and so, it is exact on Σ . This implies that dz is also exact on $\mathbb{C}/\langle i \rangle$, which is impossible. This contradiction proves that Σ is a torus.

Finally, consider on Σ the Weierstrass pair $\left(g_1(\xi, w) = \xi, dh_1 = \frac{d\xi}{w}\right)$.

The metric associated to this pair is $\left(\frac{1}{2}(|\xi| + |\xi|^{-1})\frac{|d\xi|}{|w|}\right)^2$, which can be easily proven to be positive definite and complete in $\Sigma - \{(0, 0), (\infty, \infty)\}$. Note that $g_1 \circ \pi = g$ and $\pi^*\left(\frac{d\xi}{w}\right) = dz$. This implies that the Weierstrass pair (g, dz) of M can be induced on the twice punctured torus $\Sigma - \{(0, 0), (\infty, \infty)\}$, from where the lemma follows easily. \square

PROPOSITION 6.16. *If the Shiffman function S_M of an embedded surface $M \in \mathcal{M}$ is linear, then M is a Riemann minimal example.*

SKETCH OF PROOF. Let $\mathcal{M}_1 \subset \mathcal{M}$ be the subset of surfaces which are singly-periodic and their smallest orientable quotient is a properly embedded, twice-punctured minimal torus in a quotient of \mathbb{R}^3 by a translation. By Lemma 6.15, our proposition reduces to proving that \mathcal{M}_1 coincides with the family of Riemann minimal examples. This is a particular case of the main theorem in the 1998 paper [66] by Meeks, Pérez and Ros, and we now give some ideas of its proof.

Consider the flux map $h: \mathcal{M}_1 \rightarrow (0, \infty)$ which associates to each surface $M \in \mathcal{M}_1$ the positive number $h(M)$ such that $F = (h(M), 0, 1)$ is the flux vector of M . The goal is to show that h is a homeomorphism if we endow \mathcal{M}_1 with the uniform topology on compact sets. This is a consequence of the following steps.

- (A) The family $\mathcal{R} \subset \mathcal{M}_1$ of Riemann minimal examples is a connected component of \mathcal{M}_1 .
- (B) $h: \mathcal{M}_1 \rightarrow (0, \infty)$ is a proper map.
- (C) $h: \mathcal{M}_1 \rightarrow (0, \infty)$ is an open map.
- (D) There exists $\varepsilon > 0$ such that if $M \in \mathcal{M}_1$ and $h(M) \in (0, \varepsilon)$, then $M \in \mathcal{R}$.

We briefly indicate how the above four steps can be proved. In order to show (A), first note that \mathcal{R} is a path-connected set in \mathcal{M}_1 (by construction), and \mathcal{R} is closed in \mathcal{M}_1 (since Riemann minimal examples are characterized in \mathcal{M}_1 by being foliated by circles and lines in parallel planes, a condition which is preserved under limits in the uniform topology on compact sets). Hence (A) will follow if we check that \mathcal{R} is open in \mathcal{M}_1 . We will give two different reasons of why this openness holds. The first one is based on the fact that each Riemann minimal example R is non-degenerate in the sense that its space of periodic, bounded Jacobi functions reduces to the linear Jacobi functions (as follows from Montiel and Ros [77] since all the branch values of the Gauss map of R lie on a spherical equator), together with the property that being non-degenerate is an open condition (see Pérez [83]). The second proof is by contradiction: otherwise we find surfaces $\{\Sigma_n\}_n \subset \mathcal{M}_1 - \mathcal{R}$ which converge on compact subsets of \mathbb{R}^3 to some Riemann minimal example

$R \in \mathcal{R}$. Since $\Sigma_n \notin \mathcal{R}$, then the Shiffman functions S_{Σ_n} of the Σ_n cannot be identically zero. After normalizing $\widehat{S}_{\Sigma_n} = \frac{1}{\sup_{\Sigma_n} |S_{\Sigma_n}|} S_{\Sigma_n}$ (recall that S_{Σ_n} is bounded on Σ_n , hence $\sup_{\Sigma_n} |S_{\Sigma_n}|$ exists), we have a sequence of bounded Jacobi functions, for which a subsequence converges to a periodic Jacobi function \widehat{S}_∞ on R . Recall that the zeros of the Shiffman function of a surface are the critical points of the curvature of the horizontal level sections of this surface. By the Four Vertex Theorem, each \widehat{S}_{Σ_n} has at least four zeros on each compact horizontal section of Σ_n (counted with multiplicity), and thus, the same holds for \widehat{S}_∞ on each compact horizontal section of R . On the other hand, \widehat{S}_∞ is a periodic, bounded Jacobi function on R , hence it is linear. Now one finishes by checking that a linear Jacobi function on a Riemann minimal example has at most two zeros on each horizontal circle, and that these zeros are simple (see Assertion 9.5 in Meeks, Pérez and Ros [64]).

The properness of h in step **(B)** is a direct consequence of the curvature estimates in Theorem 6.1 (actually one only needs a weaker version of this curvature estimates, namely for *singly-periodic* minimal planar domains, which was proved for the first time in Theorem 4 of [66]).

The openness property **(C)** can be proved as follows. Consider the space $\mathcal{W}_1 = \{(\Sigma, g, [\alpha])\}$ where Σ is a compact Riemann surface of genus one, $g: \Sigma \rightarrow \mathbb{C} \cup \{\infty\}$ is a degree two meromorphic function with a double zero p and a double pole q , and $[\alpha]$ is a homology class in $H_1(\Sigma - \{p, q\}, \mathbb{Z})$ which is non-trivial in $H_1(\Sigma, \mathbb{Z})$. We denote the elements in \mathcal{W}_1 simply by g . There exists a natural inclusion $\mathcal{W}_1 \subset \mathcal{W}$ that unwraps both Σ and g so that $[\alpha]$ remains in the lifted cylinder that cyclically covers Σ (elements in \mathcal{W}_1 can be viewed as singly-periodic elements in \mathcal{W}). The space \mathcal{W}_1 is a two-dimensional complex manifold with local charts given by $g \mapsto (a_1 + a_2, a_1 \cdot a_2)$, where $a_1, a_2 \in \mathbb{C} - \{0\}$ are the (possibly equal) branch values of $g \in \mathcal{W}_1$ close to a given element $g_0 \in \mathcal{W}_1$ (in a chart, we can forget about the homology class $[\alpha]$ associated to g after identification with that of g_0). Given $g \in \mathcal{W}_1$, we associate a unique holomorphic differential ϕ on Σ by the equation $\int_\alpha \phi = i$. Consider the related *period map* $\text{Per}_1: \mathcal{W}_1 \rightarrow \mathbb{C}^2$ given by $\text{Per}_1(g) = \left(\int_\alpha \frac{\phi}{g}, \int_\alpha g \phi \right)$ (compare with (17) and recall that we are considering singly-periodic minimal surfaces with only two ends, so in order to express the period problem we do not need to impose the vanishing of the residue of $\frac{\phi}{g}$ at p or of $g \phi$ at q because the residues of a meromorphic differential on a compact Riemann surface add up to zero). Then, the space of elements $g \in \mathcal{W}_1$ such that (g, ϕ) is the Weierstrass pair of an *immersed* minimal surface are $\mathcal{M}_1^{\text{imm}} = \text{Per}_1^{-1}(\{(a, \bar{a}) \mid a \in \mathbb{C}\})$. Since Per_1 is a holomorphic map, for $a \in \mathbb{C}$ fixed the set $\mathcal{M}_1^{\text{imm}}(a) = \text{Per}_1^{-1}(a)$ is a complex analytic subvariety of \mathcal{W}_1 . As the limit of embedded surfaces is embedded, the subset $\mathcal{M}_1 \subset \mathcal{M}_1^{\text{imm}}$ of *embedded* surfaces is closed in $\mathcal{M}_1^{\text{imm}}$. An application the maximum principle at infinity (Theorem 2.3) gives that \mathcal{M}_1 is also open in $\mathcal{M}_1^{\text{imm}}$. In particular, the set $\mathcal{M}_1(a) = \mathcal{M}_1^{\text{imm}}(a) \cap \mathcal{M}_1$ is a complex analytic subvariety of \mathcal{W}_1 . By the uniform curvature estimates for *embedded* (possibly

non-singly-periodic) surfaces in Theorem 6.1 and subsequent uniform local area estimates, we deduce that $\mathcal{M}_1(a)$ is compact. Since the only compact, complex analytic subvarieties of \mathcal{W}_1 are finite sets, we deduce that $\mathcal{M}_1(a)$ is finite. Thus, given $M \in \mathcal{M}_1$, there exists an open neighborhood U of M in \mathcal{W}_1 such that $U \cap \mathcal{M}_1(a) = U \cap \mathcal{M}_1^{\text{imm}}(a) = \{M\}$. In this setting, the openness theorem for finite holomorphic maps (Chapter 5.2 of Griffiths and Harris [38]) gives that Per_1 is an open map locally around M . Finally, the formula $\text{Per}_1(g) = (i\bar{h}, -ih, 0, 0)$ where $h = h(M)$ and g is the Gauss map of $M \in \mathcal{M}_1$, relates the period map Per_1 with the flux map $h: \mathcal{M}_1 \rightarrow (0, \infty)$, from where the desired openness for h follows.

