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The stability of matter and quantum electrodynamics

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Abstract. Much progress has been made in the last few decades in developing the necessary mathematics for understanding the full implications of the quantum-mechanical many-body problem and why the material world appears to be as stable as it is despite the serious $-1/|x|$ singularity of the Coulomb potential that attracts negative electrons to positive atomic nuclei. Many problems remain, however, especially the understanding of the interaction of matter and the quantized radiation field discovered by Planck in 1900. A short review of some of the main topics, recent progress, and open problems will be given.

1. Introduction

This paper [**1**] is a brief survey of the quantum-mechanical many-body problem, especially the question of the interaction of matter with radiation. The quantum-mechanical revolution of the 1920's brought with it many successes, but also a few problems that have yet to be resolved. The realization that there were a few problems with the simple textbook theory surfaced three or four decades ago. Since then some of the mathematical questions have been answered, but some big ones remain. This brief overview might, it is hoped, encourage some mathematicians to look into this fascinating topic.

We begin with a sketch of the topics that will concern us here.

1.1. Triumph of Quantum Mechanics. One of the basic problems of classical physics (after the discovery of the point electron by Thomson and of the (essentially) point nucleus by Rutherford) was the stability of atoms. Why do the electrons in an atom not fall into the

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nucleus? Quantum mechanics explained this fact. It starts with the classical Hamiltonian of the system (nonrelativistic kinetic energy for the electrons plus Coulomb's law of electrostatic energy among the charged particles). By virtue of the non-commutativity of the kinetic and potential energies in quantum mechanics the stability of an atom – in the sense of a finite lower bound to the energy – was a consequence of the fact that any attempt to make the electrostatic energy very negative would require the localization of an electron close to the nucleus and this, in turn, would result in an even greater, positive, kinetic energy.

Thus, the basic stability problem for an atom was solved by an inequality that says that $\langle 1/|x| \rangle$, the expected value of $1/|x|$, can be made large only at the expense of making the kinetic energy, which is proportional to $\langle p^2 \rangle$, even larger. A fundamental hypothesis of quantum mechanics is that p is represented by the differential operator $-i\hbar\nabla$ with $\hbar = h/2\pi$ and $h =$Planck's constant.

In elementary presentations of the subject it is often said that the mathematical inequality that ensures this fact is the famous uncertainty principle of Heisenberg (proved by Weyl), which states that $\langle p^2 \rangle \langle x^2 \rangle \geq (9/8)\hbar^2$. While this principle is mathematically rigorous it is actually insufficient for the purpose, as explained, e.g., in [**21, 23**], and thus gives only a heuristic explanation of the power of quantum mechanics to prevent collapse. A more powerful inequality, such as Sobolev's inequality (9), is needed (see, e.g., [**24**]). The utility of the latter is made possible by Schrödinger's representation of quantum mechanics (which earlier was a somewhat abstract theory of operators on a Hilbert space) as a theory of differential operators on the space of square integrable functions on $\mathbb{R}^3$. The importance of Schrödinger's representation is sometimes underestimated by formalists, but it is of crucial importance because it permits the use of functional analytic methods, especially inequalities such as Sobolev's, which are not easily visible on the Hilbert space level. These methods are essential for the developments reported here.

To summarize, the understanding of the stability of atoms and ordinary matter requires a formulation of quantum mechanics with two ingredients:

- A Hamiltonian formulation in order to have a clear notion of a lowest possible (ground state) energy. Lagrangian formulations, while popular, do not always lend themselves to the identification of that quintessential quantum mechanical notion of a ground state energy. In quantum mechanics a Hamiltonian is not a function on phase space but rather a (pseudo-) differential operator.

- A formulation in terms of concrete function spaces instead of abstract Hilbert spaces so that the power of mathematical analysis can be fully exploited.

1.2. Some Basic Definitions. As usual, we shall denote the lowest energy (eigenvalue of a Hamiltonian operator) of a quantum mechanical system by E_0. (More generally, E_0 denotes the infimum of the spectrum of the Hamiltonian H in case this infimum is not an eigenvalue of H or is $-\infty$.) Our intention is to investigate arbitrarily large systems, not just atoms. In general we suppose that the system of interest is composed of N electrons and K nuclei of various kinds. Of course we could include other kinds of particles but N and K will suffice here. $N = 1$ for a hydrogen atom and $N = 10^{23}$ for a mole of hydrogen. We shall use the following terminology for two notions of stability:

$$E_0 > -\infty \qquad \text{Stability of the first kind,} \tag{1}$$

$$E_0 > C(N+K) \qquad \text{Stability of the second kind} \tag{2}$$

for some constant $C \leq 0$ that is independent of N and K, but which may depend on the physical parameters of the system (such as the electron charge and mass). Usually, $C < 0$, which means that there is a positive binding energy per particle.

Stability of the second kind is absolutely essential if quantum mechanics is going to reproduce some of the basic features of the ordinary material world: The energy of ordinary matter is extensive (i.e., it is proportional to the number of particles), the thermodynamic limit exists (i.e., the $N \to \infty$ limit exists, see Sect. 8) and the laws of thermodynamics hold. Bringing two stones together might produce a spark, but not an explosion with a release of energy comparable to the energy in each stone. Stability of the second kind does not guarantee the existence of the thermodynamic limit for the free energy, but it is an essential ingredient [**19, 20**], [**21**, Sect. V].

It turns out that stability of the second kind cannot be taken for granted, as Dyson discovered [**9**]. If Coulomb forces are involved, then *the Pauli exclusion principle is essential.* (This means that the L^2 functions of N variables, $\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, $\mathbf{x}_i \in \mathbb{R}^3$, is antisymmetric under all transpositions $\mathbf{x}_i \leftrightarrow \mathbf{x}_j$. Particles, like electrons, whose functions Ψ obey this principle are called *fermions*. Particles whose Ψ functions are symmetric under permutations are called *bosons*.)

