WHEN IS A FIXED NUMBER OF OBSERVATIONS OPTIMAL?

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1. Introduction

Many of the classical fixed sample size tests and estimates have sequential counterparts which are more economical, needing on the average fewer observations to ensure a given performance. It turns out, however, that under some circumstances, admittedly artificial, a sample of fixed, nonrandom size is optimal.

We determine here rather inclusive conditions ensuring that for a sequence of partial sums of independent, identically distributed random variables, a fixed sample size is optimal with respect to a given nonnegative payoff function.

2. Notations

Let $X_1, X_2, \cdots$ be independent and identically distributed replicates of a random variable X, and set $S_0 = 0$, $S_n = X_1 + X_2 + \cdots + X_n$, $n \geqq 1$.

Let M be the set of all real numbers α for which $\varphi(\alpha)$, the moment generating function of X, is finite: $\varphi(\alpha) = E(\exp\{\alpha X\})$ and $M = \{\alpha | \varphi(\alpha) < \infty\}$. The set M is an interval containing $\alpha = 0$, and may consist of all the real numbers, a subinterval of them, or the sole value zero.

The nonnegative function $r_n(x)$, $n = 0, 1, \cdots$, x real, will be called the payoff function in the sense that if one stops observations after n trials his income is $r_n(S_n)$.

The optimal stopping problem is to determine a stopping time N, if possible, such that

$$E(r_N(S_N)) = \sup_T E(r_T(S_T)), \tag{1}$$

where the sup on the right is taken over all stopping times T. When such a stopping time N exists, we denote its "value" by V; that is, V is the maximal expected payoff given by (1); $V = E(r_N(S_N))$.

The pair (n, x) is called *accessible* if S_n is contained in every neighborhood of x with positive probability. Clearly, the value of $r_n(x)$ at inaccessible points is irrelevant.

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3. The main result

THEOREM. *The fixed integer n_0 is an optimal stopping time for S_n if there exists a measure μ over the set M such that*

$$\int_M \exp\{\alpha x\}\, \varphi^{-n}(\alpha)\mu(d\alpha) \geqq r_n(x), \qquad n = 0, 1, 2, \cdots, \qquad -\infty < x < \infty, \tag{2}$$

with equality holding in (2) at all x for which the pair (n_0, x) is accessible. The value is

$$V = E(r_{n_0}(S_{n_0})) = \mu(M). \tag{3}$$

To prove this theorem, we introduce the space-time chain (n, S_n), $n = 0, 1, \cdots$, and the harmonic functions $h_n(x)$ with respect to it. These are functions with the property that $h_n(x) = E(h_{n+1}(x + X))$, and it is known that any such function can be represented by the integral on the left side of (2), for an appropriate μ. This fact is proved for discrete valued random variables in [1] and [3], and is easily extended to the present case. The set M in the theorem is called the Martin boundary.

Suppose now that μ and n_0 are as stated in the theorem. Then for the corresponding harmonic function $h_n(x)$, the sequence $h_n(S_n)$ is a martingale, and if N is any bounded stopping rule, $h_0(S_0) = \mu(M) = E(h_N(S_N))$. Thus, denoting by $a \wedge b$ the minimum of a and b we have, by Fatou's lemma for any stopping time T,

$$\begin{aligned} \mu(M) = E(h_{T\wedge n}(S_{T\wedge n})) &= \lim_{n\to\infty} E(h_{T\wedge n}(S_{T\wedge n})) \\ &\geqq E(h_T(S_T)) \geqq E(r_T(S_T)). \end{aligned} \tag{4}$$

Consequently, for any T, $E(r_T(S_T)) \leqq \mu(A)$, and by the definition of accessibility, equality holds everywhere in (4) for the stop rule given by $T = n_0$. Thus, $T = n_0$ is optimal and $V = E(r_{n_0}(S_{n_0})) = \mu(M)$.

There is a certain sense in which the conditions of the theorem are necessary in order that $T = n_0$ be optimal, but we do not discuss them here.

4. Some examples

Let the variables be normally distributed $N(\theta, 1)$ with unknown mean θ and variance 1. Suppose we always estimate θ by taking the sample mean $\overline{X}_n = (1/n)S_n$. Suppose the payoff for stopping after n trials with a sample mean $\overline{X}_n$ is

$$r_n(S_n) = \exp\left\{\frac{n^2}{2(n+1)}(\overline{X}_n - \theta)^2\right\} d_n, \tag{5}$$

where d_n is some numerical sequence. Then if the sequence $d_n(n+1)^{1/2}$ attains its supremum at $n = n_0$, the fixed stopping time n_0 is optimal and the value is $V = d_{n_0}(n_0 + 1)^{1/2}$.

To prove this assertion let $W = d_{n_0}(n_0 + 1)^{1/2} = \sup d_n (n+1)^{1/2}$, and note that $\varphi(\alpha) = \exp\{\theta\alpha + \alpha^2/2\}$. If we take

$$\mu(d\alpha) = \frac{W}{(2\pi)^{1/2}} \exp\left\{-\frac{\alpha^2}{2}\right\} d\alpha, \qquad -\infty < \alpha < \infty, \tag{6}$$

we obtain for the integral on the left side of equation (2)

$$
\begin{aligned}
(7) \qquad h_n(x) &= \frac{W}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left\{\alpha x - n\alpha\theta - \frac{n\alpha^2}{2} - \frac{\alpha^2}{2}\right\} d\alpha \\
&= \frac{W}{(n+1)^{1/2}} \exp\left\{\frac{(x-n\theta)^2}{2(n+1)}\right\} \\
&= \frac{W}{(n+1)^{1/2}} \frac{r_n(x)}{d_n}.
\end{aligned}
$$

Hence, $h_n(x) \geqq r_n(x)$ if and only if $W \geqq d_n\,(n+1)^{1/2}$. By the definition of W this is true, equality holds when $n = n_0$, and $V = W$.

As a second example, consider the case of the exponential payoff $r_n(S_n) = \exp\{aS_n\}\,d_n$, where $a \in M$ and d_n is a numerical sequence. Dynkin [2] has shown in this case that a fixed number of trials is optimal, using quite different methods.

This is a special case of the theorem when μ assigns a mass c to the point a, and μ assigns zero measure to any set not containing a.

Then

$$
(8) \qquad h_n(x) = \int \exp\{\alpha x\}\, \varphi^{-n}(\alpha)\mu(d\alpha) = c\exp\{\alpha x\}\,\varphi^{-n}(a).
$$

The condition $h_n(x) \geqq r_n(x)$ becomes

$$
(9) \qquad c\exp\{ax\}\,\varphi^{-n}(a) \geqq \exp\{ax\}\,d_n,
$$

or $c \geqq \varphi^n(a)\,d_n$, $n = 0, 1, 2, \cdots$.

If we suppose that $\varphi^n(a)\,d_n$ assumes its supremum at $n = n_0$, and we set $c = \varphi^{n_0}(a)\,d_{n_0}$, the stopping rule $T = n_0$ is optimal by the theorem, and $V = c$.

REFERENCES

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