COVERAGE OF GENERALIZED CHESS BOARDS BY RANDOMLY PLACED ROOKS

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1. Introduction

At a recent colloquium on combinatorial structures, H. Kamps and J. van Lint presented a paper [2] on the minimal number of rooks $\sigma(n, k)$ required to "cover" a generalized chessboard; the latter is represented by R_k^n, the set of n vectors (or cells) with components in the ring of integers mod k. To explain the notion of "cover" we first define the Hamming distance $d_H(\mathbf{x}, \mathbf{y})$ between two vectors ("squares" of the chessboard) as the number of components in which they differ; under the metric d_H, the board R_k^n is a metric space. The familiar chessboard is R_8^2. Then the rook domain or region *covered* by a rook at x is the unit sphere

$$B(x, 1) = \{y \in R_k^n | d_H(\mathbf{x}, \mathbf{y}) \leqq 1\}. \tag{1.1}$$

Kamps and van Lint gave the following table of $\sigma(n, k)$ which represents almost all the known results to date for the above deterministic problem.

TABLE I

KNOWN VALUES OF $\sigma(n, k)$

$k \backslash n$	3	4	5	6	7	8	$\cdots$	13
2	2	4	7	12	16	2^5		
3	5	9	3^3					3^{10}
4	8	24	4^3					
5	13			5^4				
6	18	72						
7	25					7^6		

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The only general results known (see their references) are

$$\sigma(2, k) = k, \tag{1.2}$$

$$\sigma(3, k) = [(k^2 + 1)/2], \tag{1.3}$$

where $[x]$ = integer part of x, and

$$\sigma(n, k) = \frac{k^n}{1 + n(k - 1)}, \tag{1.4}$$

for $n > 3$, provided

(a) the right side of (1.4) is an integer and

(b) the integer k is the power of a prime.

For example, from (1.4), $\sigma(4, 3) = 9$ and from (1.3), we have $\sigma(3, 3) = 5$. Many values of $\sigma(n, k)$ were computed by R. Stanton [4], Stanton and J. Kalbfleisch [5], [6], and others.

We consider two stochastic versions of the rook coverage problem. Rooks are placed in cells (vectors) sequentially and independently with uniform probabilities. We consider the distribution (in particular, the expectation) of the number of rooks Y required to cover R_k^n for the first time. In the multinomial case (case M), the cells have constant probability k^{-n} and repetition of occupancy is permitted. In the hypergeometric case (case H) each successive occupancy is permitted only in one of the currently unoccupied cells, with uniform probability over these cells.

By introducing the stochastic version of the problem, we feel that the problem has been broadened in an interesting and nontrivial manner. Indeed, although the deterministic problem is trivial for $n = 2$, the corresponding stochastic problem is by no means trivial. Moreover, it is hoped that the more general approach used in the stochastic version would lead to further extensions in the deterministic version, especially in the case of higher dimensions.

2. Exact solution for the multinomial case with $n = 2$

Consider a two dimensional $k \times k$ chessboard. For case M, let Y_M denote the random number of rooks required to cover the $k \times k$ board and let y denote values of Y_M. The event "covering a row (column)" is equivalent to "occupying a row (column)."

Coverage of the board R_k^2 is characterized by occupancy either of all the rows *or* of all the columns. We also use the fact that for any given number of rooks N, the number of rows occupied is independent of the number of columns occupied. Finally, occupancy of rows (similarly for columns) is a direct consequence of of the classical Maxwell-Boltzmann statistics (see, for example, p. 59 of Feller [1]). In particular, the probability that all k rows are occupied by x randomly placed rooks is given exactly by

$$F_k(x) = \sum_{\alpha=0}^{k} (-1)^\alpha \binom{k}{\alpha} \left(1 - \frac{\alpha}{k}\right)^x, \tag{2.1}$$

and the same result holds for columns. By virtue of the independence of row and column occupancy, the cumulative distribution function (c.d.f.) $G_k(y)$ of Y_M is given by