Finally, to prove property **(D)** one firstly studies the boundary points of \mathcal{W}_1 which correspond to limits of surfaces $M \in \mathcal{M}_1$ such that $h(M)$ converges to zero (in this case the surfaces $M \in \mathcal{M}_1$ degenerate into a singular object consisting of a stack of two vertical catenoids). In second place one parametrizes \mathcal{W}_1 around the related boundary points, and notices that these singular objects form an analytic subvariety of a polydisc in \mathbb{C}^2 (say centered at the origin, which corresponds to the stack of two catenoids). Furthermore, not only \mathcal{W}_1 but also the Period map Per_1 can be holomorphically extended through these singular points, and the Jacobian of the extended Period map at the origin is a bijection. Now **(D)** follows from the Inverse Function Theorem¹⁷ applied to the extension of Per_1 to the polydisc. \square

To finish the classification of the properly embedded minimal surfaces with two limit ends modulo Theorem 6.11, we only need to show the following result.

PROPOSITION 6.17. *The Shiffman function of every surface $M \in \mathcal{M}$ is linear.*

PROOF. Consider a surface $M \in \mathcal{M}$. According to the notation at the beginning of this Section 6, the heights of the planar ends of $M \in \mathcal{M}$ are

$$(30) \quad \dots < \Re(p_{-1}) < \Re(q_{-1}) < \Re(p_0) < \Re(q_0) < \Re(p_1) < \Re(q_1) < \dots$$

Consider the positive functions $h_j: \mathcal{M} \rightarrow \mathbb{R}$ for $j \in \mathbb{N}$ given by

$$\begin{aligned} h_1(M) &= \Re(q_0 - p_0), & h_2(M) &= \Re(p_0 - q_{-1}), & h_3(M) &= \Re(p_1 - p_0), \\ h_4(M) &= \Re(p_0 - p_{-1}) \dots \end{aligned}$$

thus the functions h_j associate to each surface $M \in \mathcal{M}$ the absolute value of the relative height of its planar ends with respect to one of these ends, namely p_0 . First note that h_j is continuous (we endow \mathcal{M} with the topology of the uniform convergence on compact subsets of \mathbb{R}^3).

Let \mathcal{M}_F be the set of surfaces in \mathcal{M} with the same flux vector F as M . By the uniform curvature estimates in Theorem 6.1 and subsequent uniform local area estimates, \mathcal{M}_F is compact. Thus, the set $\mathcal{M}_F(1) = \{M' \in$

¹⁷A simpler proof of property **(D)** which does not use the periodicity of the surfaces can be found in Theorem 36 of Meeks and Pérez [60].

$\mathcal{M}_F \mid \{h_1(M') = \max_{\mathcal{M}_F} h_1\}$ is non-empty. Consider the restriction of h_2 to $\mathcal{M}_F(1)$, and as before maximize h_2 on $\mathcal{M}_F(1)$, hence the set $\mathcal{M}_F(2) = \{M' \in \mathcal{M}_F(1) \mid h_2(M') = \max_{\mathcal{M}_F(1)}(h_2)\}$ is non-empty. Repeating the argument, induction lets us for each $j \in \mathbb{N}$ maximize h_{j+1} in $\mathcal{M}_F(j) = \{M' \in \mathcal{M}_F(j-1) \mid h_j(M') = \max_{\mathcal{M}_F(j-1)}(h_j)\} \neq \emptyset$. Since the compact subsets $\mathcal{M}_F(j)$ satisfy $\mathcal{M}_F(j) \supset \mathcal{M}_F(j+1)$ for all j , this collection of closed sets satisfies the finite intersection property. By the compactness of \mathcal{M}_F , we conclude that $\bigcap_{j \in \mathbb{N}} \mathcal{M}_F(j) \neq \emptyset$. Thus there exists a surface $M_{\max} \in \mathcal{M}_F$ that maximizes each of the functions h_{j+1} in $\mathcal{M}_F(j)$ for all $j \geq 1$. In the same way, we find a surface $M_{\min} \in \mathcal{M}_F$ that minimizes the functions h_{j+1} on $\mathcal{M}_F(j)$ for all $j \geq 1$.

Next we prove that if a surface $M_0 \in \mathcal{M}_F$ maximizes all the functions h_j as in the previous paragraph, then its Shiffman function S_{M_0} is linear (for minimizing surfaces the argument is similar). By our assumption, Theorem 6.11 holds for M_0 . Hence there exists a curve of functions $t \in \mathbb{D}(\varepsilon) \mapsto g_t \in \mathcal{M}_{\text{imm}} \subset \mathcal{W}$ such that the zeros $p_j(t)$ and poles $q_j(t)$ of g_t depend holomorphically on t , satisfying items *i*), *ii*) and *iii*) of Theorem 6.11 for $M = M_0$. With the notation of that theorem, let $\psi_t: (\mathbb{C}/\langle i \rangle) - \{p_j(t), q_j(t)\}_j \rightarrow \mathbb{R}^3$ be the parametrization of M_t given by $\psi_t(z) = \Re \int_{z_0}^z \Psi_t$, where $z_0 \in (\mathbb{C}/\langle i \rangle) - \{p_j(t), q_j(t) \mid j \in \mathbb{Z}, |t| < \varepsilon\}$ and $\Psi_t = \left(\frac{1}{2}(\frac{1}{g_t} - g_t), \frac{i}{2}(\frac{1}{g_t} + g_t), 1 \right) dz$. Item *iii*) of Theorem 6.11 together with Corollary 6.14 and equation (26) imply that equation (21) holds, i.e. $\langle \frac{d}{dt} \Big|_t \int^z \Psi_t, N_t \rangle$ is, up to a multiplicative constant, the complex valued Shiffman function $S_{M_t} + iS_{M_t}^*$ of M_t plus a linear function of the Gauss map N_t of M_t . Furthermore, equation (24) applied to $\dot{g}_t = \frac{d}{dt}g_t$ gives that \dot{g}_t lies in the kernel of the differential of the period map Per at g_t , for all t , see Remark 6.13. Hence, the (complex) period map remains constant along $t \mapsto g_t$, which implies that $M_t \in \mathcal{M}_F$ for all t . This condition gives that the harmonic function $t \in \mathbb{D}(\varepsilon) \mapsto h_1(M_t) = \Re(q_0(t) - p_0(t))$ attains a maximum at $t = 0$, so it is constant. Since the function $t \in \mathbb{D}(\varepsilon) \mapsto q_0(t) - p_0(t)$ is holomorphic with constant real part, then $q_0(t) - p_0(t)$ does not depend on t . The same argument applies to each function $t \mapsto h_j(M_t)$ with $j \in \mathbb{N}$, hence for every t all the planar ends $p_j(t), q_j(t)$ of M_t are placed at

$$p_j(t) = p_0(t) + p_j - p_0, \quad q_j(t) = p_0(t) + q_j - p_0.$$

Geometrically, this means that the maps ψ_t coincide with ψ_0 up to translations in the parameter domain and in \mathbb{R}^3 . Therefore, the normal part of the variational field of $t \mapsto \psi_t$ is linear, and thus the Shiffman function of M_0 is linear.

Finally, we prove that the Shiffman function of every $M \in \mathcal{M}$ is linear. Given $M \in \mathcal{M}$, let $F = (h, 0, 1)$ be its flux vector. By the arguments above, we find embedded minimal surfaces $M_{\max}, M_{\min} \in \mathcal{M}_F$ such that M_{\max} (resp. M_{\min}) maximizes (resp. minimizes) all the functions h_j in the above sense, $j \in \mathbb{N}$. Furthermore, the arguments in the last paragraph imply that

the Shiffman functions of M_{\max}, M_{\min} are linear. By Proposition 6.16, both M_{\max}, M_{\min} are Riemann minimal examples. But \mathcal{M}_F contains at most one Riemann minimal example, since the flux is a parameter for the space of Riemann examples. This implies that $M_{\max} = M_{\min}$. On the other hand, the vertical distance between the ends p_0, q_0 of M is bounded above (resp. below) by the distance between the corresponding ends of M_{\max} (resp. of M_{\min}). So, the vertical distance between the ends p_0, q_0 of M is maximum, or equivalently, M maximizes h_1 on \mathcal{M}_F . Analogously, M maximizes all the functions h_j and so, its Shiffman function S_M is linear. \square

7. Infinitely many ends III: The KdV equation

In this section we will prove Theorem 6.11, which finishes our classification of the properly embedded minimal planar domains in \mathbb{R}^3 . The main tool in this proof is the *Korteweg-de Vries equation* (KdV) and its *hierarchy*, two cornerstones in integrable systems theory.

