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End Invariants and the Classification of Hyperbolic 3-Manifolds:

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1. Introduction

These notes are a biased guide to some recent developments in the deformation theory of hyperbolic 3-manifolds and Kleinian groups. This field has its roots in the work of Poincaré and Klein, and connects to topology via Thurston's geometrization program, to analysis via the Ahlfors-Bers quasiconformal theory, and to complex dynamics via the work of Thurston, Sullivan and others. It encompasses many techniques and ideas and may be too big a subject for a single account. We will focus on the geometric study of ends of hyperbolic 3-manifolds and boundaries of deformation spaces, and in particular on the techniques that led to the recent solution by Brock, Canary and the author [**82, 23**] of the incompressible-boundary case of Thurston's "Ending Lamination Conjecture".

The space of hyperbolic structures on a fixed 3-manifold M is studied by considering representations of $\pi_1(M)$ into the isometry group of hyperbolic 3-space $\mathbb{H}^3$, up to a natural equivalence by conjugation. In this space, called the character variety, the injective representations with discrete image correspond to hyperbolic 3-manifolds that are homotopy-equivalent to M (see §2 for more details).

Mostow/Prasad rigidity tells us that when M is closed or admits a hyperbolic structure of finite volume, the hyperbolic structure is unique, and hence the deformation space is a single point. On the other hand, in the infinite volume case the deformation space has nontrivial interior in the character variety.

This interior has been fairly completely described, in a complex-analytic sense, by the work of Ahlfors, Bers, Kra, Marden, Maskit and Sullivan (see e.g. [**3, 8, 9, 69, 61, 67, 100**]. The boundary continues to pose challenging questions, although considerable progress has been made in recent years. In many ways the general picture we have of this deformation space is analogous to that of the Mandelbrot set in complex dynamics, or more generally to the bifurcation locus in a family of rational maps of the Riemann sphere.

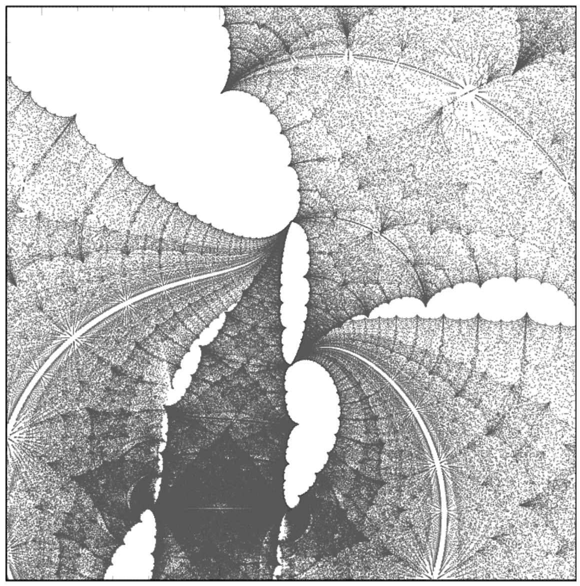

FIGURE 1. A 2-dimensional slice of the 4-dimensional deformation space associated to a once-punctured torus. The interior is the solid white region. (Courtesy C. McMullen and D. Wright)

We can summarize the current state of knowledge in terms of the following basic problems and questions:

- **Classification:** Describe elements in the deformation space in terms of their *asymptotic geometry*. For elements in the interior this amounts to the Ahlfors-Bers parametrization in terms of Teichmüller parameters. For elements on the boundary this corresponds to Thurston's Ending Lamination Conjecture.
- **Density of the interior:** Bers, Sullivan and Thurston conjectured that the deformation space is the closure of its interior. This is analogous in dynamics to the density of hyperbolic (or Axiom A) systems – false in general but still conjectured and open for rational maps. Density was established,

for the incompressible boundary case without cusps, by Brock-Bromberg, and is also a consequence of the Ending Lamination Conjecture.

- **Bumping:** Analogous to bifurcation in dynamics, this is the phenomenon, discovered by Anderson-Canary, that different components of the interior of the deformation space can have intersecting closures. This in particular implies that the topological type of a hyperbolic 3-manifold can change discontinuously in the representation space. Bumping and associated phenomena are the chief remaining obstructions to having a complete topological picture of the deformation space.
- **Tameness:** Marden conjectured that a hyperbolic 3-manifold with finitely-generated fundamental group is *tame*, i.e. homeomorphic to the interior of a compact 3-manifold. This is known for manifolds whose cores have incompressible boundaries but for the compressible boundary case is open, and is the chief obstruction to completing the classification problem.

There are other questions about the *geometry* of the deformation space, e.g. Miyachi's work on the shape of cusps [**84**], Kapovich's work on hyperbolic fixed points of 'renormalization' maps [**54**], or Keen-Series' work on pleating coordinates [**56**], which we will not address at all in this account. For a lovely semi-popular and well-illustrated treatment of many aspects of the deformation theory, see the book *Indra's Pearls* [**34**] by Mumford-Series-Wright.

In §§2–4 we will discuss, in a general way, the structure of the deformation theory and the current state of the problems listed above. In §§5–8 we will give a tutorial-style discussion of some central geometric tools that are used in the analysis of hyperbolic 3-manifolds: Margulis tubes, geometric limits, pleated surfaces, laminations and the complex of curves. In §6, in particular, we will discuss a collection of examples of Kleinian surface groups that illustrate some of the richness of the geometric structures that can occur. In §9 we will indicate in a broad outline how these ideas are put together to give a proof of the Ending Lamination Conjecture. We will conclude in §10 with a discussion of some corollaries of this work, and of some remaining open questions.

2. Hyperbolic geometry and deformations

Hyperbolic geometry. Hyperbolic n-space $\mathbb{H}^n$ is the unique complete simply connected Riemannian n-manifold with all sectional curvatures $K \equiv -1$. A standard model of $\mathbb{H}^n$ is given by the upper half-space $U = \{(x_1, \dots, x_n) : x_n > 0\}$, with the metric

$$\frac{dx_1^2 + \cdots dx_n^2}{x_n^2}.$$

The isometries of $\mathbb{H}^n$ are the Möbius transformations that preserve U. For $n = 2$, $\mathbb{H}^2$ can be identified with $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and the orientation-preserving isometries are just $\mathrm{PSL}_2(\mathbb{R})$, acting by $z \mapsto \frac{az+b}{cz+d}$. For $n = 3$ the orientation preserving isometries are $\mathrm{PSL}_2(\mathbb{C})$, acting on the upper half-space $\{(z,t) : z \in \mathbb{C}, t > 0\}$ by an extension of the natural action on $\mathbb{C}$. In fact the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a natural boundary for $\mathbb{H}^3$, and the group acts there by conformal homeomorphisms.

A hyperbolic structure on an n-manifold N is a complete metric of constant sectional curvature $K = -1$ (usually considered up to diffeomorphism isotopic to the identity). Every such structure is obtained as the quotient $N \cong \mathbb{H}^n/\Gamma$ where Γ

acts discretely on hyperbolic n-space $\mathbb{H}^n$ by isometries. When $n = 2$ Γ is called a *Fuchsian group*, and when $n = 3$ it is called a *Kleinian group.*

Geometrization. Thurston conjectured in the 1970's that all compact 3-manifolds admit canonical decompositions into "geometric pieces," where the geometry is one of eight possible types coming from three-dimensional homogeneous spaces: Euclidean space $\mathbb{E}^3$, the sphere S^3, hyperbolic space $\mathbb{H}^3$, and the "fibred geometries", $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{PSL}_2}(\mathbb{R})$, the 3-dimensional solvable group Sol, and the 3-dimensional (nilpotent) Heisenberg group Nil. By far the most interesting and diverse family is the hyperbolic manifolds. See Scott [**95**] for a thorough discussion.

In particular, let M be the interior of a compact oriented 3-manifold $\overline{M}$. The hyperbolic part of Thurston's conjecture states:

Hyperbolization Conjecture. *M admits a complete hyperbolic structure if and only if*

(1) *$\pi_1(M)$ is infinite or M is the 3-ball,*
(2) *M is irreducible, and*
(3) *M is atoroidal.*

M is irreducible if every embedded 2-sphere bounds a ball, and is atoroidal if no immersed torus is π_1-injective unless it is homotopic into $\partial\overline{M}$. It is easy to see that these conditions are necessary. Thurston proved this conjecture in many cases, and in particular in the case that $\partial\overline{M} \neq \emptyset$, which is the case that concerns us the most in these lectures. (The work of G. Perelman, announced not long after these lectures were given, promises to prove the Geometrization Conjecture in its full generality.)

Rigidity. If M admits a hyperbolic structure, one naturally asks about the set of all such structures, and this will be our concern in what follows. The theorems of Mostow [**86**] and Prasad [**93**] tell us that

Mostow/Prasad Rigidity. *If $n \geq 3$ and a hyperbolic n-manifold M is closed or has finite volume, then its hyperbolic structure is unique.*

In particular for oriented 3-manifolds, $vol(M) < \infty$ is equivalent to the condition that $\partial\overline{M}$ is a union of tori. In this case the ends of M are *cusps:* a neighborhood of a component T^2 of $\partial\overline{M}$ can be parameterized as $T^2 \times (0, \infty)$, where the second coordinate denotes distance and the cross-sectional tori $T^2 \times \{t\}$ contract exponentially with t.

This uniqueness of the hyperbolic structure is of course of great value in topology: any geometric property of the hyperbolic metric is automatically a topological invariant. Since, for example, many knot complements admit hyperbolic structures, this places hyperbolic geometry squarely in the center of 3 dimensional topology.

Deformations. When $\partial\overline{M}$ contains surfaces of genus 2 or greater, there is a very rich deformation theory for hyperbolic structures on M.

Let us first return for a moment to the 2-dimensional case. If S is the interior of a compact surface it is well-known that S admits a hyperbolic structure if and only if $\chi(S) < 0$. Assume for simplicity that S is oriented. Since a hyperbolic structure gives an identification of $\pi_1(S)$ with a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, the space of all such structures can be studied by first considering the quotient (in the sense of algebraic geometry)

$$X(S) = Hom(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))//\mathrm{PSL}_2(\mathbb{R}).$$

Here $\mathrm{PSL}_2(\mathbb{R})$ acts on $Hom(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ by conjugation of the image, since conjugate representations certainly give the same geometric structure. $X(S)$ is known as the *character variety* of $\pi_1(S)$, since it can be parametrized by traces of the images of generators (see e.g. Shalen [**96**]). Within $X(S)$ we consider those (equivalence classes of) representations that are injective with discrete images, and so that the quotient of $\mathbb{H}^2$ by the resulting group action has finite area, and is homeomorphic to S (when S is closed the last two conditions are automatic). This is called the *Teichmüller space* of S, denoted $\mathcal{T}(S)$. It is a well-known space that has been studied from various points of view since Riemann, and is in particular homeomorphic to $\mathbb{R}^{6g-6+2b}$, where g is the genus of S and b is the number of boundary components.

$\mathcal{T}(S)$ can also be identified with the space of Riemann surface structures on S modulo diffeomorphism homotopic to the identity. The equivalence with the previous definition is via the Uniformization Theorem.

For our 3-manifold M we can consider a similar definition:

$$X(M) = Hom(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C}).$$

Within this variety lies $AH(M)$, those equivalence classes $[\rho]$ where $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ is injective and has discrete image. (The terminology "AH", due to Thurston, indicates the Algebraic topology on the set of Hyperbolic structures.) The quotient manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is equipped with a homotopy class of homotopy-equivalences $M \to N_\rho$, determined by $[\rho]$. We can think of $AH(M)$ as the set of hyperbolic 3-manifolds equipped with such homotopy-equivalences, called markings.

$AH(M)$ is a closed subset (excluding the elementary cases where $\pi_1(M)$ is abelian): this is a theorem of Jørgensen, using his criterion for discreteness [**53**] (or alternatively the "Margulis lemma" applied to $\mathrm{PSL}_2(\mathbb{C})$, see Thurston [**103**]). It is a consequence of the Ahlfors-Bers quasiconformal deformation theory that $AH(M)$ has a non-empty interior if $\partial\overline{M}$ has components of genus 2 or more.

To gain some appreciation for this fact, consider first a natural decomposition of the Riemann sphere imposed by the dynamics of the action of a discrete group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$. The *limit set* Λ is the smallest closed invariant subset of $\widehat{\mathbb{C}}$, and the action of Γ on Λ is chaotic in various senses. The complement $\Omega = \widehat{\mathbb{C}} \setminus \Lambda$ is known as the *domain of discontinuity*, and the quotient Ω/Λ is a Riemann surface, sometimes denoted $\partial_\infty N$.

Let us consider the special (but central) case that $\overline{M} = S \times [-1, 1]$ where S is a closed surface of genus $g \geq 2$. A point in $\mathcal{T}(S)$ is represented by $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$, which can equally well be considered as a representation of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$. The quotient $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to M. The limit set of $\rho(\pi_1(M))$ is the circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and Ω is a pair of disks $\Omega_\pm$. The quotient $\Omega/\rho(\pi_1(S))$ is a pair of Riemann surfaces conformally equivalent to the original point in $\mathcal{T}(S)$.

