NECESSARY CONDITIONS FOR DISCRETE PARAMETER STOCHASTIC OPTIMIZATION PROBLEMS

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1. Introduction

Consider the following formal optimization problem. Let $\{\xi_i\}$ denote a sequence of random vectors, and define the sequence (1.1) of n dimensional vectors $\{X_i, i = 0, \cdots, k\}$, $X_i = \{X_i^1, \cdots, X_i^n\}$, where k is a fixed integer and u_i is a control, which is an element of an abstract set $\tilde{U}_i$:

$$X_{i+1} = X_i + f_i(X_i, u_i, \xi_i). \tag{1.1}$$

The object is to find the $\{u_i\}$ which minimizes

$$\begin{aligned} EX_k^0 &\equiv \sum_{i=0}^{k-1} f_i^0(X_i, u_i, \xi_i), \\ X_{i+1}^0 &= X_i^0 + f_i^0(X_i, u_i, \xi_i), \qquad X_i^0 \text{ fixed}, \end{aligned} \tag{1.2}$$

subject to certain constraints. Sometimes it is convenient to augment the vector X_i by adding X_i^0, the "cost" component. Then, we write ${}^+\underline{X}_i = (X_i^0, X_i)$, $\underline{f}_i = (f_i, f_i^0)$ and

$$\underline{X}_{i+1} = \underline{X}_i + \underline{f}_i(X_i, u_i, \xi_i). \tag{1.1'}$$

The constraints are

$$r_0(X_0) \equiv E\tilde{r}_0(X_0) = 0, \qquad q_0(\underline{X}_0) \equiv E\tilde{q}_0(\underline{X}_0, E\underline{X}_0) \leqq 0, \tag{1.3}$$

$$\begin{aligned} q_i(X_k) &\equiv E\tilde{q}_i(X_i, EX_i) \leqq 0, \qquad i = 1, \cdots, k, \\ r_k(X_k) &\equiv E\tilde{r}_k(X_k, EX_k) = 0, \end{aligned} \tag{1.4}$$

where $\tilde{r}_0$, $\tilde{q}_0$, $\tilde{r}_k$, and $\tilde{q}_i$ are vector valued functions. The q_0 is allowed to depend on X_0^0 in order to fix or limit X_0^0 in some way. That is, some component of $\tilde{q}_0(\underline{X}_0)$ may be $\tilde{q}_0^0(\underline{X}_0) = -X_0^0 \leqq 0$.

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The constraints $E\tilde{q}_i(X_i, EX_i) \leqq 0$ of (1.4) can be used to model or approximate a variety of constraints. For example, we can approximate the constraint $X_n \in A$ with probability 1 by letting q_n be the expectation of a suitably smooth approximation to the indicator of A. The constraint $P\{X_n \notin A, \text{ some } n = 1, \cdots, k\} \leqq \varepsilon$ can be modelled letting $\tilde{g}(\cdot)$ denote a suitably smooth approximation to the indicator of A and admitting the constraint $g(X_1, \cdots, X_k) = E \max_{k \geqq n \geqq 1} \tilde{g}(X_n) \geqq 1 - \varepsilon$. Note that g may have a "convex differential," although not necessarily a linear differential. See the comment after Theorem 3.1.

Necessary conditions for optimality in the form of Kuhn-Tucker conditions or Lagrange multiplier rules are well developed for very general deterministic discrete and continuous parameter problems [4], [11], and much of the recent work depends heavily on abstractions of the well-known geometric methods of nonlinear programming. In this paper, we apply some of the recent developments in abstract programming to obtain necessary conditions for (local) optimality for several discrete parameter optimization problems. The results are only typical of the possibilities and do not exhaust them. Hopefully, the results will suggest useful computational procedures, although our investigations along these lines are only beginning.

In [8] and [9], the author derived some necessary conditions for optimality for a class of continuous parameter stochastic problems, and in [10] for a discrete problem. The results in [8] and [9] are true "maximum principles" or "minimum principles" in the sense used in control theory, while the result in [5] is a necessary condition for a stationary point. Subsequent work was reported in [1], [2], [3], [5], [12], [13]. The development in [3], for an essentially linear problem (f_i linear) with a convex cost, and where the u_i are real numbers, seems to be the only work in which programming ideas are explicitly used. However, the programming approach gives better results with reasonable effort. Indeed, by properly identifying quantities in the abstract work [11] with quantities in the stochastic problems, we obtain and extend most previous discrete parameter results. Continuous parameter results will be reported elsewhere.

Section 2 cites the basic results from [11], which will be heavily used in the sequel. Sections 3 to 5 deal with the discrete parameter problem. In Section 2, the u_i are measurable with respect to given σ-algebras $\tilde{\mathscr{B}}_i$; in Section 3, the u_i are allowed to depend explicitly on the states, X_i, and so forth; and in Section 5 a maximum principle is derived, analogous to the deterministic discrete parameter maximum principle [4].

2. Mathematical background

This section describes a somewhat weakened version of a result of Neustadt [11], on an abstract variational problem which underlies the development of the sequel. Let $\mathscr{T}$ be a Banach space which contains the sets B and Q. The structures introduced next are abstract counterparts of these used in nonlinear programming in Euclidean space. The terminology is slightly changed from that of [11].

DEFINITION 2.1. *Let Z be a convex cone with vertex $\{0\}$ in $\mathscr{T}$. If ρ is an arbitrary ray of Z, let there be a cone Z_ρ with a nonempty interior and vertex $\{0\}$ and ρ internal to Z_ρ, and also a neighborhood N_ρ of $\{0\}$, such that $Z_\rho \cap N_\rho \subset B$. Then Z is an internal cone to B at $\{0\}$.*

DEFINITION 2.2. *Let P^ν denote the set $\{\beta: \beta_i \geqq 0, \Sigma_1^\nu \beta_i \leqq 1\}$. Let K be a convex set in $\mathscr{T}$ which contains $\{0\}$ and some point other than $\{0\}$. Let $w_1, \cdots, w_\nu$ be in K and let N be an arbitrary neighborhood of $\{0\}$. Let there exist an $\varepsilon_0 > 0$ (depending on $\nu, w_1, \cdots, w_\nu$, and N) so that, for each ε in $(0, \varepsilon_0]$, there is a continuous map $\zeta_\varepsilon(\beta)$ from P^ν to $\mathscr{T}$ with the property*

$$\zeta_\varepsilon(\beta) \subset \left\{\varepsilon\left(\sum_{i=1}^{\nu} \beta_i w_i + N\right)\right\} \cap Q. \tag{2.1}$$

Then K is a first order convex approximation to Q.

2.1. *A basic optimization problem.* Let $\mathscr{T}$ contain the set Q'. Find the element $\hat{w}$ in Q' which minimizes $\varphi_0(w)$ subject to the constraints $\varphi_i(w) = 0$, $i = 1, \cdots, m$, $\varphi_{-i}(w) \leqq 0$, $i = 1, \cdots, t$. We say that $\hat{w}$ is a *local solution* to the optimization problem (or, more loosely, the *optimal solution*) if, for some neighborhood N of $\{0\}$, $\varphi_0(w) \geqq \varphi_0(\hat{w})$ for all w in $\hat{w} + N$ which satisfy the constraints. Let $\hat{w}$ denote the optimal solution. The constraints φ_{-i} for which $\hat{\varphi}_{-i} \equiv \varphi_{-i}(\hat{w}) = 0$ for $i = 1, \cdots, t$ are called the *active constraints*. Define the set of indices $J = \{i: \varphi_{-i}(\hat{w}) = 0, i > 0\} \cup \{0\}$.

2.2. *The basic necessary condition for optimality.* First we collect some assumptions.

ASSUMPTION 2.1. *The $\varphi_i(w)$, $i \geqq 1$, are continuous at $\hat{w}$ and have Fréchet derivatives ℓ_i at $\hat{w}$, and $\ell_1, \cdots, \ell_m$ are continuous and linearly independent.*

Thus, $[\varphi_i(\hat{w} + \varepsilon w) - \varphi_i(\hat{w})]/\varepsilon - \ell_i(w) \to 0$ uniformly for w in any bounded neighborhood of $\mathscr{T}$.

ASSUMPTION 2.2. *There is a neighborhood N of $\{0\}$ in $\mathscr{T}$ so that, for all* inactive *constraints, we still have $\varphi_{-i}(\hat{w} + w) < 0$ for $w \in N$.*

ASSUMPTION 2.3. *Let the active constraints and also φ_0 be continuous at $\hat{w}$. Let*

$$\frac{\varphi_{-i}(\hat{w} + \varepsilon w) - \varphi_{-i}(\hat{w})}{\varepsilon} \to c_i(w) \tag{2.2}$$

for all w in $\mathscr{T}$, and uniformly for w in any bounded neighborhood of $\{0\}$, where $c_i(w)$ is a continuous and convex functional. There is some w and $j \in J$ for which $c_j(w) > 0$. There is a w for which $c_j(w) < 0$ for all $j \in J$.

A case of particular importance is where the $c_i(w)$ are linear. Then we substitute the stronger Assumption 2.3′.

ASSUMPTION 2.3′. *Let the* active *constraints and also φ_0 be continuous at $\hat{w}$ and have Fréchet derivatives c_i at $\hat{w}$ (corresponding to φ_{-i}) which are continuous, and suppose that there is a $w \in \mathscr{T}$ for which $c_i(w) < 0$ for all $i \in J$.*

We now have a particular case of Neustadt [11], Theorem 4.2. The *local solution* here is called a *totally regular local solution* in [11].

THEOREM 2.1. *Let Assumptions* 2.1 *to* 2.3 *hold. Let* $\hat{w}$ *be a local solution to the optimization problem. Then there exist* $\alpha_1, \cdots, \alpha_m, \alpha_0, \alpha_{-1}, \cdots, \alpha_{-t}$ *not all zero with* $\alpha_{-1} \leqq 0$ *for* $i \geqq 0$, *so that*

$$\sum_{i=1}^{m} \alpha_i \ell_i(w) + \sum_{i \in J} \alpha_{-i} c_i(w) \leqq 0 \tag{2.3}$$

for all w *in* $\bar{K}$, *where* K *is a first order convex approximation to* $Q' - \hat{w} \equiv Q$, *and* $\bar{K}$ *is the closure of* K *in* $\mathscr{T}$.

