BIRTH AND DEATH OF MARKOV PROCESSES

P. A. MEYER
UNIVERSITY OF STRASBOURG
R. T. SMYTHE
UNIVERSITY OF WASHINGTON
J. B. WALSH
UNIVERSITY OF STRASBOURG

1. Introduction

We must start with some basic notation and definitions. To avoid stopping the process at time 0, we shall deal with one particular situation, leaving it to the specialist to check whether our conclusions remain true when all hypotheses are deleted. Let E be a locally compact space with countable base. Some $\Delta \in E$ has been singled out for infamous purposes. Let Ω be the set of all mappings $\omega : \mathbf{R}_+ \twoheadrightarrow E$ which are right continuous and possess a "lifetime" ζ (possibly 0 or $+\infty$), namely,

$$\omega(t) \neq \Delta \text{ for } t < \zeta(\omega), \qquad \omega(t) = \Delta \text{ for } t > \zeta(\omega). \tag{1.1}$$

We set as usual $X_t(\omega) = \omega(t)$, $X_\infty(\omega) = \Delta$, and provide Ω with the natural family of σ-fields $(\mathscr{F}_t^0)_{t \leqq \infty}$ of the process (X_t). Given now a Hunt transition semigroup $(P_t)_{t \geqq 0}$ on E, with Δ as an absorbing point, we can define as usual measures P^μ, P^x on Ω, for which the process (X_t) is Markovian, with the transition semigroup (P_t) and initial measures μ, ε_x. The assumptions concerning left limits in the definition of Hunt processes will be superfluous most of the time. We postpone all other definitions to the main text.

We are interested in operations on the sample paths which preserve the homogeneous Markov character of the process, with possible alteration of the semigroup. Known examples of these operations are: turning a set into an absorbing barrier; restarting the process at a stopping time (we shall use the terminology "optional r.v." rather than "stopping time"); killing the process at a terminal time; reversing time at an L time (L times are called cooptional random variables below); clock changing relative to a continuous additive functional. Our purpose here consists in giving two more examples of such transformations.

Let us say informally that a positive random variable R is a *birth time* for the process if the process $(X_{R+t})_{t \geqq 0}$ starting at time R is, for every law P^μ, a homogeneous Markov process (its transition semigroup may depend on R, but not on μ). Similarly, replacing the process starting at R by the process killed at R, we get the notion of a *death time*. Then two of the preceding examples can be stated as follows:

(a) optional times are birth times,

(b) terminal times are death times.

We are going to prove here that:

(c) coterminal times are birth times,

(d) cooptional times are death times.

Thus, for instance, the last exit time from a Borel set, which was historically the first example of an L time (or cooptional r.v.), turns out to be the model for coterminal times also. According to (c) and (d), it is both a birth time and a death time.

Properties (c) and (d) are dual to (a) and (b), since time reversal from ζ exchanges optional for cooptional, terminal for coterminal and birth for death. Unfortunately, full proofs using time reversal would be very cumbersome, owing to the impossibility of reversing on $\{\zeta = \infty\}$. Therefore, we have used time reversal only as a guide to intuition.

2. Cooptional random variables and killing

Let us complete our notations. As usual, $\mathscr{F}^\mu$ denotes the σ-field $\mathscr{F}^0$ completed with respect to P^μ, and $\mathscr{F}_t^\mu$ is $\mathscr{F}_t^0$ augmented with all subsets of $\mathscr{F}^\mu$ of measure 0; we set $\mathscr{F} = \bigcap_\mu \mathscr{F}^\mu$, $\mathscr{F}_t = \bigcap_\mu \mathscr{F}_t^\mu$. It is well known that the family $(\mathscr{F}_t)$ is right continuous.

The *shift operator* θ_t is defined as usual by $X_s(\theta_t\omega) = X_{s+t}(\omega)$ for all $s \geqq 0$, and the *killing operator* k_t is defined by

$$X_s(k_t\omega) = \begin{cases} X_s(\omega) & \text{if } s < t, \\ \Delta & \text{if } s \geqq t. \end{cases} \tag{2.1}$$

We have $k_s \circ k_t = k_{s \wedge t}$.

DEFINITION 2.1. *A positive random variable L on $(\Omega, \mathscr{F})$ is cooptional if and only if we have identically in $t \geqq 0$*

$$L(\theta_t\omega) = \big(L(\omega) - t\big)^+. \tag{2.2}$$

One generally allows a negligible set on which (2.2) doesn't hold. For the sake of simplicity, we demand a true identity. If (2.2) isn't an identity, one can generally restrict Ω to some shift invariant and killing invariant subset Ω_0 of full measure on which (2.2) holds identically, and the extension of our results becomes trivial.