Charged bosons are *not stable* because for them $E_0 \sim -N^{7/5}$ (nonrelativistically) and $E_0 = -\infty$ for large, but finite N (relativistically, see Sect. 3.2). While positively charged bosons exist in the form of atomic nuclei, negatively charged, long-lived bosons do not exist in nature. This is a good thing in view of the instability just mentioned.

1.3. The Electromagnetic Field. A second big problem handed down from classical physics was the 'electromagnetic mass' of the electron. This poor creature has to drag around an infinite amount of electromagnetic energy that Maxwell burdened it with. Moreover, the electromagnetic field itself is quantized – indeed, that fact alone started the whole revolution [**36**].

While quantum mechanics accounted for stability with Coulomb forces and Schrödinger led us to think seriously about the 'wave function of the universe', physicists shied away from talking about the wave function of the particles in the universe *and* the electromagnetic field in the universe. It is noteworthy that physicists are happy to discuss the quantum mechanical many-body problem with external electromagnetic fields non-perturbatively, but this is rarely done with the quantized field. The quantized field cannot be avoided because it is needed for a correct description of atomic radiation, the laser, etc. However, the interaction of matter with the quantized field is almost always treated perturbatively or else in the context of highly simplified models (e.g., with two-level atoms for lasers).

The quantized electromagnetic field greatly complicates the stability of matter question. It requires, ultimately, mass and charge renormalizations. At present such a complete theory does not exist, but a theory *must* exist because matter exists and because we have strong experimental evidence about the manner in which the electromagnetic field interacts with matter, i.e., we know the essential features of a Hamiltonian that adequately accounts for the low energy processes that exist in every day life. In short, nature tells us that it must be possible to formulate a self-consistent quantum electrodynamics (QED) *non-perturbatively,* (perhaps with an ultraviolet, or high frequency, cutoff of the field at a few MeV). It should not be necessary to have recourse to quantum chromodynamics (QCD) or some other high energy theory to explain ordinary matter.

Physics and other natural sciences are successful because physical phenomena associated with each range of energy and other parameters are explainable to a good, if not perfect, accuracy by an appropriate self-consistent theory. This is true whether it be hydrodynamics, celestial dynamics, statistical mechanics, etc. If low energy physics (atomic and condensed matter physics) is not explainable by a self-consistent, non-perturbative theory on its own level one can speak of an epistemological crisis.

Some readers might say that QED is in good shape. After all, it accurately predicts the outcome of some very high precision experiments (Lamb shift, g-factor of the electron). But the theory does not really work well when faced with the problem, which is explored here, of understanding the many-body ($N \approx 10^{23}$) problem and the stable low energy world in which we spend our everyday lives.

1.4. Relativistic Mechanics. When the classical kinetic energy of a particle, $p^2/2m$, is replaced by its relativistic version $\sqrt{p^2c^2 + m^2c^4}$ the stability question becomes much more complicated, as will be seen later. It turns out that even stability of the first kind is not easy to obtain and it depends on the values of the physical constants, notably the fine structure constant

$$\alpha = e^2/\hbar c = 1/137.04 \ , \tag{3}$$

where $-e$ is the electric charge of the electron.

For ordinary matter relativistic effects are not dominant but they are noticeable. In large atoms these effects severely change the innermost electrons and this has a noticeable effect on the overall electron density profile. Therefore, some version of relativistic mechanics is needed, which means, presumably, that we must know how to replace $p^2/2m$ by the Dirac operator (see (18)).

The combination of relativistic mechanics plus the electromagnetic field (in addition to the Coulomb interaction) makes the stability problem difficult and uncertain. Major aspects of this problem have been worked out in the last few years (about 35) and that is the subject of this paper.

2. Nonrelativistic Matter without the Magnetic Field

Maxwell's equations define the electric and magnetic fields in terms of potentials. While the equations determine the fields, the potentials are not determined uniquely; the choice of potentials is called the choice of gauge. We work in the 'Coulomb' gauge for the electromagnetic field. Despite the assertion that quantum mechanics and quantum field theory are gauge invariant, it seems to be essential to use this gauge, even though its relativistic covariance is not as transparent as that of the Lorentz gauge. The reason is the following.

The Coulomb gauge is the gauge in which electrostatic part of the interaction of matter with the electromagnetic field is just the conventional Coulomb "action at a distance" potential V_c given by (4) below (in energy units mc^2 and length units the Compton wavelength $\hbar/mc$). This part of the interaction depends only on the coordinates of the particles and not on their velocities. The dependence of the interaction on velocities, or currents, comes about through the magnetic part of the interaction. Despite appearances, this picture is fully Lorentz invariant (even if it is not gauge invariant).

$$V_c = -\sum_{i=1}^{N}\sum_{k=1}^{K} \frac{Z_k}{|\mathbf{x}_i - \mathbf{R}_k|} + \sum_{1\le i<j\le N} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} + \sum_{1\le k<l\le K} \frac{Z_k Z_l}{|\mathbf{R}_k - \mathbf{R}_l|}. \tag{4}$$

The first sum is the interaction of the electrons (with dynamical coordinates $\mathbf{x}_i$) and fixed nuclei located at $\mathbf{R}_k$ of positive charge Z_k times the (negative) electron charge e. The second is the electron-electron repulsion and the third is the nucleus-nucleus repulsion. The nuclei are fixed because they are so massive relative to the electron that their motion is irrelevant. It could be included, however, but it would change nothing essential. Likewise, there is no nuclear structure factor because if it were essential for stability then the size of atoms would be the size of nuclei, about 10^{-13} cm, instead of about 10^{-8} cm, contrary to what is observed.

Although the nuclei are fixed points the constant C in the stability of matter (2) is required to be independent of the $\mathbf{R}_k$'s. Likewise (1) requires that E_0 have a finite lower bound that is independent of the $\mathbf{R}_k$'s.

For simplicity of exposition we shall assume here that all the Z_k are identical, i.e., $Z_k = Z$.

The magnetic field, which will be introduced later, is described by a vector potential $\mathbf{A}(x)$ which is a dynamical variable in the Coulomb gauge. The magnetic field is $\mathbf{B} = \mathrm{curl}\mathbf{A}$.