Poincaré and Klein were aware that such a group could be perturbed slightly in $\mathrm{PSL}_2(\mathbb{C})$ to yield a group that is no longer conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Instead, such a small perturbation ρ' will have a limit set that is a Jordan curve but

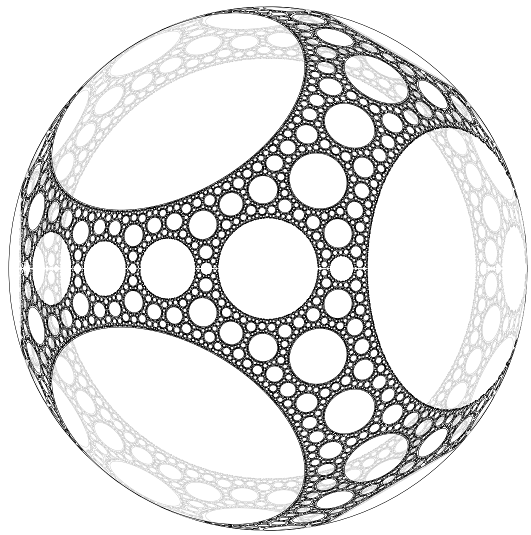

FIGURE 2. The limit set of a Kleinian group (courtesy C. McMullen)

not a circle. The quotients $\Omega'_\pm/\rho'(\pi_1(M))$ are a pair of Riemann surfaces equipped with an identification with S, hence give rise to a new pair of points in $\mathcal{T}(S)$. Bers [**8**] proved the astonishing result that

Simultaneous Uniformization Theorem. *Every pair of points in $\mathcal{T}(S)$ are realized as $\Omega_\pm/\rho(\pi_1(S))$ for some $\rho \in AH(S \times [-1,1])$. This correspondence yields a homeomorphism*

$$QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S).$$

Here $QF(S) \subset AH(M)$ is the subset consisting of representations whose action on $\widehat{\mathbb{C}}$ is topologically (in fact quasiconformally) conjugate to the Fuchsian action. These are known as *quasifuchsian groups*. Bers used his work with Ahlfors [**2**] on solutions to the Beltrami equation to show that *any* deformation of the conformal structures at infinity, i.e. of the surfaces $\Omega_\pm/\rho(\pi_1(S))$, can be extended to a deformation of the entire group action.

The work of Marden [**66**] and Sullivan [**100**] shows that $QF(S)$ is in fact the interior of $AH(S)$. This picture generalizes to any compact $\overline{M}$: Each component of $int(AH(M))$ is a region of *structural stability*: two points in the same component correspond to marked hyperbolic manifolds that are in the same marked homeomorphism class, and whose associated actions on $\widehat{\mathbb{C}}$ are quasiconformally conjugate. The Simultaneous Uniformization Theorem generalizes to give a parameterization of each component in terms of a boundary Teichmüller space (see Kra [**61**] and Maskit [**68**]).

In general, quasiconformally conjugate actions on $\widehat{\mathbb{C}}$ correspond to quotient manifolds that are *quasi-isometric* (via e.g. the extension theorem of Douady-Earle [**35**]). Mostow rigidity depends on the fact that, when the manifolds are compact, no quasiconformal deformation of their group actions is possible. Sullivan proved

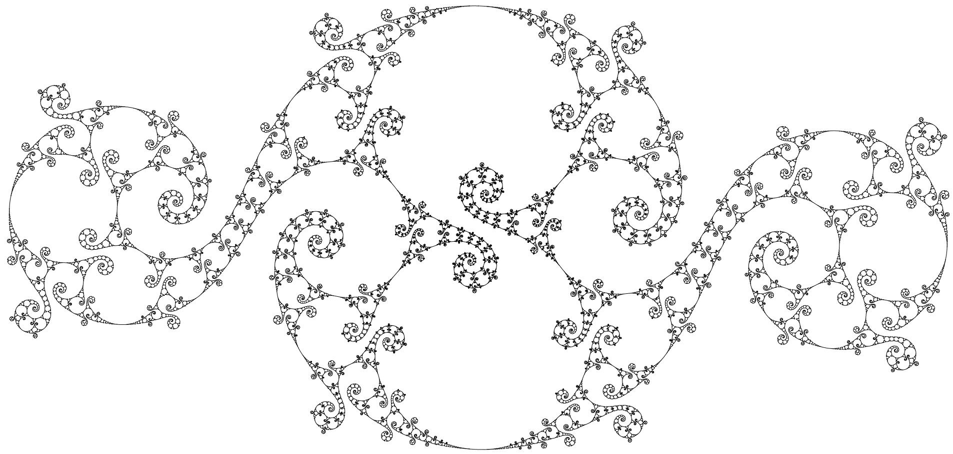

FIGURE 3. This Jordan curve is the limit set of a quasifuchsian group (courtesy D. Wright).

a strong generalization of Mostow rigidity to finitely generated groups with infinite volume quotients, showing in essence that any quasiconformal deformation of a group must be supported on its domain of discontinuity:

Sullivan Rigidity. [**99**] *Let $f : N_1 \to N_2$ be a quasi-isometry between hyperbolic 3-manifolds with finitely-generated fundamental groups, that extends to a conformal homeomorphism from $\partial_\infty N_1$ to $\partial_\infty N_2$. Then f is homotopic to an isometry.*

Equivalently, a quasiconformal conjugacy in $\widehat{\mathbb{C}}$ between two finitely generated Kleinian groups which is conformal on the domain of discontinuity is a Möbius transformation.

3. Cores and ends of hyperbolic 3-manifolds

Let us consider a particular element of $AH(M)$, with quotient manifold N. We can understand, in a coarse sense, the hyperbolic structure of N by considering its decomposition into a *compact core* and *ends*. Scott [**94**] showed that any 3-manifold N with finitely-generated fundamental group contains a compact 3-dimensional submanifold K whose inclusion into N is a homotopy-equivalence. The components of $N \setminus K$ are then in one-to-one correspondence with the ends of N as a topological space, where each component is a neighborhood of an end. Let us assume, as we may by changing our choice of M, that K is homeomorphic to $\overline{M}$.

N also has a *convex core* C_N, which is the smallest closed *convex* submanifold of N whose inclusion is a homotopy-equivalence. The structure of the ends of N is classified by how they intersect with the convex core.

To simplify the discussion, assume from now on that the hyperbolic structure on N has *no cusps*: that is, no element of the associated group $\Gamma = \rho(\pi_1(N))$ is parabolic. In this case we have a dichotomy for the structure of each end of N:

- An end E is *geometrically finite* if it has a neighborhood that is outside the convex core. Such a neighborhood can then be foliated by a family S_t of surfaces, isotopic to a component S of ∂K, so that each S_t is convex, and the size of S_t grows exponentially with distance from K. The metrics on S_t, after conformal rescaling to constant size, converge to a unique hyperbolic metric on S, which therefore gives a point ν_E in the Teichmüller space $\mathcal{T}(S)$.

Equivalently, such an end corresponds to a component of $\partial_\infty N = \Omega/\rho(\pi_1(M))$, which is homeomorphic to S, and ν_E is the natural conformal structure inherited from Ω.

- An end E is *geometrically infinite* if it has a neighborhood that is completely contained in the convex core. (In this case there is no component of $\partial_\infty N$ corresponding to E).

That these are the only two possibilities is a consequence of Ahlfors' Finiteness Theorem [**3**], which implies that the hyperbolic area of ∂C_N is finite, from which it follows that ∂C_N cannot partition a neighborhood of an end into two unbounded pieces.

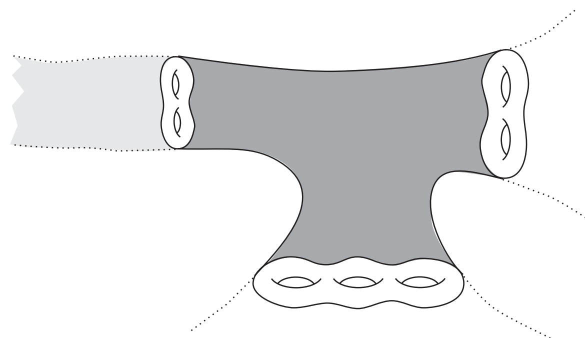

FIGURE 4. Schematic of a compact core (dark grey). The convex core includes the additional geometrically infinite end on the left (light grey).

The geometry of a geometrically finite end is determined, up to uniform bilipschitz equivalence, from any of the surfaces S_t (see Epstein-Marden [**36**]), or equivalently from the limiting structure $\nu_E \in \mathcal{T}(S)$. Furthermore, the general form of the Ahlfors-Bers theory implies that, if N has only geometrically finite ends, then it is uniquely determined by the corresponding points in the Teichmüller space.

Note that, in this description, we did not need to know (because of Scott's theorem) that N was actually homeomorphic to M – the discussion goes through just as well only with the assumption that $\pi_1(N)$ is finitely generated. This is a useful observation for considering arbitrary points in $AH(M)$, which may not yield manifolds homeomorphic to the interior of any compact manifold. The property of being the interior of a compact manifold is known as *topological tameness,* and it is worth recording here Marden's conjecture from [**66**]:

Tameness Conjecture. *A hyperbolic 3-manifold with finitely generated fundamental group is topologically tame.*

Our description of geometrically finite ends implies in particular that any manifold all of whose ends are geometrically finite is topologically tame – as originally shown by Marden in [**66**].

Geometrically infinite ends. What can we say about the structure of geometrically infinite ends? Groups with such ends were constructed by Bers-Maskit [**11**] (and shown to be geometrically infinite by Greenberg [**40**]), but very little was understood about their geometry.

It turns out that the state of our knowledge depends on whether the boundary of $\overline{M}$ is compressible or not – incompressibility means that the map $\pi_1(S) \to \pi_1(M)$ induced by the inclusion of any component S of $\partial\overline{M}$ is injective.

If $\partial\overline{M}$ is incompressible, then the work of Thurston and Bonahon [**102, 13**] gives a preliminary description of the geometry and topology of the geometrically infinite ends of N. Thurston defined the notion of a *simply degenerate* end, and showed that such an end determines a unique *lamination* on the corresponding boundary component of $\overline{M}$, which is called an ending lamination. Thurston also showed that if every geometrically infinite end of M is simply degenerate, then in fact M is topologically tame. Bonahon showed that indeed every geometrically infinite end of M *is* simply degenerate (provided $\partial\overline{M}$ is incompressible). Thus Marden's conjecture is established in the setting of incompressible boundary. We will discuss ending laminations in more detail in the next section.

When $\partial\overline{M}$ is compressible, much less is known. Canary showed [**29**] that, if M is topologically tame, then a suitable notion of simply degenerate ends and ending laminations can be defined, so that the same theory carries through with suitable modifications. There has also been a string of results [**27, 90, 5, 38, 22**] that establish the tameness of certain limits of geometrically finite manifolds. However the general tameness conjecture remains open, and is at this point the biggest remaining problem in the deformation theory of hyperbolic 3-manifolds.

4. Laminations and degenerate ends

4.1. Geodesic laminations on surfaces

Consider, by way of motivation, the case of a torus T^2. The set of simple closed curves in T^2, up to homotopy, can be described via the extended rational numbers $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, by associating each curve to its slope, in a chosen identification of $H_1(T^2)$ with $\mathbb{Z}^2$. The natural completion of this countable set to $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ can be given a geometric interpretation in terms of *foliations* on T^2, whose slopes can be irrational.

A surface of higher genus does not admit non-singular foliations because of the Poincaré index theorem, but the related notions of singular measured foliations and measured geodesic laminations serve as good generalizations of the situation for the torus. We will describe this using the language of laminations, but see e.g. [**39, 63**] for the foliation point of view.

Fix a closed hyperbolic surface S. A *geodesic lamination* in S is a compact set, foliated by complete geodesics. A simple closed geodesic is the simplest example, but one can obtain more complicated examples by taking Hausdorff limits of sequences of simple closed geodesics whose lengths go to ∞. In fact the space $\mathcal{GL}(S)$ of all geodesic laminations in S is compact in the topology of Hausdorff convergence.

A *transverse measure* on a geodesic lamination is a family of Borel measures on arcs transverse to the lamination, invariant by holonomy; that is, by sliding along the leaves. (See Figure 5).

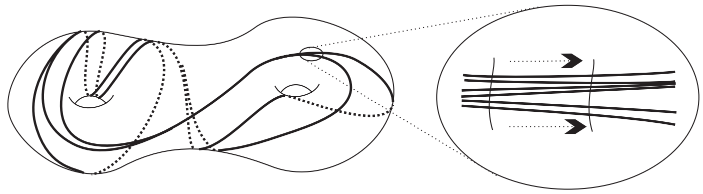

FIGURE 5. A geodesic lamination with closeup showing a transversal sliding along the leaves.

