# CRITICAL AGE DEPENDENT BRANCHING PROCESSES

HOWARD J. WEINER
UNIVERSITY OF CALIFORNIA, DAVIS

## 1. Introduction

This paper is a survey of some recent work which generalizes standard results in the Bellman-Harris single type critical age dependent branching process, especially the asymptotic probability of nonextinction of the process, and a limiting conditional exponential limit law. Also included are new results combining existing extensions and suggestions for further research and techniques in relaxing conditions on the processes.

## 2. Definition

The classical Bellman-Harris age dependent branching process ([8], Chapter 6) is defined as follows. At time 0, one new born cell starts the process with nonlattice lifetime distribution function $G(t)$, with $G(0) = 0$. At the end of its life, the cell disappears and is replaced by $k$ daughter cells with probability $p_k$, $k = 0, 1, 2, 3, \cdots$. Each daughter cell behaves independent of all other cells, and has the lifetime distribution $G(t)$. Denote by $h(s)$ the generating function

$$h(s) = \sum_{k=0}^{\infty} p_k s^k. \tag{2.1}$$

If $h'(1) \equiv m$, the mean number of daughter cells born to a parent cell, then the cases $m > 1$, $m = 1$, and $m < 1$ form a trichotomy for the behavior of the process in crucial respects, where $m = 1$ is the critical case (see [8]). We will consider now results for $Z(t)$, the number of cells alive at $t$.

## 3. Early results

When $m = 1$, $h^{(2)}(1) > 0$ and $h^{(3)}(1) < \infty$, and $G(t) = 1 - \exp\{-\lambda t\}$, Sevast'janov (see [8], Chapter 5) showed, by consideration of a differential equation satisfied by the generating function $F(s, t) \equiv \sum_{k=0}^{\infty} P[Z(t) = k]s^k$, that

$$\lim_{t\to\infty} tP[Z(t) > 0] = 2[\lambda h^{(2)}(1)]^{-1} \tag{3.1}$$

and

$$\lim_{t\to\infty} P[2(\lambda h^{(2)}(1)t)^{-1}Z(t) > u | Z(t) > 0] = \exp\{-u\}, \tag{3.2}$$

for $u \geqq 0$.

The first generalization of the result (3.1) was a basic paper by Chover and Ney [1] which stated that for $m = 1$, $1 - G(t) = 0(t^{-3})$, and $h^{(3)}(1) < \infty$, that

$$\lim_{t\to\infty} tP[Z(t) > 0] = \frac{2\int_0^\infty u\, dG(u)}{h^{(2)}(1)} \equiv b. \tag{3.3}$$

Goldstein [7] has obtained this result by using the corresponding discrete time result first given by Kolmogorov and Yaglom (see [8], Chapter 5) to approximate the $P[Z(t) > 0]$ above and below.

By means of Abelian and Tauberian arguments satisfied by the integral equations for $E(Z^n(t))$, $n \geqq 1$, Weiner [20] showed that

$$\lim_{t\to\infty} t^{-(n-1)}E[Z^n(t)] = n!b^{-(n-1)}. \tag{3.4}$$

Since

$$E[(bt^{-1}Z(t))^n|Z(t) > 0] \equiv \frac{E[bt^{-1}Z(t))^n]}{P[Z(t) > 0]} \rightarrow n! \tag{3.5}$$

by (3.4) and (3.5) above, and since $n!$ is the $n$th moment of an exponential law with parameter 1, Carleman's moment theorem yields that

$$\lim_{t\to\infty} P[bt^{-1}Z(t) > u|Z(t) > 0] = \exp\{-u\}. \tag{3.6}$$

Sevast'janov [17] also claimed to prove (3.3) and (3.6) by different methods, but his proof contained a gap [20] which was acknowledged and presumably corrected.

## 4. One dimensional generalizations

Durham [3] has defined a generalization of the age dependent branching process in which each cell is allowed to give birth to daughter cells throughout its lifetime. This process is the same as the classical Bellman-Harris process except that we define $N(t)$ to be the number of daughter cells born to the initial parent cell by time $t$, if the parent cell lives longer than $t$. Assume $N(t) \uparrow N < \infty$, where $N$ is a bona fide random variable. The case $EN = 1$ corresponds to the critical case.