7.1. Relationship between the KdV equation and the Shiffman function. Recall that by Corollary 6.14, the Shiffman function S_M of a quasiperiodic immersed minimal surface of Riemann type M admits a globally defined conjugate Jacobi function S_M^* on $\mathbb{C}/\langle i \rangle$, and $f = f_S = S_M + iS_M^* \in \mathcal{J}_{\mathbb{C}}(g)$ is given by equation (23) with $h = h_S = \frac{i}{2} \frac{(g')^2}{g^3}$. This function h , when plugged into equation (22), produces the quasiperiodic meromorphic function $\dot{g}_S \in T_g\mathcal{W}$ given by equation (28). By definition of $T_g\mathcal{W}$, \dot{g}_S is the derivative at $t=0$ of a holomorphic curve $t \in \mathbb{D}(\varepsilon) \mapsto g_t \in \mathcal{W}$ with $g_0 = g$. What we want to prove is that such a holomorphic curve $t \mapsto g_t$ can be chosen so that for all t , the pair (g_t, dz) is the Weierstrass data of a minimal surface $M_t \in \mathcal{M}$ and

$$(31) \quad \left. \frac{d}{dt} \right|_t g_t = \frac{i}{2} \left(g_t''' - 3 \frac{g_t' g_t''}{g_t} + \frac{3}{2} \frac{(g_t')^3}{g_t^2} \right).$$

Therefore, one could think of the above problem as finding an integral curve of a vector field in a manifold. Unfortunately, this approach is unsatisfactory: on one hand, the right-hand-side of (28) does not give an element of $T_g\mathcal{W}$ for all $g \in \mathcal{W}$ (it can be proved that $g''' - 3 \frac{g' g''}{g} + \frac{3}{2} \frac{(g')^3}{g^2} \in T_g\mathcal{W}$ provided that (g, dz) closes periods at the zeros and poles of g ; in particular, this holds when $g \in \mathcal{M}_{\text{imm}}$). On the other hand, \mathcal{M}_{imm} is not known to have a manifold structure, so the general theory of integral curves of vector fields does not apply to \mathcal{M}_{imm} .

A more satisfactory viewpoint is to consider (31) as an evolution equation with respect to the complex time t . Therefore, one could apply general PDE techniques to find solutions $g_t = g_t(z)$ of this initial value problem, only defined *a priori* locally around a point $z_0 \in (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$ with the initial condition $g_0 = g$. Such solutions are not necessarily global on $\mathbb{C}/\langle i \rangle$, might develop essential singularities (we need the g_t to be *meromorphic* in z on $\mathbb{C}/\langle i \rangle$ in order g_t to lie in \mathcal{W}), and even if g_t were meromorphic

on $\mathbb{C}/\langle i \rangle$, it is not clear that g_t would have only double zeros and poles and other properties necessary to give rise to minimal surfaces M_t in \mathcal{M} . (Meromorphic) KdV theory will be crucial to solve all these problems as we next explain.

The change of variables

$$(32) \quad u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}.$$

transforms the evolution equation (31) into one of the standard forms of the KdV equation¹⁸

$$(33) \quad \frac{\partial u}{\partial t} = -u''' - 6uu'.$$

The change of variables (32) has an easy explanation. The three terms in the right-hand-side of (28) are rational expressions of derivatives of g with respect to z , with some common homogeneity (order three in derivatives and degree one under multiplication $g \mapsto \lambda g$). One way of transforming (28) into a polynomial expression is by means of the change of variables $x = g'/g$, which gives an equation of mKdV type,¹⁹ namely $\dot{x} = \frac{i}{2}(x''' - \frac{3}{2}x^2x')$. It is a standard fact that mKdV equations in x can be transformed into KdV equations in u through the so called *Miura transformations*, $x \mapsto u = ax' + bx^2$ with a, b suitable constants (see for example [37] page 273). Equation (32) is just the composition of $g \mapsto x$ with a Miura transformation. As KdV theory is more standard than mKdV theory we have opted to deal with the KdV, although it would have been possible to perform entirely all what follows directly with the mKdV equation.

A well-known condition on the initial condition $u(z)$ which guarantees that the Cauchy problem (33) can be solved globally producing a holomorphic curve $t \mapsto u_t$ of meromorphic functions $u_t(z)$ on $\mathbb{C}/\langle i \rangle$, is that $u(z)$ is an *algebro-geometric* potential of the KdV equation. Next we describe this notion, and we will postpone to section 7.3 the property that the holomorphic integration of (28) amounts to solving globally in $\mathbb{C}/\langle i \rangle$ the Cauchy problem for equation (33) with initial condition u given by (32) (this reduction is not direct, since the law $g \mapsto u$ in (32) might not be invertible).

7.2. Algebro-geometric potentials for the KdV equation. The KdV equation (33) is just one of the terms in a sequence of evolution equations of u , called the *KdV hierarchy*:

$$(34) \quad \left\{ \frac{\partial u}{\partial t_n} = -\partial_z \mathcal{P}_{n+1}(u) \right\}_{n \geq 0},$$

where $\partial_z = \frac{\partial}{\partial z}$ and $\mathcal{P}_{n+1}(u)$ is a differential operator given by a polynomial expression of u and its derivatives up to order $2n$, defined by the recurrence

¹⁸In the literature it is usual to find different other KdV equations, with different coefficients for u''' , uu' ; all of them are equivalent after a change of variables.

¹⁹mKdV is an abbreviation for *modified Korteweg-de Vries*.

law

$$(35) \quad \begin{cases} \partial_z \mathcal{P}_{n+1}(u) = (\partial_{zzz} + 4u \partial_z + 2u') \mathcal{P}_n(u), \\ \mathcal{P}_0(u) = \frac{1}{2}. \end{cases}$$

The first operators $\mathcal{P}_j(u)$ and evolution equations of the KdV hierarchy are given by

$$(36) \quad \begin{array}{l} \mathcal{P}_1(u) = u \\ \mathcal{P}_2(u) = u'' + 3u^2 \\ \mathcal{P}_3(u) = u^{(4)} + 10uu'' \\ \quad \quad \quad + 5(u')^2 + 10u^3 \\ \vdots \end{array} \quad \text{(KdV)} \quad \left| \begin{array}{l} \frac{\partial u}{\partial t_0} = -u' \\ \frac{\partial u}{\partial t_1} = -u''' - 6uu' \\ \frac{\partial u}{\partial t_2} = -u^{(5)} - 10uu''' \\ \quad \quad \quad - 20u'u'' - 30u^2u' \\ \vdots \end{array} \right. \quad \text{(KdV)}$$

The Cauchy problem for the n -th equation of the KdV hierarchy consists of finding a solution $u(z, t)$ of $\frac{\partial u}{\partial t_n} = -\partial_z \mathcal{P}_{n+1}(u)$ with prescribed initial condition $u(z, 0) = u(z)$.

DEFINITION 7.1. Given a meromorphic function $u = u(z)$ defined on an open set of \mathbb{C} , the right-hand-side of (34) gives a sequence of functions of z , each of which is a polynomial expression in u and its derivatives, which we will call *infinitesimal flows* of u . With an abuse of notation, we will denote the n -th infinitesimal flow of u by $\frac{\partial u}{\partial t_n}$ (though u only depends on z). The function $u(z)$ is said to be an *algebro-geometric potential of the KdV equation* (or simply *algebro-geometric*) if there exists an infinitesimal flow $\frac{\partial u}{\partial t_n}$ which is a complex linear combination of the lower order infinitesimal flows.