Thurston introduced this notion and defined the space $\mathcal{ML}(S)$ of all such *measured laminations*. This space has a natural topology, coming from weak-* convergence of the measures, which makes it homeomorphic to $\mathbb{R}^{6g-6}$ where g is the genus of S. The hyperbolic metric we chose for S turns out to be irrelevant: different choices yield canonically homeomorphic spaces. The *projective lamination space*, denoted $\mathcal{PML}(S)$, is the quotient of $\mathcal{ML}(S) \setminus \{0\}$ by the $\mathbb{R}_+$ action that multiplies measures by constants (0 denotes the empty lamination). $\mathcal{PML}(S)$ is homeomorphic to the $6g-7$-sphere, and contains the simple closed geodesics as a dense set of "rational" points, in analogy with the torus. See Bonahon [**14**], Casson-Bleiler [**33**] and Penner-Harer [**92**] for more details.

Let us also define the space $\mathcal{UML}(S)$ of "unmeasured laminations", which is just the quotient of $\mathcal{ML}(S) \setminus \{0\}$ under the equivalence relation that forgets the measures.

This space is different from $\mathcal{GL}(S)$ in two ways. First, not all laminations admit transverse measures of full support; $\mathcal{UML}(S)$ contains only laminations that are the supports of measures. Second, the topology of Hausdorff convergence in $\mathcal{GL}(S)$ is different from the quotient topology on $\mathcal{UML}(S)$.

$\mathcal{UML}(S)$ is also different from $\mathcal{PML}(S)$, and the difference is due to the fact that there exist laminations which support a continuum of projectively inequivalent measures. The natural map $\mathcal{PML}(S) \to \mathcal{UML}(S)$ collapses all these measures to a point.

Although $\mathcal{UML}(S)$ is not a Hausdorff space, it contains a natural subset that will be useful to us, and which is Hausdorff. This is the set of *filling laminations*: A measured lamination λ is called filling if it has non-trivial intersection with every essential closed curve in S. Equivalently λ is filling if it cuts S into a union of ideal hyperbolic polygons (if S is not closed then these can be once-punctured polygons).

The filling laminations are in fact generic in $\mathcal{PML}(S)$, just as the irrational numbers are generic among real numbers. Their image in $\mathcal{UML}(S)$, which we denote by $\mathcal{EL}(S)$, is a Hausdorff space (see Klarreich [**58**]) and we will revisit it in the next section.

4.2. Simply degenerate ends

If an end of a hyperbolic 3-manifold is not geometrically finite, both its geometry and topology are a priori rather mysterious. Thurston discovered a way to "tame" the study of ends by considering the placement of closed geodesics within them (and more usefully, of "pleated surfaces", which we will discuss in §7).

Thurston defined an end E of N (in the incompressible-boundary case) to be *simply degenerate* if there is a sequence of simple closed curves α_i on the associated surface S whose geodesic representatives α_i^* in N are eventually contained in any neighborhood of E – we say that α_i^* *exit the end*. Note the contrast between this and the property of being geometrically finite: The existence of the convex surfaces S_t means that, outside S_t, there cannot be *any* closed geodesics. This is because the exterior of a convex surface admits a distance-decreasing projection back to the surface, and hence the shortest representative of any homotopy class of closed curves must lie inside the convex core C_N.

If such a sequence α_i exists, it turns out that its asymptotic behavior as a subset of $\mathcal{UML}(S)$ is quite simple. Thurston proved that a simply degenerate end always has a unique lamination which is the limit point of any such sequence. Furthermore, he showed:

THEOREM 4.1. [**102**] *If $\partial\overline{M}$ is incompressible and N is the quotient manifold of a point in $AH(M)$ without cusps, then, whenever E is a simply degenerate end of N facing a component S of the compact core boundary,*

(1) *There is a unique lamination $\nu_E \in \mathcal{UML}(S)$ such that a sequence α_i of simple closed curves in S converges to ν_E in $\mathcal{UML}(S)$ if and only if α_i^* exit the end E.*
(2) *There exists a sequence α_i^* exiting E such that $\ell_N(\alpha_i^*) \leq L_0$ where L_0 depends only on the topological type of S.*
(3) *ν_E is* filling*: it has the property that any homotopically non-trivial simple closed curve in S intersects ν_E nontrivially.*
(4) *E is* topologically tame*: it has a neighborhood homeomorphic to $S \times (0, \infty)$.*

(For a few remarks on the case with cusps see §5).

Thurston called ν_E the *ending lamination* of E. He also proved that every limit of geometrically finite elements of $AH(M)$ has ends which are either geometrically finite or simply degenerate. Ends which are either one or the other are called *geometrically tame*. A few years later, Bonahon proved the foundational theorem:

THEOREM 4.2. [**13**] *If $\partial\overline{M}$ is incompressible and $N \in AH(M)$, then every end of N is geometrically tame.*

In other words the ending laminations are well-defined in the setting that will concern us in these notes. To unify notation, the *end invariant* of a geometrically tame end facing a component S of $\partial\overline{M}$ is the associated $\nu_S \in \mathcal{T}(S)$ if the end is geometrically finite, and the associated ending lamination in $\mathcal{EL}(S)$ if it is simply degenerate.

Thurston's Double Limit Theorem (see [**101**] and Ohshika [**89**]), together with the quasiconformal deformation theory, implies that every filling lamination actually occurs as an ending lamination (this justifies our name $\mathcal{EL}(S)$ for the set of filling laminations). We state this a bit more carefully, and in the case without cusps:

THEOREM 4.3. *Let $\overline{M}$ be compact with incompressible boundary and let*

$$(\nu_S \in \mathcal{EL}(S) \cup \mathcal{T}(S))_S$$

be a set of choices of end invariants for components S of ∂M, with the restriction that, when $\overline{M} = S \times [-1, 1]$, $\nu_{S\times\{-1\}} \neq \nu_{S\times\{1\}}$ if they lie in $\mathcal{EL}(S)$. Then there exists $N \in AH(M)$ homeomorphic to M, whose associated end invariants are the (ν_S).

In other words, all possible combinations of end invariants are obtained (the restriction when N is a product is necessary: Ending laminations for opposite ends can never be the same).

Thus a complete classification of $AH(M)$ would follow from the following conjecture of Thurston:

Ending Lamination Conjecture. *An element of $AH(M)$ is uniquely determined by its list of end invariants.*

The proof of this conjecture, in the case that ∂M is incompressible (in particular the case $M = S \times [0,1]$) will appear in [**82**] and [**23**].

5. Tubes, cusps and geometric limits

A central feature of the study of hyperbolic 3-manifolds is the notion of a *thick-thin decomposition*, and we will take a little time now to discuss this decomposition and the structure of the components of the thin part.

Hyperbolic tubes. A *hyperbolic tube* is the quotient of an r-neighborhood of a geodesic in $\mathbb{H}^3$ by a translation or a screw motion. Our goal here is to explain how the geometry of a hyperbolic tube is controlled by the geometry of its *marked boundary.*

More explicitly, given $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, and $r > 0$, we define $\mathbb{T}(\lambda, r)$ to be the quotient of the r-neighborhood of the vertical line L above $0 \in \mathbb{C}$ in the upper half-space model of $\mathbb{H}^3$ by the loxodromic $\gamma : z \mapsto e^{\lambda}z$. The action of γ in the third coordinate is just multiplication by $|e^{\lambda}|$, and in particular γ translates the axis L by a hyperbolic distance $\operatorname{Re}\lambda$, and rotates around it by angle $\operatorname{Im}\lambda$. Any hyperbolic tube is isometric to some $\mathbb{T}(\lambda, r)$, but we note that the imaginary part of λ is, so far, only determined modulo 2π.

Marked boundaries and Teichmüller parameters. If T is an oriented Euclidean torus, a *marking* of it is an ordered pair (α, β) of homotopy classes of unoriented simple closed curves with intersection number 1. There is a unique $t > 0$ and $\omega \in \mathbb{H}^2 = \{z : \operatorname{Im} z > 0\}$ such that T can be identified with $\mathbb{C}/t(\mathbb{Z} + \omega\mathbb{Z})$ by an orientation-preserving isometry, so that the images of $\mathbb{R}$ and $\omega\mathbb{R}$ are in the classes α and β, respectively. The parameter ω describes the conformal structure of T as a point in the Teichmüller space $\mathcal{T}(T) \equiv \mathbb{H}^2$.

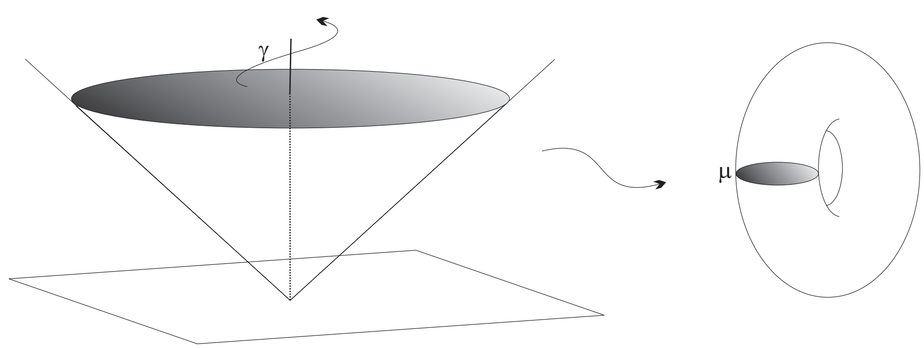

FIGURE 6. A hyperbolic tube has a Euclidean torus boundary. The meridian disk is shaded.

The boundary torus of a hyperbolic tube $\mathbb{T}$ inherits a Euclidean metric and an orientation from $\mathbb{T}$, and it admits an almost uniquely defined marking: Let μ denote the homotopy class of a meridian of the torus (i.e. the boundary of an essential disk in $\mathbb{T}$) and let α denote a homotopy class in $\partial\mathbb{T}$ of simple curves homotopic to the core curve of $\mathbb{T}$. While μ is unique, α is only defined up to multiples of μ. Fixing such a choice (α, μ), we obtain *boundary parameters* (ω, t) as above.

The freedom of choice of $\lambda \pmod{2\pi}$ is related to the freedom of choice of α mod μ, and indeed there is a natural way to choose α given λ. We then obtain (see [**82**, §3]) a bijective correspondence between the parameters (λ, r) and the boundary data (ω, t).

Parabolic tubes. A rank-1 parabolic tube is the quotient of a horoball in $\mathbb{H}^3$ by a cyclic group of parabolic transformations. Explicitly, consider the parabolic transformation $z \mapsto z + t$, acting on the region of height ≥ 1 in the upper half space model of $\mathbb{H}^3$. The quotient is homeomorphic to $S^1 \times \mathbb{R} \times [0, \infty)$, and we note that any rank-1 parabolic tube is obtained in this way for some t.

A rank-2 parabolic tube is the quotient of a horoball by a parabolic group isomorphic to $\mathbb{Z}^2$. It is homeomorphic to $T^2 \times [0, \infty)$ and can be obtained as the quotient of the region of height ≥ 1 by the group generated by $z \mapsto z + t$ and $z \mapsto z + t\omega$ with $\omega \in \mathbb{H}^2$. As for hyperbolic tubes, once a marking (α, μ) for the torus boundary is chosen the parameters (ω, t) are uniquely determined; but note that there is no natural choice of either μ or α in this case.

Thick-thin decomposition. By Margulis' Lemma, or Jørgensen's inequality (see e.g. Thurston [**103**] or Kapovich [**55**]), there is a universal $\epsilon_0 > 0$ such that the *ϵ-thin part* of a hyperbolic 3-manifold N, i.e. the set $N_{(0,\epsilon]} = \{x : inj(x) \leq \epsilon\}$, is a disjoint union of hyperbolic and parabolic tubes, whenever $\epsilon \leq \epsilon_0$. The parabolic tubes are usually called *cusps*.

While the structure of each individual tube is relatively simple, the real challenge in deformation theory is understanding how these tubes are organized within a given hyperbolic 3-manifold. We will see some indication of the complexity of this in the examples of Section 6.

Geometric limits. The *geometric limit* of a sequence of Kleinian groups Γ_i is their Gromov-Hausorff limit as subsets of $\mathrm{PSL}_2(\mathbb{C})$. Fixing an origin $0 \in \mathbb{H}^3$, if the translation lengths $d(0, \gamma(0))$ are bounded away from 0 and ∞ for all $1 \neq \gamma \in \Gamma_i$, for all i, then such a limit always exists (after taking a subsequence) and is a nontrivial and discrete group Γ_∞. The equivalent statement for the quotient manifolds $N_i = \mathbb{H}^3/\Gamma_i$ is this: let the *basepoint* x_i of N_i be the image of 0 under the quotient map. Then geometric convergence implies that, for any $R > 0$, the R-neighborhood of x_i in N_i is eventually diffeomorphic to the R-neighborhood of x_∞ in N_∞, by a map that is locally converging to an isometry as $i \to \infty$. The opposite implication also holds, up to possibly conjugating the groups by rotations fixing 0. (See e.g. [**30**]).