OBSERVATION. Let $\varphi_i(\cdot) = 0, i > 0$. If there is a $w \in K$ for which $c_j(w) < 0$ for all active j, then $\alpha_0 < 0$, and we can set $\alpha_0 = -1$.

Define

$$\begin{aligned} B &= \{w: \varphi_{-i}(\hat{w} + w) < \varphi_{-i}(\hat{w}), i \in J\} \cup \{0\}, \\ \pi &= \{w: \ell_i(\hat{w} + w) = 0, i = 1, \cdots, m\}. \end{aligned} \tag{2.4}$$

Then Theorem 2.1 is essentially a consequence of the result (see [11]) that the intersection of π and any internal cone to B can be separated from $K \cap \pi$ by a continuous linear functional.

3. The stochastic variational formula when the controls are measurable over fixed σ-algebras

In the first part of this section, a stochastic optimization problem will be treated in a fairly general way. We introduce only those assumptions which are required to apply Theorem 2.1. Then, more specific conditions which guarantee some of these assumptions are introduced.

3.1. *A stochastic optimization problem. Definitions and assumptions.* Let $\xi_0, \cdots, \xi_i, \cdots$ be a sequence of random variables, where $\xi_0, \cdots, \xi_i$ are measurable on the σ-algebra $\mathscr{B}(\xi_0, \cdots, \xi_i)$, and define the random sequence $\{\underline{X}_i\}$ by (1.1′). The measures on the $\mathscr{B}(\xi_0, \cdots, \xi_i)$ do not depend on the selected control sequence; the ξ_i are of the nature of "exogenous inputs." We seek the $\underline{X}_0, \cdots, \underline{X}_k$, $u_0, \cdots, u_{k-1}$ which minimizes (1.2) subject to the constraints (1.3) and (1.4).

3.2. *The admissible controls.* For a vector Y with components Y^i write $|Y| = \Sigma_i |Y^i|$ and $\|Y\|_q = \Sigma_i E^{1/q}|Y^i|^q$. Denote $L_q(\mathscr{B})$ the Banach space of $\mathscr{B}$ measurable random functions Y with norm $\|Y\|_q$. Let $\underline{L}_q(\mathscr{B})$ be the Banach space of $n + 1$ dimensional vectors $\underline{X}_i = (X_i^0, X_i)$ with norm $\|\underline{X}_i\|_q \equiv E|X_i^0| + \|X_i\|_q$. For a random matrix $M = \{M_{ij}\}$, define $\|M\|_q = \Sigma_{i,j} \|M_{ij}\|_q$. Suppose that $\{\tilde{\mathscr{B}}_i\}$ and $\mathscr{B}_0$ are a sequence of given σ-algebras, and U_i a sequence of convex sets. The $\tilde{\mathscr{B}}_i$, $\mathscr{B}_0$ and the measures on them do not depend on the chosen controls. In this section the admissible control set, denoted by $\tilde{U}_i$, are the random variables in $L_{p'}(\tilde{\mathscr{B}}_i)$ which take values in U_i for given $p' \geqq 1$. Then the $\underline{X}_i$ are measurable over $\mathscr{B}_i$, where $\mathscr{B}_i \equiv \mathscr{B}_{i-1} \cup \tilde{\mathscr{B}}_{i-1} \cup \mathscr{B}(\xi_{i-1})$ and $\underline{X}_0$ is a random variable measurable over the given σ-algebra $\mathscr{B}_0$. The set of admissible controls covers at least the three cases:

(i) the u_i depend explicitly on some function of the $\xi_0, \cdots, \xi_{i-1}$;

(ii) the u_i depend explicitly on noise corrupted observations of the $\xi_0, \cdots, \xi_{i-1}$, where the corrupting noise does not depend on the selected control sequence;

(iii) a randomized version of (i) and (ii).

It is well known from linear programming on Markov chains that a randomized control may give a smaller cost in a constrained stochastic optimization problem, than a nonrandomized control. Our controls can be randomized by a suitable choice of $\tilde{\mathscr{B}}_i$. Let $\tilde{v}_0, v_0, \cdots, v_k$ denote a sequence of independent random variables, which are also independent of the $\{\xi_i\}$ sequence and each of which has, say, a uniform distribution on [0, 1]. (We suppose that the underlying probability space is big enough to carry these random variables.) Suppose that the data field $\hat{\mathscr{B}}_i \subset \mathscr{B}(\xi_0, \cdots, \xi_{i-1})$ is available to the controller at time i. (That is, $\hat{\mathscr{B}}_i$ measures the information upon which the control depends.) Randomization is achieved by letting $\tilde{\mathscr{B}}_i = \hat{\mathscr{B}}_i \cup \mathscr{B}(v_i)$ and $\mathscr{B}_0 = \mathscr{B}(\tilde{v}_0)$. To determine the actual control value $u_i(\omega)$, we need to draw a value of v_i at random.

3.3. *Assumptions and notation.* Notation will frequently be abused by using the same term for a function and for its values. Let $u_i \in \tilde{U}_i$. Let IC_i denote the *pointwise internal cone* to $\tilde{U}_i - \hat{u}_i$ at $\{0\}$; that is, IC_i is a convex cone of random variables in $L_{p'}(\tilde{\mathscr{B}}_i)$ with the property that, if $\delta u_i^s \in IC_i$, for $s = 1, \cdots, v$, then

$$\hat{u}_i + \varepsilon \sum_{s=1}^{v} \beta_s \delta u_i^s \in U_i \text{ for all } \omega \text{ for } \beta_s \geqq 0, \sum_s \beta_s \leqq 1 \text{ and } 0 \leqq \varepsilon \leqq \varepsilon_0, \tag{3.1}$$

where $\varepsilon_0 > 0$ may depend on the δu_i^s. Also, $\delta u_i^s \in L_{p'}(\tilde{\mathscr{B}}_i)$.

Let $\delta u^s = (\delta u_0^s, \cdots, \delta u_{k-1}^s) \in IC_u \equiv IC_0 \times \cdots \times IC_{k-1}$. Write

$$\begin{aligned} \delta u_i(\beta) &\equiv \sum_{s=1}^{v} \beta_s \delta u_i^s, \qquad \delta u(\beta) \equiv \sum_{s=1}^{v} \beta_s \delta u^s, \\ \delta \underline{X}_{i+1}^s &= \delta \underline{X}_i^s + \hat{\underline{f}}_{i,x} \cdot \delta X_i^s + \hat{\underline{f}}_{i,u} \cdot \delta u_i^s, \\ \delta \underline{X}_i(\beta) &= \sum_s \beta_s \delta \underline{X}_i^s. \end{aligned} \tag{3.2}$$

We have

$$\underline{X}_{i+1}(\beta) = \underline{X}_i(\beta) + \underline{f}_i(X_i(\beta), \hat{u}_i + \varepsilon \delta u_i(\beta), \xi_i). \tag{3.3}$$

Let $r_{0,x}$ denote the matrix $\partial r_0(x)/\partial x$ and $\hat{r}_{0,x}$ denote $r_{0,x}$ evaluated at $\hat{x}_0$. Let $\tilde{q}_{i,e}$ denote $\partial \tilde{q}_i(x, e)/\partial e$, $i > 0$, the derivatives with respect to the second vector argument of $\tilde{q}_i(\cdot, \cdot)$. We also use $\hat{q}_{i,x} = \tilde{q}_{i,x}(\hat{X}_i, E\hat{X}_i)$ and $q_{0,x} = \partial q_0(\underline{x})/\partial \underline{x}$. Also

$$\underline{f}_{i,x} = \frac{\partial \underline{f}(x, u, \xi)}{\partial x}, \qquad \underline{f}_{i,x} = \frac{\partial \underline{f}(x, u, \xi)}{\partial x}, \tag{3.4}$$

and $\hat{q}_i = q_i(\hat{X}_i)$.

Fix $\delta u_i^s \in IC_i$ for all $i = 0, \cdots, k - 1$ and $s = 1, \cdots, \ell$.

ASSUMPTION 3.1. *Assume* $u_i \in \tilde{U}_i$, *and for any sequence* $u_i \in \tilde{U}_i$, *and any* $\underline{X}_0$ *satisfying the constraints, assume that the* $\underline{X}_i$ *given by* (1.1′) *are in* $\underline{L}_p(\mathscr{B}_i)$ *for given* $p \geqq 1$ *and* $i = 0, \cdots, k$. *The* $\delta\underline{X}_i$ *given by* (3.14) *are in* $\underline{L}_p(\mathscr{B}_i)$ *for any* $\delta u_i^s \in IC_i$.

ASSUMPTION 3.2. *The* IC_i *contain at least one point other than the origin.*

ASSUMPTION 3.3. *For* $\varepsilon_0 \geqq \varepsilon > 0$, *where* $\varepsilon_0 > 0$ *depends on the* δu_i^s, *suppose that the* $X_i(\beta)$ *given by* (3.3) *are continuous in* β *in* $L_p(\mathscr{B}_i)$, *and that*

$$\|X_i(\beta) - \hat{X}_i - \varepsilon\delta X_i(\beta)\|_p = o(\varepsilon) \tag{3.5}$$

uniformly in $\beta = (\beta_1, \cdots, \beta_m)$, *for* $\beta_s \geqq 0$, $\Sigma_s \beta_s = 1$.