Let $M(t) \equiv EN(t)$ and $A \equiv \frac{1}{2}E(N(N-1))$. If $a = A\int_0^\infty u\, dG(u) \Big/ \left(\int_0^\infty t\, dM(t)\right)^2$, and $m_1 = \int_0^\infty u\, dG(u) \Big/ \int_0^\infty t\, dM(t)$, then

$$\lim_{t\to\infty} t^{-(n-1)}E[Z^n(t)] = n!m_1a^{n-1}. \tag{4.1}$$

Suppose that $\infty > E[N(N-1)] - E[N(t)(N(t)-1)] = O(t^{-2})$ and that $1 - G(t) = O(t^{-2})$. Then, using an extension of the method of Chover and Ney [1],

$$\lim_{t\to\infty} tP[Z(t) > 0] = \frac{b}{A}. \tag{4.2}$$

These two results (4.1) and (4.2) along with $EN^n < \infty$ for all $n \geqq 1$ yield, by the argument in (3.5), that

$$\lim_{t\to\infty} P[(at)^{-1}Z(t) > u | Z(t) > 0] = \exp\{-u\}. \tag{4.3}$$

Fildes [5] has considered an extension to the Bellman-Harris process, where the lifetime distributions of different generations may differ, with the lifetime distributions of each member of the $n$th generation denoted by $G_n(t)$. The basic results are that for $m = 1$, $1 - G_n(t) = o(t^{-2-\delta})$ uniformly in $n$, $\lim_{n\to\infty} G_n(t) \to G(t)$ with $\mu = \int_0^\infty t\, dG(t)$ and if $X_n$ are independent with $P[X_n < t] = G_n(t)$, and $(1/n)\sum_{\ell=1}^{n} X_\ell \xrightarrow{P} \mu$ and if $h^{(n)}(1) < \infty$ for all $n$, then if $b = 2\mu(h^{(2)}(1))^{-1}$, and letting $Z_\ell(t)$ denote the number of cells alive at $t$ starting with one cell newborn at $t = 0$ of generation $\ell$, then

$$\lim_{t\to\infty} tP[Z_\ell(t) > 0] = b \tag{4.4}$$

and

$$\lim_{t\to\infty} P[bt^{-1}Z_\ell(t) > u | Z(t) > 0] = \exp\{-u\}. \tag{4.5}$$

The proofs of (4.4) and (4.5) are formally the same as for (3.4) and (3.6). We recall that Durham's proofs for (4.2) and (4.3) were along these lines.

Hence, one may state a theorem combining the extensions of Durham and Fildes as follows. We consider an age dependent branching process with generation dependent cell lifetime distributions and with each cell giving birth to offspring throughout its lifetime. Explicitly, the process starts at $t = 0$ with one new born cell of generation $\ell$. Let $N_\ell(t)$ be the number of daughter cells born to a parent cell of generation $\ell$, given that the parent cell is alive at $t$. Then we can state the following.

THEOREM 4.1. *Assume* $N_\ell(t) \uparrow N_\ell < \infty$ *as* $t \uparrow \infty$ *and that all* $\ell$ *and* $\{N_\ell\}$ *are independent random variables. Assume* $G_\ell(t) \to G(t)$ *as* $\ell \to \infty$, *with* $0 < \mu = \int_0^\infty t\, dG(t) < \infty$, *and that* $1 - G_\ell(t) = o(t^{-2-\delta})$ *uniformly in* $\ell$ *as* $t \to \infty$. *Assume that* $N_\ell(t) \to N(t)$ *for all* $t$ *as* $\ell \to \infty$ *and that* $\lim_{t\to\infty} N(t) = \lim_{\ell\to\infty} N_\ell = N < \infty$ *a bona fide random variable. Assume further that if* $\{X_n\}$ *are independent random variables with* $P[X_n < t] = G_n(t)$, *that* $(1/n)\sum_{\ell=1}^{n} X_\ell \xrightarrow{P} \mu$. *Assume that* $\lim_{n\to\infty} (1/n)\sum_{\ell=1}^{n} N_\ell = EN = 1$, *the critical case.*