The notion of geometric limit is extremely useful in the study of hyperbolic 3-manifolds – a lot of our available geometric control comes from compactness or contradiction arguments involving geometric limits.

The simplest types of Kleinian groups are the cyclic groups giving rise to hyperbolic tubes as above. It is instructive to consider geometric limits in this setting. Consider a sequence of hyperbolic tubes $\mathbb{T}_i$ with boundary parameters $(\omega_i, 1)$. If

$\omega_i \to \omega$ with $\operatorname{Im}\omega > 0$ then it is fairly clear that the tubes (and the corresponding groups) converge geometrically to the tube with parameters $(\omega, 1)$. The case $\operatorname{Im}\omega = 0$ involves a non-discrete limit and will not interest us here. More interesting is the case that $|\omega_i| \to \infty$. Here we find that the radii $r_i \to \infty$, and (after restricting to a subsequence and choosing appropriate basepoints) the tubes converge geometrically to a parabolic tube. Which kind of parabolic tube depends on the sequence: If $\operatorname{Im}\omega_i \to \infty$, then the limit is a rank-1 tube. But if $\operatorname{Re}\omega_i \to \infty$ and $\operatorname{Im}\omega_i$ is bounded, the limit is a rank-2 tube. This is because in this case the tori defined by $(\omega_i, 1)$ remain in a bounded set of shapes *up to homeomorphism*, or equivalently the projections of ω_i to the moduli space of the torus remain bounded.

This phenomenon was first discovered by Jørgensen [**52**], and also plays a very important role in Thurston's "Dehn-filling theorem" and its generalizations, which show that hyperbolic 3-manifolds with rank-2 cusps can be geometrically approximated by hyperbolic manifolds where the cusp tubes are replaced by solid tori.

End invariants in the presence of cusps. If N is allowed to have cusps the description of ends from §3 becomes a bit more complicated, but is in principle the same. The main idea is to consider N_0, the complement in N of the (interiors of) parabolic Margulis tubes. This manifold has annuli (and/or tori) in its boundary, and a *relative compact core* K (see McCullough [**72**] and Kulkarni-Shalen [**62**]) which meets ∂N_0 in compact annuli and tori. The components of $N_0 \setminus K$ determine the ends of N_0. Assuming that the components of $\partial K \setminus \partial N_0$ are incompressible, each one will support an end invariant as before, which is either a filling lamination or an element of a Teichmüller space.

The ending lamination conjecture now becomes the statement that all this data – parabolic elements, ending laminations and Teichmüller data – determine N uniquely.

An important example comes from surface groups, if we consider a compact surface $\overline{S}$ with boundary and an element $[\rho] \in AH(\overline{S} \times [-1, 1])$ such that ρ takes $\partial\overline{S}$ to parabolic elements. Then the cusps associated to these elements cut N into $\overline{S} \times \mathbb{R}$. Note that the *relative* boundary components of N_0 will be incompressible in this case, in spite of the fact that $\overline{S} \times [0, 1]$ is a handlebody.

6. Examples of surface groups

The following collection of examples gives a taste of the huge variety of possibilities for the geometry of Kleinian surface groups.

6.1. Fuchsian and Quasifuchsian groups

If Γ preserves a totally geodesic copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$ as well as its transverse orientation, then $N = \mathbb{H}^3/\Gamma$ contains a totally geodesic copy of the surface $S = \mathbb{H}^2/\Gamma$, and the metric on N can be explicitly described in the coordinates $S \times \mathbb{R}$, as

$$(\cosh^2 t)ds^2 + dt^2 \tag{6.1}$$

where t is the $\mathbb{R}$ parameter, measuring distance from S, and ds is the hyperbolic metric on S. Γ is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$, and is called a Fuchsian group, as described in §2.

For a general quasi-Fuchsian group, the geometry outside the convex core can be modeled, up to uniform bilipschitz distortion, by half of the Fuchsian example, namely $S \times [0, \infty)$ with the metric (6.1) (see Epstein-Marden [**36**]). From now on we will concentrate on describing the metric inside the convex core.

6.2. Periodic manifolds

Our next example is a well-known type of manifold with geometrically infinite ends (the convex core is the whole manifold.)

Let $\varphi : S \to S$ be a pseudo-Anosov homeomorphism of the closed surface S (this means that φ leaves no finite set of non-boundary curves invariant up to isotopy). The mapping torus of φ is

$$M_\varphi = S \times \mathbb{R}/\langle (x,t) \mapsto (\varphi(x), t+1) \rangle,$$

a surface bundle over S^1 with fibre S and monodromy φ. Thurston [**101**] showed, as part of his hyperbolization theorem, that M_φ admits a hyperbolic structure which we'll call N_φ (see also Otal [**91**] and McMullen [**74**]). Let $N \cong S \times \mathbb{R}$ be the infinite cyclic cover of N_φ, "unwrapping" the circle direction (Figure 7). After identifying S with some lift of the fibre, we obtain an isomorphism $\rho : \pi_1(S) \to \pi_1(N) \subset \mathrm{PSL}_2(\mathbb{C})$, which gives us an element of $AH(S \times [-1, 1])$.

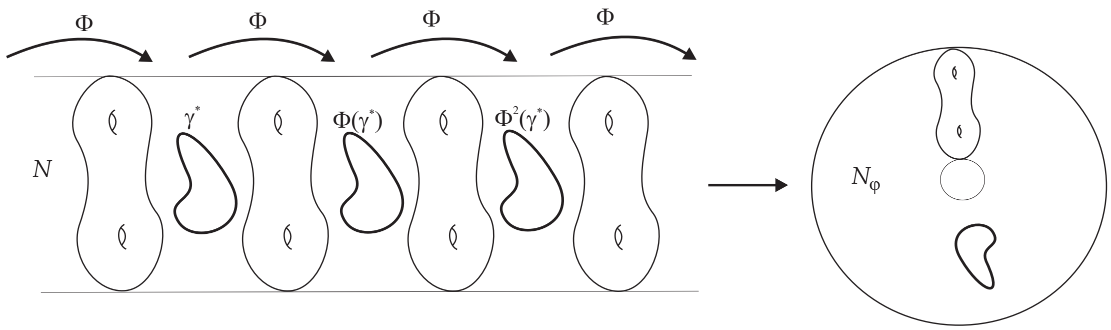

FIGURE 7. N covers the surface bundle N_φ.

The deck translation $\Phi : N \to N$ of the covering, acting as an isometry of the lifted metric, induces $\Phi_* = \varphi_* : \pi_1(S) \to \pi_1(S)$. Note that N has two ends, neither of which can be geometrically finite: for if one of the ends were foliated by convex surfaces S_t, which are expanding exponentially, the isometry Φ could not exist.

We next consider the action of φ on S, which, after isotopy, can be arranged (by Thurston's classification theorem for surface automorphisms, see e.g. [**39, 33**]) to have the following structure: There is a transverse pair (ν_+, ν_-) of laminations so that φ preserves the leaves of both ν_+ and ν_-, stretching the former and contracting the latter. Every simple closed curve γ in S intersects this system of leaves nontrivially, and it follows that repeated application of φ tends to make γ more and more parallel to ν_+ (and application of φ^{-1} brings it more in the direction of ν_-). Indeed in $\mathcal{PML}(S)$ (and in $\mathcal{UML}(S)$) the sequences $\{\varphi^n(\gamma)\}$ and $\{\varphi^{-n}(\gamma)\}$ converge to ν_+ and ν_-, respectively.

We can see $\nu_\pm$ directly in the asymptotic geometry of N: For a curve γ in S, let γ^* be its geodesic representative in N. Now consider $\Phi^n(\gamma^*)$ – these are all geodesics of the same length, marching off to infinity in both directions as $n \to \pm\infty$, and note that $\Phi^n(\gamma^*) = \varphi^n(\gamma)^*$. So, we have a sequence of simple curves in S, converging to

ν_+ as $n \to \infty$, whose geodesic representatives "exit the + end" of N (similarly as $n \to \infty$ they converge to ν_- and the geodesics exit the other end). It follows that $\nu_\pm$ are the ending laminations of ρ.

The way that Thurston actually proved the existence of this example was to obtain it as a limit of quasi-Fuchsian groups. For $X, Y \in \mathcal{T}(S)$, let $qf(X,Y)$ denote the quasi-Fuchsian group (up to conjugation) whose surfaces at infinity are X and Y. That is, qf is the Ahlfors-Bers simultaneous uniformization map from §2.

Now if we fix X and consider the sequence

$$\rho_n = qf(\varphi^{-n}(X), \varphi^n(X)),$$

Thurston showed that this converges in $AH(S \times [-1,1])$ to the representation ρ. At the n^{th} stage, the convex core of ρ_n is roughly (up to uniform bilipschitz diffeomorphism) a concatenation of $2n$ fundamental domains of Φ's action on the limiting manifold (see McMullen [**74**]).

6.3. Drilling out a curve

Kerckhoff-Thurston [**57**] constructed the following sequence of quasi-Fuchsian representations ρ_n, which is related to Jørgenson's original observation that the geometric limit of a sequence of hyperbolic tubes can be a rank-2 parabolic tube.

Let γ be an essential curve in S, and let D_γ be a Dehn twist on γ. Then consider the sequence

$$\rho_n = qf(X, D_\gamma^n(X)).$$

The bottom surface of the convex core of N_{ρ_n} is (by Sullivan's theorem [**36**]) uniformly bilipschitz equivalent to X, and so is the top, but by a map which is in the homotopy class of D_γ^n. Kerckhoff and Thurston showed that a uniform bilipschitz model for the geometry of the convex core of N_{ρ_n} can be described as follows:

Let $M = S \times [-1,1]$ and let A be a closed annular collar of γ in S. Let U be the solid torus $A \times [-1/2, 1/2]$ (see figure 8). Place a fixed metric on M for which the torus ∂U is Euclidean. We can describe the meridian of U by choosing an arc a in A connecting the two components of ∂A, and letting

$$\mu = \partial(a \times [-1/2, 1/2]).$$

Now remove U from M, and replace it by a new solid torus U_n, whose meridian μ_n is obtained from μ by twisting its top arc, $a \times \{1/2\}$, n times around the core of $A \times \{1/2\}$. The new manifold is homeomorphic to the old, but the geometry will be different.

Letting α be the core of $A \times \{1/2\}$, we have a marking (α, μ_n) for ∂U. As discussed in §5, there is a unique hyperbolic tube whose marked boundary has the same geometry as ∂U with this marking, and we can identify U_n with this tube. The resulting manifold thus has a very short representative γ_n of γ, surrounded by a deep Margulis tube.

As $n \to \infty$, this sequence of models M_n (and hence also the convex cores of the hyperbolic manifolds) converges in the sense of Gromov-Hausdorff to a manifold in which U has been replaced by a *parabolic tube* with torus boundary. (This is just the same as the limit discussed in §5, where the boundary parameters ω_n satisfy $\mathrm{Im}\,\omega_n$ bounded and $\mathrm{Re}\,\omega_n \to \infty$.) Equivalently, the geometric limit is homeomorphic to $S \times [-1,1]$ minus the level curve $\gamma \times \{0\}$.

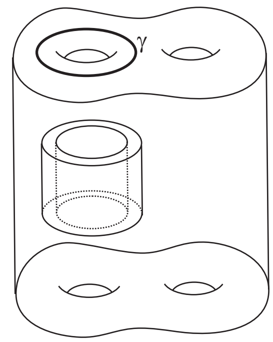

FIGURE 8. The model for the Kerckhoff-Thurston "drilling" construction.

This example begins to show the role that Margulis tubes play in the structure of surface groups. It also indicates that geometric limits of surface groups do not themselves have to be surface groups.

6.4. Drilling out infinitely many curves

Thurston [**101**] and Bonahon-Otal [**15**] produced iterations of the Kerckhoff-Thurston construction which illustrate that the geometric limit of a sequence of quasi-Fuchsian groups can in fact be infinitely generated.

In this example, consider an infinite sequence $\{\gamma_1, \gamma_2, \ldots\}$ of curves in S, and let D_i be Dehn twists about γ_i. Suppose that the sequence has the following two properties:

- *Eventual filling property:* For each i there is some $j > i$ such that the curves $\gamma_i, \ldots, \gamma_j$ *fill* the surface, meaning that every essential curve in S must intersect one of these curves.
- *Non-annular property:* If $\gamma_i = \gamma_j$ for $i < j$ then there must be some $i < k < j$ such that γ_k intersects γ_i essentially.

Then the sequence

$$\rho_n = qf(X, D_1^{e(n)} \circ \cdots \circ D_n^{e(n)}(X))$$

(where $e(n) \to \infty$ sufficiently fast) has a geometric limit which is homeomorphic to

$$S \times \mathbb{R} \setminus \bigcup_{i=1}^{\infty} \gamma_i \times \{i\}.$$

The missing curves $\gamma_i \times \{i\}$ correspond to rank 2 cusps, and the non-annularity condition is necessary because of the property of hyperbolic manifolds that curves in distinct cusps cannot be homotopic. The "eventual filling" condition is not necessary for the construction, but in its absence the structure of the limit can be even more complicated.