The most classical examples of cooptional times are *last exit times*: if A is Borel in E, one sets $L_A(\omega) = \sup\{t \geqq 0 : X_t(\omega) \in A\}$ with the usual convention that $\sup(\varnothing) = 0$. The terminology "cooptional" requires some justification, provided by the following remark. Let T be a random variable on $(\Omega, \mathscr{F}^0)$, such that $0 \leqq T \leqq \zeta$. Then T is optional relative to the family $(\mathscr{F}_{t+}^0)$ if and only if the identity $T \circ k_t = T \wedge t$ holds, and this is dual to (2.2) by time reversal at ζ.

We collect in the following proposition some useful and simple results on cooptional random variables.

PROPOSITION 2.1. (i) *If L is cooptional, so is $(L - t)^+$ for $t \geqq 0$.*

(ii) *If L and L' are cooptional, so are $L \wedge L'$ and $L \vee L'$.*

(iii) *Let L be cooptional, and $\mathscr{G}_L$ be the set of all $A \in \mathscr{F}$ such that for every $u \geqq 0$, $A \cap \{L > u\} = \theta_u^{-1}(A) \cap \{L > u\}$. Then $\mathscr{G}_L$ is a σ-field, and X_L is $\mathscr{G}_L$ measurable (and also $\{L = \infty\} \in \mathscr{G}_L$). If $A \in \mathscr{G}_L$, the random variable L^A defined by $L^A = L$ on A, $L^A = 0$ on A^c is cooptional.*

The proofs are very easy, and we give no details.

DEFINITION 2.2. *The excessive function associated to the cooptional random variable L is $c_L(\cdot) = P^{\cdot}\{L > 0\}$.*

We have $P_t c_L(x) = P^x\{L \circ \theta_t > 0\} = P^x\{L > t\}$. This implies immediately that c_L is excessive. We shall write c instead of c_L, to simplify notation, and define as usual the c path semigroup Q_t as

$$Q_t(x, dy) = \begin{cases} \dfrac{1}{c(x)} P_t(x, dy) c(y) & \text{if } c(x) \neq 0, \\ \varepsilon_\Delta(dy) & \text{if } c(x) = 0. \end{cases} \tag{2.3}$$

We give now the main theorem of this section. The result is quite easy, but it has one surprising feature: we are *killing* the process (X_t), and get a new process (Y_t) whose transition semigroup *is not dominated* by (P_t). This is shocking at first, but becomes quite easy to understand if one notes that if (X_t) starts at x, then the initial measure of (Y_t) is not ε_x, but $c(x)\varepsilon_x + \big(1 - c(x)\big)\varepsilon_\Delta$.

THEOREM 2.1. *Let L be cooptional, and (Y_t) be the process (X_t) killed at time L*

$$Y_t(\omega) = \begin{cases} X_t(\omega) & \text{if } t < L(\omega), \\ \Delta & \text{if } t \geqq L(\omega). \end{cases} \tag{2.4}$$

If Ω is given the measure P^μ, the process (Y_t) is Markovian with (Q_t) as transition semigroup, and $c \cdot \mu + \langle \mu, 1 - c\rangle \varepsilon_\Delta$ as initial measure.

PROOF. Let us define the σ-field $\mathscr{H}_t$ as the set of all $A \in \mathscr{F}$ such that there exists $A_t \in \mathscr{F}_t$ satisfying $A \cap \{t < L\} = A_t \cap \{t < L\}$. The family $(\mathscr{H}_s)$ is increasing, and Y_t is $\mathscr{H}_t$ measurable.

Let s and t be two epochs such that $s < t$, ϕ be a bounded $\mathscr{H}_s$ measurable random variable, and f be Borel and bounded on E, equal to 0 at Δ. The theorem amounts to saying that

$$E^\mu[\phi \cdot f \circ Y_t] = E^\mu[\phi \cdot Q_{t-s}(Y_s, f)]. \tag{2.5}$$

Let us compute the left side. It is equal to

$$E^\mu[\phi \cdot f \circ X_t \cdot I_{\{t<L\}}] = E^\mu[\phi_s \cdot f \circ X_t \cdot I_{\{t<L\}}] = E^\mu[\phi_s \cdot f \circ X_t \cdot I_{\{L\circ\theta_t > 0\}}], \tag{2.6}$$

where ϕ_s is (according to the definition of $\mathscr{H}_s$) some $\mathscr{F}_s$ measurable and bounded r.v. equal to ϕ on $\{s < L\}$. Taking conditional expectations with respect to $\mathscr{F}_t$, we get

$$E^\mu[\phi_s \cdot f \circ X_t \cdot P^{X_t}\{L > 0\}] = E^\mu[\phi_s \cdot ((fc) \circ X_t)]. \tag{2.7}$$