ASSUMPTION 3.4. *For a real number* K_1,

$$\begin{aligned} E|q_i(X_i)| &\leqq K_1(1 + E|X_i|^p), \qquad i = 1, \cdots, k, \\ E|r_i(X_i)| &\leqq K_1(1 + E|X_i|^p). \end{aligned} \tag{3.6}$$

ASSUMPTION 3.5. *Let* $\tilde{q}_{i,x}$, $\tilde{q}_{i,e}$, $\tilde{r}_{i,x}$, *and* $\tilde{r}_{i,e}$ *exist and be continuous, and* $\|\hat{q}_{i,e}\|_1 < \infty$, $\|\hat{q}_{i,x}\|_{p/(p-1)} < \infty$. *Let* N_i *denote an arbitrary bounded neighborhood of* $\{0\}$ in $\mathscr{T}$. *Then all the following tend to zero as* $\varepsilon \to 0$, *uniformly for* $\underline{v}_i$ *in* N_i (*and also for* $\tilde{r}_{i,x}$, $\tilde{r}_{i,e}$ *replacing* $\tilde{q}_{i,x}$, *and* $\tilde{q}_{i,e}$, *respectively*),

$$\begin{aligned} &\|\tilde{q}_{i,e}(\hat{X}_i + \varepsilon\underline{v}_i, E\hat{X}_i + \varepsilon Ev_i) - \tilde{q}_{i,e}(\hat{X}_i, E\hat{X}_i)\|_1, \\ &\|\tilde{q}_{i,x}(\hat{X}_i + \varepsilon\underline{v}_i, E\hat{X}_i + \varepsilon Ev_i) - \tilde{q}_{i,x}(\hat{X}_i, E\hat{X}_i)\|_{p/(p-1)}. \end{aligned} \tag{3.7}$$

ASSUMPTION 3.6. *Define the linear maps* $\hat{R}_0$, $\hat{R}_k$ (*from* $y_0 \in L_p(\mathscr{B}_0)$ *and* $y_k \in L_p(\mathscr{B}_k)$ *to the appropriate Euclidean space*), *and suppose that the components are linearly independent for each i. Then*

$$\hat{R}_i \cdot y_i \equiv E[\hat{r}_{i,x} \cdot y_i + \hat{r}_{i,e} E y_i]. \tag{3.8}$$

ASSUMPTION 3.7. *For the inactive constraints* q_i^j, *suppose that there is a neighborhood* N_i *of the origin in* $L_p(\mathscr{B}_i)$ *for* $i > 0$ *and in* $\underline{L}_p(\mathscr{B}_0)$ *for* $i = 0$, *for which* $q_i^j(\hat{X}_i + y_i) < 0$, $q_0^j(\hat{\underline{X}}_0 + \underline{y}_0) < 0$, *for* $y_i \in N_i$, $i > 0$, $\underline{y}_0 \in N_0$. *Suppose that there is an* X_i *in* $L_p(\mathscr{B}_i)$, $i > 0$, *and* $\underline{X}_0 \in \underline{L}_p(\mathscr{B}_0)$ *so that*

$$\begin{aligned} E[\tilde{q}_{i,x}^j \cdot X_i + \tilde{q}_{i,e} E X_i] &< 0 \qquad \text{for all active } q_i^j, \\ E[\tilde{q}_{0,\underline{x}}^j \cdot \underline{X}_0 + \tilde{q}_{0,\underline{e}} E\underline{X}_0] &< 0 \qquad \text{for all active } q_0^j. \end{aligned} \tag{3.9}$$

ASSUMPTION 3.8. *Assume that* $f_{i,x}^0$, $f_{i,u}^0$ *are continuous in* x *and* u *and* $\|\hat{f}_{i,x}^0\|_{p/(p-1)} < \infty$ *and* $\|\hat{f}_{i,u}^0\|_{p'/(p'-1)} < \infty$. *For a real* K_1,

$$|f_i^0(X_i, u_i, \xi_i)| \leqq K_1(1 + |X_i|^p + |u_i|^{p'}) \tag{3.10}$$

and

$$\begin{aligned} &\|f_{i,x}^0(\hat{X}_i + \varepsilon v_i, \hat{u}_i + \varepsilon\delta u_i(\beta)) - \hat{f}_{i,x}^0\|_{p/(p-1)} \to 0, \\ &\|f_{i,u}^0(\hat{X}_i + \varepsilon v_i, \hat{u}_i + \varepsilon\delta u_i(\beta)) - \hat{f}_{i,u}^0\|_{p'/(p'-1)} \to 0, \end{aligned} \tag{3.11}$$

as $\varepsilon \to 0$, *uniformly for* v_i *in* N_i *and in* β, *for* $i = 0, \cdots, k - 1$.

3.4. *Identification with the definition in Section* 2. Define $\mathscr{T}$ to be the space in which $\underline{X}_0, \cdots, \underline{X}_k$ lie, namely, $\mathscr{T} = \underline{L}_p(\mathscr{B}_0) \times \cdots \times \underline{L}_p(\mathscr{B}_k)$, and let Q' denote the set of all sequences in $\mathscr{T}$ which are solution to (1.1′) for the class of allowed controls and initial conditions.

Assumption 3.8 implies that (3.5) can be replaced by

$$\|\underline{X}_i(\beta) - \hat{\underline{X}}_i - \varepsilon\delta\underline{X}_i(\beta)\|_p = o(\varepsilon), \tag{3.12}$$

since, by (3.5), we can show that

$$\begin{aligned} E|f_i^0(X_i(\beta), \hat{u}_i + \varepsilon\delta u_i(\beta), \xi_i) - f_i^0(\hat{X}_i, \hat{u}_i, \xi_i) - \varepsilon\hat{f}^0_{i,x}\cdot\delta X_i(\beta) - \varepsilon\hat{f}^0_{i,x}\cdot\delta u_i(\beta)| \\ \leqq \varepsilon E|f^0_{i,x}(\hat{X}_i + \theta_{\varepsilon,\beta}(X_i(\beta) - \hat{X}_i), \hat{u}_i + \varepsilon\theta_{\varepsilon,\beta}\delta u_i(\beta), \xi_i) - \hat{f}^0_{i,x}|\cdot|\delta X_i(\beta)| \\ + \varepsilon|f^0_{i,u}(\hat{X}_i + \theta_{\varepsilon,\beta}(X_i(\beta) - \hat{X}_i), \hat{u}_i + \varepsilon\theta_{\varepsilon,\beta}\delta u_i(\beta), \xi_i) - \hat{f}^0_{i,u}|\cdot|\delta u_i(\beta)|, \end{aligned} \tag{3.13}$$

where $\theta_{\varepsilon,\beta}$ is a random variable in $[0, 1]$, and we can complete the assertion by using Hölder's inequality. Then it is straightforward to verify that the set $K \in \mathscr{T}$ (given by (3.2) or (3.14)) of all vectors $\delta\underline{X}_0, \cdots, \delta\underline{X}_k$ corresponding to $\delta u_i \in IC_i$, $\delta\underline{X}_0 \in \mathscr{B}_0$, is a first order convex approximation to $Q \equiv Q' - \{\hat{\underline{X}}_0, \cdots, \hat{\underline{X}}_k\} \subset \mathscr{T}$. One can write

$$\begin{aligned} \delta\underline{X}_{i+1} &= \delta\underline{X}_i + \hat{\underline{f}}_{i,\underline{x}}\delta\underline{X}_i + \hat{\underline{f}}_{i,u}\delta u_i, \\ \delta\underline{X}_i &\equiv \sum_{j=1}^{i} F(j, i)\hat{\underline{f}}_{j-1,u}\delta u_{j-1} + F(0, i)\delta\underline{X}_0, \\ F(j, i) &= (I + \hat{\underline{f}}_{i-1,x}) \cdots (I + \hat{\underline{f}}_{j,x}), \qquad j < i, \\ F(i, i) &= I. \end{aligned} \tag{3.14}$$

$$\hat{\underline{f}}_{i,\underline{x}} = \left[\begin{array}{c|c} 0 & \hat{f}^0_{i,x^1}, \cdots, \hat{f}^0_{i,x^n} \\ \hline 0 & \\ \vdots & \hat{f}_{i,x} \\ 0 & \end{array}\right]. \tag{3.15}$$

Identify the components of r_0 and r_k with $\varphi_1, \cdots$, and φ_{-i}, $i > 0$, with the components of the q_i, $i \geqq 0$. Also $\varphi_0 \equiv EX_k^0$. The $\hat{R}_i$ of Assumption 3.6 is the Fréchet derivative of the vector valued map $r_i(X_i)$. The following $\hat{Q}_i$, $i \geqq 0$,

$$\begin{aligned} \hat{Q}_i\cdot y_i &\equiv E[\hat{q}_{i,x}\cdot y_i + \hat{q}_{i,e}Ey_i]. \\ \hat{Q}_0\cdot \underline{y}_0 &\equiv E[\hat{q}_{i,\underline{x}}\cdot \underline{y}_0 + \hat{q}_{0,\underline{e}}E\underline{y}_0] \end{aligned} \tag{3.16}$$

are the Fréchet derivatives of the vector valued maps q_i at $\hat{\underline{X}}_i$. Thus, Assumption 2.1 is implied by Assumptions 3.4, 3.5, and 3.6. Assumption 3.7 implies Assumptions 2.2, and 2.3′ is implied by 3.1 and 3.4 through 3.8.

That $\hat{Q}_i$ is a Fréchet derivative can be seen from the following brief calculation. Let N_i denote an arbitrary bounded neighborhood of $\{0\}$ in $L_p(\mathscr{B}_i)$. There are random variables $\theta \in [0, 1]$ (depending on ε, v_i) so that, for $i > 0$,

$$\begin{aligned}(3.17)\qquad e \equiv \varepsilon^{-1}|E\tilde{q}_i(X_i + \varepsilon v_i, EX_i + \varepsilon Ev_i) - E\tilde{q}_i(X_i, EX_i) \\ - \varepsilon E\tilde{q}_{i,x}(X_i, EX_i)\cdot v_i - \varepsilon E\tilde{q}_{i,e}(X_i, EX_i)EV_i| \\ \leqq |E[\tilde{q}_{i,x}(X_i + \varepsilon\theta v_i, EX_i + \varepsilon\theta Ev_i) - \tilde{q}_{i,x}(X_i, EX_i)]v_i \\ + E[\tilde{q}_{i,e}(X_i + \varepsilon\theta v_i, EX_i + \varepsilon\theta v_i) - \tilde{q}_{i,e}(X_i, EX_i)]Ev_i|.\end{aligned}$$

By using Assumption 3.5 and Hölder's inequality, we can show that $e \to 0$ as $\varepsilon \to 0$ uniformly in v_i, completing the calculation.

Note that, for the Fréchet derivatives of the equality constraints to be linearly independent, it is enough to consider $r_0(X)$ and $r_k(X_k)$ separately, since r_0 does not depend on X_k and r_k does not depend on X_0.