A model for the n^{th} manifold in this sequence can be constructed, similarly to the previous example, by starting with $S \times [0, n+1]$ and performing appropriate integer Dehn surgeries on solid tori centered on $\gamma_i \times \{i\}$, giving the new solid tori the geometry of the appropriate hyperbolic tubes.

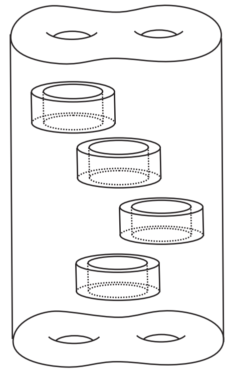

FIGURE 9. The convex hull can have many Margulis tubes, which in the geometric limit become cusps.

6.5. Brock's example

Brock [20] gave an example that illustrates that exotic geometric limits do not just involve appearance of rank-2 cusps, as in the previous two cases.

For this example, let $R \subset S$ be a (closed) proper essential subsurface, not an annulus or a 3-holed sphere. Let $\phi_R : R \to R$ be a pseudo-Anosov and let $\phi : S \to S$ be a map which is the identity outside R and equal to ϕ_R in R. This is called a *partial pseudo-Anosov*.

If we now let

$$\rho_n = qf(X, \phi^n(X)),$$

Brock showed that the geometric limit of the manifolds N_{ρ_n} is homeomorphic to

$$S \times \mathbb{R} \setminus R \times \{0\}.$$

Moreover, the following is a uniform bilipschitz model for the convex core of the n^{th} manifold: Let Q denote the periodic hyperbolic 3-manifold homeomorphic to $int(R) \times \mathbb{R}$ which covers the mapping torus of ϕ_R, as in §6.2. Note that because R has boundary, Q should have a parabolic tube of rank 1 for each component of ∂R. Let Q_n denote the union of n successive fundamental domains of the deck translation of Q, minus the cusp tubes associated to ∂R. Thus Q_n is homeomorphic to $R \times [0, n]$, endowed with a metric such that each block $R \times [i, i+1]$ is isometric to $R \times [0, 1]$ by a map homotopic to ϕ_R^i.

Now remove from $S \times [-1, 1]$ a tubular neighborhood of $\partial R \times \{0\}$, and then cut out a small collar neighborhood of $R \times \{0\}$ and replace it by Q_n. Refill the tube associated to each component of ∂R by an appropriate hyperbolic tube – this is done using the natural marking as in the previous examples; In this case ω_n for each torus will have large imaginary part, corresponding to the height of Q_n.

Thus as $n \to \infty$, each tube of ∂R will become a cusp, but a rank-1 cusp now instead of a rank-2 cusp. If we take a geometric limit with basepoint, say, on the boundary of the convex core, then the distance to the top and bottom of Q_n stays bounded, but its middle grows ever farther, and hence in the geometric limit Q_n will become two eventually periodic geometrically infinite ends of the form $R \times [0, \infty)$.

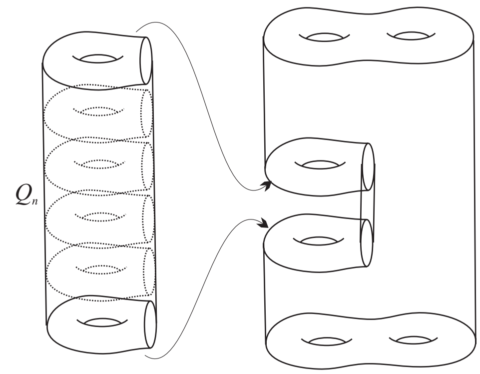

FIGURE 10. The bilipschitz model of the partial pseudo-Anosov example

6.6. Other constructions and limits

It is clear from these examples that many other variations are possible. For example a combination of the iterated "drill holes" construction of §6.4 and Brock's partial pseudo-Anosov construction should yield sequences that develop many different kinds of Margulis tubes and almost-periodic regions, and whose geometric limits can be the complement of many, even infinitely many, level surfaces and curves. Indeed the homeomorphism type of a general geometric limit of quasi-Fuchsian groups can be quite intricate, a fact which is expressed in the structure of the "model manifolds" that we will discuss in Section 9.2. See also Soma [**97**], and the discussion in §10.

6.7. Bounded and unbounded geometry

It is worth dwelling a bit more on the example N from §6.2 which cyclically covers the pseudo-Anosov mapping torus N_φ, and its relation to the general case. First, since N admits a compact quotient, its geometry is bounded in a fairly strong sense. In particular its injectivity radii are bounded away from 0. Let us say that N in $AH(M)$ has *bounded geometry* if it admits a positive lower bound on injectivity radius (in the case with cusps we say that there is a positive infimum on the lengths of closed geodesics). This case is considerably simpler than the general case. In [**76, 77**] we showed that the Ending Lamination Conjecture holds in the setting of manifolds with bounded geometry (and no cusps).

Furthermore a bounded-geometry manifold homeomorphic to $S \times \mathbb{R}$ has structure similar to that of the pseudo-Anosov example, but not quite periodic: Part of the proof in [**76, 77**] involved showing that such a manifold is modeled, up to bilipschitz homeomorphism, by the "universal curve" over a Teichmüller geodesic. That is, there is a geodesic g in $\mathcal{T}(S)$ and a metric on $S \times \mathbb{R}$ such that $S \times \{t\}$ is isometric to the hyperbolic surface parameterized by $g(t)$ (where t is an arclength parameter for g). This metric on $S \times \mathbb{R}$ is within uniform bilipschitz distortion of the hyperbolic metric. In the pseudo-Anosov case, g is the axis of φ (as constructed by Bers in [**10**]). See also Mosher [**85**] and Bowditch [**17**] for new alternative proofs.

The examples in §§6.3–6.5 illustrate what happens when the bounded-geometry condition is dropped: the Margulis tubes cut the manifold into a number of different types of pieces, the universal curve model for the geometry no longer holds, and sequences of such manifolds can have exotic geometric limits (whereas it follows from Thurston's work that the geometric limit of a sequence of surface groups with uniformly bounded geometry is always a surface group). On the other hand the manifold decomposes into pieces that are either tubes, or bounded-geometry manifolds of lower genus (the Q_n in §6.5), or have bounded complexity. This behavior, properly formalized, turns out to hold in the general case.

McMullen [**73**] showed that unbounded-geometry manifolds are generic, in an appropriate sense, in the boundary of a Bers slice (a particular middle-dimensional slice of $AH(S \times [-1, 1])$). This was generalized by Canary-Culler-Hersonsky-Shalen [**26**] to other deformation spaces.

Uniform control of the geometry of elements of $AH(S \times [-1, 1])$ can only be possible if a way is found to predict (in terms of end invariants) the occurrence and configuration of Margulis tubes of short geodesics. In the sections that follow we will outline the mechanism for deducing, from the structure of the end invariants of N, the set of short geodesics in N and their topological arrangement.

7. Pleated surfaces

The geometry of a hyperbolic surface is rather simple and explicitly controlled by its topology. The Gauss-Bonnet theorem fixes its area; injectivity radius is bounded above; the thick-thin decomposition has a simple description in terms of collars and pieces of bounded complexity. On the other hand the hyperbolic metrics vary in a moduli or Teichmüller space so there is great flexibility. Thurston's idea of pleated surfaces harnesses this control coupled with flexibility by considering useful isometric maps of hyperbolic surfaces into hyperbolic 3-manifolds.

A pleated surface is something like a piecewise-geodesic surface, except that there are no "corner points" and the bending lines are typically infinitely many. More formally, it is a map $f : S \to N$ where N is a hyperbolic 3-manifold, together with a hyperbolic metric σ_f on the surface S, such that path lengths are preserved by f, and every point in S either has a neighborhood that is mapped totally geodesically or is on a leaf of a *geodesic lamination* λ_f whose leaves are mapped totally geodesically. Fairly detailed accounts of the theory of pleated surfaces can be found in [**30**] and [**36**].

Spinning. Typically we will have a fixed homotopy class of maps $[h : S \to N]$ in mind, and will look for pleated surfaces within this class. Starting with any essential simple closed curve γ on S for which $h(\gamma)$ is not homotopic to a point or into a cusp, there are pleated surfaces $f \sim h$ such that $\gamma \subset \lambda_f$. Suppose for simplicity that γ is part of a pants decomposition P of S. We can complete P to a lamination of S by adding leaves in each pair of pants that spiral toward the boundary (see Figure 11).

We can deform h to get a map taking each component of P to its unique geodesic representative. Assuming appropriate topological conditions on h (being an isomorphism on π_1 suffices) each of the infinite spiraling leaves can be realized in N as a leaf that spirals around the geodesic representatives of P. The remainder is a collection of ideal triangles that can only be mapped in one way. It is not hard to see that the resulting surface inherits a hyperbolic metric.

Other examples can be obtained from this one by a process of taking limits – given a sequence P_n converging to some lamination μ the associated sequence of

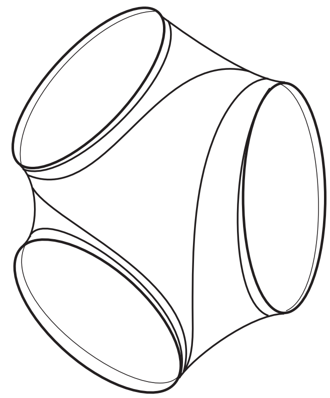

FIGURE 11. A lamination in a pair of pants with leaves spinning toward the boundary.

pleated surfaces, if convergent, will yield a pleated surface which maps μ geodesically. This convergence is made possible by a nice *compactness property* of pleated surfaces – a sequence of pleated surfaces to a fixed manifold N, after taking a subsequence, either leaves every compact set, or converges in a suitable sense to a pleated surface, or degenerates in a predictable way, developing thin parts that enter cusps of N.

Pleated surfaces also arise naturally as boundaries of convex hulls of hyperbolic 3-manifolds. Thurston pointed out that these boundaries are embedded pleated surfaces with the added condition of convexity.

A lamination μ is *realizable* in a homotopy class $[h]$ if it is mapped geodesically by some pleated surface in $[h]$. If h is the inclusion map of an incompressible component of the compact core, non-realizability of μ turns out to be equivalent to the statement that μ has a component which is either homotopic to a cusp or is an ending lamination of N.

Pleated surfaces and ends. If E is an end of N facing an incompressible component S of the compact core, then we can begin to see the consequences of Thurston's simple degeneracy condition using pleated surfaces. A sequence α_i of simple curves in S whose geodesic representatives α_i^* exit E gives rise, via the spinning construction, to a sequence of pleated surfaces $f_i : S \to N$ (homotopic to the inclusion) whose images exit E (one must deal with the possibility that part of $f_i(S)$ is far out in E and part remains near the core – this is understood in terms of the thick-thin decomposition of the surface). Part (2) of Theorem 4.1, the existence of a sequence of *bounded* curves exiting the end, is now a consequence of the fact that the shortest curve in each hyperbolic metric σ_{f_i} has uniformly bounded length.

Because laminations can be organized in a continuum, namely in the sphere $\mathcal{PML}(S)$ (§4.1), it becomes possible to *interpolate continuously* between any two pleated surfaces. Not all the intermediate surfaces in such an interpolation are pleated – one must slightly generalize the category while keeping most of the geometric control. With this, Thurston showed that a simply degenerate end E can be "swept" by a family of these almost-pleated surfaces going all the way to infinity. This implies for example upper bounds on injectivity radius anywhere in E, as well as the topological tameness of E (part (4) of Theorem 4.1). Thurston also used this to control the growth of harmonic functions on N, thus obtaining a proof of *Ahlfors' Measure Conjecture* for the case that the core has incompressible boundary (see §10).

Injectivity and Efficiency. One can get finer geometric control from a pleated surface if one can control the extent to which it is "folded". The presence of a pleated surface $f : S \to N$ easily yields upper bounds: if a curve α has length L in σ_f then L bounds the length of the geodesic representative of $f(\alpha)$. In his proof of the Double Limit Theorem [**101**], which yielded the geometrization of manifolds that fibre over S^1, Thurston needed to obtain *lower bounds* of the form

$$\ell_N(f(\alpha)^*) \geq \ell_{\sigma_f}(\alpha) - C(\lambda_f, \alpha)$$

where $C(\lambda_f, \alpha)$ is a combinatorial constant determined by α and the lamination λ_f, and independent of the geometric data. That is, he needed to find in $f(S)$ an "efficient" representative of the curve α.