There is a basic physical distinction between electric and magnetic forces which does not seem to be well known, but which motivates this choice of gauge. In electrostatics "like charges repel" while in magnetostatics "like currents attract". A consequence of these facts is that the correct magnetostatic interaction energy can be obtained by minimizing the energy functional $\frac{1}{2}\int B^2 - \int \mathbf{j}\cdot\mathbf{A}$ with respect to the vector field $\mathbf{A}$, where $\mathbf{j}$ is the electric current density. The positive electrostatic energy, on the other hand, *cannot* be obtained by a minimization principle with respect to the field (e.g., minimizing $\frac{1}{2}\int |\boldsymbol{\nabla}\phi|^2 - \int \phi\varrho$ with respect to ϕ).

The Coulomb gauge, which puts in the electrostatics correctly, by hand, so to speak, and allows us to minimize the total energy with respect to the $\mathbf{A}$ field, is the gauge that gives us the correct physics and is consistent with the "quintessential quantum mechanical notion of a ground state energy" mentioned in Sect. 1.1. In any other gauge one would have to look for a critical point of a Hamiltonian rather than a true global minimum.

The type of Hamiltonian that we wish to consider in this section is

$$H_N = T_N + \alpha V_c \ . \tag{5}$$

Here, T_N is the kinetic energy of the N electrons and has the form

$$T_N = \sum_{i=1}^{N} T_i \ , \tag{6}$$

where T_i acts on the coordinate of the i^{th} electron. The nonrelativistic choice is $T = p^2$ with $\mathbf{p} = -i\boldsymbol{\nabla}$ and $p^2 = -\Delta$ in appropriate units.

2.1. Nonrelativistic Stability for Fermions. The problem of stability of the second kind for nonrelativistic quantum mechanics was recognized in the early days by a few physicists, e.g., Onsager, but not by many. It was not solved until 1967 in one of the most beautiful papers in mathematical physics by Dyson and Lenard [**10**].

They found that the Pauli principle, i.e., Fermi-Dirac statistics, is essential. Mathematically, this means that the Hilbert space is the subspace of antisymmetric functions, i.e., $\mathcal{H}^{\mathrm{phys}} = \wedge^N L^2(\mathbb{R}^3;\mathbb{C}^2)$. This is how the Pauli principle is interpreted post-Schrödinger; Pauli invented his principle a year earlier, however!

Their value for C in (2) was rather high, about -10^{15} eV (electron volts) for $Z = 1$. (The ground state energy of a hydrogen atom is -13 eV.) The situation was improved later by Thirring and myself [**33**] to about -20 eV for $Z = 1$ by introducing an inequality that holds only for the kinetic energy of fermions (not bosons) in an arbitrary state Ψ.

$$\langle \Psi, T_N \Psi\rangle \geq (\mathrm{const.}) \int_{\mathbb{R}^3} \varrho_\Psi(\mathbf{x})^{5/3}\, \mathrm{d}^3\mathbf{x}, \tag{7}$$

where ϱ_Ψ is the one-body density in the (normalized) fermionic wave function Ψ (of space and spin) given by an integration over $(N-1)$ coordinates and N spins as follows.

$$\varrho_\Psi(\mathbf{x}) = N \sum_{\sigma_1,\ldots,\sigma_N} \int_{\mathbb{R}^{3(N-1)}} |\Psi(\mathbf{x},\, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \ldots \sigma_N)|^2\, \mathrm{d}^3\mathbf{x}_2 \cdots \mathrm{d}^3\mathbf{x}_N. \tag{8}$$

Inequality (7) allows one simply to reduce the quantum mechanical stability problem to the stability of Thomas-Fermi theory, which was worked out earlier by Simon and myself [**32**].

The older inequality of Sobolev, mentioned in Sect. 1.1,

$$\langle \Psi, T_N \Psi\rangle \geq (\mathrm{const.}) \left(\int_{\mathbb{R}^3} \varrho_\Psi(\mathbf{x})^3\, \mathrm{d}^3\mathbf{x}\right)^{1/3}, \tag{9}$$

is not as useful as (7) for the many-body problem because its right side is proportional to N instead of $N^{5/3}$. It is, however, strong enough to yield the stability of a system, like an atom, that has only a few electrons.

It is amazing that from the birth of quantum mechanics until 1967 none of the luminaries of physics had quantified the fact that electrostatics plus the uncertainty principle *do not suffice* for stability of the second kind, and thereby make thermodynamics possible (although they do suffice for the first kind). See Sect. 2.2. It was noted, however, that the Pauli principle was responsible for the large sizes of atoms and bulk matter (see, e.g., [**9, 10**]).

2.2. Nonrelativistic Instability for Bosons. What goes wrong if we have charged bosons instead of fermions? Stability of the first kind (1) holds in the nonrelativistic case, but (2) fails. If we assume the nuclei are infinitely massive, as before, and $N = KZ$ then $E_0 \sim -N^{5/3}$ [**10, 22**]. To remedy the situation we can let the nuclei have finite mass (e.g., the same mass as the negative particles). Then, as Dyson showed [**9**], $E_0 \leq -(\text{const.})N^{7/5}$. This calculation was highly non-trivial! Dyson had to construct a variational function with pairing of the Bogolubov type in a rigorous fashion and this took several pages.

Thus, finite nuclear mass improves the situation, but not enough. The question whether $N^{7/5}$ is the correct power law remained open for many years. A lower bound of this type was needed and that was finally obtained in [**6**].

The results of this Section 2 can be summarized by saying that stability of the hydrogen atom is one thing but stability of many-body physics is something else!

3. Relativistic Kinematics (no magnetic field)

The next step is to try to get some idea of the effects of relativistic kinematics, which means replacing p^2 by $\sqrt{p^2+1}$ in non-quantum physics. (Recall that $mc^2 = 1$ in our units.) The simplest way to do this is to substitute $\sqrt{p^2+1}$ for T in (6). The Dirac operator will be discussed later on, but for now this choice of T will suffice. Actually, it was Dirac's choice before he discovered his operator and it works well in some cases. For example, Chandrasekhar used it successfully, and accurately, to calculate the collapse of white dwarfs (and later, neutron stars).