Taking conditional expectations with respect to $\mathscr{F}_s$, we write this last integral as $E^\mu[\phi_s \cdot P_{t-s}(X_s, fc)]$. On the other hand, $P_{t-s}(fc) = 0$ on $\{c = 0\}$ since the function c is excessive, and the expectation can be written

$$E^\mu[\phi_s \cdot I_{\{c>0\}} \cdot P_{t-s}(X_s, fc)] = E^\mu[\phi_s \cdot c \circ X_s \cdot Q_{t-s}(X_s, f)] \tag{2.8}$$

On the other hand, the right side of (2.5) is equal to

$$\begin{aligned} E^\mu[\phi \cdot Q_{t-s}(X_s, f) \cdot I_{\{s<L\}}] &= E^\mu[\phi_s \cdot Q_{t-s}(X_s, f) \cdot I_{\{s<L\}}] \\ &= E^\mu[\phi_s \cdot Q_{t-s}(X_s, f) \cdot c \circ X_s] \end{aligned} \tag{2.9}$$

(by the same reasoning as above). The theorem is proved.

We shall leave the main subject for a moment, to discuss the relation between the existence of cooptional times and the transience of the process. Let us say that a cooptional time L is *trivial* if for all x either $P^x\{L = 0\} = 1$, or $P^x\{L = \infty\} = 1$, and denote by U the potential operator $\int_0^\infty P_t\, dt$. We say that an excessive function h is *nearly constant* if for all x $P^x\{h \circ X_t = h(x) \text{ for all } t\} = 1$.

THEOREM 2.2. *The following statements are equivalent:*

(i) *$U(x, B)$ takes on only the values* 0 *and* ∞;

(ii) *all cooptional times are trivial.*

PROOF. Assume (i), and let h be excessive, bounded by 1. Then h cannot have a potential part, and therefore is invariant. From martingale theory, $h(x) = E^x[h_\infty]$, $h \circ X_t = E[h_\infty | \mathscr{F}_t]$, where $h_\infty = \lim_{s\to\infty} h \circ X_s$. Now $1 - h^2$ is excessive, hence invariant; therefore, $h^2(x) = E^x[h_\infty^2]$ and h_∞ must be P^x a.s. equal to $h(x)$, and h is nearly constant. This part of the reasoning is well known.

Let L be cooptional, and assume that $P^x\{L = \infty\} < 1$. Set $h(x) = P^x\{L = \infty\}$. This is an invariant function, and $h_\infty = I_{\{L=\infty\}}$. Since h_∞ is a constant P^x a.s., and is not a.s. 1, $h_\infty = 0$ P^x a.s. Then the excessive function $c_L = P^\cdot\{L > 0\}$ is such that $P_t c_L \to 0$ at x as $t \to \infty$. Therefore at x, $c_L = \lim_{t\to 0} U((c_L - P_t c_L)/t)$; since U takes on the values 0 and ∞ only, $c_L(x) = 0$. Finally, $P^x\{L = 0\} = 1$.

Conversely, assume that for some $x \in E$ and $B \subset E$, we have $0 < U(x, B) < \infty$. Then $P_t(x, U(I_B)) \to 0$ as $t \to \infty$. Hence, $U(X_t, B) \to 0$ P^x a.s. as $t \to 0$.

Define L as the last exit time from $\{U(\cdot, B) > \varepsilon\}$; then for $\varepsilon > 0$, L is nontrivial in the stronger sense that $P^x\{0 < L < \infty\} \neq 0$. Theorem 2.2 is proved.

3. Some trivialities about fields

In Section 5, it will be useful to define $\mathscr{F}_R$ for random variables R which are not optional. This has already been done by K. L. Chung and J. L. Doob in a well-known paper, but their definition is not convenient for our purposes. Therefore, we are going to shift for a while to "general abstract nonsense." This section may be safely skipped, except for a glance at the results.

DEFINITION 3.1. *Let R be any positive r.v. on $(\Omega, \mathscr{F})$. We define $\mathscr{F}_R$ as the set of all $A \in \mathscr{F}$ such that for every $t \geqq 0$ there exists $A_t \in \mathscr{F}_t$ such that $A \cap \{R < t\} = A_t \cap \{R < t\}$. We say that R is honest if R is $\mathscr{F}_R$ measurable.*

In more general situations $A \in \mathscr{F}$ should be replaced by $A \in \mathscr{F}_\infty$. If R is optional, Definition 3.1 is equivalent to the standard definition of $\mathscr{F}_R$.