Segal and Wilson [92] proved that if u is algebro-geometric, then it extends to a meromorphic function $u: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ (see also Gesztesy and Weikard [37]). We will use later two well-known properties of algebro-geometric potentials, which can be found in [37] and Weikard [96]:

LEMMA 7.2. *Let $u(z)$ be an algebro-geometric potential. Then:*

- (1) *If u has a pole at $z = z_0$, then there exists $k \in \mathbb{Z}$ such that around z_0 ,*

$$u(z) = \frac{-k(k+1)}{(z-z_0)^2} + \text{holomorphic}(z),$$

- (2) *All the solutions of the linear Schrödinger equation $y'' + uy = 0$ are meromorphic functions $y: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$.*

For our purposes, the key property of algebro-geometric potentials is that the Cauchy problem for any of the equations in the KdV hierarchy can be (uniquely) solved if the initial condition is algebro-geometric. To understand why this is true, suppose $u = u(z) : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ is algebro-geometric, with $\frac{\partial u}{\partial t_n} = c_0 \frac{\partial u}{\partial t_0} + \dots + c_{n-1} \frac{\partial u}{\partial t_{n-1}}$, $c_0, \dots, c_{n-1} \in \mathbb{C}$. Calling

$\frac{\partial}{\partial s} = \frac{\partial}{\partial t_n} - c_0 \frac{\partial}{\partial t_0} - \dots - c_{n-1} \frac{\partial}{\partial t_{n-1}}$, then the next PDE system encodes solving the k -th equation in the KdV hierarchy among functions $u(z, t)$ which are algebro-geometric with the same coefficients $c_j \in \mathbb{C}$ as $u(z)$ (note that the (A-G) below is an ODE in z , and (KdV) is a PDE in z, t):

$$(37) \quad \left. \begin{array}{l} \text{(A-G)} \quad \frac{\partial u}{\partial s} = 0, \\ \text{(KdV)} \quad \frac{\partial u}{\partial t} = -\partial_z \mathcal{P}_{k+1}(u) \end{array} \right\}$$

According to the Frobenius Theorem, the integrability condition of (37) is given by the commutativity $\frac{\partial}{\partial s} \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \frac{\partial u}{\partial s}$, which in turn reduces to $\frac{\partial}{\partial t_j} \frac{\partial u}{\partial t_k} = \frac{\partial}{\partial t_k} \frac{\partial u}{\partial t_j}$ for all $j = 0, \dots, n - 1$. This commutativity is a well-known fact in KdV theory. Therefore, given any $z_0 \in \mathbb{C}$ which is not a pole of $u(z)$, there exists a $\delta > 0$ and a unique solution $u(z, t)$, $(z, t) \in \{|z - z_0| < \delta\} \times \mathbb{D}(\delta)$, of the system (37) with initial conditions

$$(38) \quad \frac{\partial^j u}{\partial z^j}(z_0, 0) = u^{(j)}(z_0), \quad j = 0, \dots, 2n.$$

(Note that (A-G) is an ODE of order $2n + 1$). Since $u_t = u_t(z)$ is algebro-geometric by (A-G), then u_t extends meromorphically to the whole plane \mathbb{C} . As $u(z)$ satisfies both equations (A-G) and (38), we have $u(z, 0) = u(z)$. Consequently, we have solved the Cauchy problem for the k -th equation of the KdV hierarchy with algebro-geometric initial condition $u(z)$.

7.3. Proof of Theorem 6.11 provided that u is algebro-geometric. Coming back to our setting where $u(z)$ is given by (32) for a given $g \in \mathcal{M}_{\text{imm}}$, suppose that u is algebro-geometric. By the arguments in the last section, we can solve the Cauchy problem for the KdV equation (which is the k -th equation in the KdV hierarchy for $k = 1$), obtaining a family $t \in \mathbb{D}(\delta) \mapsto u_t(z) = u(z, t)$ of algebro-geometric meromorphic functions on \mathbb{C} , with $u(z, 0) = u(z)$. It is then interesting to know how periodicity of the initial condition propagates to $u_t(z)$ (note that $u(z)$ is defined on $\mathbb{C}/\langle i \rangle$ since g is). Clearly, the uniqueness of solution of (37) with initial conditions (38) implies that if $u(z)$ is invariant by the translation of a vector $\omega \in \mathbb{C}$, then $u_t(z)$ has the same invariance for all t . In particular for each $t \in \mathbb{D}(\delta)$, $u_t(z)$ descends to the cylinder $\mathbb{C}/\langle i \rangle$ as a meromorphic function. It is now time to produce $g_t(z)$ from $u_t(z)$ via equation (32), which is our next goal.

To do this, we first choose a meromorphic function $y: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ such that $g = 1/y^2$ (the existence of y is guaranteed because g has double zeroes and double poles without residues since $g \in \mathcal{M}_{\text{imm}}$). Note that y is either periodic, $y(z + i) = y(z)$, or anti-periodic $y(z + i) = -y(z)$ (in this last case y does not descend to $\mathbb{C}/\langle i \rangle$ and this is what happens with the Riemann minimal examples, where the Gauss map g restricts to each compact horizontal section with degree one). A direct computation using (32) gives that $y'' + uy = 0$. Now consider the PDE system with unknown

$y(z, t)$:

$$(39) \quad \left. \begin{aligned} (S) \quad & y'' + uy = 0, \\ (J) \quad & \frac{\partial y}{\partial t} = \mathcal{P}_1(u)'y - 2\mathcal{P}_1(u)y' \end{aligned} \right\}$$

where the function u in (39) is the above solution $u(z, t)$ of the Cauchy problem (37)–(38). Note that (S) is a Schrödinger type ODE in the variable z ; on the contrary, (J) is a PDE in z, t . The reason why we consider the system (39) is that its integrability condition is precisely that $u(z, t)$ solves the KdV equation,²⁰ as proved by Joshi [44]. Therefore, the Frobenius theorem implies that (39) admits a unique solution $y = y(z, t)$ with initial condition $y(z, 0) = y(z)$. Since $z \mapsto u(z, t)$ is algebro-geometric for every t , part 2 of Lemma 7.2 together with equation (S) imply that $y(z, t)$ is defined on $\mathbb{C} \times \mathbb{D}(\varepsilon)$ (for some $\varepsilon > 0$) and is meromorphic in z . The uniqueness of solution of an initial value problem together with the fact that $y(z + i) = \pm y(z)$, give that $y(z + i, t) = \pm y(z, t)$, with the same choice of signs as for $y(z)$. Finally, defining

$$(40) \quad g_t(z) = g(z, t) = y^{-2}(z, t)$$

then

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{1}{y_t^2} \right) = -\frac{2}{y_t^3} \frac{\partial y_t}{\partial t} \stackrel{(J)}{=} -2 \frac{\mathcal{P}_1(u_t)'y_t - 2\mathcal{P}_1(u_t)y_t'}{y_t^3} \\ &= -2 \partial_z \left(\frac{\mathcal{P}_1(u_t)}{y_t^2} \right) = -2 \partial_z (g_t \mathcal{P}_1(u_t)) \\ &\stackrel{(36)}{=} -2 \partial_z (g_t u_t) \stackrel{(\star)}{=} -2 \partial_z \left[g_t \left(-\frac{3(g_t')^2}{4g_t^2} + \frac{g_t''}{2g_t} \right) \right] = -g_t''' + 3 \frac{g_t' g_t''}{g_t} - \frac{3}{2} \frac{(g_t')^3}{g_t^2}, \end{aligned}$$

where in (\star) we have used (S) and (40) (we cannot substitute directly (32) since *a priori* is only valid for $u(z, 0) = u(z)$). The equality in the last two lines tells us that, up to a multiplicative constant, $t \mapsto g_t$ satisfies the evolution equation in item *iii*) of Theorem 6.11. Therefore, in order to finish the proof of Theorem 6.11 we need to demonstrate items *i*), *ii*) of that theorem; these are technical issues that we next sketch.

Equation (32) implies that the poles of $u(z)$ coincide with the zeros and poles of $g(z)$, which are double. A direct computation gives that the Laurent expansion of $u(z)$ around each pole z_0 is of the form

$$(41) \quad u(z) = \frac{-2}{(z - z_0)^2} + \text{holomorphic}(z).$$

The next step consists of proving that every pole z_0 of $u(z)$ propagates holomorphically in t to a curve of poles $z_0(t)$ of $u_t(z)$ with a similar Laurent expansion as in (41). The argument is purely local: On one hand, as $u_t(z)$

²⁰This fact can be generalized to the k -th equation of the KdV hierarchy only by changing t by t_k and \mathcal{P}_1 by \mathcal{P}_k in equation (J).

is algebro-geometric for all t , item 1 of Lemma 7.2 implies that u_t admits a Laurent expansion of the type

$$u_t(z) = \frac{-k_j(k_j + 1)}{(z - a_j)^2} + \text{holomorphic}(z, t)$$

in a neighborhood of each of the poles a_1, \dots, a_m of u_t in a fixed closed disk D centered at z_0 , such that $u(z)$ does not vanish in $D - \{z_0\}$ and u_t has no zeros in ∂D (both the number of poles of u_t and the poles themselves may depend on t). Now a continuity argument with respect to t together with the fact that k_j is integer-valued, give that for $|t|$ small, $u_t(z)$ has a unique pole in D and $k_1 = 1$. Once we know that u_t has a *unique* pole in D , the holomorphic dependence of this pole with respect to t is a standard argument.