Theorem 3.1 is the main result of this section. Let P' denote the $(n + 1)$ row vector $(1, 0, \cdots, 0)$. The prime on P' denotes transpose. While r_0, r_k, q_i, $i > 0$, do not actually depend on the X_i^0, it is convenient to write (3.19) and subsequent formulas as though they did. Thus, we write $\underline{r}_k(\underline{X}_k, E\underline{X}_k)$ for $r_k(X_k, EX_k)$ and $\tilde{\underline{r}}_{k,\underline{x}}(\underline{X}_k, E\underline{X}_k)$ for

$$(3.18)\qquad \begin{bmatrix} 0 & \vdots & \\ 0 & \vdots & r_{k,x}(X_k, EX_k) \\ 0 & & \end{bmatrix}, \qquad \text{and so forth.}$$

THEOREM 3.1. *Let Assumptions* 3.1 *through* 3.8 *hold. There exists a scalar* $p^0 \leqq 0$, *and there exist vectors* α_0, α_k, *and* $\psi_i \leqq 0$, $i = 0, \cdots, k$, *not all zero, such that*

$$\begin{aligned}(3.19)\qquad p^0E\delta\underline{X}_k^0 + E\alpha_0'[\hat{\underline{r}}_{0,\underline{x}} + (E\hat{\underline{r}}_{0,\underline{e}})]\delta\underline{X}_0 + E\alpha_k'[\hat{\underline{r}}_{k,\underline{x}} + (E\hat{\underline{r}}_{k,\underline{e}})]\delta\underline{X}_k \\ + E\sum_{i=0}^{k}\psi_i'[\hat{\underline{q}}_{i,\underline{x}} + (E\hat{\underline{q}}_{i,\underline{e}})]\delta\underline{X}_i \leqq 0\end{aligned}$$

for $\delta\underline{X}_0, \cdots, \delta\underline{X}_k \in \bar{K}$, *where* $\psi_i'\hat{q}_i = 0$. *Define the vectors* $\underline{p}_k, \cdots, \underline{p}_0$:

$$\begin{aligned}\underline{p}_k = p^0P + [\hat{\underline{r}}_{k,\underline{x}}' + (E\hat{\underline{r}}_{k,\underline{e}}')]\alpha_k + [\hat{\underline{q}}_{k,\underline{x}}' + (E\hat{\underline{q}}_{k,\underline{e}}')]\psi_k, \\ (3.20)\qquad \underline{p}_{i-1} = (I + \hat{\underline{f}}_{i-1,\underline{x}}')\underline{p}_i + [\hat{\underline{q}}_{i-1,\underline{x}}' + (E\hat{\underline{q}}_{i-1,\underline{e}}')]\psi_{i-1} \\ + [\hat{\underline{r}}_{i-1,\underline{x}}' + (E\hat{\underline{r}}_{i-1,\underline{e}}')]\alpha_{i-1}, \qquad k \geqq i \geqq 1.\end{aligned}$$

Then

$$(3.21)\qquad E[\underline{p}_i'\hat{\underline{f}}_{i-1,u} \mid \bar{\mathscr{B}}_{i-1}]\delta u_{i-1} \leqq 0$$

for all $\delta u_{i-1} \in \overline{IC}_{i-1}$ *and*

$$(3.22)\qquad E[\underline{p}_0 \mid \mathscr{B}_0] = 0.$$

PROOF. Equation (3.19) follows from Theorem 2.1 and the discussion preceding Theorem 3.1. Equations (3.21) and (3.22) are specializations of (3.19), as follows. Let $\delta \underline{X}_0 = 0$, $\delta u_j = 0, j \neq i-1$. Then $\delta \underline{X}_j = F(i,j) f_{i-1,u} \delta u_{i-1}$, and (3.19) yields

$$E\{p^0 P' F(i,k) + \alpha_k'[\hat{\underline{r}}_{k,x} + (E\hat{\underline{r}}_{k,e})]F(i,k) + \sum_{j=1}^{k} \psi_j'[\hat{\underline{q}}_{j,x} + (E\hat{\underline{q}}_{j,e})]F(i,j)\} \hat{\underline{f}}_{i-1,u} \cdot \delta u_{i-1} \leqq 0. \tag{3.23}$$

The bracketed term in (3.23) is $\underline{p}_i'$. The closure of the first order convex approximation given by (3.2) and (3.3) is merely the set of solutions $(\delta \underline{X}_0, \cdots, \delta \underline{X}_k)$ of (3.2) and (3.3) which can be obtained by using $\{\delta u_i\}$ in the closure in $L_{p'}(\tilde{\mathscr{B}}_i)$ of $\{IC_i\}$. Thus,

$$E[\underline{p}_i' \hat{\underline{f}}_{i-1,u} \delta u_{i-1}] \leqq 0 \tag{3.24}$$

for all $\delta u_{i-1} \in \overline{IC}_{i-1}$. Let $B \in \tilde{\mathscr{B}}_{i-1}$ and suppose that (χ_B is the characteristic function of B)

$$E \chi_B \underline{p}_i' \hat{\underline{f}}_{i-1,u} \delta u_{i-1} > 0. \tag{3.25}$$

Then $\delta \tilde{u}_{i-1} \equiv \chi_B \delta u_{i-1} \in \overline{IC}_{i-1}$ and we have $E \underline{p}_i' \hat{\underline{f}}_{i-1,u} \cdot \delta \tilde{u}_{i-1} > 0$, which contradicts (3.24). Thus, (3.21) holds.

Next, let $\delta u_i = 0$, $i = 0, \cdots, k-1$. Then substituting $\delta \underline{X}_i = F(0,i) \delta \underline{X}_0$ into (3.19) yields

$$E \underline{p}_0' \delta \underline{X}_0 \leqq 0 \tag{3.26}$$

for all $\delta \underline{X}_0$ in $\underline{L}_p(\mathscr{B}_0)$. Using the argument which proved (3.21) and the fact that $-\delta \underline{X}_0 \in \underline{L}_p(\mathscr{B}_0)$ if $\delta \underline{X}_0$ is, gives (3.22). *Q.E.D.*

3.5. *Remark on generalizations.* The spaces $\underline{L}_p(\mathscr{B}_i)$ can easily be replaced by less restrictive spaces where, for example, each of the components X_i^j has its own integrability property, (that is, $X_i^j \in L_{p_{ji}}(\mathscr{B}_{ji})$). Assumption 2.3 requires only that the $c_i(x)$ be smooth and convex, whereas the "derivatives" Q_i of the $q_0, \cdots, q_k, EX_k^0$, were linear operators. The "convex" derivatives of Assumption 2.3 arise, for example, where, the cost to be minimized, or the state space constraints take the form $E \max_i \|X_i - t_i\|$, and Theorem 3.1 can be extended to include constraints or costs of these forms. Constraints of the type $P\{X_n \in A\} > 1 - \varepsilon$ can conceivably be inserted into the definition of Q', but we do not know how to find a first order convex approximation to such a constrained Q'.

For illustrative purposes, we verify Assumption 3.3 under a specific set of conditions on the f_i.

THEOREM 3.2. *Let $u_i \in \tilde{U}_i$ with $p' \geqq p \geqq 1$, and $\tilde{\mathscr{B}}_i \subset \mathscr{B}(\xi_0, \cdots, \xi_{i-1}) \cup \mathscr{B}(v_i)$, where the independent sequence $\{v_i\}$ is independent of the independent sequence of matrices $\{\xi_i\}$ and*

$$X_{i+1} = X_i + f_i(X_i, u_i, \xi_i) = g_i(X_i, u_i) + \xi_i h_i(X_i, u_i). \tag{3.27}$$

The moments satisfy $E|\xi_i|^q < \infty$ *for all* $q = 1, 2, \cdots$. *Let* g_i *and* h_i *be continuous with bounded and continuous derivatives in* X_i, u_i. *Then Assumption* 3.3 *holds.*

PROOF. From the following estimate, for some real K,

$$|X_{i+1}| \leqq |X_i| + K(|X_i| + |u_i| + 1) + K(|X_i| + |u_i| + 1)|\xi_i|), \tag{3.28}$$

we can deduce that all moments of $|X_i|$ exist up to order p', and similarly for the moments of the δX_i given by $\delta X_{i+1} = \delta X_i + \hat{f}_{i,x}\delta X_i + \hat{f}_{i,u}\delta u_i$, or for the moments of $\delta X_i(\beta)$.

Fix $\varepsilon > 0$ and write

$$X_{i+1}(\beta) = X_i(\beta) + g_i[X_i(\beta), \hat{u}_i + \varepsilon\delta u_i(\beta)] + \xi_i h_i[X_i(\beta), \hat{u}_i + \varepsilon\delta u_i(\beta)]. \tag{3.29}$$

From the relation, for some real K,

$$\begin{aligned} |X_{i+1}(\beta) - X_{i+1}(\tilde{\beta})| \leqq K|X_i(\beta) - X_i(\tilde{\beta})|(1 + |\xi_i|) \\ + \varepsilon K|\delta u_i(\beta) - \delta u_i(\tilde{\beta})|(1 + |\xi_i|), \end{aligned} \tag{3.30}$$

and the relations $|\delta u_i(\beta) - \delta u_i(\tilde{\beta})| \to 0$ in $L_{p'}(\tilde{\mathscr{B}}_i)$ as $\tilde{\beta} \to \beta$, we conclude that $X_i(\beta)$ is a continuous $L_p(\mathscr{B}_{i-1})$ valued function of β, for any $\varepsilon > 0$. Next, define the sequence $Y_i = X_i(\beta) - \hat{X}_i$,

$$\begin{aligned} Y_{i+1} = Y_i + [g_i(\hat{X}_i + Y_i, \hat{u}_i + \varepsilon\delta u_i(\beta)) + \xi_i h_i(\hat{X}_i + Y_i, \hat{u}_i + \varepsilon\delta u_i(\beta))] \\ - [g_i(\hat{X}_i, \hat{u}_i) + \xi_i h_i(\hat{X}_i, \hat{u}_i)]. \end{aligned} \tag{3.31}$$