The main issue turned out to be preventing different leaves of λ_f from having images in N that are too close together and nearly parallel. Thurston proved the *Uniform Injectivity Theorem*, which provides such control under suitable topological restrictions (again f being an isomorphism on π_1 suffices). This theorem is proved by recourse to geometric limits: If it fails then there is a sequence of examples $f_i : S \to N_i$, in which leaves that remain apart in S get closer and more parallel in N_i. Extracting a suitable limit of the manifolds and maps Thurston obtained a map in which two leaves have images that coincide, which he then showed contradicted the topological restrictions.

The Uniform Injectivity Theorem also plays a major role in the proof of the Ending Lamination Conjecture; see Section 9.

8. The complex of curves

We have already seen, in the examples of §6 and in the discussion of pleated surfaces in §7, the important geometric role played by simple closed curves. A central ingredient in a complete analysis of the structure of Kleinian surface groups is the intrinsic combinatorial structure of the set of all simple closed curves on a surface (up to homotopy), as encoded by Harvey's *complex of curves*.

In this section let S be any compact surface with genus $g \geq 0$ and $n \geq 0$ boundary components. Let $\xi(S) = 3g + n$.

We define the complex of curves $\mathcal{C}(S)$ as follows when $\xi(S) > 4$: the vertices of $\mathcal{C}(S)$ are the essential homotopy classes of simple closed curves in S, and k-simplices are $(k+1)$-tuples $[\alpha_0, \ldots, \alpha_k]$ that have simultaneously disjoint representatives.

If $\xi(S) = 4$ then $(g, n) = (0, 4)$ or $(1, 1)$, and no two nonhomotopic essential simple curves can be disjoint. We redefine $\mathcal{C}(S)$ so that $[\alpha_0, \alpha_1]$ is an edge whenever α_0 and α_1 have geometric intersection number 1 (for $(1, 1)$) or 2 (for $(0, 4)$). The same can be done when $(g, n) = (1, 0)$. Other than that, $\mathcal{C}(S)$ is empty when $\xi(S) \leq 3$.

These complexes were formulated by Harvey in [**44**], and later applied by Harer [**42, 43**], Ivanov [**49, 51, 50**] and more recently Luo [**65, 64**] Korkmaz [**60**] and Hempel [**45**] to the study of the mapping class group of a surface, and to Heegaard decompositions of 3-manifolds.

In joint work with Masur, we studied $\mathcal{C}(S)$ as a metric space, by endowing each edge with length 1. We showed that

THEOREM 8.1. [**70**] *Whenever* $\xi(S) \geq 4$, $\mathcal{C}(S)$ *is an infinite-diameter* δ*-hyperbolic space.*

(See Bowditch [**16**] for a new and much improved proof). δ-hyperbolicity is a "coarse geometry" notion, introduced by Gromov [**41**] and Cannon [**31**], to capture some of the large-scale properties of spaces such as $\mathbb{H}^n$, and metric trees. Geodesics in a δ-hyperbolic space X tend to diverge exponentially fast, and there is a robust notion of *boundary at infinity* $\partial_\infty X$, which roughly speaking is the set of asymptotic classes of infinite quasi-geodesic rays. The union $X \cup \partial_\infty X$ admits a natural topology. It is natural to ask what the boundary at infinity of $\mathcal{C}(S)$ would be. Because $\mathcal{C}(S)$ is not locally finite it is not a proper metric space, and $\partial_\infty \mathcal{C}(S)$ does not have as many nice properties as less exotic examples. In particular it is not compact.

It is not unreasonable to expect that $\partial_\infty \mathcal{C}(S)$ should have something to do with laminations on S. After all, any infinite sequence of simple closed curves should accumulate on some set of laminations. A few moments of thought indicate that not all laminations should correspond to endpoints at infinity. For example, if λ is a lamination in S which is disjoint from a curve α, we can approximate λ (say in $\mathcal{UML}(S)$) by a sequence $\{\gamma_i\}$ of vertices in $\mathcal{C}(S)$, which are themselves disjoint from α. Thus $d(\alpha, \gamma_i) = 1$ and hence $\{\gamma_i\}$ are *not* a sequence going to infinity in $\mathcal{C}(S)$. If λ were *filling* then this particular phenomenon couldn't occur, and this suggests we should direct our attention to the filling laminations in S.

An additional issue that arises is whether points in $\partial\mathcal{C}(S)$ should correspond to *measured* or *unmeasured* laminations. This is a subtle point, since there are filling laminations which support more than one projective class of measures. Klarreich [**58**] resolved this question in a nice way:

THEOREM 8.2. [**58**] *The boundary* $\partial_\infty \mathcal{C}(S)$ *is homeomorphic to* $\mathcal{EL}(S)$*, in such a way that convergence to a point at infinity in* $\mathcal{C}(S) \cup \partial_\infty \mathcal{C}(S)$ *corresponds to convergence in* $\mathcal{UML}(S)$.

Recall that $\mathcal{EL}(S)$ are the filling laminations in $\mathcal{UML}(S)$, and by Theorem 4.3 are exactly those laminations that occur as ending laminations for manifolds without cusps. Thus our ending laminations appear as points at infinity in the complex of curves.

8.0.1. The torus and the Farey graph. When S is the torus (or one-holed torus or 4-holed sphere), the complex of curves and its boundary recapitulate the notion of approximation of irrational numbers by rationals via continued fraction expansions.

As indicated in §4.1, the vertices of $\mathcal{C}(S)$ in this case are the extended rationals $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, associating to each curve its slope in $H_1(S)$. It is a nice exercise to check that the definition of $\mathcal{C}(S)$ in this case yields the "Farey Graph," which is embedded in the disk with vertices at the rational points of the circle. (See Figure 12, and [**79, 78**] for more details.)

In this case, the boundary of $\mathcal{C}(S)$ is just the irrational points in the circle, with convergence at infinity being the normal one in $\mathbb{R}$. Furthermore, there is a natural class of quasi-geodesic rays: Any irrational $r \in \mathbb{R}\backslash\mathbb{Q}$ has a unique continued-fraction

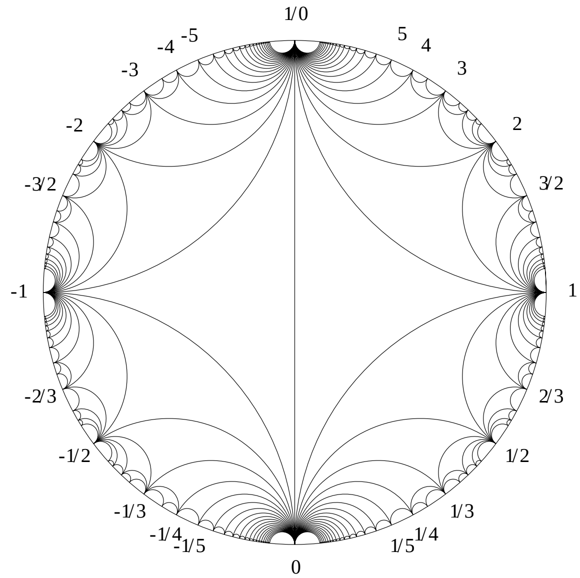

FIGURE 12. The Farey graph in the disk is the complex of curves of the torus

expansion of the form

$$r = [n_0, n_1, n_2, \ldots] = n_0 + \cfrac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \cdots}}}$$

where $n_0 \in \mathbb{Z}$ and $n_i \in \mathbb{N}$ for $i > 0$. The sequence of partial expansions $\{[n_0, \ldots, n_k]\}$ forms a quasigeodesic ray converging to r.

The role of the coefficients n_i is played, for general surfaces, by the "subsurface coefficients" discussed in §9.2.

9. The proof of the Ending Lamination Conjecture

We will discuss now the main ideas that go into the proof of Thurston's conjecture. For another expository account of part of this argument (particularly §9.1 and §9.2) see [**83**].

To simplify this discussion, we will restrict to the case of *doubly degnerate surface groups* (although we will also discuss some examples from a more general class). That is, let us fix an element $[\rho] \in AH(S \times [-1, 1])$, and let $N = N_\rho$ denote the quotient manifold. Suppose also, for simplicity, that N has no cusps, and both ends of N are simply degenerate, with ending laminations $\nu_\pm$.

Our goal is to determine the geometry of N uniquely from $\nu_\pm$, and this can be divided more or less into these steps:

(1) Quasiconvexity and a-priori length bounds: Using hyperbolicity of $\mathcal{C}(S)$ and Thurston's Uniform Injectivity theorem for pleated surfaces, we analyze the set of vertices of $\mathcal{C}(S)$ whose length in N is bounded. In particular we obtain an interpolation between ν_- and ν_+ via a "hierarchy of geodesics" passing through this bounded-length vertex set.

(2) Lipschitz Model: We apply the hierarchical structure of $\mathcal{C}(S)$ to build a *model* for N which admits a Lipschitz map into N.
(3) Embeddings and Bi-Lipschitz bounds: we improve the Lipschitz map to a bi-Lipschitz homeomorphism. This together with Sullivan's rigidity theorem is sufficient to establish the conjecture.

9.1. Length functions and quasiconvexity

Part (2) of Thurston's Theorem 4.1 on simply degenerate ends tells us that the set of vertices

$$\mathcal{C}(\rho, L_0) \equiv \{\alpha \in \mathcal{C}_0(S) : \ell_N(\alpha^*) \leq L_0\},$$

for suitable L_0, accumulates onto the ending laminations $\nu_\pm$, which we now think of as points in $\partial_\infty \mathcal{C}(S)$. Because our goal is to control the geometry of N, we would like to know that these accumulation points at infinity for $\mathcal{C}(\rho, L_0)$ suffice to determine $\mathcal{C}(\rho, L_0)$, at least roughly.

Our first result in this direction, from [**80**], is:

THEOREM 9.1. *The set $\mathcal{C}(\rho, L_0)$ is r-quasiconvex in $\mathcal{C}(S)$ where r depends only on the topological type of S.*

A set $C \subset X$ is r-quasiconvex if any geodesic of X with endpoints in C remains in an r-neighborhood of C. If X is δ-hyperbolic it is relatively easy to prove that a subset C is quasi-convex: all that is needed is a (coarse) Lipschitz retraction, i.e. a map

$$\Pi : X \to C$$

which is a uniformly bounded distance from the identity on C, and satisfies

$$d(\Pi(x), \Pi(y)) \leq K d(x, y) + D$$

with uniform K, D (see Lemma 3.3 of [**80**]).

Such a map

$$\Pi : \mathcal{C}(S) \to \mathcal{C}(\rho, L_0)$$

is constructed by the use of pleated surfaces. For any vertex α we consider the set of pleated surfaces $f : S \to N$ in the homotopy class of ρ which take α to α^*. The image $\Pi(\alpha)$ is obtained by selecting some curve of length bounded by L_0 (one always exists) on one of these surfaces. In order to obtain the coarse Lipschitz property for Π, it turns out to be sufficient to show that two different pleated surfaces mapping α to α^* yield curves of length $\leq L_0$ that are not too far apart in $\mathcal{C}(S)$. This is done using the Uniform Injectivity result discussed in §7. A bit more precisely: if f and g are both pleated surfaces taking α to α^*, then one can find a curve β in S which is formed of an arc along α of bounded length (in both σ_f and σ_g) composed with a "jump" of bounded length in σ_f. The Uniform Injectivity theorem is applied here to show that the jump is small also in σ_g.

Digression: motivation from harmonic maps. The idea for the map Π came from a natural construction in the *analytical* setting: Given a homotopy class $[h : S \to N]$ (with mild topological restrictions) where N is a hyperbolic 3-manifold, there exists a map

$$\Phi : \mathcal{T}(S) \to \mathcal{T}(S)$$

where $\mathcal{T}(S)$ is the Teichmüller space of S, defined as follows: Fixing a Riemannian metric on S, the Dirichlet energy of each $f : S \to N$, defined as

$$\mathcal{E}(f) = \frac{1}{2} \int_S |df|^2 \, dA,$$

is minimized in the homotopy class by a unique map, called a *harmonic map*. Since $\mathcal{E}(f)$ is in fact invariant under conformal changes of the domain metric, this harmonic map actually depends only on the choice of a point $\sigma \in \mathcal{T}(S)$. If we then pull back the metric of N via the harmonic map to S, we obtain a new point in $\mathcal{T}(S)$, and this is $\Phi(\sigma)$.

It is not hard to see that the Dirichlet energy is bounded on the image of Φ, and one may attempt to study the geometry of N by understanding the Dirichlet energy function on $\mathcal{T}(S)$ and the self-map Φ. In the case of manifolds with *bounded geometry* (lower bounds on injectivity radius) this was somewhat successful, and this was the basis of the solution of the Ending Lamination Conjecture in this case [**76, 77**]. However, in the presence of very short curves in N the analytical approach runs into trouble. It is hard to control the map Φ because when a harmonic map enters a very thin part its dependence on the domain structure becomes very loose. The combinatorics of the thin parts themselves begin to play an important role, and this is why the complex of curves must be considered.

The map Π, then, is a combinatorial analogue of Φ, in which we have replaced harmonic maps with pleated surfaces, relinquished precise analytic control, and obtained instead the coarse structure of δ-hyperbolicity and quasiconvexity.