In order to obtain the desired holomorphicity of the curves of zeros and poles of $g_t(z)$ with respect to t (which is item *i*) of Theorem 6.11), we need to know that these zeros and poles of g_t (or of y_t) coincide with the poles of u_t : we already know that the series expansion of u_t around each of its poles $z_0(t)$ is of the form (41) with $z_0(t)$ instead of z_0 (of course, the holomorphic term in the right-hand-side also depends on t). Using that $y_t'' + u_t y_t = 0$ (equation (S)), it is straightforward to expand y_t around $z_0(t)$ thereby proving that outside of the poles of u_t , the function y_t is holomorphic and its zeros are simple, while at each pole of u_t , the function y_t has either a simple pole or a double zero. Another continuity argument shows that the possibility of y_t having a double zero at a pole of u_t cannot occur (for $t = 0$ it does not occur, since $y(z) = 1/\sqrt{g(z)}$ has only single zeros and poles). Therefore the zeros and poles of g_t are double, and coincide with the poles of u_t . In particular, the holomorphic dependence with respect to t in item *i*) of Theorem 6.11 holds.

The quasiperiodicity of $u_t(z)$ is guaranteed by that of $u(z)$ together with the uniqueness of the solution of (37) (with $k = 1$). In turn, this quasiperiodicity of $u_t(z)$ implies the same property for $y_t(z)$ and thus for $g_t(z)$. To finish the proof of items *i*), *ii*), *iii*) of Theorem 6.11, it only remains to show that $g_t \in \mathcal{M}_{\text{imm}}$. This follows from the fact that the map $t \mapsto \text{Per}(g_t)$ is constant, see equation (24) and Remark 6.13.

Finally, the last sentence in the statement of Theorem 6.11 follows from the maximum principle for minimal surfaces. This finishes the sketch of proof of Theorem 6.11.

7.4. Why u is algebro-geometric if $g \in \mathcal{M}_{\text{imm}}$. In the last section we proved Theorem 6.11 under the additional assumption that the function $u = u(z)$ given by (32) is an algebro-geometric potential of the KdV equation. We will devote this section to explaining why this assumption for u holds for all $g \in \mathcal{M}_{\text{imm}}$. Briefly, the desired property for u follows from two facts: firstly, that each infinitesimal flow $\frac{\partial u}{\partial t_n}$ of u given in Definition 7.1 produces a complex valued, bounded Jacobi function v_n on M (for instance,

$\frac{\partial u}{\partial t_1}$ produces $S_M + iS_M^*$) which extends smoothly across the zeros and poles of g to a function in the bounded kernel of an operator of the type $\Delta + V$ on $\mathbb{S}^1 \times \mathbb{R}$, where Δ is the laplacian in the product metric and $V: \mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{R}$ is a bounded potential. And secondly, that the bounded kernel of such a Schrödinger operator is finite-dimensional. We will now develop the details of this sketch.

From now on, we consider a function $g \in \mathcal{M}_{\text{imm}}$ and let u the meromorphic function on $\mathbb{C}/\langle i \rangle$ given by (32). Similarly as in Definition 7.1, we define the *infinitesimal flows for g* as the sequence of meromorphic functions on $\mathbb{C}/\langle i \rangle$ given by

$$(42) \quad \left\{ \frac{\partial g}{\partial t_n} = -2 \partial_z (g \mathcal{P}_n(u)) \right\}_{n \geq 0} .$$

Hence, each $\frac{\partial g}{\partial t_n}$ is a rational expression in g and its derivatives up to some order. By substituting (35) and (32) in (42), one can compute explicitly the infinitesimal flows for g ; for instance, $\frac{\partial g}{\partial t_0} = -g'$ is the infinitesimal deformation of g in \mathcal{W} given by translations in the parameter domain (see Remark 6.9), and $\frac{\partial g}{\partial t_1} = -g''' + 3\frac{g'g''}{g} - \frac{3}{2}\frac{(g')^3}{g^2}$ is, up to a multiplicative constant, the infinitesimal deformation \dot{g}_S given in equation (28), which corresponds to the (complex valued) Shiffman function. In particular, both $\frac{\partial g}{\partial t_0}, \frac{\partial g}{\partial t_1}$ belong to $T_g \mathcal{W}$.

For each $n \geq 0$, the infinitesimal flow $\frac{\partial g}{\partial t_n}$ for g satisfies the following two key properties:

- (1) *There exists a meromorphic function h_n on $\mathbb{C}/\langle i \rangle$ which is a rational expression of g and its derivatives up to some order (depending on n), such that $\frac{\partial g}{\partial t_n} = \partial_z \left(\frac{g^3 h'_n}{2g'} \right)$.*

The proof of this property reduces to integrating the equality $-2\partial_z (g \mathcal{P}_n(u)) = \partial_z \left(\frac{g^3 h'_n}{2g'} \right)$, which is an ODE in z with unknown h_n (the recurrence law (35) for the operators \mathcal{P}_n is useful here).

- (2) *The principal divisor D of the meromorphic function $\frac{\partial g}{\partial t_n}$ satisfies $D \geq \prod_j p_j q_j^{-3}$, where the principal divisor of g is $(g) = \prod_j p_j^2 q_j^{-2}$. Therefore, $\frac{\partial g}{\partial t_n} \in T_g \mathcal{W}$.*

This can be deduced from the local expansions of g (which has double zeros and double poles without residue since $g \in \mathcal{M}_{\text{imm}}$), of u (given by equation (41)) and of $\mathcal{P}_n(u)$ (proven by induction on n).

The properties 1, 2 above together with Proposition 6.12 produce, for each $n \geq 0$, a complex valued Jacobi function $f(h_n) = \frac{g^2 h'_n}{g'} + \frac{2gh_n}{1+|g|^2} \in \mathcal{J}_{\mathbb{C}}(g)$ which is bounded on $\mathbb{C}/\langle i \rangle$. To continue our argument, we need the following finiteness result, whose proof we postpone till the end of this section.

THEOREM 7.3. *Let $M \subset \mathbb{R}^3$ be a quasiperiodic, immersed minimal surface of Riemann type. Then, the linear space of bounded Jacobi functions on M is finite dimensional.*

Assuming that Theorem 7.3 holds, we finish the proof that u is algebrogeometric. We had constructed a sequence of bounded Jacobi functions $f(h_n) \in \mathcal{J}_{\mathbb{C}}(g)$. By Theorem 7.3, only finitely many of them can be linearly independent, hence there exists $n \in \mathbb{N}$ such that $f(h_n) \in \text{Span}\{f(h_0), \dots, f(h_{n-1})\}$. Since the linear map $h \mapsto f(h)$ given by equation (23) is injective, we conclude that $h_n \in \text{Span}\{h_0, \dots, h_{n-1}\}$. As the map $h \mapsto \dot{g}(h)$ given by (22) is linear, then $\frac{\partial g}{\partial t_n} \in \text{Span}\{\frac{\partial g}{\partial t_0}, \dots, \frac{\partial g}{\partial t_{n-1}}\}$. Finally, the equation

$$\frac{\partial u}{\partial t_n} = \frac{\partial}{\partial t_n} \left(-\frac{3(g')^2}{4g^2} + \frac{g''}{2g} \right),$$

implies that $\frac{\partial u}{\partial t_n} \in \text{Span}\{\frac{\partial u}{\partial t_0}, \dots, \frac{\partial u}{\partial t_{n-1}}\}$, which proves that u is algebrogeometric.

We finish this section with a sketch of the proof of Theorem 7.3 stated above. Suppose that $M \subset \mathbb{R}^3$ is a quasiperiodic, immersed minimal surface of Riemann type. Then, M is conformally equivalent to $(\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$ where $g \in \mathcal{M}_{\text{imm}}$ is the Gauss map of M . Take global coordinates (θ, t) on $\mathbb{S}^1 \times \mathbb{R}$ and consider the product metric $d\theta^2 \times dt^2$, which is conformal to the metric ds^2 on M induced by the usual inner product of \mathbb{R}^3 : $ds^2 = \lambda^2(d\theta^2 + dt^2)$. This conformality allows us to relate the Jacobi operator $L = \Delta - 2K$ of M (here K is the Gaussian curvature of M) to a Schrödinger operator $L_M = (\Delta_{\mathbb{S}^1} + \partial_t^2) + V$ on $\mathbb{S}^1 \times \mathbb{R}$ by means of the formula $L = \lambda^{-2}L_M$, where the potential V is equal to the square of the norm of the differential of the Gauss map of M (with respect to $d\theta^2 \times dt^2$). The quasiperiodicity of M implies that V_M is globally bounded on $\mathbb{S}^1 \times \mathbb{R}$.