From (3.31), we can easily show that $E^{1/p}|Y_i|^p = O(\varepsilon)$, uniformly in β. Next, $Z_i \equiv Y_i - \varepsilon\delta X_i$ satisfies, for random $\theta_i \in [0, 1]$, which may depend on ε and β,

$$\begin{aligned} Z_0 &= 0 \\ Z_{i+1} &= Z_i + [\hat{g}_{i,x} + \xi_i\hat{h}_{i,x}]Z_i \\ &\quad + [g_{i,x}(\hat{X}_i + \theta_i Y_i, \hat{u}_i + \varepsilon\theta_i\delta u_i(\beta)) - \hat{g}_{i,x}]Y_i \\ &\quad + \xi_i[h_{i,x}(\hat{X}_i + \theta_i Y_i, \hat{u}_i + \varepsilon\theta_i\delta u_i(\beta)) - \hat{h}_{i,x}]Y_i \\ &\quad + \varepsilon[g_{i,u}(\hat{X}_i + \theta_i Y_i, \hat{u}_i + \varepsilon\theta_i\delta u_i(\beta)) - \hat{g}_{i,u}]\delta u_i(\beta) \\ &\quad + \varepsilon\xi_i[h_{i,u}(\hat{X}_i + \theta_i Y_i, \hat{u}_i + \varepsilon\theta_i\delta u_i(\beta)) - \hat{h}_{i,u}]\delta u_i(\beta). \end{aligned} \tag{3.32}$$

This expression together with $E^{1/p}|Y_i|^p = O(\varepsilon)$, implies that $E^{1/p}|Z_i|^p = O(\varepsilon)$. The proof is straightforward and only the following observation is needed.

$$\left(\frac{Y_i}{\varepsilon}\right)^p [g_{i,x}(\hat{X}_i + \theta_i Y_i, \hat{u}_i + \varepsilon\theta_i\delta u_i(\beta)) - \hat{g}_{i,x}]^p \tag{3.33}$$

is uniformly integrable with parameters ε and β, and goes to zero as $\varepsilon \to 0$ with probability 1. Thus, the expectation of the term goes to zero as $\varepsilon \to 0$, uniformly in β. *Q.E.D.*

4. The multiplier rule when the control depends explicitly on the state

In Section 3, the controls u_i were measurable over the fixed σ-algebras $\tilde{\mathscr{B}}_i$, and did not depend explicitly on the state. If we allow the controls u_i to depend on the X_i, then some condition must be imposed on the u_i which guarantees that replacing $u_i(X_i)$ by $u_i(X_i + \delta X_i) + \varepsilon \delta u_i(X_i + \delta X_i)$ in (1.1) (where $X_{i+1} + \delta X_{i+1} = X_i + \delta X_i + f(X_i + \delta X_i, u_i + \varepsilon \delta u_i, \xi_i)$) alters the paths only the order of ε. In Section 3, $u_i(X_i + \delta X_i) = u_i(X_i)$. Thus, some smoothness on the u_i is required. In Theorem 4.1, we assume the form (3.27).

For simplicity of notation, it is assumed that u_i depends explicitly on X_i, and is not randomized. Subsequently, several extensions are stated.

4.1. *Assumptions and notation.* Let $p = p'$ and let $\mathscr{T}$ be as in Section 3, where $\mathscr{B}_i = \mathscr{B}(\xi_0, \cdots, \xi_{i-1})$ and $\mathscr{B}_0$ is the trivial σ-algebra.

ASSUMPTION 4.1. *Let U_i be a convex set, and let $\tilde{U}_i$ denote the convex set of controls which can be used at time i. We have $u_i \in \tilde{U}_i$ if $u_{i,x}$ is bounded and continuous, and $u_i(x) \in U_i$ for each x.*

Again, let $\hat{X}_0, \cdots, \hat{X}_k$, $\hat{u}_0(\hat{X}_0) = \hat{u}_0, \cdots, \hat{u}_{k-1}(\hat{X}_{k-1}) = \hat{u}_{k-1}$ denote the optimal solution. Assume that IC_i, the internal cone to $\tilde{U}_i - \hat{u}_i$ at $\{0\}$ exists and contains some point other than $\{0\}$. Then, for any $\delta u_i^s \in IC_i$, $\delta u_{i,x}^s$ is bounded and continuous and $\hat{u}_i(x) + \varepsilon \sum_{s=1}^{v} \beta_s \delta u_i^s(x) \in U_i$ for sufficiently small ε, for all x and $\beta = (\beta_1, \cdots, \beta_v) \in P^v$.

ASSUMPTION 4.2. *Assume that $h_{i,x}$, $g_{i,x}$, $h_{i,u}$, $g_{i,u}$ are bounded and are continuous in their arguments. The $\{\xi_i\}$ are mutually independent, and all of their moments exist.*

ASSUMPTION 4.3. *Assume that $f_{i,x}^0$, $f_{i,u}^0$ are continuous in their variables and, for some real $K < \infty$,*

$$\begin{aligned} |f_i^0(x, u)| &\leqq K(1 + |x|^p + |u|^p), \\ |f_{i,x}^0(x, u)| + |f_{i,u}^0(x, u)| &\leqq K(1 + |x|^{p-1} + |u|^{p-1}). \end{aligned} \tag{4.1}$$

Define $\delta \underline{X}_0(\beta) = \Sigma_s \beta_s \delta \underline{X}_0^s$,

$$\delta u_i(\beta, X_i) = \sum_s \beta_s \delta u_i^s(X_i), \qquad \delta u_i^s(x) \in IC_i$$

$$\delta \underline{X}_{i+1} = \delta \underline{X}_i + [\hat{\underline{f}}_{i,\underline{x}} + \hat{\underline{f}}_{i,u} \cdot \hat{u}_{i,\underline{x}}] \delta \underline{X}_i + \hat{\underline{f}}_{i,u} \cdot \delta \hat{u}_i, \tag{4.2}$$

where we write $\delta \hat{u}_i$ for $\delta u_i(\wedge_i)$ and also $\delta \underline{X}_i(\beta)$ for $\delta \underline{X}_i$ if $\delta \hat{u}_i$ takes the form $\delta u_i(\beta, \hat{X}_i)$. With

$$\begin{aligned} F_u(j, i) &\equiv (I + \hat{\underline{f}}_{i-1,\underline{x}} + \hat{\underline{f}}_{i-1,u} \hat{u}_{i-1,\underline{x}}) \cdots (I + \hat{\underline{f}}_{j,\underline{x}} + \hat{\underline{f}}_{j,u} \hat{u}_{j,\underline{x}}), \quad j \leqq i, \\ F_u(i, i) &= I, \end{aligned} \tag{4.3}$$

we have

$$\delta \underline{X}_{i+1} = F_u(i, i+1) \delta \underline{X}_i + \hat{\underline{f}}_{i,u} \cdot \delta \hat{u}_i \tag{4.4}$$

and

$$\delta \underline{X}_i = \sum_{j=1}^{i} F_u(j, i)\hat{\underline{f}}_{j-1,u}\delta\hat{u}_{j-1} + F_u(0, i)\delta\underline{X}_0. \tag{4.5}$$

We will use the notation $\hat{\underline{f}}_i = \underline{f}(\hat{X}_i, \hat{u}_i(\hat{X}_i))$, and so forth. If arguments of a function are other than $\hat{X}_i$, $\hat{u}_i(\hat{X}_i)$, or $\hat{\underline{X}}_i$, they will be explicitly inserted.

THEOREM 4.1. *Let Assumptions* 4.1, 4.2, 4.3, and 3.4, 3.5, 3.6, 3.7 *hold. Define* $\underline{p}_k$ *by* (3.20) *and* $\underline{p}_i$, $i < k$, *by*

$$\begin{aligned} \underline{p}_{i-1} = (I + \hat{\underline{f}}'_{i-1,\underline{x}} + \hat{\underline{u}}'_{i-1,\underline{x}}\hat{\underline{f}}'_{i-1,u})\underline{p}_i + [\underline{q}'_{i-1,\underline{x}} + (E\hat{\underline{q}}'_{i-1,\underline{e}})]\psi_{i-1} \\ + [\hat{\underline{r}}'_{i-1,\underline{x}} + (E\hat{\underline{r}}_{i-1,\underline{e}})]\alpha_{i-1}. \end{aligned} \tag{4.6}$$

Then (4.7) *and* (4.8), *the analogs of* (3.21) *and* (3.22), *hold, for all* $\delta\hat{u}_{i-1} \in \overline{IC}_{i-1}$,

$$E[\underline{p}_0|\mathscr{B}_0] = E\underline{p}_0 = 0, \tag{4.7}$$

$$E[\underline{p}'_i\hat{\underline{f}}'_{i-1,u}|\hat{X}_{i-1}]\delta\hat{u}_{i-1} \leqq 0. \tag{4.8}$$

PROOF. First we verify that Assumption 3.3 holds. By Assumption 4.2,

$$|X_{i+1}| \leqq K(1 + |\xi_i|)(|X_i| + |u_i(X_i)|) \tag{4.9}$$

and, since $|u_i(x)| \leqq K(1 + |x|)$, all moments of X_i exist; similarly, so do all moments of δX_i, where δX_i is given by (4.2) for $\delta u_i \in IC_i$ and δX_0 is an arbitrary n vector.

Next, fix both $\varepsilon > 0$ and the δu_i^s, and write

$$X_{i+1}(\beta) = X_i(\beta) + f_i[X_i(\beta), \hat{u}_i(X_i(\beta)) + \varepsilon\delta u_i(\beta, X_i(\beta))]. \tag{4.10}$$

Using the Lipschitz conditions on f_i, namely,

$$|f_i(a, b, \xi) - f_i(\tilde{a}, \tilde{b}, \xi)| \leqq K(1 + |\xi|)(|a - \tilde{a}| + |b - \tilde{b}|), \tag{4.11}$$

and the bounds $(|\beta - \tilde{\beta}| = \Sigma_s|\beta_s - \tilde{\beta}_s|)$,

$$\begin{aligned} |\delta X_0(\beta) - \delta X_0(\tilde{\beta})| &\leqq K|\beta - \tilde{\beta}|, \\ |\delta u_i(\beta, x) - \delta u_i(\tilde{\beta}, \tilde{x})| &\leqq \sum_s |\beta_s\delta u_i^s(x) - \tilde{\beta}_s\delta u_i^s(x)| \\ &\leqq \sum_s \{|\beta_s - \tilde{\beta}_s|\cdot|\delta u_i^s(x)| + |\delta u_i^s(x) - \delta u_i^s(\tilde{x})|\tilde{\beta}_s\}, \\ |\delta u_i^s(x) - \delta u_i^s(\tilde{x})| &\leqq K|x - \tilde{x}|, \end{aligned} \tag{4.12}$$

we have that $\|X_i(\beta) - X_i(\tilde{\beta})\|_p \to 0$ as $|\beta - \tilde{\beta}| \to 0$ for any $p \geqq 1$, and any $\varepsilon > 0$. Thus, the $X_i(\beta)$ given by (4.10) are continuous in β in the $L_p(\mathscr{B}_i)$ sense.