*Define* $M(t) \equiv EN(t)$; $b = \int_0^\infty t\, dM(t)$; $m_1 = \mu/b$; $A = E(\frac{1}{2}N(N-1))$; *and* $a = m_1A/b$. *Let* $E(N_\ell(t)(N_\ell(t) - 1)(N_\ell(t) - 2) | \textit{cell lives longer than } t) = B_\ell(t)$. *If* $B_\ell(t) \to B_\ell < \infty$ *as* $t \to \infty$ *and* $B_\ell \to B$ *as* $\ell \to \infty$, *and if* $EN^k < \infty$ *for all* $k \geqq 1$, $B - B_\ell(t) = o(1/t^2)$ *uniformly in* $\ell$ *for* $t$ *sufficiently large and* $1 - G_\ell(t) = o(1/t^2)$ *uniformly in* $\ell$ *for* $t$ *sufficiently large, then*

$$\lim_{t\to\infty} tP[Z_\ell(t) > 0] = \frac{b}{A} \tag{4.6}$$

*and*

$$\lim_{t\to\infty} P[(at)^{-1}Z_t(t) > u | Z_t(t) > 0] = \exp\{-u\}. \tag{4.7}$$

Sevast'janov has considered an extension of the Bellman-Harris process to the case of variable offspring generating functions depending on the age at death of the parent.

Let a Bellman-Harris process start with one cell at time 0 with lifetime distribution $G(t)$. If the parent cell dies at time $u$, then $n$ independent daughter cells, each proceeding as the parent cell with lifetime distribution $G(t)$, are born with probability $p_n(u)$, where $p_n(u) > 0$ and $\sum_{n=0}^{\infty} p_n(u) = 1$ for all $u > 0$.

Let $h(u, s) = \sum_{n=0}^{\infty} p_n(u)s^n$. Let

$$\begin{aligned} a(u) &= \frac{\partial h}{\partial s}\bigg|_{s=1}, \\ b(u) &= \frac{\partial^2 h}{\partial s^2}\bigg|_{s=1}, \\ c(u) &= \frac{\partial^3 h}{\partial s^3}\bigg|_{s=1}. \end{aligned} \tag{4.8}$$

Then Sevast'janov has shown by Taylor expansions and approximations in the basic integral equation for $F(s, t) = \sum_{k=0}^{\infty} P[Z(t) = k]s^k$ the following results.

THEOREM 4.2 [18]. *Assume* $\int_0^\infty a(u)\, dG(u) = 1$ (*critical case*); $\infty > \int_0^\infty b(u)\, dG(u) > 0; \int_0^\infty u^3\, dG(u) < \infty; \int_0^\infty u^3 c(u)\, dG(u) < \infty$; *and* $\int_0^\infty c(u)\, dG(u) < \infty$. *Then*

$$\lim_{t\to\infty} tP[Z(t) > 0] = \frac{2\int_0^\infty ua(u)\, dG(u)}{\int_0^\infty b(u)\, dG(u)}. \tag{4.9}$$

THEOREM 4.3 [19]. *In addition to the assumptions of Theorem 4.2, assume further that*

$$\begin{aligned} EZ(t) &= \frac{\int_0^\infty u\, dG(u)\cdot \int_0^\infty a(u)\, dG(u)}{\int_0^\infty ua(u)\, dG(u)} + o\left(\frac{1}{t}\right), \\ E[(Z(t))(Z(t)-1)] &= \frac{\int_0^\infty b(u)\, dG(u)\left(\int_0^\infty u\, dG(u)\right)^2 \left(\int_0^\infty a(u)\, dG(u)\right)^3}{\left(\int_0^\infty ua(u)\, dG(u)\right)^3}\, t \\ &\qquad + B_2 + o(1), \end{aligned} \tag{4.10}$$

*where* $B_2$ *is some constant, then*

$$\lim_{t\to\infty} P\left[\frac{Z(t)}{E(Z(t)|Z(t) > 0)} > u | Z(t) > 0\right] = \exp\{-u\}. \tag{4.11}$$

It is conjectured that adding the assumption that $(\partial^n/\partial s^n)h|_{s=1} < \infty$ for all $n$, and lumping together the extensions and assumptions of Fildes, Durham, and

Sevast'janov into a generalization of the Bellman-Harris critical age dependent branching process, to allow for generation dependent lifetimes, births of daughter cells throughout the life of a parent, and parent age at death dependent daughter cell generating functions, one could formulate a theorem giving that

$$\lim_{t \to \infty} tP[Z(t) > 0] = c \tag{4.12}$$

and that

$$\lim_{t \to \infty} P\left[\frac{Z(t)}{E(Z(t)|Z(t) > 0)} > u|Z(t) > 0\right] = \exp\{-u\}, \tag{4.13}$$

but this will not be attempted here.