$$G_k(y) = 1 - [1 - F_k(y)]^2. \tag{2.2}$$

The corresponding probability law $g_k(y)$ of Y_M is obtained by taking differences in equation (2.2). Expectations are then obtained from $g_k(y)$ or by summing the complement of $G_k(y)$ over $y \geqq 0$: this yields the two equivalent exact expressions

$$\begin{aligned} E\{Y_M\} &= k + \sum_{\beta=k}^{\infty}\left[\sum_{\alpha=1}^{k-1}(-1)^{\alpha}\binom{k}{\alpha}\left(1-\frac{\alpha}{k}\right)^{\beta}\right]^2 \\ E\{Y_M\} &= k + \frac{1}{k^{2(k-1)}}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(-1)^{i+j}\frac{\binom{k}{i}\binom{k}{j}(ij)^k}{k^2-ij}, \end{aligned} \tag{2.3}$$

both of which are useful for computing (see Table II).

3. Exact solution of the hypergeometric case for $n = 2$

Here rooks are placed one at a time, independently, and with uniform probability in the unoccupied cells. This case requires extensive modification of the solution strategy, mainly due to the loss of independence between row occupancies and column occupancies. We employ the method of inclusion-exclusion and Fréchet sums ([1], p. 99) but the basic events have to be defined carefully.

First, we note that the k^2 vector space (chessboard) is *not* covered by y rooks if and only if at least one cell is not covered and this, in turn, holds if and only if at least one row is not occupied *and* at least one column is not occupied. The event that one particular cell is not covered, in positive terms, requires that all y rooks currently placed are in some $(k-1) \times (k-1)$ product subspace defined by the offending cell. Intersections of these subspaces are again product subspaces, which may be indexed by the deleted rows and columns. Thus, we define our basic events $E_{ij}^{(y)}$ as the event (row i and column j are not covered when y rooks are randomly placed). We now proceed to apply the Fréchet sum technique as follows.

In this hypergeometric setup, y rooks can be placed without repetition in $\binom{k^2}{y}$ ways. They can fall in a product subspace avoiding r specified rows and c specified columns in $\binom{(k-r)(k-c)}{y}$ ways and the probability of this event (not necessarily basic) is given by

$$\frac{\binom{(k-r)(k-c)}{y}}{\binom{k^2}{y}}, \qquad y = 0, 1, 2, \cdots. \tag{3.1}$$

Since the r rows and c columns can be specified in $\binom{k}{r}\binom{k}{c}$ ways, the Fréchet sums, for a fixed total $t = r + c, r \geqq 1, c \geqq 1$, of rows and columns not covered, are given by

$$S_t(y) = \sum_{r=1}^{t-1} \frac{\binom{k}{r}\binom{k}{t-r}\binom{(k-r)(k-t+r)}{y}}{\binom{k^2}{y}}. \tag{3.2}$$

According to the discussion above, if a cell is not covered then the sum t of the number of rows and columns not covered is at least 2 and clearly $t \leqq 2k - f(y)$ where $f(y)$ is the minimum total of rows and columns that y rooks can occupy. Hence, the probability of realization of at least one of the basic events is

$$1 - H_k(y) = \sum_{t=2}^{2k-f(y)} (-1)^t S_t(y), \tag{3.3}$$

where $H_k(y)$ is the c.d.f. of the number Y_H of rooks required for coverage in case H.

The expected value of Y_H is obtained by summing (3.3) over $y \geqq 0$. In this sum, the first k terms are all equal to 1. Since $Y_H \leqq 1 + (k-1)^2$, it follows that $1 - H_k(y) = 0$ for $y \geqq 1 + (k-1)^2$, and hence,

$$E\{Y_H\} = k + \sum_{y=k}^{(k-1)^2} \big(1 - H_k(y)\big). \tag{3.4}$$

This completes the exact solution for $E\{Y_H\}$ in case H (see Table II).