Write

$$Y_{i+1} = Y_i + f_i[\hat{X}_i + Y_i, \hat{u}_i(\hat{X}_i + Y_i) + \varepsilon\delta u_i(\beta, \hat{X}_i + Y_i)] - \hat{f}_i \tag{4.13}$$

(see (3.31)). Again, using the bounds on $f_{i,x}, f_{i,u}$ and $\hat{u}_{i,x}$, (for example, $|f_{i,x}(x, u)| \leqq K(1 + |\xi_i|)(|x| + |u|)$) and the bound on $\hat{u}_{i,x}$ and $\delta u_{i,x}$, it is straightforward to show that $\|Y_i\|_p = O(\varepsilon)$ for any $p \geqq 1$.

Next, defining $Z_i = Y_i - \varepsilon\delta X_i(\beta)$, as in Theorem 3.2, we can show that $\|Z_i\|_p = o(\varepsilon)$ uniformly in β. Thus, Assumption 3.3 holds.

Next, we show that $X_i^0(\beta)$ is continuous in β in the $L_1(\mathscr{B}_i)$ sense for any $\varepsilon > 0$. This follows from (4.14) by an application of Assumption 4.3, Hölder's inequality, the Lipschitz conditions on $\hat{u}(x)$ and $\delta u(\beta, x)$, and the continuity of $X_i(\beta)$ in β in the $L_p(\mathscr{B}_i)$ sense. We have

$$\begin{aligned}(4.14)\quad & f_i^0[X_i(\beta), \hat{u}_i(X_i(\beta)) + \varepsilon\delta u_i(\beta, X_i(\beta))] \\ &\qquad - f_i^0[\tilde{X}_i(\tilde{\beta}), \hat{u}_i(\tilde{X}_i(\tilde{\beta})) + \varepsilon\delta u_i(\tilde{\beta}, \tilde{X}_i(\tilde{\beta}))] \\ &= f_{i,x}^0(\alpha_1, \alpha_2)(\tilde{X}_i(\tilde{\beta}) - X_i(\beta)) \\ &\quad + f_{i,u}^0(\alpha_1, \alpha_2)[\hat{u}_i(\tilde{X}_i(\tilde{\beta})) - \hat{u}_i(X_i(\beta)) \\ &\qquad + \varepsilon\, \delta u_i(\tilde{\beta}, \tilde{X}_i(\tilde{\beta})) - \varepsilon\delta u_i(\beta, X_i(\beta))],\end{aligned}$$

where, for some random θ_i with values in $[0, 1]$,

$$\begin{aligned}&\alpha_1 = X_i(\beta) + \theta_i(\tilde{X}_i(\tilde{\beta}) - X_i(\beta)),\\ (4.15)\quad &\alpha_2 = \hat{u}_i(X_i(\beta)) + \varepsilon\delta u_i(\beta, X_i(\beta))\\ &\quad + \theta_i[\hat{u}_i(\tilde{X}_i(\tilde{\beta})) - \hat{u}_i(X_i(\beta)) + \varepsilon\delta u_i(\tilde{\beta}, \tilde{X}_i(\tilde{\beta})) - \varepsilon\delta u_i(\beta, X_i(\beta))].\end{aligned}$$

We will not complete the details (which are quite straightforward), but it can be shown that $\|Z_i^0 - Y_i^0\|_1 = o(\varepsilon)$. Thus, the set $\{\delta\underline{X}_0, \cdots, \delta\underline{X}_k\}$ given by (4.2) is a first order convex approximation K to $Q' - \{\hat{\underline{X}}_0, \cdots, \hat{\underline{X}}_k\}$.

Now, (3.19) holds for $(\delta\underline{X}_0, \cdots, \delta\underline{X}_k)$ in $\bar{K}$, the closure of K in $\mathscr{T}$. By specializing (3.19), we get (4.7) and $E\underline{p}_i' \underline{f}_{i-1,u}' \delta\hat{u}_{i-1} \leqq 0$ for $\delta\hat{u}_{i-1} \in IC_{i-1}$. But $\bar{K}$ contains those $(\delta\underline{X}_0, \cdots, \delta\underline{X}_k)$ which can be obtained by using the $\delta u_i(\cdot)$ in the $L_p(\mathscr{B}_i)$ closure $\overline{IC}_i$ of IC_i and $\overline{IC}_i$ contains pointwise limits of uniformly bounded sequences in $\overline{IC}_i$. Thus, if $\chi_A(\cdot)$ is the characteristic function of an n dimensional Borel set A and $\delta u_i(\cdot) \in IC_i$, then $\chi_A(\cdot)\delta u_i(\cdot) \equiv \delta\tilde{u}_i(\cdot) \in \overline{IC}_i$. Equation (4.8) is obtained by combining the last statement together with the argument which led from (3.24) to (3.21). *Q.E.D.*

4.2. *Extensions.* Let $y_i(\cdot)$ be a continuous vector valued function with uniformly bounded and continuous derivatives. Let u_i depend on $y_i(X_i)$, rather than on X_i directly. Then Theorem 4.1 remains true if the $\hat{u}_{i,\underline{x}}$ term in (4.6) is replaced by $\hat{u}_{i,y} \cdot \hat{y}_{i,\underline{x}}$, the conditioning in (4.8) is on $y_i(\hat{X}_i)$, and the $\delta u_i(\cdot)$ are functions of $y_i(X_i)$.

If the control has the form $u_i[y_i(X_i, X_{i-1}, \cdots, X_0)]$, it is still possible to derive a multiplier result, but the expressions are considerably more complicated, since $\delta\underline{X}_i$ may depend explicitly on $\delta\underline{X}_{i-1}, \cdots, \delta\underline{X}_0$.

The controls and initial condition can be randomized in the following way. Let $\tilde{v}_0, v_0, \cdots, v_{k-1}$ be independent random variables with values in $[0, 1]$ and which are independent of the $\{\xi_i\}$ sequence. Let $\mathscr{B}_0 = \mathscr{B}(\tilde{v}_0)$. In addition to the conditions in Theorem 4.1, let u_i depend on X_i and v_i. Suppose that $u_i(x, v_i)$ is differentiable in x and measurable in both variables, and that $u_{i,x}(x, v)$ is bounded and continuous, uniformly in v in $[0, 1]$. Also $u_i(x, v) \in U_i$, a convex set. Then Theorem 4.1 remains true if the conditioning on $\hat{X}_{i-1}$ in (4.8) is replaced by conditioning on $\hat{X}_{i-1}$ and v_{i-1}.

5. A stochastic maximum principle

For the *continuous time deterministic problem*, where $\dot{\underline{x}} = \underline{f}(x, u)$ and $\underline{p}_t$ denotes the adjoint vector, relation (3.24) is $\underline{p}_t' \underline{f}(\hat{x}_t, u_t) \leqq \underline{p}_t' \underline{f}(\hat{x}_t, \hat{u}_t)$ for all $u_t \in U_t$ or, equivalently, $\hat{u}_t$ is the u which maximizes $\underline{p}_t' f(\hat{x}_t, u)$. Under a convexity condition, Halkin [6] and Holtzman [7] have proved a similar relation for the discrete time deterministic case. The stochastic analogy of this result is straightforward to derive, and we closely follow the treatment in Canon, Cullum, and Polak ([4], pp. 84–93).

For the sake of concreteness, we treat essentially the analog of Theorem 3.1, with a more specific form of Assumption 3.3, although generalizations are possible.

DEFINITION 5.1. *With the $\tilde{U}_i$ defined in Section* 3, *and system* (1.1′) *with constraints* (1.3), (1.4), *the control problem is directionally convex if, for each* $0 \leqq \lambda \leqq 1$ *and* u_i', u_i'' *in* $\tilde{U}_i$, *there is a* $u_i(\lambda) \in \tilde{U}_i$ *so that, with probability* 1, *for each* $X_i \in L_p(\mathscr{B}_i)$,

$$\begin{aligned} \lambda f_i(X_i, u_i', \xi_i) + (1 - \lambda) f_i(X_i, u_i'', \xi_i) &= f_i\big(X_i, u_i(\lambda), \xi_i\big), \\ \lambda f_i^0(X_i, u_i') + (1 - \lambda) f_i^0(X_i, u_i'') &\geqq f_i^0\big(X_i, u_i(\lambda)\big). \end{aligned} \tag{5.1}$$

EXAMPLE 5.1. A common and important example of a directionally convex problem is

$$\begin{aligned} f_i(x, u, \xi) &= g_i(x, \xi) + k_i(x, \xi) u, \\ f_i^0(x, u) &= g_i^0(x) + u' Q u, \end{aligned} \tag{5.2}$$

where Q is nonnegative definite. Then $u_i(\lambda) = \lambda u_i' + (1 - \lambda) u_i''$.

5.1. *A comment on Theorem* 2.1. Using the notation of Section 2, let B_i denote the set $\{w: \varphi_i(\hat{w} + w) < \varphi_i(\hat{w})\} \cup \{0\}$, and let Z_i denote a nonempty internal cone to B_i. Define

$$Z' = \Big[\bigcap_{i>0} \{w: \ell_i(w) = 0\}\Big] \bigcap_{i \in J} Z_i. \tag{5.3}$$

and assume that it contains a point other than $\{0\}$. Theorem 2.1 is a consequence of the fact that, if $\hat{w}$ is optimal, then Z' and K (a first order convex approximation to $Q = Q' - \hat{w}$) can be separated by a continuous linear functional. (See Theorems 2.1 and 4.2 in [11].) Indeed, the proofs of Theorems 2.1 and 4.2 in [11] imply that if Theorem 2.1 does not hold at a given $\hat{w}$, (namely, if there is a ray which is internal to both K and Z'), then for any neighborhood N of $\{0\}$ in $\mathscr{T}$, there is a $\tilde{w} \in Q' \cap \{N + \hat{w}\}$ which satisfies the constraints for which $\varphi_0(\tilde{w}) < \varphi_0(\hat{w})$. Thus, if Theorem 2.1 does not hold at $\hat{w}$, then $\hat{w}$ is *not* an optimal solution.