## 5. Multitype processes

We will consider a branching process with $m > 1$ distinguishable particle types as follows. At time 0, one newly born cell of type $i$ is born, $i = 1, 2, \cdots, m$. Cell type $i$ lives a random lifetime with continuous distribution function $G_i(t)$, $G_i(0+) = 0$. At the end of its life, cell $i$ is replaced by $j_1$ new cells of type 1, $j_2$ new cells of type 2, $\cdots$, $j_m$ new cells of type $m$ with probability $p_{ij_1 \cdots j \cdots}$, and we define the generating functions

$$h_i(s_1, \cdots, s_m) \equiv h_i(s) = \sum_{j_1 \cdots j \cdots} p_{ij_1 \cdots j_m} s_1^{j_1} s_2^{j_2} \cdots s_m^{j_m} \equiv \sum_j p_{ij} s^j \tag{5.1}$$

for $i = 1, \cdots, m$, where $s = (s_1, \cdots, s_m)$, $j = (j_1, \cdots, j_m)$ and $s^j \equiv s_1^{j_1}, \cdots, s_m^{j_m}$. Each new daughter cell proceeds independently of the state of the system, with each cell type $j$ governed by $G_j(t)$ and $h_j(s)$.

We will assume second moments of $h_i(s)$, $i = 1, \cdots, m$ to exist. Define $m_{ij} \equiv (\partial h_i(s)/\partial s_j)|_{s=1}$, where $1 = (1, \cdots, 1)$, an $m \times 1$ row vector, and let $M = (m_{ij})$ be the $m \times m$ matrix of first moments of the offspring distribution. Assume $m_{ij} > 0$ for all $i, j$.

Definition 5.1. *Let $Z_{ij}(t)$ = number of cells of type $j$ alive at $t$ given that the process started at time 0 with one new cell type $i$, $1 \leqq i, j \leqq m$.*

Definition 5.2. *Let $Z_i(t) = (Z_{i1}(t), Z_{i2}(t), \cdots, Z_{im}(t))$ denote the row vector of the numbers of cells alive at $t$ given that the process started at time $t = 0$ with one cell of type $i$.*

Definition 5.3. *Define*

$$P_i(t) = P[Z_i(t) > 0]. \tag{5.2}$$

We have from Frobenius' theory [12] that, assuming

$$M = (m_{ij}) = \frac{\partial h_i(1)}{\partial s_j}, \qquad 0 < m_{ij} < \infty, \tag{5.3}$$

is a positive matrix (it suffices that $M^n > 0$ for some $n$), we can make the basic assumption of criticality.

Definition 5.4. *Let $\rho$ be that positive eigenvalue of $M$ such that $\rho \geqq |v|$, where $v$ is any other eigenvalue of $M$.*

The basic assumption of criticality of the branching process throughout this paper is $\rho = 1$.

It follows that there are strictly positive eigenvectors $u > 0$, $v > 0$ such that

$$Mu = u, \qquad vM = v, \qquad \sum_{i=1}^{m} u_i \equiv u \cdot 1 = 1 \tag{5.4}$$

and

$$u \cdot v \equiv \sum_{i=1}^{n} u_i v_i = 1. \tag{5.5}$$

ASSUMPTION 5.1. *The second moments of* $h(s) \equiv (h_1(s), \cdots, h_m(s))$ *exist at* 1.

DEFINITION 5.5. *The sum*

$$Q(u) = \frac{1}{2} \sum_{i=1}^{m} \sum_{\ell=1}^{m} \sum_{r=1}^{m} \frac{\partial^2 h_i(1)}{\partial s_\ell \partial s_r} u_\ell u_r v_i < \infty \tag{5.6}$$

*and is strictly positive.*

ASSUMPTION 5.2. *The partial derivative*

$$\frac{\partial^2 h_i(1)}{\partial s_j \partial s_k} \equiv a_{ijk} > 0 \qquad \text{for all } 1 \leqq i, j, k \leqq m. \tag{5.7}$$

ASSUMPTION 5.3. *All moments of* $h_i(s)$ *exist at* $s = 1$.