4. Asymptotic evaluations

In case M, we have from (2.1) asymptotically ($k \to \infty$)

$$F_k(x) = \sum_{\alpha=0}^{k} (-1)^\alpha \binom{k}{\alpha}\left(1 - \frac{\alpha}{k}\right)^x \sim (1 - e^{-x/k})^k \sim \exp\{-ke^{-x/k}\}. \tag{4.1}$$

Using the normalizing transformation ([1], p. 106),

$$X = k \log k + kZ, \tag{4.2}$$

we obtain for large k the limiting c.d.f. of Z (which takes on values z)

$$V_k(z) = \exp\{-e^{-z}\}, \qquad -\infty < z < \infty, \tag{4.3}$$

the (standardized) extreme value distribution.

In our application, Y_M is the smaller of two independent chance variables each having the same c.d.f. $F_k(x)$ and it follows from (4.2) that for $k \to \infty$,

$$E\{Y_M\} \sim k \log k + kE\{Z_{1:2}\} = k(C + \log k), \tag{4.4}$$

where $Z_{1:2}$ is the smaller of two independent chance variables with c.d.f. $V_k(z)$

in (4.3) and $E\{Z_{1:2}\} = C = -0.1159315$ by the table of J. Lieblein and H. Salzer [3].

In case H we no longer have independence of row and column coverage and have to resort to an "*ad hoc* method" to obtain a useful approximation which is as good as the approximation already obtained for case M. Indeed, one reason for considering the two cases together in the same paper is that we suspected that asymptotically the expectations for case M and case H would be the same to the first order approximation.

We make use of the fact that if we delete repetitions in placing Y_M rooks at random by the multinomial scheme, then the remaining observations Y_H are formally indistinguishable from a hypergeometric sample sequence. The difference $D = Y_M - Y_H$ is the redundancy in the multinomial sampling and our evaluation of $E\{Y_H\}$ arises by using

$$(4.5) \qquad E\{Y_H\} = E\{Y_M\} - E\{D\}.$$

To evaluate $E\{D\}$, we first write $D = \Sigma_{i=1}^k \Sigma_{j=1}^k D_{ij}$, where D_{ij} is the redundancy due to extra rooks placed in the (i, j) cell. The total number of rooks placed in the (i, j) cell under multinomial sampling is approximately binomial with parameters Y_M and $1/k^2$. Our "*ad hoc* method" is to replace Y_M by $E\{Y_M\}$ in evaluating $E\{D_{ij}\}$; we justify this by noting that the error introduced in the last expressions of (4.7) and (4.8) below is of the order of magnitude

$$(4.6) \qquad O\left(\frac{E\{Y_M\}}{k^2}\right) = O\left(\frac{C + \log k}{k}\right) \to 0$$

as $k \to \infty$. We now obtain

$$(4.7) \qquad \begin{aligned} E\{D_{ij}\} &\sim \sum_{\alpha=2}^{E\{Y_M\}} (\alpha - 1)\binom{E\{Y_M\}}{\alpha}\left(\frac{1}{k^2}\right)^\alpha\left(1 - \frac{1}{k^2}\right)^{E\{Y_M\}-\alpha} \\ &= \frac{1}{k^2}E\{Y_M\} - 1 + \left(1 - \frac{1}{k^2}\right)^{k(C+\log k)}. \end{aligned}$$

Using (4.4) for $E\{Y_M\}$ and expanding the last term in (4.7) gives

$$(4.8) \qquad E\{D_{ij}\} \sim \frac{1}{2}\left(\frac{C + \log k}{k}\right)^2 + O\left(\frac{\log^3 k}{k^3}\right)$$

and the error term in (4.8) can also be disregarded. Thus, for the total set of k^2 cells we have from (4.8),

$$(4.9) \qquad E\{D\} \sim \frac{1}{2}(C + \log k)^2 + O\left(\frac{\log^3 k}{k}\right),$$

and hence by (4.5),

$$(4.10) \qquad E\{Y_H\} \sim k(C + \log k) - \frac{1}{2}(C + \log k)^2,$$

where the error, which tends to zero as $k \to \infty$, is now omitted.