5.2. *A transformation of the control problem.* The stochastic optimization problem of Section 3 is equivalent to the following problem. Find the $\underline{X}_i$, $\underline{v}_i$ satisfying $\underline{v}_i \in \underline{f}_i(X_i, \tilde{U}_i, \xi_i)$ and $\underline{X}_{i+1} = \underline{X}_i + \underline{v}_i$, for which $r_0(X_0) = r_k(X_k) = 0$, $q_0(\underline{X}_0) \leqq 0$, $q_i(X_i) \leqq 0$, $i > 0$, and for which $E\,\Sigma_{i=0}^{k-1} v_i^0$ is a minimum. Denote the optimizing variables by $\hat{\underline{X}}_0, \cdots, \hat{\underline{X}}_k, \hat{\underline{v}}_0, \cdots, \hat{\underline{v}}_{k-1}$.

Since the variables to be chosen are now $\underline{X}_0, \cdots, \underline{X}_k, \underline{v}_0, \cdots, \underline{v}_{k-1}$, with both $\underline{X}_i$ and $\underline{v}_i$ in $\underline{L}_p(\mathscr{B}_i)$, redefine $\mathscr{T}$ to be

$$\mathscr{T} = \underline{L}_p(\mathscr{B}_0) \times \cdots \times \underline{L}_p(\mathscr{B}_k) \times \underline{L}_p(\mathscr{B}_1) \times \cdots \times \underline{L}_p(\mathscr{B}_{k-1}). \tag{5.4}$$

Let the problem be directionally convex, and define

$$\tilde{Q}' = \{\underline{X}_0, \cdots, \underline{X}_k, \underline{v}_0, \cdots, \underline{v}_{k-1} : v_i \in \operatorname{co} \underline{f}_i(X_i, \tilde{U}_i, \xi_i), \underline{X}_{i+1} = \underline{X}_i + \underline{v}_i\}, \tag{5.5}$$

where co S is the convex hull of the set S. Namely, $\operatorname{co} \underline{f}_i(X_i, \tilde{U}_i, \xi_i)$ is the convex hull of the set of random variables $\{\underline{f}_i(X_i, u_i, \xi_i), u_i \in \tilde{U}_i\}$. Let $\tilde{K}$ denote a first order convex approximation to

$$\tilde{Q}' - \{\hat{\underline{X}}_0, \cdots, \hat{\underline{X}}_k, \hat{\underline{v}}_0, \cdots, \hat{\underline{v}}_{k-1}\} = \tilde{Q}' - \hat{w} \equiv \tilde{Q}. \tag{5.6}$$

Suppose that the inequality in Theorem 2.1 does not hold for some suitable set of constants where $\tilde{K}$ replaces K (using the identification of terms and boundedness and continuity conditions in Section 3). Then the comment of the last subsection implies that there is a ray which is internal to both Z' and $\tilde{K}$, a neighborhood N of $\hat{w}$, and a $\tilde{w} = \{\tilde{\underline{X}}_0, \cdots, \tilde{\underline{X}}_k, \tilde{\underline{v}}_0, \cdots, \tilde{\underline{v}}_{k-1}\} \in \tilde{Q} \cap \{N + \hat{w}\}$ for which the constraints hold and

$$\begin{aligned} \varphi_0(\tilde{w}) &= E \sum_{i=0}^{k-1} \tilde{v}_i^0 < E \sum_{i=0}^{k-1} \hat{v}_i^0 = \varphi_0(\hat{w}), \\ \tilde{\underline{X}}_{i+1} &= \tilde{\underline{X}}_i + \tilde{\underline{v}}_i. \end{aligned} \tag{5.7}$$

There are $u_i^s \in \tilde{U}_i$, $\lambda_i^s \geqq 0$, and $\Sigma_s\, \lambda_i^s = 1$ so that

$$\begin{aligned} \tilde{v}_i^0 &= \sum_s \lambda_i^s f_i^0(\tilde{X}_i, u_i^s), \\ \tilde{v}_i &= \sum_s \lambda_i^s f_i(\tilde{X}_i, u_i^s, \xi_i). \end{aligned} \tag{5.8}$$

By directional convexity, there is a $\tilde{u}_i \in \tilde{U}_i$ for which

$$\begin{aligned} \tilde{v}_i &= f_i(\tilde{X}_i, \tilde{u}_i, \xi_i), \\ \tilde{v}_i^0 &\leqq f_i^0(\tilde{X}_i, \tilde{u}_i). \end{aligned} \tag{5.9}$$

Thus, by combining (5.8) and (5.9), one gets $\tilde{X}_{i+1} = \tilde{X}_i + f_i(\tilde{X}_i, \tilde{u}_i, \xi_i)$ and

$$E \sum_{i=0}^{k-1} f_i^0(\tilde{X}_i, \tilde{u}_i) < E \sum_{i=0}^{k-1} f_i^0(\hat{X}_i, \hat{u}_i), \tag{5.10}$$

which contradicts the optimality of $\{\hat{\underline{X}}_i, \hat{u}_i\}$. Thus, the inequality in Theorem 2.1 holds for $\tilde{K}$ replacing K. Also, (3.19) holds for all $\delta\underline{X}_i$ for which $\{\delta\underline{X}_0, \cdots, \delta\underline{X}_k, \delta\underline{v}_0, \cdots, \delta\underline{v}_{k-1}\} \in \tilde{K}$.

Define the set $\tilde{K} \in \mathscr{T}$:

$$\begin{aligned}\tilde{K} = \{\delta\underline{X}_0, \cdots, \delta\underline{X}_k, \delta\underline{v}_0, \cdots, \delta\underline{v}_{k-1} : \delta\underline{X}_{i+1} = \delta\underline{X}_i + \delta\underline{v}_i, \quad \text{such that} \\ \lambda[\delta\underline{v}_i - \hat{\underline{f}}_{i,\underline{x}} \cdot \delta\underline{X}_i] \in \text{co}\, \underline{f}_i(\hat{X}_i, \tilde{U}_i, \xi_i) - \hat{\underline{v}}_i, \delta\underline{X}_0 \in L_p(\mathscr{B}_0)\}\end{aligned} \tag{5.11}$$

for sufficiently small λ. Theorem 5.1 gives conditions under which $\tilde{K}$ is a first order convex approximation to $\tilde{Q}$.

Let

$$\lambda[\delta\underline{v}_i^s - \hat{\underline{f}}_{i,\underline{x}} \cdot \delta\underline{X}_i^s] \in \text{co}\, \underline{f}_i(\hat{X}_i, \tilde{U}_i, \xi_i) - \hat{\underline{v}}_i \tag{5.12}$$

for $s = 1, \cdots, \nu$, and all sufficiently small λ. The elements ($\lambda_i^\ell \geqq 0$, $\Sigma_\ell \lambda_i^\ell = 1$)

$$\begin{aligned}\delta\underline{X}_{i+1}^s &= \delta\underline{X}_i^s + \hat{\underline{f}}_{i,\underline{x}} \cdot \delta\underline{X}_i^s + [\sum_\ell \lambda_i^\ell \underline{f}_i(\hat{X}_i, u_i^{\ell,s}, \xi_i) - \hat{\underline{v}}_i] \\ &\equiv \delta\underline{X}_i^s + \delta\underline{v}_i^s,\end{aligned} \tag{5.13}$$

and $\delta\underline{v}_i^s$ and their convex combinations for $\beta_s \geqq 0$, $\Sigma_s \beta_s = 1$, namely,

$$\begin{aligned}\delta\underline{X}_{i+1}(\beta) &= \delta\underline{X}_i(\beta) + \hat{\underline{f}}_{i,\underline{x}} \cdot \delta\underline{X}_i(\beta) + \sum_s \beta_s[\sum_\ell \lambda_i^\ell \underline{f}_i(\hat{X}_i, u_i^{\ell,s}, \xi_i) - \hat{\underline{v}}_i] \\ &\equiv \delta\underline{X}_i(\beta) + \delta\underline{v}_i(\beta),\end{aligned} \tag{5.14}$$

and $\delta\underline{v}_i(\beta) = \Sigma_s \beta_s \delta\underline{v}_i^s$ are in $\tilde{K}$. We may write

$$\begin{aligned}\delta\underline{X}_{i+1}(\beta) &= [I + \hat{\underline{f}}_{i,\underline{x}}]\delta\underline{X}_i(\beta) + \delta\underline{W}_i(\beta), \\ \delta\underline{W}_i(\beta) &= \sum_s \beta_s[\sum_\ell \lambda_i^\ell \underline{f}_i(\hat{X}_i, u_i^{\ell,s}, \xi_i) - \hat{\underline{v}}_i], \\ \delta\hat{X}_i(\beta) &= \sum_{j=1}^{i} F(j, i)\delta\underline{W}_{j=1}(\beta) + F(0, i)\delta\underline{X}_0(\beta).\end{aligned} \tag{5.15}$$

THEOREM 5.1. *Let Assumptions* 3.4 *through* 3.7 *hold and assume that the control problem is directionally convex. Also make the following assumptions:*

(i) $\tilde{U}_i$ *is the convex set of functions in* $L_p(\tilde{\mathscr{B}}_i)$ *with values in the convex set* U_i; IC_i *contains some point other than zero;*

(ii) *the* $\{\xi_i\}$ *are mutually independent and all of their moments are finite;*

(iii) $|f_i(x, u, \xi)| \leqq K(1 + |\xi|)(1 + |u| + |x|)$ *and* $|f_i^0(x, u)| \leqq K(1 + |u|^{p'} + |x|^p)$ *for a real* K;

(iv) $|f_i(x, u, \xi) - f_i(\tilde{x}, u, \xi)| \leqq K(1 + |\xi|)(|x - \tilde{x}|)$ *and* $f_i^0(X_i, u_i)$ *is continuous in* X_i *in the* $\|\cdot\|_p$ *norm for any* u_i *in* $L_{p'}(\tilde{\mathscr{B}}_i)$;