ASSUMPTION 5.4.

$$\lim_{t \to \infty} t^2 (1 - G_i(t)) = 0, \qquad 1 \leqq i \leqq m. \tag{5.8}$$

DEFINITION 5.6. *Let* $\mu_i \equiv \int_0^\infty t \, dG_i(t)$.

Then we have the following.

THEOREM 5.1 [21]. *Under all assumptions except Assumption* 5.3 *we have,*

$$\lim_{t \to \infty} t P_i(t) = \left( \sum_{\ell=1}^{m} \mu_\ell u_\ell v_\ell \right) \frac{u_i}{Q(u)}. \tag{5.9}$$

This behavior of the extinction rate contrasts with the cases $\rho > 1$, where the process becomes extinct with probability less than one [17], and with the case $\rho < 1$, where the process becomes extinct with probability one at an exponential rate [16].

Goldstein [7] has also obtained this result, but by using upper and lower bounds for $P_i(t)$ based on the corresponding multitype Galton-Watson result in discrete time [11].

Using (5.9), one can then, by moment methods similar to the one dimensiona case, and a Frobenius decomposition used by Mode [13], obtain Theorem 5.2.

THEOREM 5.2 [21]. *Let* $Z_i(t)$ *be the vector of the number of cells alive at time* $t$, $i = 1, 2, \cdots, m$, *starting with one new cell of type* $i$ *in a critical multitype branching process satisfying all the assumptions stated. Then for all* $r = (r_1, \cdots, r_m)$, $r_i \geqq 0$, $i = 1, \cdots, m$,

$$\lim_{t \to \infty} P\left[ \frac{Z_i(t)}{t} > r \,|\, Z_i(t) > 0 \right] = \exp \left\{ \frac{-\left( \sum_{j=1}^{m} u_j v_j \mu_j \right)^2}{Q(u)} \max_{1 \leqq k \leqq m} \frac{r_k}{v_k \mu_k} \right\}. \tag{5.10}$$

A special case of these results has been obtained by Nair [15] (see also Mode [14]).

## 6. Related work

Goldstein's heavy use of imbedded Galton-Watson process results to obtain similar results for the corresponding age dependent cases suggests that this method perhaps can be used to obtain further results on other extensions of the Bellman-Harris process from corresponding discrete time results.

We briefly indicate some recent critical case Galton-Watson discrete time results which may be so used. Foster [6] has obtained limiting conditional distributions for a Galton-Watson process subject to immigration, as has Heathcote [9]. Fearn [4] has shown that a Galton-Watson process with generation dependent offspring distributions $h_n(s)$ = generating function of the number of offspring born to a cell in the $n$th generation has the following property. If $m_n \equiv h_n'(1) \to 1$ sufficiently fast, $Z_n$ denotes the number of cells alive at time $n$ starting with one cell in generation 0, then if $\operatorname{Var}(Z_n/EZ_n) \to \infty$,

$$P[Z_n > 0] \sim \frac{2}{\operatorname{Var}(Z_n/EZ_n)}, \tag{6.1}$$

generalizing the discrete time result of Kolmogorov and Yaglom (see [7], Chapter 5).

Another generalization concerns sequences of Bellman-Harris processes. Let $Z_n(t)$ denote the number of cells alive at $t$ in an ordinary Bellman-Harris process starting with $n$ new born cells alive at $t = 0$, and lifetime distribution $G(t)$. Assume also that the daughter cell generating function for each parent in this "$n$th process" is $h_n(s)$, where $h_n^{(1)}(1) \equiv m_n = 1 + \alpha/n + o(1/n)$, and $h_n^{(2)}(1) \to 2\beta$ as $n \to \infty$, and $h_n^{(3)}(1) < C$ all $n$.

Then Jagers [10] has shown that

$$\frac{1}{n} Z_n(nt) \xrightarrow{L} X(t), \tag{6.2}$$

where $X(t)$ is a diffusion process of a simple kind. If tightness conditions [2] can be established under reasonable conditions, then perhaps study of $X(t)$ can lead to boundary crossing results about the underlying Bellman-Harris processes. Work is proceeding along these lines.

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