Since we are interested only in stability, we may, and shall, substitute $|\mathbf{p}| = \sqrt{-\Delta}$ for T. The error thus introduced is bounded by a constant times N since $|\mathbf{p}| < \sqrt{p^2+1} < |\mathbf{p}| + 1$ (as an operator inequality). Our Hamiltonian is now $H_N = \sum_{i=1}^N |\mathbf{p}_i| + \alpha V_c$.

3.1. One-Electron Atom. The touchstone of quantum mechanics is the Hamiltonian for 'hydrogen' which is, in our case,

$$H = |\mathbf{p}| - Z\alpha/|\mathbf{x}| = \sqrt{-\Delta} - Z\alpha/|\mathbf{x}| \; . \tag{10}$$

It is well known (also to Dirac) that the analogous operator with $|\mathbf{p}|$ replaced by the Dirac operator (18) ceases to make sense when $Z\alpha > 1$. Something similar happens for (10).

$$E_0 = \begin{cases} 0 & \text{if } Z\alpha \leq 2/\pi; \\ -\infty & \text{if } Z\alpha > 2/\pi. \end{cases} \tag{11}$$

The reason for this behavior is that both $|\mathbf{p}|$ and $|\mathbf{x}|^{-1}$ scale in the same way. Either the first term in (10) wins or the second does.

A result similar to (11) was obtained in [**11**] for the free Dirac operator $D(0)$ in place of $|\mathbf{p}|$, but with the wave function Ψ restricted to lie in the positive spectral subspace of $D(0)$. Here, the critical value is $\alpha Z \leq (4\pi)/(4+\pi^2) > 2/\pi$.

The moral to be drawn from this is that relativistic kinematics plus quantum mechanics is a 'critical' theory (in the mathematical sense). This fact will plague any relativistic theory of electrons and the electromagnetic field – primitive or sophisticated.

3.2. Many Electrons and Nuclei. When there are many electrons is it true that the condition $Z\alpha \leq const.$ is the only one that has to be considered? The answer is no! One *also* needs the condition that α itself must be small, regardless of how small Z might be. This fact can be called a 'discovery' but actually it is an overdue realization of some basic physical ideas. It should have been realized shortly after Dirac's theory in 1927, but it does not seem to have been noted until 1983 [**8**].

The underlying physical heuristics is the following. With α fixed, suppose $Z\alpha = 10^{-6} \ll 1$, so that an atom is stable, but suppose that we have 2×10^6 such nuclei. By bringing them together at a common point we will have a nucleus with $Z\alpha = 2$ and one electron suffices to cause collapse into it. Then (1) fails. What prevents this from happening, presumably, is the nucleus-nucleus repulsion energy which goes to $+\infty$ as the nuclei come together. But this repulsion energy is proportional to $(Z\alpha)^2/\alpha$ and, therefore, if we regard $Z\alpha$ as fixed we see that $1/\alpha$ must be large enough in order to prevent collapse.

Whether or not the reader believes this argument, the mathematical fact is that there is a fixed, finite number $\alpha_c \leq 2.72$ ([**34**]) so that when $\alpha > \alpha_c$ (1) fails for *every* positive Z and for every $N \geq 1$ (with or without the Pauli principle).

The open question was whether (2) holds for *all* N and K if $Z\alpha$ and α are both small enough. The breakthrough was due to Conlon [**5**] who proved (2), for fermions, if $Z = 1$ and $\alpha < 10^{-200}$. The situation was improved by Fefferman and de la Llave [**13**] to $Z = 1$ and $\alpha < 0.16$. Finally, the expected correct condition $Z\alpha \leq 2/\pi$ and $\alpha < 1/94$ was obtained in [**34**]. (This paper contains a detailed history up to 1988.) The situation was further improved in [**29**]. The multi-particle version of the use of the free Dirac operator, as in Sect. 3.1, was treated in [**18**].

Finally, it has to be noted that charged bosons are *always* unstable of the first kind (not merely the second kind, as in the nonrelativistic case) for *every* choice of $Z > 0, \alpha > 0$. E.g., there is instability if $Z^{2/3}\alpha N^{1/3} > 36$ ([**34**]).

We are indeed fortunate that there are no stable, negatively charged bosons.

4. Interaction of Matter with Classical Magnetic Fields

The magnetic field $\mathbf{B}$ is defined by a vector potential $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x}) = \operatorname{curl} \mathbf{A}(\mathbf{x})$. In this section we take a first step (warmup exercise) by regarding $\mathbf{A}$ as classical, but indeterminate, and we introduce the classical field energy

$$H_f = \frac{1}{8\pi} \int_{\mathbb{R}^3} \mathbf{B}(\mathbf{x})^2 \mathrm{d}x \ . \tag{12}$$

The Hamiltonian is now

$$H_N(\mathbf{A}) = T_N(\mathbf{A}) + \alpha V_c + H_f \ , \tag{13}$$

in which the kinetic energy operator has the form (6) but depends on $\mathbf{A}$. We now define E_0 to be the infimum of $\langle \Psi,\ H_N(\mathbf{A})\Psi\rangle$ both with respect to Ψ *and with respect to* $\mathbf{A}$.

4.1. Nonrelativistic Matter with Magnetic Field. The simplest situation is merely 'minimal coupling' without spin, namely,

$$T(\mathbf{A}) = |\mathbf{p} + \sqrt{\alpha}\mathbf{A}(\mathbf{x})|^2 \tag{14}$$

This choice does not change any of our previous results qualitatively. The field energy is not needed for stability. On the one-particle level, we have the 'diamagnetic inequality' $\langle \phi,\ |\mathbf{p} + \mathbf{A}(\mathbf{x})|^2\phi\rangle \geq \langle |\phi|,\ p^2|\phi|\rangle$. The same holds for $|\mathbf{p} + \mathbf{A}(\mathbf{x})|$ and $|\mathbf{p}|$. More importantly, inequality (7) for fermions continues to hold (with the same constant) with $T(\mathbf{A})$ in place of p^2. (There is an inequality similar to (7) for $|\mathbf{p}|$, with 5/3 replaced by 4/3, which also continues to hold with minimal substitution [**7**].)