The quasiconvexity theorem is not enough, however, to control the set $\mathcal{C}(\rho, L_0)$. Since $\mathcal{C}(S)$ is locally infinite, distance bounds are not easy to use. Finer control is obtained by looking at subsurfaces of S. If W is an essential subsurface of S, then there is a projection

$$\pi_W : \mathcal{C}(S) \to \mathcal{C}(W) \cup \{\emptyset\}$$

defined roughly as follows. If α intersects W essentially, select an essential arc of intersection and a curve γ in W disjoint from it. This is $\pi_W(\alpha)$ (the choices in this construction yield a bounded-diameter set in $\mathcal{C}(W)$). If α has no essential intersection with W then $\pi_W(\alpha) = \emptyset$. Let $d_W(\alpha, \beta)$ denote $d_{\mathcal{C}(W)}(\pi_W(\alpha), \pi_W(\beta))$ whenever these projections are nonempty.

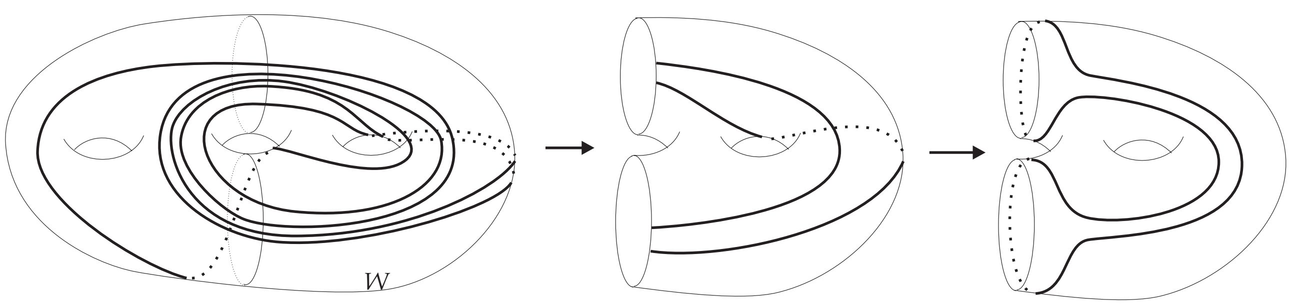

FIGURE 13. The subsurface projection π_W

The "relative quasiconvexity" theorem is the following:

THEOREM 9.2. [**82**] *The set $\pi_W(\mathcal{C}(\rho, L))$ is r-quasiconvex for each subsurface W of S, and a uniform r.*

This theorem does produce enough control to get a priori bounds on the set $\mathcal{C}(\rho, L_0)$. We will describe some of this in the next section.

9.2. Hierarchies and model manifolds

We can use the combinatorial structure of $\mathcal{C}(S)$ and the points at infinity $\nu_\pm$ to construct a certain manifold M_ν, equipped with a piecewise-Riemannian metric, and meant to be a bilipschitz model for the geometry of N.

We begin by letting g be an infinite geodesic in the 1-skeleton $\mathcal{C}_1(S)$ of $\mathcal{C}(S)$, such that the vertices of g converge in one direction to ν_- and in the other to ν_+. The existence of such a geodesic for two points on the boundary of a δ-hyperbolic space is easy to obtain by a limiting argument when the space is proper (bounded balls are compact). In the case of $\mathcal{C}_1(S)$ the limiting step is not automatic, but the machinery developed in Masur-Minsky [**71**] provides sufficient control to carry out such a limit.

We then build up g to obtain a "hierarchy of geodesics", the main construction of [**71**]. To give a hint of this procedure, let u, v, w be three succesive vertices of g, and let Y be the component of $S \setminus v$ containing u and w (if they were in different components they would have distance 1, contradicting the assumption that g is a geodesic). Then u and w can be joined by a new geodesic h in $\mathcal{C}_1(Y)$, and all the vertices in h are in the link of v. We can repeat this at every vertex, obtaining a "thickening" of g into a collection of geodesics in links of vertices, and then repeat this inductively for each of those geodesics. The resulting object, a collection of geodesics supported in curve complexes of subsurfaces of S, is called a "hierarchy of geodesics". The construction is in fact considerably complicated by considerations of how to deal with endpoints. In the case of a 5-holed sphere the construction is especially simple to describe, and we refer to the expository article [**83**] for more details.

The structure of this hierarchy is controlled by $\nu_\pm$. In particular, for any essential subsurface $Y \subset S$ the projections $\pi_Y(\nu_\pm)$ (which are defined similarly to projections of simple closed curves) control whether or not Y appears as the support of one of the geodesics in the hierarchy: we show in [**71**] that if $d_Y(\nu_+, \nu_-)$ is sufficiently large then Y must be the support of a geodesic in the hierarchy with length roughly estimated by $d_Y(\nu_+, \nu_-)$.

On the hyperbolic geometry side, we showed in [**80**] that, if $d_Y(\nu_+, \nu_-)$ is large, then $\ell_N(\partial Y^*)$ is very small.

These results, together with the relative quasiconvexity theorem 9.2, allow us to show that

LEMMA 9.3. *If W is an essential subsurace of S then*

$$d_W(\alpha, \Pi(\alpha)) \leq D \tag{9.1}$$

over all vertices α appearing in H, for which $\pi_W(\alpha)$ and $\pi_W(\Pi(\alpha))$ are non-empty.

This roughly means that α and the bounded-length curves in $\Pi(\alpha)$ are not too far apart in a combinatorial sense. We can in fact use this to deduce an a priori length bound on α itself:

THEOREM 9.4. [**82**] *Given* $[\rho] \in AH(S \times [-1,1])$ *with ending laminations* $\nu_\pm$, *let* H *be a hierarchy constructed from a geodesic* g *in* $\mathcal{C}_1(S)$ *joining* ν_- *and* ν_+. *Then all the vertices that appear in* H *are contained in* $\mathcal{C}(\rho, L)$, *where* L *depends only on the topology of* S.

The hierarchy and the a priori bounds allow us to construct our model manifold M_ν and its Lipschitz map to N_ρ. M_ν, which can be identified with $S \times \mathbb{R}$, is a union of pieces called "blocks" and "tubes". The tubes are solid tori of the form (annulus)×(interval), where the homotopy classes of the annuli are exactly the vertices of $\mathcal{C}(S)$ that occur in the hierarchy construction. Each tube is given the structure of a hyperbolic tube as in §5, which is controlled by a Teichmüller coefficient $\omega \in \mathbb{H}^2$, defined with respect to a natural marking for ∂U. In fact these coefficients can be estimated from the data of the hierarchy, and hence from the subsurface projection distances, $\{d_Y(\nu_-, \nu_+)\}$.

The *blocks* are pieces that can be identified with $W \times [0,1]$ where W denotes a subsurface of S, and their geometric structures fall into a finite set of isometry types. Different blocks are glued along subsurfaces of the boundary in $W \times \{0\}$ and $W \times \{1\}$, so that the unglued parts of the boundary are annuli; these annuli form the boundaries of the tubes.

Theorem 9.4 gives us enough control to map the blocks of M_ν with Lipschitz bounds into N. The tubes U with large ω are shown to correspond to short curves in N.

Furthermore the the map can be adjusted so that each U with large ω maps properly onto the corresponding Margulis tube in N, and conversely that the complement of this set of tubes maps to the complement of the corresponding Margulis tubes.

In summary, we have a "Lipschitz model" for the thick part of N. It gives upper bounds for the lengths of closed geodesics in N, as well as a topological description of the arrangement of the components of the thin part (collars of short geodesics). However, it is not yet a bilipschitz model.

9.3. Examples

Let us consider what the model manifold would look like in some of the examples from §6.

The periodic case. If N_ρ is the cyclic cover of a fibred manifold, as described in §6.2, then $\nu_\pm$ are the fixed points of the pseudo-Anosov monodromy φ, and in the construction of the hierarchy we start with a geodesic g which is a "pseudo-axis" for φ: $\varphi(g)$ and g are, if not identical, at least within bounded distance of each other. The hierarchy in this case has a bounded structure: All geodesics that occur in its construction have uniformly bounded lengths (essentially because of the φ-periodicity), and as a result the model manifold has bounded geometry: all the tubes have bounded ω coefficients, and it follows that there is some positive lower bound on the length of the geodesics at their cores. The deck translation of the covering acts on the model by a bilipschitz homeomorphism, showing that the model structure is almost periodic. Of course we know that N_ρ itself is exactly periodic, but the hierarchy and model constructions are only rough approximations. In view of Sullivan's rigidity theorem (§9.4), this rough approximation suffices for our needs.

The drilling example. If ρ_n is the manifold $qf(X, D^n_\gamma(X))$ where D_γ is a Dehn twist, as described in §6.3, then the model manifold contains a bounded number of blocks and tubes of bounded ω, and one tube associated with γ, with coefficient $\omega \approx n + i$. This is essentially the model described in §6.3.

In Brock's example in §6.5, where $\rho_n = qf(X, \phi^n(X))$ with ϕ a partial pseudo-Anosov supported in $R \subset S$, the hierarchy will contain a geodesic in the curve complex $\mathcal{C}(R)$, and this will give an approximately periodic sequence of blocks corresponding to the piece Q_n described in §6.5. Again the tubes associated to ∂R will arise in the model.

In all the rest of the examples of §6 something similar happens: the structure of $qf(X, Y)$, in particular those tubes with large coefficient and small core length, is detected by the hierarchy construction and thus appears explicitly in the model manifold.

The punctured torus. The case that S is a one-holed torus, treated separately in [**79**], is especially simple. In this case the hierarchy corresponds to a geodesic in the Farey graph (§8.0.1), and the data for the model manifold is carried in the associated continued-fraction coefficients. For example if $\nu_- = \infty$ and ν_+ has continued-fraction expansion $[n_0, n_1, n_2, \ldots]$, then the continued-fraction approximants $[n_0, \ldots, n_k]$ represent the slopes of the curves associated to the tubes in the model, whose tube coefficients ω_k are estimated by $n_k + i$.

9.4. Bilipschitz bounds

In Brock-Canary-Minsky [**23**] we show that the Lipschitz model of the previous section can be converted into a bilipschitz homeomorphism

$$f : M_\nu \to N_\rho,$$

with uniform bounds. This is done by successively adjusting the map on pieces of the model, and maintaining geometric control by a variety of geometric limit arguments.

Embedding of horizontal subsurfaces. Let $f_0 : M_\nu \to N$ be the Lipschitz map described in §9.2. Within M_ν there are many "horizontal slices", which are surfaces F isotopic to level subsurfaces $Z \times \{t\} \subset S \times \mathbb{R}$, which have bounded geometry in the model metric, and are properly embedded in $M_\nu \setminus \mathcal{U}$. For example, in Brock's construction in §6.5, The subpiece Q_n contains many bounded-geometry surfaces isotopic to $R \times \{t\}$ with boundary in the hyperbolic tube(s) associated to ∂R. There are also the bottom and top boundaries of the convex core, which can be represented by horizontal slices of the form $S \times \{t\}$.

The restriction of f_0 to such a surface F is uniformly Lipschitz and homotopic to an embedding, and the trick is to deform it to a *uniformly bilipschitz embedding.* Using the techniques of Anderson-Canary [**5**] and Anderson-Canary-Culler-Shalen [**6**], we are able to show that the only obstruction to this is the possibility that $f_0|_F$ is "wrapped" around a Margulis tube of N: If $\mathbb{T}$ is a standard Margulis tube in N we say that a map $g : F \to N$ is wrapped around $\mathbb{T}$ if it cannot be deformed to $+\infty$ or $-\infty$ in $N \cong S \times \mathbb{R}$ in the complement of $\mathbb{T}$.

The argument is by contradiction. If there is a sequence of successively worse examples, i.e. Lipschitz maps $g_i : F_i \to N_i$ which are homotopic to embeddings and satisfy the topological condition of not being wrapped around Margulis tubes of N_i, but which do not admit uniformly bounded homotopies to embeddings, we

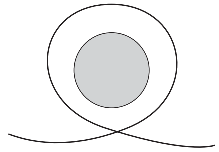

FIGURE 14. A surface wrapped around a Margulis tube. Here $\mathbb{T}$ is represented by a disk and the surface by a curve; crossing this figure by S^1 yields a correct configuration in a neighborhood of $\mathbb{T}$.

can extract a geometric limit $g : F \to N$ in which some of the Margulis tubes have become parabolic cusps, and the limit map is not homotopic to an embedding at all. At this point we consider the way in which the subgroup $g_*(\pi_1(F))$ and its conjugates lie inside $\pi_1(N)$, and the techniques of [**5, 6**] produce a contradiction to the assumption that the original maps were not wrapped.

In the case of our model map, it is possible to show directly that for all of the horizontal slices F, $f_0|_F$ is not wrapped around any Margulis tubes. This is because we know that the Margulis tubes are all images of tubes in $\mathcal{U}$, and the complement of the tubes in $\mathcal{U}$ (with large ω) maps to the complement of the Margulis tubes. Since in the model it is easy to show that each of the slices can be pushed to ∞ or to $-\infty$ in the complement of the model tubes, the same follows for its image in N.