TABLE II

EXPECTED VALUE OF THE NUMBER OF RANDOM ROOKS REQUIRED TO COVER THE k^2 CHESSBOARD

k	$E\{Y_M\}$	Approximation to $E\{Y_M\}$ based on (A.9)	$E\{Y_H\}$	Approximation to $E\{Y_H\}$ based on (A.10)
2	2.3333333333	1.5115	2.0000000	1.8619
3	4.1821428571	3.7886	3.5000000	3.6634
4	6,3655677654	6.1561	5.3522478	5.5832
5	8.7938685820	8.6870	7.4723892	7.7268
6	11.4171670989	11.1376	9.8091916	10.0743
7	14.2030879491	14.2070	12.3278253	12.5990
8	17.1286506847	17.1658	15.0029299	15.2784
9	20.1766249904	20.2393	17.8152024	18.0941
10	23.3335906237	23.4163	20.7494692	21.0315
11	26.5887915430	26.6878	23.7935002	24.0784
12	29.9334107812	30.0458	26.9372363	27.2250
13	33.3600877782	33.4837		30.4628
14	36.8625841610	36.9958		33.7848
15	40.4355447768	40.5770		37.1847
16	44,074322209	44.2229		40.6573
17	47.77484495	47.9297		44.1980
18	51.5335164	51.6940		47.8025
19	55.3471359	55.5125		51.4674
20	59.212836	59.3827		55.1892
21	63.12803	63.3019		58.9652
22	67.09038	67.2679		62.7925
23	71.09771	71.2786		66.6689
24	75.1481	75.3321		70.5921
25	79.2396	79.4267		74.5600
26	83.3704	83.5607		78.5710
27	87.539	87.7327		82.6231
28	91.743	91.9413		86.7150
29	95.981	96.1852		90.8450
30	100.250	100.4632		95.0119

Table II gives exact values of $E\{Y_M\}$ for $k = 2(1)30$ using (2.3) and approximate values based on (A.9). It also gives exact values of $E\{Y_H\}$ for $k = 2(1)12$ using (3.4) and approximate values based on (A.10). Roundoff errors in this table are estimated to be at most one in the last digit shown.

5. Coverage of k^n board for $n > 2$

Define a skeletal axis centered at cell C as the n mutually perpendicular lines of cells parallel to the sides of the hypercube and having the cell C in common; for $n = 3$ denote the cell by $C_{\alpha,\beta,\gamma}$, $\alpha, \beta, \gamma = 1, 2, \cdots, k$, and the corresponding skeletal axis by $C^{\alpha,\beta,\gamma}$. For any n, a cell $C_{\alpha,\beta,\gamma}$ is not covered if and only if the skeletal axis $C^{\alpha,\beta,\gamma}$ has no occupancies. Hence, we can use as our basic sets for an inclusion-exclusion argument the sets $C^{\alpha,\beta,\gamma}$, $\alpha, \beta, \gamma = 1, 2, \cdots, k$. However, the intersections of these skeletal axes are not simple and the corresponding analysis is complicated even for $n = 3$. A complete discussion of this analysis will not be considered here. Thus, the stochastic problem becomes more difficult as n increases as it does in the deterministic case of Kamps and van Lint [2]. Theodore Levy, a student of one of the authors at Michigan State University, is working on a class of such problems; the results are not yet very encouraging.

6. Use of independence in higher dimensions

It is of some interest to find a way to generalize the independence of row occupancy and column occupancy that was used above for $n = 2$. For this purpose, we define a piece that starts at a cell C in n dimensions and moves (anywhere) inside any Hamming sphere centered at C and of radius $n - 1$. For $n = 2$, this reduces to the usual rook move. For $n = 3$ and starting at cell C, the piece moves inside the horizontal plane (H plane) through C or inside the north-south plane (NS plane) through C or inside the east-west plane (EW plane) through C. Hence, one such piece covers all the cells in three mutually perpendicular slabs that contain the starting cell..