The situation gets much more interesting if spin is included. This takes us a bit closer to the relativistic case. The kinetic energy operator is the Pauli operator

$$T^P(\mathbf{A}) = |\mathbf{p} + \sqrt{\alpha}\ \mathbf{A}(\mathbf{x})|^2 + \sqrt{\alpha}\ \mathbf{B}(\mathbf{x})\cdot\boldsymbol{\sigma} \ , \tag{15}$$

where $\boldsymbol{\sigma}$ is the vector of 2×2 Pauli spin matrices and $L^2(\mathbb{R}^3)$ is replaced by $L^2(\mathbb{R}^3;\mathbb{C}^3)$

4.1.1. *One-Electron Atom.* The stability problem with $T^P(\mathbf{A})$ is complicated, even for a one-electron atom. Without the field energy H_f the Hamiltonian is unbounded below. (For fixed $\mathbf{A}$ it is bounded but the energy tends to $-\infty$ like $-(\log B)^2$ for a homogeneous field [**2**].) The field energy saves the day, but the result is surprising [**14**] (recall that we must minimize the energy with respect to Ψ *and* $\mathbf{A}$):

$$|\mathbf{p} + \sqrt{\alpha}\ \mathbf{A}(\mathbf{x})|^2 + \sqrt{\alpha}\ \mathbf{B}(\mathbf{x})\cdot\boldsymbol{\sigma} - Z\alpha/|\mathbf{x}| + H_f \tag{16}$$

is bounded below if and only if $Z\alpha^2 \leq C$, where C is some constant that can be bounded as $1 < C < 9\pi^2/8$.

The proof of instability [**35**] is difficult and requires the construction of a zero mode (soliton) for the Pauli operator, i.e., a finite energy magnetic field and a *square integrable* ψ such that

$$T^P(\mathbf{A})\psi = 0 \ . \tag{17}$$

The usual kinetic energy $|\mathbf{p}+\mathbf{A}(\mathbf{x})|^2$ has no such zero mode for any $\mathbf{A}$, even when 0 is the bottom of its spectrum.

The original magnetic field [**35**] that did the job in (17) is independently interesting, geometrically (many others have been found since then).

$$\mathbf{B}(\mathbf{x}) = \frac{12}{(1+|\mathbf{x}|^2)^3}[(1-x^2)\mathbf{w} + 2(\mathbf{w}\cdot\mathbf{x})\mathbf{x} + 2\mathbf{w}\wedge\mathbf{x}]$$

with $|\mathbf{w}| = 1$. The field lines of this magnetic field form a family of curves, which, when stereographically projected onto the 3-dimensional unit sphere, become the great circles in what what is known as the Hopf fibration.

Thus, we begin to see that nonrelativistic matter with magnetic fields behaves like relativistic matter without fields – to some extent.

The moral of this story is that a magnetic field, which we might think of as possibly self-generated, can cause an electron to fall into the nucleus. The uncertainty principle cannot prevent this, not even for an atom!

4.1.2. *Many Electrons and Many Nuclei.* In analogy with the relativistic (no magnetic field) case, we can see that stability of the first kind fails if $Z\alpha^2$ *or* α is too large. The heuristic reasoning is the same and the proof is similar.

We can also hope that stability of the second kind holds if both $Z\alpha^2$ *and* α are small enough. The problem is complicated by the fact that it is the field energy H_f that will prevent collapse, but there there is only one field energy while there are $N \gg 1$ electrons.

The hope was finally realized, however. Fefferman [**12**] proved stability of the second kind for $H_N(\mathbf{A})$ with the Pauli $T^P(\mathbf{A})$ for $Z = 1$ and "α sufficiently small". A few months later it was proved [**30**] for $Z\alpha^2 \leq 0.04$ and $\alpha \leq 0.06$. With $\alpha = 1/137$ this amounts to $Z \leq 1050$. This very large Z region of stability is comforting because it means that perturbation theory (in $\mathbf{A}$) can be reliably used for this particular problem.

Using the results in [**30**], Bugliaro, Fröhlich and Graf [**3**] proved stability of the same nonrelativistic Hamiltonian – but with an ultraviolet cutoff, quantized magnetic field whose field energy is described below. (Note: No cutoffs are needed for classical fields.)

There is also the very important work of Bach, Fröhlich, and Sigal [**4**] who showed that this nonrelativistic Hamiltonian with ultraviolet cutoff, quantized field *and* with sufficiently small values of the parameters has other properties that one expects. E.g., the excited states of atoms dissolve into resonances and only the ground state is stable. The infrared singularity notwithstanding, the ground state actually exists (the bottom of the spectrum is an eigenvalue); this was shown in [**4**] for small parameters and in [**15**], [**27**] for all values of the parameters. (See Sect. 7.)

5. Relativity Plus Magnetic Fields

As a next step in our efforts to understand QED and the many-body problem we introduce relativity theory along with the classical magnetic field.

5.1. Relativity Plus Classical Magnetic Fields. Originally, Dirac and others thought of replacing $T^P(\mathbf{A})$ by $\sqrt{T^P(\mathbf{A})+1}$ but this was not successful mathematically and does not seem to conform to experiment. Consequently, we introduce the Dirac operator for T in (6), (13)

$$D(\mathbf{A}) = \boldsymbol{\alpha}\cdot\mathbf{p} + \sqrt{\alpha}\ \boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x}) + \beta m\ , \tag{18}$$

where $\boldsymbol{\alpha}$ and β denote the 4×4 Dirac matrices and $\sqrt{\alpha}$ is the electron charge as before. (This notation of $\boldsymbol{\alpha}$ and α is historical and is not mine.) The Hilbert space for N electrons is now changed to

$$\mathcal{H} = \wedge^N L^2(\mathbb{R}^3;\mathbb{C}^4)\ . \tag{19}$$

The well known problem with $D(\mathbf{A})$ is that it is unbounded below, and so we cannot hope to have stability of the first kind, even with $Z = 0$. Let us imitate QED (but without pair production or renormalization) by restricting the electron wave function to lie in the positive spectral subspace of a Dirac operator.