We conclude that f_0 on any horizontal slice can be made an embedding, in a uniform way.

Disjoint surfaces and partial orders. Thus fixing any set of disjoint horizontal slices, the map f_0 can be deformed in a uniform way to a map f_1 which is an embedding on each of these slices. However, there is nothing preventing these individual embeddings from intersecting, and it is not possible to fix these intersections by the simple expedient of performing surgeries on them: such surgeries increase the bilipschitz constants, and since there may be infinitely many horizontal slices a bad intersection pattern could cause the constants to grow without bound.

The solution to this is indirect. By the Lipschitz bounds on f_0 (and hence f_1) and the fact that each block in M_ν contains a bounded-length curve in a distinct homotopy class, it is not hard to see that there is a uniform upper bound on the number of horizontal slices whose images intersect the image of a given one. We use this fact to obtain a decomposition of M_ν with *local* disjointness properties. That is, we can carefully select among all the horizontal slices in M_ν to get a subset that cuts $M_\nu \setminus \mathcal{U}$ into pieces of bounded diameter and geometry, and such that the slices in the boundary of each piece have disjoint images. This requires careful use of the structure of the hierarchy to organize these slices, and an analysis of a "topological partial order" among embedded surfaces in $S \times \mathbb{R}$.

In Brock's example of §6.5, the selected slices would be some set of disjoint horizontal slices of type $R \times \{t\}$ in Q_n, chosen with appropriate spacing, as well as the top and bottom surfaces of the convex hull. This cuts M_ν into a large number of pieces of type $R \times [0, 1]$ (with bounded geometry), which decompose Q_n, and a piece that can be identified with $M_\nu \setminus Q_n$ (and of course the Margulis tubes associated to ∂R).

The next step is to show that the topological partial order among the slices in M_ν is preserved by the map f_1, at least among those that are in the boundary of a particular complementary piece. The constants used in the selection of the subsets of slices must be chosen so that this additional condition of order preservation is satisfied. The proof that this is possible is, again, an argument by contradiction and passage to a geometric limit, using the structure of the lipschitz model map.

The order preservation property implies that the images of slices from our selected subset cut the hyperbolic manifold N into regions that are homeomorphic (preserving orientation) to the complementary regions of the cuts in M_ν. A final argument by contradiction and geometric limit shows that these homeomorphisms can be made uniformly bilipschitz. Thus they piece together to a bilipschitz map of $M_\nu \setminus \mathcal{U}$ to the thick part of N. Bilipschitz extension of the map to the tubes of $\mathcal{U}$ is now a fairly simple matter.

Applying Sullivan rigidity. Now suppose that ρ_1 and ρ_2 have the same pair of ending laminations $\nu_\pm$. Then there are bilipschitz homeomorphisms f_1 and f_2 from the same model manifold M_ν to N_1 and N_2, respectively. Thus $f_2 \circ f_1^{-1} : N_1 \to N_2$ is a bilipschitz homeomorphism. Since $\partial_\infty N_1$ and $\partial_\infty N_2$ are empty in the case we are considering, Sullivan's rigidity theorem implies that f must be homotopic to an isometry. This establishes the Ending Lamination Conjecture. (The case with nonempty ∂_∞ or with cusps is handled with standard modifications to this argument).

10. Other theorems and conjectures

The Ending Lamination Conjecture has long been known to imply the Bers-Sullivan-Thurston Density Conjecture, which can be stated this way:

Density Conjecture. *$AH(M)$ is the closure of its interior.*

In other words, this conjecture states that every hyperbolic 3-manifold with finitely generated fundamental group can be obtained as a limit of geometrically finite manifolds.

Interestingly, a new and unexpected proof was recently found, at least in the non-cusped cases (and with incompressible $\partial\overline{M}$), by Bromberg [**24**] and Brock-Bromberg [**21**]. This argument uses a branched-cover construction to convert a degenerate manifold with very short geodesics into a geometrically finite cone-manifold, and then deforms this to a smooth geometrically finite hyperbolic manifold using machinery developed by Hodgson-Kerckhoff [**46, 47**]. If there are infinitely many sufficiently short curves then this can be repeated to obtain a sequence of elements in the interior converging to the desired manifold. If not, then the manifold has bounded geometry, and this case is dealt with in [**81**].

The proof of the Density Conjecture via the Ending Lamination Theorem uses Thurston's Double Limit Theorem together with the Ahlfors-Bers quasiconformal deformation theory. These imply, as stated in Theorem 4.3 for the case without cusps, that any "legal" combination of end invariants can be obtained, and in fact constructed as a limit of points from the interior of $AH(M)$. Thus given an arbitrary point $[\rho]$ of $AH(M)$, a point $[\rho']$ on the closure of the interior of $AH(M)$ can be obtained with the same invariants. The Ending Lamination Theorem implies that $[\rho] = [\rho']$.

The Density Conjecture remains open in the compressible-boundary case. The main hurdle appears to be the Tameness Conjecture, which continues to resist attempts at its solution.

Another application of the solution to the ending lamination conjecture is the following further extension of the rigidity theory:

Topological Rigidity. *If* $[\rho_1], [\rho_2] \in AH(M)$ *where* $\partial\overline{M}$ *is incompressible, and the actions of* ρ_1 *and* ρ_2 *on* $\widehat{\mathbb{C}}$ *are topologically conjugate, then their actions are in fact quasiconformally conjugate.*

The point here is that a topological conjugacy at infinity implies that the two representations have homeomorphic compact cores, and the same ending laminations. The solution to the ending lamination conjecture then implies the quotient manifolds admit a bilipschitz homeomorphism, which extends to a quasiconformal conjugacy at infinity. The significance of this, as we saw in Section 9.4, is that it puts us in a position to apply Sullivan's Rigidity Theorem.

Geometric limits of hyperbolic 3-manifolds play a central role in the proof of our theorem, and indeed in the entire theory. Conversely, the existence of a bilipschitz model manifold gives us the kind of uniform control that allows us to describe fairly well the geometry of geometric limits. Here is a theorem about these geometric limits:

Topology of geometric limits. *Let* $\overline{M}$ *have incompressible boundary, and let* N_i *be a sequence of hyperbolic 3-manifolds homeomorphic to* M*, which converge geometrically to* N_∞*. Then* N_∞ *is homeomorphic to an open subset of* M*.*

Although this seems to be rather a weak statement, it does not follow from any kind of general considerations. In fact the topological type of a general geometric limit can be quite wild. Soma [**97**] has also proved this theorem, and additionally given a precise characterization of the possible topological types that occur as geometric limits

One way to measure the complexity of a hyperbolic 3-manifold is to consider volume growth functions. Fixing $\epsilon > 0$, let CN_ϵ be the ϵ-thick part of N, intersected with its convex core C_N. Let d_ϵ denote the metric induced by lengths of paths in CN_ϵ. Fix a basepoint x of CN_ϵ and define

$$g_N(r) = vol\{y \in CN_\epsilon : d_\epsilon(x, y) \le r\}.$$

When N is geometrically finite, CN_ϵ is compact, so $g_N(r)$ is bounded. If N has a degenerate end with bounded geometry, then $g_N(r)$ grows linearly with r. An analysis of the geometry of the model manifolds verifies a conjecture of McMullen:

Polynomial volume growth. *Let* N *be any element of* $AH(M)$*, where* $\partial\overline{M}$ *is incompressible. Then*

$$g_N(r) = O(r^k)$$

where k *can be computed from* $\partial\overline{M}$*.*

For instance if $M = S \times [-1, 1]$ for a closed surface S of genus g then $k = 3g - 3$. This exponent is sharp, as can be shown by direct constructions.

Structure of the limit set. Ahlfors realized the importance of the limit set for the study of deformations of Kleinian groups. He formulated

Ahlfors' Measure Conjecture. *The limit set of a finitely generated Kleinian group is either the whole sphere, or has zero Lebesgue measure.*

Ahlfors proved this conjecture for geometrically finite groups [**1**]. The original motivation was the knowledge that no quasiconformal deformations can be supported on a set of zero measure. Although this application is now superseded by Sullivan's rigidity theorem (which applies even when the limit set is the whole sphere), the conjecture remains of interest. Thurston proved that the conjecture follows from geometric tameness, and hence Bonahon's theorem implies it in the incompressible-boundary setting. Later on, Canary [**29**] showed that in fact the assumption of topological tameness, by a clever branched-cover argument, suffices to make Thurston's techniques apply even in the case of compressible boundary, and Ahlfors' measure conjecture holds in this case also. Thus, this conjecture and a number of others would now follow from a solution of Marden's tameness conjecture. Recently, Brock-Bromberg-Evans-Souto [**22**] established the measure conjecture for all limits of geometrically finite manifolds, hence reducing it to the Density Conjecture. This proof relies on theorems establishing the Tameness Conjecture for certain limits of geometrically finite manifolds, extending results of Canary-Minsky [**27**], Ohshika [**88, 90**] and Evans [**37**].

A stronger version of Ahlfors' conjecture states that, when the limit set has full measure, the group acts ergodically on it. This was verified as part of the techniques of the previous paragraph, in all the topologically tame cases.

The measure-theoretic understanding of the limit set has also been further refined – by Patterson and Sullivan [**98**] (see also Nicholls [**87**]), who constructed invariant conformal densities on the limit set and related its Hausdorff dimension to the spectral theory of the quotient manifold in the geometrically finite case. These results were generalized by Canary [**28**] to the topologically tame category, and by others. Of particular note is the result of Bishop-Jones [**12**], who show that, for manifolds of infinite volume and finitely generated fundamental group, the Hausdorff dimension of the limit set is 2 if and only if the manifold is geometrically infinite.

In a different direction, it is of interest to know if the limit set is *locally connected*. This is true for example for quasifuchsian groups, whose limit sets are Jordan curves, but is a particularly difficult question for degenerate surface groups, i.e. geometrically infinite elements of $AH(S \times [-1, 1])$. More generally, local connectivity would be a consequence of a conjectural topological model for Kleinian group actions on the sphere. Cannon-Thurston [**32**] established the correctness of this model for groups closely related to the pseudo-Anosov examples of §6.2, and the author [**77**] extended this to bounded-geometry groups without cusps. Klarreich [**59**] extended this, still in the bounded geometry setting without cusps, to other manifolds M, and Bowditch [**18**] treated the bounded-geometry case with cusps.

McMullen proved local connectivity for surface groups where S is a one-holed torus, with an argument completely different from Cannon-Thurston's original approach, and utilizing the model manifold constructed in Minsky [**79**] in an earlier solution of the Ending Lamination Conjecture for that case. Now that there is a good description of a model manifold for the general case, it is interesting to see if McMullen's argument can be generalized. This looks like a fairly hard problem.

Topological structure of $AH(M)$. The Ending Lamination Conjecture provides a unique description of every point in $AH(M)$ (when $\partial\overline{M}$ is incompressible), but it does *not* give a description of the topology of $AH(M)$. This is due to two related facts: The ending invariants do not vary continuously in $AH(M)$ in any of the standard topologies (see Brock [**19**]), and there is a phenomenon of "bumping": the intersection of closures of components of the interior of $AH(M)$, which is not accounted for by ending laminations (see Anderson-Canary [**4**], Anderson-Canary-McCullough [**7**], Holt [**48**]). There is also a phenomenon of "self-bumping" of single components (see McMullen [**75**], Bromberg-Holt [**25**]): A component of the interior self-bumps at a boundary point p if its intersection with all sufficiently small neighborhoods of p is disconnected. Figure 1 suggests self-bumping of a quasifuchsian deformation space, although it has not to the author's knowledge been shown rigorously that the features of this figure are in fact true self-bumping, and not artifacts of the choice of 2-dimensional slice.

All known bumping phenomena involve essential annuli in $\overline{M}$. Given such an annulus A a sequence of elements of $AH(M)$ can be found whose geometric limit is homeomorphic to M minus a core curve of A in the interior – this gives rise to a rank 2 cusp. Then an exotic immersion of $\overline{M}$ into this cusped manifold yields a sequence of elements of $AH(M)$ which converge to the closure of a different component, or to the same component but from the "wrong" direction.

Although many different bumping and self-bumping points have been found, a topological model for all of $AH(M)$ remains out of reach. It is an indication of the incomplete nature of our knowledge that the following question is still without an answer in general:

Question. *Is the closure of a Bers slice homeomorphic to a closed ball?*

A Bers slice is the part of $int(AH(S \times [-1,1]))$ parameterized by $\{X\} \times \mathcal{T}(S)$ in the Ahlfors-Bers coordinates. Hence it is homeomorphic to the interior of a ball. A weaker question is whether there is no bumping of a Bers slice with itself, which seems quite plausible. The answer to this question is also known to be affirmative when $\mathcal{T}(S)$ is one complex dimensional, e.g. when S is a once-punctured torus [**79**].

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