The cube R_k^3 will be covered as soon as either all L slabs or all NS slabs or all EW slabs are occupied. Hence, the same argument as for $n = 2$ (case M) gives for general n (case M) the exact solution for the c.d.f. of Y_M.

$$G_k(y) = 1 - [1 - F_k(y)]^n, \tag{6.1}$$

where $F_k(y)$ is given by (2.1). For $n = 3$, the expectation becomes

$$\begin{aligned} E\{Y_M\} &= k - \sum_{\beta=k}^{\infty}\left[\sum_{\alpha=1}^{k-1}(-1)^{\alpha}\binom{k}{\alpha}\left(1-\frac{\alpha}{k}\right)^{\beta}\right]^3 \\ &= k - \frac{1}{k^{3(k-1)}}\sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{k-1}\sum_{\gamma=1}^{k-1}(-1)^{k-\alpha-\beta-\gamma}\frac{\binom{k}{\alpha}\binom{k}{\beta}\binom{k}{\gamma}(\alpha\beta\gamma)^k}{k^3-\alpha\beta\gamma}, \end{aligned} \tag{6.2}$$

both of which can be used for computing.

In the corresponding asymptotic ($k \to \infty$) evaluation for $n = 3$, we need the expectation of the smallest of three independent observations on the c.d.f. (4.3); this is given in [3] as -0.4036136. This analysis is easily generalized to any number of dimensions n. This type of solution became possible only after we defined a "super piece" that moved in more than one dimension. No similar analysis was found for the original definition of a rook move in the Hamming sphere of radius 1.

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APPENDIX

A more careful evaluation of $E\{Y_M\}$ starts as in (4.1) but, letting $x' = x - 1$ and $k' = k - 1$ we now write

$$\text{(A.1)}\qquad F_k(x) = \sum_{\alpha=0}^{k} (-1)^\alpha \binom{k}{\alpha}\left(1 - \frac{\alpha}{k}\right)^x = \sum_{\alpha=0}^{k'} (-1)^\alpha \binom{k'}{\alpha}\left(1 - \frac{\alpha}{k}\right)^{x'}.$$

Letting $x'' = x - 2$, we use the approximation

$$\text{(A.2)}\qquad \left(1 - \frac{\alpha}{k}\right)^{x'} \sim \left(\exp\left\{-\frac{\alpha}{k}\right\} - \frac{1}{2}\frac{\alpha^2}{k^2}\right)^{x'}$$
$$\sim \exp\left\{-\frac{\alpha x'}{k}\right\} - \frac{x'}{2}\frac{\alpha^2}{k^2}\exp\left\{-\frac{\alpha x''}{k}\right\}.$$

Substituting this in (A.1) and summing gives

$$\text{(A.3)}\qquad F_k(x) \sim \left(1 - \exp\left\{-\frac{x'}{k}\right\}\right)^{k'}$$
$$- \frac{x'k'(k'-1)}{2k^2}\exp\left\{-\frac{2x''}{k}\right\}\left(1 - \exp\left\{-\frac{x''}{k}\right\}\right)^{k'-2}$$
$$+ \frac{x'k'}{2k^2}\exp\left\{-\frac{x''}{k}\right\}\left(1 - \exp\left\{-\frac{x''}{k}\right\}\right)^{k'-1},$$

which is correct up to terms of order $\log k/k$ and $1/k$ in $F_k(x)$. These in turn yield all the terms in the final answer for $E(Y_M\}$ of order $\log k$ and 1, so that the new error will go to zero as $k \to \infty$.

Letting $X' = k \log k' + kZ$, we obtain for large k

$$\text{(A.4)}\qquad F_k(x) \sim \exp\{-e^{-z}\} + \frac{(z + \log k')}{2k} e^{-z}\exp\{-e^{-z}\}(1 - e^{-z}),$$

where the leading term is the same as in (4.3) and we have dropped terms that approach zero in the final result. Using the same method as in (2.3) above, we replace the sum by an integral and obtain

$$\text{(A.5)}\qquad E\{Y_M\} \sim k + k\int_a^\infty \left[1 - \exp\{-e^{-z}\} - \frac{(z + \log k')}{2k} e^{-z}\exp\{-e^{-z}\}(1 - e^{-z})\right]^2 dz$$
$$\sim k + k\int_a^\infty (1 - \exp\{-e^{-z}\})^2\, dz$$
$$- \int_a^\infty (1 - \exp\{-e^{-z}\})(1 - e^{-z})$$
$$\cdot (z + \log k')\, e^{-z}\exp\{-e^{-z}\}\, dz,$$

where $a = 1 - \log k' - 3/2k$ and we have taken a as the value of z when $x = k - \frac{1}{2}$ to get a valid approximation. Using the Euler-MacLaurin sum formula, it can be shown that this leads a correct asymptotic approximation for $k \to \infty$; we omit this proof.