Which Dirac operator?

There are two natural operators in the problem. One is $D(0)$, the free Dirac operator. The other is $D(\mathbf{A})$ that is used in the Hamiltonian. In almost all formulations of QED the electron is defined by the positive spectral subspace of $D(0)$. Thus, we can define

$$\mathcal{H}^{\rm phys} = P^+\ \mathcal{H} = \Pi_{i=1}^N \pi_i\, \mathcal{H}\ , \tag{20}$$

where $P^+ = \Pi_{i=1}^N \pi_i$, and π_i is the projector of onto the positive spectral subspace of $D_i(0) = \boldsymbol{\alpha}\cdot\mathbf{p}_i + \beta m$, the free Dirac operator for the $i^{\rm th}$ electron. We then restrict the allowed wave functions in the variational principle to those Ψ satisfying

$$\Psi = P^+\ \Psi \qquad \textit{i.e.},\ \Psi \in \mathcal{H}^{\rm phys}\ . \tag{21}$$

Another way to say this is that we replace the Hamiltonian (13) by $P^+\, H_N\, P^+$ on $\mathcal{H}$ and look for the bottom of its spectrum.

It turns out that this prescription leads to disaster! While the use of $D(0)$ makes sense for an atom, it fails miserably for the many-fermion problem, as discovered in [**31**] and refined in [**16**]. The result is:

For all $\alpha > 0$ *in* (18) (*with or without the Coulomb term* αV_c) *one can find* N *large enough so that* $E_0 = -\infty$.

In other words, the term $\sqrt{\alpha}\,\boldsymbol{\alpha}\cdot\mathbf{A}$ in the Dirac operator can cause an instability that the field energy cannot prevent.

It turns out, however, that the situation is saved if one uses the positive spectral subspace of the Dirac operator $D(\mathbf{A})$ to define an electron. (This makes the concept of an electron $\mathbf{A}$ dependent, but when we make the vector potential into a dynamical quantity in the next section, this will be less peculiar since there will be no definite vector potential but only a fluctuating quantity.) The definition of the physical Hilbert space is as in (20) but with π_i being the projector onto the positive subspace of the full Dirac operator $D_i(\mathbf{A}) = \boldsymbol{\alpha}\cdot\mathbf{p}_i + \sqrt{\alpha}\,\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x}_i) + \beta m$. Note that these π_i projectors commute with each other and hence their product P^+ is a projector.

The result [**31**] for this model ((13) with the Dirac operator and the restriction to the positive spectral subspace of $D(\mathbf{A})$) is reminiscent of the situations we have encountered before:

If α *and* Z *are small enough stability of the second kind holds for this model.*

Typical stability values that are rigorously established [**31**] are $Z \leq 56$ with $\alpha = 1/137$ or $\alpha \leq 1/8.2$ with $Z = 1$.

6. Quantized Electromagnetic Fields

Let us now try to analyze some of the problems connected with the quantization of the electromagnetic field. The great discovery of Max Planck [**36**], which was the first step in the new quantum theory, was that the energy of the electromagnetic field came in quantized units. The energy unit of electromagnetic waves of frequency ν is $h\nu$, and in terms of wave number k (i.e., the wave is proportional to $\exp(ik\cdot x)$) it is $\hbar c|k|$ since $2\pi\nu/|k| = c =$ speed of light.

We begin with the problem of generalizing the results in the previous subsection to the quantized field.

6.1. Relativity Plus Quantized Magnetic Field. The obvious next step is to try to imitate the strategy of Sect. 5.1 but with the quantized $\mathbf{A}$ field. This was done in [**25**]. The quantized A field is described by an operator-valued Fourier transform as

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2\pi}\sum_{\lambda=1}^{2}\int_{|\mathbf{k}|\leq\Lambda}\frac{\boldsymbol{\varepsilon}_\lambda(\mathbf{k})}{\sqrt{|\mathbf{k}|}}\Big[a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_\lambda(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\Big]\mathrm{d}^3\mathbf{k}, \tag{22}$$

where Λ is the ultraviolet cutoff on the wave-numbers $|\mathbf{k}|$. The operators a_λ, a_λ^* satisfy the usual canonical commutation relations

$$[a_\lambda(\mathbf{k}), a_\nu^*(\mathbf{q})] = \delta(\mathbf{k}-\mathbf{q})\delta_{\lambda,\nu}\ ,\ [a_\lambda(\mathbf{k}), a_\nu(\mathbf{q})] = 0, \quad \text{etc} \tag{23}$$

and the vectors $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$ are two orthonormal polarization vectors perpendicular to $\mathbf{k}$ and to each other.

The field energy H_f is now given by a normal-ordered version of (12)

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |\mathbf{k}|\ a_\lambda^*(\mathbf{k}) a_\lambda(\mathbf{k}) \mathrm{d}^3\mathbf{k}. \tag{24}$$

The Dirac operator is the same as before, (18). Note that $D_i(\mathbf{A})$ and $D_j(\mathbf{A})$ still commute with each other (since $\mathbf{A}(\mathbf{x})$ commutes with $\mathbf{A}(\mathbf{y})$). This is important because it allows us to imitate Sect. 5.1.

In analogy with (19) we define

$$\mathcal{H} = \wedge^N L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}\ , \tag{25}$$

where $\mathcal{F}$ is the Fock space for the photon field. We can then define the *physical* Hilbert space as before

$$\mathcal{H}^{\text{phys}} = \Pi\ \mathcal{H} = \Pi_{i=1}^N \pi_i\, \mathcal{H}\ , \tag{26}$$

where the projectors π_i project onto the positive spectral subspace of either $D_i(0)$ or $D_i(\mathbf{A})$.