By elliptic regularity, any bounded Jacobi function v on M extends smoothly through the zeros and poles of g to a function \hat{v} in the kernel of L_M , such that \hat{v} is bounded at both ends of $\mathbb{S}^1 \times \mathbb{R}$. Therefore, the space of bounded Jacobi functions on M (i.e. bounded functions in the kernel of L) identifies naturally with the bounded kernel of L_M , and thus Theorem 7.3 follows from the following standard technical result, whose proof we omit here (see Assertion 5.3 in [64] for a proof due to Frank Pacard, based on the paper by Lockhart and McOwen [48]).

ASSERTION 7.4. *Let (Σ, h) be a compact Riemannian manifold and $V \in L^\infty(\Sigma \times \mathbb{R})$. Assume that there exists $j_0 \in \mathbb{N}$ such that²¹*

$$(43) \quad 4\|V\|_{L^\infty(\Sigma \times \mathbb{R})} \leq \lambda_{j_0+1} - \lambda_{j_0},$$

where $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$ is the spectrum of $-\Delta_h$ on Σ . Then, the bounded kernel of $\Delta_h + \partial_t^2 + V$ on $\Sigma \times \mathbb{R}$ is finite dimensional.

²¹In the case where Σ is the standard \mathbb{S}^1 , then $\lambda_j = j^2$ and $\lambda_{j+1} - \lambda_j = 2j + 1$, so the hypothesis (43) is fulfilled.

8. The asymptotics of the ends of finite genus surfaces

In this section we briefly describe the asymptotic behavior of the ends of a properly embedded minimal surface M in \mathbb{R}^3 with finite genus, possibly with compact boundary. Collin’s solution of the Nitsche Conjecture [22] implies that if M has more than one end, then each annular end is asymptotic to the end of a plane or catenoid.

Next assume that M has just one end. On the last page of their paper [71], Meeks and Rosenberg claimed that their proof of the uniqueness of the helicoid could be modified to prove the following statement:

THEOREM 8.1. *Any non-planar, properly embedded minimal surface M in \mathbb{R}^3 with one end, finite topology and infinite total curvature (without boundary) satisfies the following properties:*

- (1) M is conformally a compact Riemann surface \overline{M} punctured in a single point.
- (2) After a rotation in \mathbb{R}^3 , the Weierstrass pair (g, dh) of M satisfies that both differentials $\frac{dg}{g}, dh$ extend meromorphically to \overline{M} .
- (3) M is asymptotic to a helicoid.

The technical nature of the proof by Meeks and Rosenberg of the uniqueness of the helicoid and the absence of a detailed proof of their claimed generalization above to the case of (non-zero) finite genus, motivated subsequent investigations. Bernstein and Breiner [3] have recently given a proof of Theorem 8.1, based on arguments from the original paper by Meeks and Rosenberg together with a more careful analysis of the multigraph structure of the end of a surface M under the hypotheses of Theorem 8.1, using Colding-Minicozzi theory. With these two ingredients, Bernstein and Breiner were able to prove that M has finite type (see Definition 8.2 below for the notion of a minimal surface of finite type). Once M is proven to have finite type, one can use former results by Hauswirth, Pérez and Romon [40] on the geometry of complete embedded, minimal ends of finite type to complete the proof of Theorem 8.1.

DEFINITION 8.2 (Finite Type). A minimal immersion $X: M \rightarrow \mathbb{R}^3$ is said to have *finite type* if it satisfies the following two properties.

- (1) The underlying Riemann surface to M is conformally diffeomorphic to a compact Riemann surface \overline{M} with (possibly empty) compact boundary, punctured in a finite non-empty set $\mathcal{E} \subset \text{Int}(M)$.
- (2) Given an end $e \in \mathcal{E}$ of M , there exists a rotation of the surface in space such that if (g, dh) is the Weierstrass data of M after this rotation, then the meromorphic one-forms $\frac{dg}{g}$ and dh extend across the puncture e to meromorphic one-forms on a neighborhood of e in \overline{M} .

In their survey [57], the authors of these notes outlined the proof by Meeks and Rosenberg of the uniqueness of the helicoid and at the end of

this outline they mentioned how some difficult parts of the proof could be simplified, as for instance the facts that the correct conformal structure is \mathbb{C} or that the height differential can be assumed to be dz with z being the natural coordinate on \mathbb{C} , see also Footnote 8 above. In the recent paper [58], Meeks and Pérez not only give the aforementioned simplification of the proof of the uniqueness of the helicoid, but also tackle the more general problem of describing the asymptotic behavior, conformal structure and analytic representation of an annular end of any *complete*, injectively immersed minimal surface M in \mathbb{R}^3 with *compact boundary* and finite topology. Before proceeding to explain the results in [58], it is necessary to make a remark about properness versus completeness. Although not explicitly stated in the paper [21] by Colding and Minicozzi, the results contained there imply that such an M is *properly embedded* in \mathbb{R}^3 (we also remark that in Meeks, Pérez and Ros [61] the following more general result is proven: If M is a complete, connected, injectively immersed minimal surface of finite genus, compact boundary and a countable number of ends in \mathbb{R}^3 , then M is proper).

We next state the first main result in [58].

THEOREM 8.3. *Let $E \subset \mathbb{R}^3$ be a complete, embedded minimal annulus with infinite total curvature and compact boundary. Then, the following properties hold:*

- (1) E is properly embedded in \mathbb{R}^3 .
- (2) E is conformally diffeomorphic to $D(\infty, R) = \{z \in \mathbb{C} \mid R \leq |z|\}$.
- (3) After a suitable homothety and rigid motion and possibly replacing M by a subend, then:
 - (a) The height differential $dh = dx_3 + idx_3^*$ extends meromorphically across infinity with a double pole.
 - (b) The stereographically projected Gauss map $g: D(\infty, R) \rightarrow \mathbb{C} \cup \{\infty\}$ of M can be expressed as $g(z) = e^{iz+f(z)}$ for some holomorphic function f in $D(\infty, R)$ with $f(\infty) = 0$.
 - (c) E is asymptotic to the end of a helicoid if and only if it has zero flux.

Note that Theorem 8.1 above is a direct consequence of Theorem 8.3 (the zero flux condition in item 3(c) of Theorem 8.3 follows from Stokes' theorem).

The main ideas in the proof of Theorem 8.3 are the following ones. First one proves that the sequence of surfaces $\{\lambda_n E\}_n$ has locally positive injectivity radius in $\mathbb{R}^3 - \{\vec{0}\}$, for every sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ with $\lambda_n \searrow 0$ as $n \rightarrow \infty$ (this means that for every $q \in \mathbb{R}^3 - \{\vec{0}\}$, there exists $\varepsilon_q > 0$ and $n_q \in \mathbb{N}$ such that for $n > n_q$, the injectivity radius function of $\lambda_n E$ restricted to $\{x \in \mathbb{R}^3 \mid |x - q| < \varepsilon_q\} \cap (\lambda_n E)$ is a sequence of functions which is uniformly bounded away from zero; this property is obtained after a blow-up argument on the scale of topology similar to the one explained during the proof of Proposition 5.7). The second step in the proof is to apply a result about singular minimal laminations (namely item 7 of Theorem 1.5 of Meeks, Pérez

and Ros [65]) to conclude that after extracting a subsequence, the surfaces $\lambda_n E$ converge as $n \rightarrow \infty$ to a foliation \mathcal{F} of \mathbb{R}^3 by parallel planes and the convergence is C^1 away from one straight line orthogonal to the planes in \mathcal{F} (the presence of boundary prevents us to use Colding-Minicozzi theory in our setting to get this). Under shrinkings, the boundary of $\lambda_n E$ collapses into the origin $\vec{0}$, which in turn implies that the singular set of convergence of the $\lambda_n E$ to \mathcal{F} must be a line passing through $\vec{0}$. Furthermore, this limit foliation \mathcal{F} is independent of the sequence of positive numbers λ_n , hence it can be assumed from now on that the planes in this foliation are horizontal.