(v) *$f_{i,x}(x, u)$ is uniformly bounded and is continuous in x for each vector u and $f^0_{i,x}(x, u)$ is continuous in x in the $\|\cdot\|_{p/(p-1)}$ norm for each fixed u in $L_{p'}(\tilde{\mathscr{B}}_i)$.*

Then, for $\underline{p}_k$, $\underline{p}_i$ given by (3.20), *equation* (3.22) *holds and* (3.21) *is replaced by the maximum principle.*

$$E[\underline{p}'_{i+1}\underline{f}_i(\hat{X}_i, u_i, \xi_i) \mid \tilde{\mathscr{B}}_i] \leqq E[\underline{p}'_{i+1}\underline{f}_i(\hat{X}_i, \hat{u}_i, \xi_i) \mid \tilde{\mathscr{B}}_i] \tag{5.16}$$

with probability 1 *for any u_i in $\tilde{U}_i$.*

PROOF. Suppose that $\tilde{K}$ is a first order convex approximation to $\tilde{Q}$. By the discussion prior to the theorem, equation (3.19) must hold for all $\delta\underline{X}_i$ of the form (5.15). Setting $u_i^{\ell,s} = 0$ and $\delta\underline{X}_0 \neq 0$, we get (3.22) as in Theorem 3.1. Equation (5.12) follows by letting $u_j^{\ell,s} = \hat{u}_j, j \neq i$, $\delta\underline{X}_0 = 0$ and $u_i^{\ell,s} = u_i \neq \hat{u}_i$, substituting (5.15) into (3.19), and using the definitions of $\hat{\underline{v}}_i$ and $\underline{p}_j$. We have only to show that $\tilde{K}$ is a first order convex approximation to $\tilde{Q}$.

Clearly, $\tilde{K}$ is a convex cone, with typical elements $\{\delta\underline{X}_0^s, \cdots, \delta\underline{X}_k^s, \delta\underline{v}_0^s, \cdots, \delta\underline{v}_{k-1}^s\}$, and their convex combinations $\{\delta\underline{X}_0(\beta), \cdots, \delta\underline{X}_k(\beta), \delta\underline{v}_0(\beta), \cdots, \delta\underline{v}_{k-1}(\beta)\}$ are given by (5.14). Consider the mapping $\{\underline{X}_0(\beta), \cdots, \underline{X}_k(\beta), \underline{v}_0(\beta), \cdots, \underline{v}_{k-1}(\beta)\}$ from P^ν to $\mathscr{T}$, for the fixed sequence of controls $\{u_i^{\ell,s}\}$:

$$\begin{aligned} \underline{X}_{i+1}(\beta) &= \underline{X}_i(\beta) + \underline{v}_i(\beta), \\ \underline{v}_i(\beta) &= \underline{f}_i(X_i(\beta), \hat{u}_i, \xi_i) \\ &\quad + \varepsilon \sum_s \beta_s\Big[\sum_\ell \lambda_i^\ell \underline{f}_i(X_i(\beta), u_i^{\ell,s}, \xi_i) - \underline{f}_i(X_i(\beta), \hat{u}_i, \xi_i)\Big] \\ \underline{X}_0(\beta) &= \hat{\underline{X}}_0 + \varepsilon\delta\underline{X}_0(\beta), \end{aligned} \tag{5.17}$$

where $\lambda_i^\ell \geqq 0$ and $\Sigma_\ell \lambda_i^\ell = 1$. Under (iv) of the theorem, the maps $\underline{X}_i(\beta)$ to $\underline{L}_p(\mathscr{B}_i)$ and $\underline{v}_i(\beta)$ to $\underline{L}_p(\mathscr{B}_{i+1})$ are continuous functions of β, for $\beta \in P^\nu$, and any $1 > \varepsilon > 0$. Thus, the composite map (taking $\{\underline{X}_0(\beta), \cdots, \underline{X}_k(\beta), \underline{v}_0(\beta), \cdots, \underline{v}_{k-1}(\beta)\}$ into $\mathscr{T}$) is a continuous $\mathscr{T}$ valued function of β.

Using (v) it can be shown that

$$\begin{aligned} \hat{X}_i(\beta) &= \hat{X}_i + \varepsilon\delta\hat{X}_i(\beta) + O_{1,i} \\ \underline{v}_i(\beta) &= \hat{\underline{v}}_i + \varepsilon\delta\underline{v}_i(\beta) + O_{2,i} \end{aligned} \tag{5.18}$$

where $O_{1,i}$ and $O_{2,i}$ are of the order of $o(\varepsilon)$ in $\underline{L}_p(\mathscr{B}_i)$ and $\underline{L}_p(\mathscr{B}_{i+1})$, respectively. Then, $\tilde{K}$ is indeed a first order convex approximation. The details of the last two steps involve straightforward expansions and estimates, as in Theorems 3.1, 3.2, and 4.1, and are omitted. They are probabilistic versions of the cited result ([4], pp. 84–93). *Q.E.D.*

The definition of a directionally convex problem holds if the control u_i depends on a function of the state X_i. Under directional convexity and the conditions of Theorem 4.1, Theorem 4.1 holds with equation (4.8) replaced by

$$E[\underline{p}'_{i+1}\underline{f}_i(\hat{X}_i, u_i, \xi_i) \mid \hat{X}_i] \leqq E[\underline{p}'_{i+1}\underline{f}_i(\hat{X}_i, \hat{u}_i, \xi_i) \mid \hat{X}_i]. \tag{5.19}$$

6. A relation with dynamic programming

For simplicity of presentation, this section will be largely formal. Suppose that the problem is directionally convex, and there are no constraints r_i and q_i. Let u_i depend on X_i and define the (dynamic programming) costs

$$V_i(\underset{\sim}{x}) = \inf_{u_i,\cdots,u_{k-1}} E[X_k^0 | \underset{\sim}{X}_i = \underset{\sim}{x}] = E[\hat{X}_k^0 | \underset{\sim}{X}_i = \underset{\sim}{x}],$$
$$\tilde{V}_i(x) = V_i(\underset{\sim}{x}) - x^0. \tag{6.1}$$

Define

$$W_i(\hat{X}_i; \xi_i, \cdots, \xi_{k-1}) = \hat{X}_k^0 - \hat{X}_i^0 = \sum_{j=i}^{k-1} f_i^0(\hat{X}_i, \hat{u}_i). \tag{6.2}$$

Then drop some arguments for notational simplicity and write

$$\operatorname{grad} W_i = W_{i,x} = \operatorname{grad} W_i(X_i; \xi_i, \cdots, \xi_{k-1}) \tag{6.3}$$

evaluated at $x = \hat{X}_i$; similarly, for $V_{i,\underset{\sim}{x}}$. Then $\operatorname{grad} W_k = W_{k,x} = 0$ and

$$W_{i,x} = (I + \hat{f}'_{i,x} + \hat{f}'_{i,u}\hat{u}_{i,x})W_{i+1,x} + \hat{f}^0_{i,x}. \tag{6.4}$$

Thus,

$$W_{i,x} = -p_i, \qquad (W_{i,x}, 1) = -\underset{\sim}{p}_i, \tag{6.5}$$

$$\tilde{V}_i(x) = E[W_i | \hat{X}_i = x], \tag{6.6}$$

and

$$\tilde{V}_{i,x}(x) = E(I + \hat{f}'_{i,x})\tilde{V}_{i+1,x} + E\hat{f}^0_{i,x}. \tag{6.7}$$

We must have $p^0 < 0$, since there are no constraints r_i, q_i, and not all the p^0, α_i, ψ_i can be zero. Thus, we set $p^0 = -1$.

By the principle of optimality, $EV_{i+1}(\underset{\sim}{x} + \underset{\sim}{f}_i(x, \hat{u}_i, \xi_i)) \leqq EV_{i+1}(\underset{\sim}{x} + \underset{\sim}{f}_i(x, u_\varepsilon, \xi_i))$, where u_ε is the control which, for given $u_i \neq \hat{u}_i$, satisfies

$$(1 - \varepsilon)f_i(x, \hat{u}_i, \xi_i) + \varepsilon f_i(x, u_i, \xi_i) = f_i(x, u_\varepsilon, \xi_i),$$
$$(1 - \varepsilon)f_i^0(x, \hat{u}_i) + \varepsilon f_i^0(x, u_i) \geqq f_i^0(x, u_\varepsilon). \tag{6.8}$$

Noting that $V_{i+1}(\tilde{\underset{\sim}{x}}) \leqq V_{i+1}(\underset{\sim}{x})$ if $\tilde{x} = x$, and $\tilde{x}^0 \leqq x^0$, we get

$$EV_{i+1}(\underset{\sim}{x} + \underset{\sim}{f}_i(x, \hat{u}_i, \xi_i)) \leqq EV_{i+1}(\underset{\sim}{x} + \underset{\sim}{f}_i(x, u_\varepsilon, \xi_i))$$
$$\leqq EV_{i+1}(\underset{\sim}{x} + (1 - \varepsilon)\underset{\sim}{f}_i(x, \hat{u}, \xi_i) + \varepsilon \underset{\sim}{f}_i(x, u_i, \xi_i)). \tag{6.9}$$

Thus,

$$0 \leqq EV'_{i+1,\underset{\sim}{x}}(\underset{\sim}{x} + \underset{\sim}{f}_i(x, \hat{u}_i, \xi_i))[\underset{\sim}{f}_i(x, u_i, \xi_i) - \underset{\sim}{f}_i(x, \hat{u}_i, \xi_i)], \tag{6.10}$$

where $V_{i+1,\underset{\sim}{x}} = \operatorname{grad} V_{i+1}(\underset{\sim}{x})$, evaluated at $\underset{\sim}{x} + \underset{\sim}{f}_i(x, \hat{u}_i, \xi_i)$. With the identification (6.5) and $\tilde{V}_{i+1,x}(\hat{X}_{i+1}) = E[W_{i+1,x} | \hat{X}_{i+1}]$, we get precisely the maximum principle

$$E[\underset{\sim}{p}'_{i+1}(\underset{\sim}{f}_i(\hat{X}_i, \hat{u}_i, \xi_i) - \underset{\sim}{f}_i(\hat{X}_i, u_i, \xi_i)) | \hat{X}_i] \geqq 0. \tag{6.11}$$

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