Perhaps not surprisingly, the former case leads to catastrophe, as before. This is so, even with the ultraviolet cutoff, which we did not have in Sect. 5.1. Because of the cutoff the catastrophe is milder and involves instability of the second kind instead of the first kind. This result relies on a coherent state construction in [**16**].

The latter case (use of $D(\mathbf{A})$ to define an electron) leads to stability of the second kind if Z and α are not too large. Otherwise, there is instability of the first kind. The rigorous estimates are comparable to the ones in Sect. 5.1.

Clearly, many things have yet to be done to understand the stability of matter in the context of QED. Renormalization and pair production have to be included, for example.

The results of this section suggest, however, that a significant change in the Hilbert space structure of QED might be necessary. We see that it does not seem possible to keep to the current view that the Hilbert space is a simple tensor product of a space for the electrons and a Fock space for the photons. That leads to instability for many particles (or large charge, if the idea of 'particle' is unacceptable). The 'bare' electron is not really a good physical concept and one must think of the electron as always accompanied by its electromagnetic field. Matter and the photon field are inextricably linked in the Hilbert space $\mathcal{H}^{\text{phys}}$.

The following tables [**25**] summarize some of the results of this and the previous sections

Electrons defined by projection onto the positive subspace of $D(0)$, the free Dirac operator

	Classical or quantized field without cutoff Λ $\alpha > 0$ but arbitrarily small.	Classical or quantized field with cutoff Λ $\alpha > 0$ but arbitrarily small.
Without Coulomb potential αV_c	Instability of the first kind	Instability of the second kind
With Coulomb potential αV_c	Instability of the first kind	Instability of the second kind

Electrons defined by projection onto the positive subspace of $D(\mathbf{A})$, the Dirac operator with field

	Classical field with or without cutoff Λ or quantized field with cutoff Λ
Without Coulomb potential αV_c	The Hamiltonian is positive
With Coulomb potential αV_c	Instability of the first kind when either α or $Z\alpha$ is too large
	Stability of the second kind when both α and $Z\alpha$ are small enough

6.2. Mass Renormalization. In both classical and quantum electrodynamics there is a problem of mass renormalization. This means that when a charge is accelerated its accompanying electromagnetic field is also accelerated and acts like an additional mass. The 'bare mass' of the particle (which is the mass that appears in the Hamiltonian) must be chosen so that the final, physical mass (as measured in experiments) agrees with the physically measured value.

For a point particle, the additional mass is infinity, classically. For QED it is also infinite, but the divergence is less rapid as the radius of the charge goes to zero. In any case, with a finite ultraviolet cutoff Λ the additional mass is finite, but it is far from clear that, for each $\Lambda > 0$ one can adjust the bare mass (while keeping it positive) to give the correct physical mass. Opinions differ on this point and very little is known rigorously about the problem outside of perturbation theory. See [**17**].

There are two ways to define mass renormalization. Take one particle ($N = 1$) and then `either`

1. Find the bottom of the spectrum of $T + H_f$ under the condition that the total momentum of particle plus field is p. Call it $E(p)$ and write, for small p,

$$E(p) = E(p = 0) + p^2/2m_{\text{physical}}$$

`or else`

2. Compute the binding energy of hydrogen ($N = 1, K = 1, Z = 1$). Call it E_0 and set

$$E_0 = m_{\text{physical}} c^2 \alpha^2 / 2\hbar^2$$

The first way is the usual one; the second is motivated by the earliest experiment in quantum mechanics. These two definitions are not the same. In any case, we [**26**] can now obtain non-trivial bounds on the binding energy (in the context of the Schrödinger Hamiltonian or the Pauli Hamiltonian interacting with the quantized field) and thereby get some bounds on the renormalized mass using definition 2. For large cutoff Λ, these bounds differ in their Λ dependence from what might be expected from perturbation theory.

7. Existence of Atoms in Non-relativistic QED

One of the most recent topics concerns the seemingly trivial question of the existence of atoms. In some sense this question is the opposite of the stability of matter question.

The Hamiltonian we shall use to describe an atom or molecule with N electrons is

$$H_N = \sum_{i=1}^{N} T_i^P(A) + \alpha V_c + H_f \tag{27}$$

where $T_i^P(A)$ is the Pauli kinetic energy operator (15), but A is the quantized magnetic field given by (22), and H_f is the energy of the quantized field given by (24). As before, V_c is the Coulomb potential (4) of some fixed nuclei whose total nuclear charge is denoted by $Z = \sum Z_j$.

To show the existence of stable atoms we need to establish two things about H_N

1. The ground state energy (bottom of the spectrum) of H_N is lower than that of $H_{N'}$ i.e., of a system with $N' < N$ electrons (with the remaining $N - N'$ electrons being allowed to escape to infinity). This is called *the binding condition*.

2. The bottom of the spectrum of H_N is actually an eigenvalue, i.e., Schrödinger's equation has a square integrable solution with $E =$ the bottom of the spectrum.

In the case of the Schrödinger equation without the field, problem 1. was solved by Zhislin in 1960 for the case $N < Z+1$, which includes the neutral molecule. He did this by using a localization technique, whose positive localization energy (r^{-2}) is more than offset by the Coulomb attraction ($-r^{-1}$) of a positively charged system ($Z - N'$) to a negatively charged electron. The existence of the ground state (problem 2.) follows from standard arguments because in this case the bottom of the spectrum is negative while the bottom of the essential spectrum (which, in this case, is the bottom of the continuum) starts at zero. Thus, there is a gap in the spectrum and the technique of taking weak limits easily yields a non-zero eigenfunction [**24**].