It follows from the previous paragraph that there is a solid vertical hyperboloid \mathcal{H} with axis being the x_3 -axis, such that $E - \mathcal{H}$ consists of two multigraphs over their projections to the (x_1, x_2) -plane P . By work of Colding-Minicozzi, any embedded minimal multigraph with a large number of sheets contains a submultigraph which can be approximated by the multigraph of a helicoid with an additional logarithmic term (see Corollary 14.3 in [14]). A deeper study of this multigraph structure using Colding-Minicozzi theory leads to the property that each of the two multigraphs G_1, G_2 in $E - \mathcal{H}$ contains infinite submultigraphs G'_1, G'_2 , respectively given by functions $u^1(\rho, \theta), u^2(\rho, \theta)$, such that $\frac{\partial u^i}{\partial \theta}(\rho, \theta) > 0$ (resp. < 0) for $i = 1, 2$ (this observation was also made by Bernstein and Breiner using the same arguments, see Proposition 3.3 in [3]). The positive slope property of the curves $\theta \mapsto u^i(\rho, \theta)$ is then used to prove that after passing to a subend (denoted in the same way) of E , each horizontal plane $\{x_3 = t\}$ intersects E transversely in either a proper curve at height t or in two proper arcs, each with one extremum on the boundary of E . This intersection property, together with Corollary 1.2 in Meeks and Pérez [59], imply that the conformal structure of E is a punctured disk $D(\infty, R) = \{z \in \mathbb{C} \mid |z| \geq R\}$, $R > 0$, that the height differential dh if E extends meromorphically across $z = \infty$ with a double pole and (again after passing to a subend) that the Gauss map g of E can be written as $g(z) = z^k e^{H(z)}$ for some $k \in \mathbb{Z}$ and some holomorphic function H in $D(\infty, R)$.

REMARK 8.4. *At this point in the sketch of proof of Theorem 8.3, one can give a short proof of the uniqueness of the helicoid among simply-connected, embedded, complete, non-flat minimal surfaces in \mathbb{R}^3 : It is clear from the last paragraph that in this special setting for our minimal surface M in question, M is conformally \mathbb{C} and $dh = \lambda dz$ for some $\lambda \in \mathbb{C} - \{0\}$, or after a change of coordinates, $dh = dz$. As dh has no zeros, then the Gauss map g misses $0, \infty$ on the whole surface M . Since M is simply-connected, then g lifts through the natural exponential map $e^w : \mathbb{C}^* \rightarrow \mathbb{C}$ and thus, $g(z) = e^{H(z)}$ for some entire function H . From this point one finishes the uniqueness of the helicoid as in the original proof by Meeks and Rosenberg sketched just after the statement of Theorem 4.2, proving that H is a linear function of z .*

We will finish our sketch of proof of Theorem 8.3 by indicating how to arrive to the expression $g(z) = e^{iz+f(z)}$ as in item (b) of that statement

(item (c) is a consequence of Theorem 8.5 below). To do this we first prove that $g(z) = e^{H(z)}$, or in other words, $k = 0$ in our previous expression $g(z) = z^k e^{H(z)}$. The integer k is the winding number of $g|_{\partial E}$, so it suffices to prove that this winding number vanishes. Since the winding number of $g|_{\partial E}$ is an invariant of the homotopy class of ∂E in E , we replace E by a subend E' where the winding number of $g|_{\partial E'}$ is easier to calculate and the calculation has a geometric nature. More precisely, we can choose E' so that $\partial E'$ is a simple closed curve close to the boundary of a cylinder $C(r, h) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq r^2, |x_3| \leq h\}$. Furthermore, $\partial E'$ consists of an arc α_T , which is contained in a plane parallel to the (x_1, x_2) -plane and it is close in the C^1 -sense to a line segment parallel to the x_2 -axis contained in the top disk of $C(r, h)$, a similar arc α_B near the bottom disk of $C(r, h)$ and two spiraling arcs S_1, S_2 on the boundary cylinder of $C(r, h)$, where each of these spirals winds exactly n times around the x_3 -axis for some large $n \in \mathbb{N}$. Note that the end points of α_t and α_B lie on the cylindrical sides of $C(r, h)$ and lie vertically over the points $(0, \pm r, 0)$.

By construction, the argument of the Gauss map of E' mod 2π restricted to $\alpha_T \cup \alpha_B$ lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$; to obtain this property for this portion of $\partial E'$, one uses the fact that under rescalings and translations the parts of E' in the halfspaces $\{x_3 \geq 0\}$, $\{x_3 \leq 0\}$ produce vertical right handed helicoids and also we apply the fact $\frac{\partial u}{\partial \theta} > 0$ holds on certain minimal multigraphs $u(\rho, \theta)$ with many sheets and defined in polar coordinates on annular domains in the plane. Finally, using the property $\frac{\partial u_i}{\partial \theta} > 0$ for the two multigraphs S_1, S_2 corresponding to multigraphing functions u_1, u_2 over the universal cover of $\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \geq r^2\}$, we can relate the change of the arguments of g along S_1 and S_2 to the number n of their windings around the x_3 -axis and these changes of arguments essentially cancel each other out to make the total change in the argument as g transverses $\partial E'$ to be less than 2π in absolute value; to see this holds it is helpful to notice that if under a parametrization of $\partial E'$, the winding number of S_1 is n around the x_3 -axis, then the winding number of S_2 is $-n$. Hence, the winding number of $g|_{\partial E'}$ is $k = 0$, which proves that $g(z) = e^{H(z)}$.

Once we have shown $g(z) = e^{H(z)}$, we next consider two cases, depending on whether or not $H(z)$ has an essential singularity. Following the arguments of Meeks and Rosenberg in [71], we find that the case where $H(z)$ has an essential singularity is impossible, while the case that $H(z)$ extends meromorphically across infinity can be reduced, after a change of variables, to $H(z) = iz + f(z)$ where $f(\infty) = 0$, which completes our indication as to why $g(z) = e^{iz+f(z)}$ with $f(\infty) = 0$.

In contrast to the unique asymptotic behavior of a complete, embedded minimal surface with infinite total curvature, one end and *no boundary*, the second main result of [58] shows that if we let the annular end E have non-zero flux (in particular, it is not part of a minimal surface without boundary), then we have many other asymptotic models: they are essentially given by

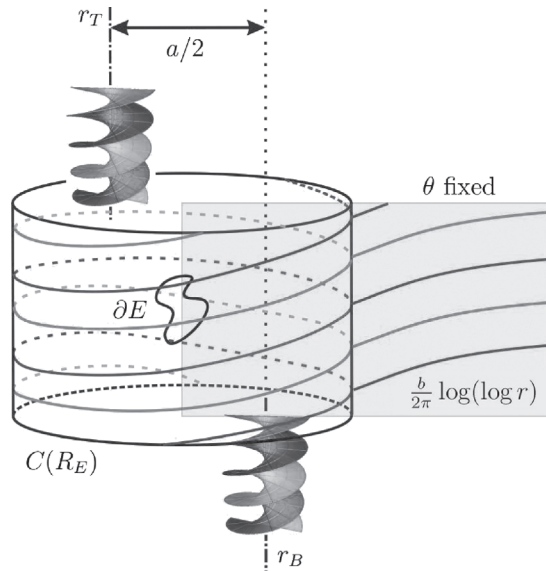


FIGURE 2. The embedded annulus E with flux vector $(a, 0, b)$ has the following description. Outside the cylinder $C(R_E)$, E consists of two horizontal multigraphs with asymptotic spacing π between them. The translated surfaces $E + (0, 0, -2\pi n - \frac{b}{2\pi} \log n)$ (resp. $E + (0, 0, 2\pi n - \frac{b}{2\pi} \log n)$) converge as $n \rightarrow \infty$ to a vertical helicoid H_T (resp. H_B) such that $H_B = H_T + (0, a/2, 0)$ (in the picture, r_T, r_B refer to the axes of these helicoids). The intersection of a vertical halfplane containing the x_3 -axis with $E - C(R_E)$ consists of an infinite number of curves, each of which is a graph of a function $u(r)$ that satisfies the property $\frac{u(r)}{\log(\log r)}$ converges to $\frac{b}{2\pi}$ as the radial distance r to the x_3 -axis tends to ∞ .

the flux vector along the boundary of E , which, after a rotation around the x_3 -axis, is $(a, 0, b) \in \mathbb{R}^3$. Therefore the really different asymptotic structures form a 2-parameter family of *canonical ends* $\{E_{a,b} \mid a, b \geq 0\}$. Here, the word *canonical* only refers to item 3 in the next statement.