Let T_1 and T_2 denote the first two terms and the last term in (A.5), respectively. We let $u = e^{-z}$ and let $b = e^{-a} = k'/\exp\{1 - 3/2k\} \sim k'/e$. Using (5.1.1), (5.1.40), and (5.1.51) of [7], we obtain

$$\text{(A.6)} \qquad \begin{aligned} T_1 &= k + k\int_0^b \frac{(1 - e^{-u})^2}{u}\,du \\ &= k + 2k\int_0^b \frac{(1 - e^{-u})}{u}\,du - k\int_0^{2b} \frac{(1 - e^{-u})}{u}\,du \\ &= k + 2k[E_1(b) + \log b + \gamma] - k[E_1(2b) + \log 2b + \gamma] \\ &\sim k + k(\gamma - \log 2 + \log b) \sim \tfrac{3}{2} + k(C + \log k'), \end{aligned}$$

where $\gamma = 0.5772156649$ is Euler's constant and $C = \gamma - \log 2$. This agrees with (4.4) in the two leading terms.

From (A.5) we obtain, for T_2,

$$\text{(A.7)} \qquad \begin{aligned} T_2 &\sim \int_0^b e^{-u}(1 - e^{-u})(1 - u)\log\left(\frac{u}{k'}\right)du \\ &= \left\{\left(\log\frac{u}{k'}\right)\left[ue^{-u} - \left(\frac{2u - 1}{4}\right)e^{-2u} - \frac{1}{4}\right]\right\}_0^b \\ &\qquad - \left\{\frac{1}{4}e^{-2u}\right\}_0^b + \frac{1}{4}\int_0^{2b}\frac{(1 - e^{-x})}{x}\,dx \sim \frac{\gamma + 1 + \log 2k'}{4}. \end{aligned}$$

Combining this with T_1 in (A.6) gives

$$\text{(A.8)} \qquad EY_M \sim k(\log k' + C) + \tfrac{1}{4}(\log 2k' + \gamma + 7),$$

which contains all terms not approaching zero in the asymptotic expansion.

The same method used above can be extended to give the (correction) terms of order $(\log 2k')^\alpha/k$ for $\alpha = 0$, 1, and 2, and we give these without proof. The complete result, including all terms of order $1/k$ and larger, is

$$\text{(A.9)} \qquad \begin{aligned} E\{Y_M\} \sim k(\log k' + C) + \frac{\log 2k' + 7 + \gamma}{4} + \frac{1}{16k}\{(\log 2k')^2 \\ + (2\gamma - 1)\log 2k' - (9 + \gamma + 2\log 2)\}. \end{aligned}$$

This is tabulated in Table II for comparisons with the exact answers; the error appears to be less than $\frac{1}{2}$ for all $k \geqq 2$.

An improvement is also possible in the approximation (4.10) for $E\{Y_H\}$. For large values of Y_M (for example in the neighborhood of $E\{Y_M\}$), the total number of rooks in the i, j cell under multinomial sampling is approximately binomial with parameters $Y_M - 1$ and $1/k^2$. The reason for using $Y_M - 1$ is that the last rook set down cannot be a duplicate.

If we go through the same analysis as in Section 4 for $E\{Y_H\}$ with Y_M replaced by $Y_M - 1$ (except in (4.5) and the definition of D) and use (A.9) for $E\{Y_M\}$, then we obtain instead of (4.10)

$$\text{(A.10)} \quad E\{Y_H\} \sim k(\log k' + C) - \frac{(\log k' + C)^2}{2} + \frac{\log 2k' + 7 + \gamma}{4} - \frac{(\log k' + C)}{4k}(\log 2k' + \gamma + 1).$$

This is the quantity tabulated in the last column of Table II.

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