When we turn on the interaction with the quantized magnetic field the situation changes significantly. One major difference is that the bottom of the essential spectrum is now the bottom of the spectrum because we can always create photons with arbitrarily small energy (recall that the energy of a photon with momentum k is $|k|$). Therefore, if a ground state exists it necessarily lies at the bottom of the essential spectrum and is not isolated. Eigenvalues in the continuum are notoriously difficult to handle, even for the simple Schrödinger operator.

A second major difference is that it is necessary to localize the A field as well as the electrons. This localization costs an energy r^{-1}, not r^{-2} as before, essentially because the field energy is proportional to $|k|$ instead of k^2. Thus, the field localization competes with the Coulomb attraction.

Problems 1. and 2. were solved in [**4**] under the condition that α and Λ are small enough.

The first general result, valid for all values of the various constants, was in [**15**], where it was shown that 2. holds whenever 1. holds.

Finally, 1. was shown to hold for all values of the constants [**27**] under the same natural condition as Zhislin's, i.e., $N < Z + 1$.

8. Thermodynamic Limit in Non-relativistic QED

We close this mini-review with with a partial result and an open problem.

Stability of the second kind for a many-body Hamiltonian, H_N of N particles is only the starting point for thermodynamics and statistical mechanics. The next step is to compute the partition function

$$\mathcal{Z} = \mathrm{Trace}\, e^{-H_N/k_BT} \ , \tag{28}$$

where k_B is Boltzmann's constant and T is the temperature. Here, the N particles are localized in a large box (a domain $\Omega \subset \mathbf{R}^3$) of volume V (Dirichlet boundary conditions on the boundary of Ω) and we want to take the "'thermodynamic" limit $N \to \infty$ with the density $\rho = N/V$ held fixed. The physical significance of $\mathcal{Z}$ is that the free energy per unit volume, f, is given by $f = -k_BTV^{-1} \ln \mathcal{Z}$.

Some years ago the problem of proving the existence of the thermodynamic limit for electrons, nuclei and other particles interacting via Coulomb forces was settled in the context of the non-relativistic Schrödinger equation [**19, 20**]. The key ingredients in this proof, in broad outline, were:

a) Stability of the second kind easily led to an upper bound on the partition function $\mathcal{Z}$, and hence a lower bound on f, the free energy per particle, independent of the shape of the sequence of Ω's.

b) A rigorous version of screening together with a variational argument for a lower bound on $\mathcal{Z}$, which led to the fact that f could only decrease as the size of the domain Ω containing the particles increases. Charge neutrality is needed for this monotonicity of f (but not for the lower bound). Since f is bounded below, this monotonicity guarantees that f has a limit as V tends to ∞.

Since then much progress has been made in understanding non-relativistic QED, as discussed in this mini-review, and it is appropriate to extend the proof of the thermodynamic limit to the QED case. This is not just an idle exercise, for several new matters of a physical nature, as well as a mathematical nature, arise. Among these is the fact that this model completely takes account of everything that we know about low energy physics except for the hyperfine interaction (for which nuclear physics is necessary) and except for the fact that the dynamics of the particles (but not the electromagnetic field) is non-relativistic. Indeed, no completely satisfactory relativistic Hamiltonian is presently available and, therefore, the fully relativistic generalization will have to await further developments. Another problem, which is yet to be resolved, is the renormalization of physical parameters in order to deal with the infinities that arise as Λ, the ultraviolet cutoff on the electromagnetic field, tends to infinity.

Otherwise, the theory is potentially complete and an example of this completeness is that it is not necessary to exclude the spin-spin inter-electron magnetic interaction, as in [**19, 20**]. The usual non-QED approximation is to mimic the interaction by a $|\mathbf{x}|^{-3}$ spin-dependent potential, which cannot possibly be stable of the first kind, and which is, therefore, normally omitted from discussion unless a hard core interaction is introduced to stabilize it. In contrast, a full theory in which the magnetic field $B(x)$ is a dynamical variable and the particles interact with the field via a $\sigma \cdot B(x)$ term (but without any explicit spin-spin interaction) is perfectly well behaved and stable and has all the right physics in the classical limit.

(We note in passing that stability of matter requires more than just the field energy to stabilize the $\sigma \cdot \mathbf{B}(\mathbf{x})$ terms. It also requires the 'kinetic' energy terms $(\mathbf{p} + e\mathbf{A}(\mathbf{x})/c)^2$. In other words, the terms $2\mathbf{p} \cdot \mathbf{A}(\mathbf{x}) + \mathbf{A}(\mathbf{x})^2$ are essential for understanding the interaction of particles with each other at small distances; the dipole-dipole approximation while correct at large distances, is certainly inadequate at short distances.)

Another major difference between the Schrödinger and the QED theories of the thermodynamic limit is the necessity of treating the thermodynamics of the field correctly. In 1900 Planck [**36**] gave us the free energy density of the pure electromagnetic field (in the whole of $\mathbf{R}^3$ at temperature T, which implies that the field cannot be confined to the container Ω without invoking artificial constraints. This requires that we first take a limit in which the size of the universe $\mathcal{U}$ tends to infinity (after subtracting the enormous pure Planck free energy) and afterward take the limit $|\Omega| \to \infty$. Obviously, the subtraction has to be done carefully and that is an exercise in itself.

In [**28**] topic a) above was achieved in the non-relativistc QED setting (with fixed ultraviolet cutoff Λ). I.e., a finite upper bound for $V^{-1} \ln Z$, or lower bound for f, was found (after taking the double limit, of course). In the previous work [**19, 20**] the upper bound required only a few lines but our QED setting presents significant difficulties that have to be overcome. While stability of the second kind is known for this QED case, as discussed in the previous sections, it is far from sufficient for obtaining the upper bound on $\mathcal{Z}$ in the manner of [**19, 20**].

It is easy to show (by simply producing one state in the Hilbert space of finite energy) that f is bounded above, but this does not mean that there is a limit *The problem of proving this $V \to \infty$ limit remains open.* The ideas hinted at in b) above do not work, for a reason that is ultimately traceable to the presence of the nonlinear $\mathbf{A}(\mathbf{x})^2$ term.

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