The image in Figure 2 describes how the flux vector $(a, 0, b)$ of E influences its geometry.

THEOREM 8.5 (Asymptotics of embedded minimal annular ends). *Given $a, b \geq 0$, there exist a positive number $R = R(a, b)$ and a properly embedded minimal annulus $E_{a,b} \subset \mathbb{R}^3$ with compact boundary and flux vector $(a, 0, b)$ along its boundary, such that the following statements hold.*

- (1) $E_{a,b} - C(R)$ consists of two disjoint multigraphs²² Σ_1, Σ_2 over $D(\infty, R)$ of smooth functions $u_1, u_2: \tilde{D}(\infty, R) \rightarrow \mathbb{R}$ such that their

²²See Footnote 7 for the notion of multigraph.

gradients satisfy $\nabla u_i(r, \theta) \rightarrow 0$ as $r \rightarrow \infty$ and the separation function $w(r, \theta) = u_1(r, \theta) - u_2(r, \theta)$ between both multigraphs converges to π as $r + |\theta| \rightarrow \infty$. Furthermore for θ fixed and $i = 1, 2$,

$$(44) \quad \lim_{r \rightarrow \infty} \frac{u_i(r, \theta)}{\log(\log(r))} = \frac{b}{2\pi}.$$

- (2) The translated surfaces $E_{a,b} + (0, 0, -2\pi n - \frac{b}{2\pi} \log n)$ (resp. $E_{a,b} + (0, 0, 2\pi n - \frac{b}{2\pi} \log n)$) converge as $n \rightarrow \infty$ to a vertical helicoid H_T (resp. H_B) such that $H_B = H_T + (0, a/2, 0)$. Note that this property together with item 1 imply that for different values of a, b , the related surfaces $E_{a,b}$ are not asymptotic after a rigid motion and homothety.
- (3) Every complete, embedded minimal annulus in \mathbb{R}^3 with compact boundary and infinite total curvature is asymptotic (up to a rigid motion and homothety) to exactly one of the surfaces $E_{a,b}$.

We will finish this section by reporting on the asymptotic behavior of any properly embedded minimal surface M in \mathbb{R}^3 with finite genus and an infinite number of ends. We have already seen that such an M has exactly two limit ends; it easily follows that each of its middle ends is asymptotic to a horizontal plane, after a fixed rotation of M . Also, after another rotation of M around a vertical axis followed by a homothety, we can assume that the flux vector associated to its limit ends has the form $F = (h, 0, 1)$, where $h > 0$. In this case M is seen to be conformally a compact Riemann surface \bar{M} punctured in a closed countable set with exactly two limit points corresponding to the two limit ends of M , and we can also describe the asymptotic behavior of the ends of M :

THEOREM 8.6 (Asymptotic Limit End Property, Meeks, Pérez and Ros [64]). *Let M be a properly embedded minimal surface in \mathbb{R}^3 with finite genus g and an infinite number of ends. Then, after a possible rotation and a homothety, the following statements hold.*

- (1) M has two limit ends. In fact, M is conformally diffeomorphic to $\bar{M} - \mathcal{E}_M$, where \bar{M} is a compact Riemann surface of genus g and $\mathcal{E}_M = \{e_n \mid n \in \mathbb{Z}\} \cup \{e_B, e_T\}$ is a countable closed subset of \bar{M} with exactly two limit points e_T and e_B . Furthermore, $\lim_{n \rightarrow -\infty} e_n = e_B$, $\lim_{n \rightarrow \infty} e_n = e_T$, and e_T (resp. e_B) corresponds to the top (resp. bottom) end of M , while every e_n with $n \in \mathbb{Z}$ corresponds to a middle end.
- (2) For each $n \in \mathbb{Z}$, there exists a punctured disk neighborhood $E_n \subset M \subset \bar{M}$ of e_n which is asymptotic in \mathbb{R}^3 to a horizontal plane P_n and which is a graph over its projection to P_n . Furthermore, the usual linear ordering on the index set \mathbb{Z} respects the linear ordering of the heights of the related planes. The ordered set of heights $H = \{h_n = x_3(P_n) \mid n \in \mathbb{Z}\}$ of these planes naturally corresponds to the set of heights of the middle ends of M .

- (3) *There exists a positive constant C_M such that if $|t| > C_M$, then the horizontal plane $\{x_3 = t\}$ intersects M in a proper arc when $t \in H$, or otherwise, $\{x_3 = t\}$ intersects M in a simple closed curve.*
- (4) *Let η denote the unitary outward conormal along the boundary of $M_t = M \cap \{x_3 \leq t\}$. Then the flux vector of M , which is defined to be*

$$F_M = \int_{\partial M_t} \eta \, ds$$

(here ds stands for the length element), is independent of the choice of t and has the form $F_M = (h, 0, 1)$, for some $h > 0$.

- (5) *Let $R_h \subset \mathbb{R}^3$ be the Riemann minimal example with horizontal tangent plane at infinity and flux vector $F = (h, 0, 1)$ along a compact horizontal section. Then, there exists a translation vector $v_T \in \mathbb{R}^3$ such that as $t \rightarrow \infty$, the function $d_+(t) = \sup\{\text{dist}(p, \mathcal{R}_h + v_T) \mid p \in M \cap \{x_3 \geq t\}\}$ is finite and decays exponentially to zero. In a similar manner, there exists $v_B \in \mathbb{R}^3$ such that as $t \rightarrow -\infty$, the function $d_-(t) = \sup\{\text{dist}(p, \mathcal{R}_h + v_B) \mid p \in M \cap \{x_3 \leq t\}\}$ is finite and decays exponentially to zero. Furthermore, $x_2(v_T) = x_2(v_B)$.*

We now briefly outline the interesting elements of the proof of Theorem 8.6. The first important ingredient is that M has bounded curvature and related uniform local area bounds in balls of fixed radius. Using these estimates and fixing an end representative E_T of the top end of M , we can find a sequence of points $p_n \in M$ diverging on E_T with Gaussian curvature bounded by above and away from zero. Our previous arguments show that a subsequence of the translated surfaces $M(n) = M - p_n$ converges as $n \rightarrow \infty$ to a translation of the Riemann minimal example R_h whose flux vector is equal to $F_M = (h, 0, 1)$. Assume after a fixed translation of M that the $M(n)$ converge to R_h . Similarly, for a bottom end representative E_B , we can find a divergent sequence of points q_n such that the surfaces $M'(n) = M - q_n$ converge to R_h as $n \rightarrow \infty$.

The second key ingredient in the proof of the theorem is to use the non-zero Shiffman function S_{E_T} of the end representative E_T , to prove that $M(n)$ converges exponentially quickly to R_h as a function of the x_3 -coordinate of the points p_n (we are assuming here that the genus of M is not zero, hence S_{E_T} is not identically zero because M is not a Riemann minimal example). This is done by first showing that the norm $|S_{E_T}|$ decays exponentially in terms of the x_3 -coordinate function of E_T . In [64] we prove that the bounded Jacobi functions on R_h are all linear and we then use this property to prove the related exponential decay of $|S_{E_T}|$. Once one has this exponential decay estimate for $|S_{E_T}|$, then elliptic theory can be used to show that E_T converges exponentially quickly in terms of x_3 -coordinates to $R_h + v_T$ for some vector $v_T \in \mathbb{R}^3$. Similarly, E_B converges exponentially to $R_h + v_B$ for some $v_B \in \mathbb{R}^3$.

Finally, a forces argument using the Divergence Theorem applied to a certain Killing field on \mathbb{R}^3 shows that the x_2 -coordinate of v_T equals the x_2 -coordinate of v_B , which completes our discussion on the proof of Theorem 8.6.

In conclusion, we remark on an important question for a possible generalization of the classification results discussed in this paper. This question asks whether or not a complete embedded minimal surface $M \subset \mathbb{R}^3$ of finite genus and compact boundary is always properly embedded. If this is the case, then we can replace in the statements of our theorems the phrase “finite genus, properly embedded minimal surfaces” by the phrase “finite genus, complete, embedded minimal surfaces”. As mentioned above, Colding and Minicozzi [21] have recently proved that this generalization holds when the surface has finite topology. Shortly afterwards, Meeks, Pérez and Ros [61] proved this result when the surface has finite genus and a countable number of ends; their result depends on the previously mentioned work in [21] and a related generalization by Meeks and Rosenberg [72] results in [21]. In relation to this problem, recall that Theorem 5.11 implies any properly embedded minimal surface has a countable number of ends. It remains open the outstanding question of whether or not a complete, embedded minimal surface of finite genus can have an uncountable number of ends.

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