Here are some obvious remarks. A real valued r.v. Y is $\mathscr{F}_R$ measurable if and only if for every t there exists some $\mathscr{F}_t$ measurable r.v. Y_t such that $Y = Y_t$ on $\{R < t\}$. Then for any optional T there exists some $\mathscr{F}_T$ measurable Y_T such that $Y = Y_T$ on $\{R < T\}$ (start with a countably valued T, and pass to the limit, using the right continuity of the family $(\mathscr{F}_t)$). Also, $\{R < T\}$ can be replaced by $\{R \leqq T\}$. Finally, note that the family $(\mathscr{F}_{R+t})_{t \geqq 0}$ is increasing and right continuous.

The random variable R is honest if and only if, for every t, R is equal on $\{R < t\}$ to some $\mathscr{F}_t$ measurable r.v. R_t. This allows a simple characterization of $\mathscr{F}_R$ when R is honest.

PROPOSITION 3.1. *Let R be honest. Then a random variable Y is $\mathscr{F}_R$ measurable if and only if there exists a progressively measurable process (Y_t) such that $Y = Y_R$ on $\{R < \infty\}$.*

PROOF. The condition is necessary: indeed, for every s rational choose some r.v. Y_s^1, which is $\mathscr{F}_s$ measurable, such that $Y = Y_s^1$ on $\{R < s\}$. Then for every t set $Y_t = \liminf_{s \downarrow t, s > t} Y_s^1$ where s is rational; this is a progressive process, and $Y = Y_R$ on $\{R < \infty\}$.

Conversely, if (Y_t) is progressive, and $Y = Y_R$, then Y is equal on $\{R < t\}$ to Y_{R_t}, which is $\mathscr{F}_t$ measurable. Therefore, Y is $\mathscr{F}_R$ measurable.

We do not need anything more about honest random variables. Let us just quote an amusing result. Consider the abstract situation of a right continuous family $(\mathscr{F}_t)$, such that $\mathscr{F}_0$ contains all sets of measure 0 for the basic measure P. Then R is honest if and only if there exists a well-measurable subset H of $\mathbf{R}_+ \times \Omega$ such that R is the *end of H*:

$$R(\omega) = \sup\{t : (t, \omega) \in H\}. \tag{3.1}$$

The class of all random variables which are ends of *predictable* sets is strictly smaller than that of honest random variables and particularly interesting, as shown by a recent paper of Azema.

4. Terminal and coterminal times, duality

DEFINITION 4.1. *A random variable $L \geqq 0$ on $(\Omega, \mathscr{F})$ is a coterminal time if*

(i) $L \circ \theta_s = (L - s)^+$ *for every* s,

(ii) $L \circ k_s = L$ *on* $\{L < s\}$,

(iii) $L \circ k_s \leqq s$ *for every* s.

The first property means that L is cooptional. The second property implies that L is honest, and would mean precisely honestness if the σ-fields had not been completed. The third property is equivalent to the inequality $L \leqq \zeta$: indeed, applying (iii) with $s = \zeta$, we get that $L \leqq \zeta$, and conversely $L \leqq \zeta$ implies $L \circ k_s \leqq \zeta \circ k_s \leqq s$.

Examples of coterminal times are: the lifetime ζ and last exit times from sets; last exit times of the left limit process (X_{t-}) from sets; last jumps of one given kind, and so forth.

We state some obvious properties of coterminal times.

PROPOSITION 4.1. *Let L be a coterminal time. Then:*

(i) $L \circ k_s \leqq L$ *for every* s;

(ii) $s < t$ *implies* $L \circ k_s \leqq L \circ k_t$;

(iii) $s < t$, $L \circ k_s = 0$, *and* $L \circ k_t > 0$ *imply* $L \circ k_t \geqq s$;

(iv) *set* $L' = \sup_{s < \infty} L \circ k_s$. *Then L' is a coterminal time, $L' \leqq L$, with equality on $\{L < \infty\}$, and $L \circ k_t = L' \circ k_t$ for every finite t.*

PROOF. (i) If $s \leqq L$, then $L \circ k_s \leqq s \leqq L$ from Definition 4.1 (iii). If $L < s$, $L \circ k_s = L$ from Definition 4.1 (ii).

(ii) Composition of (i) with k_t gives $L \circ k_s = L \circ k_s \circ k_t \leqq L \circ k_t$.

(iii) Assume the contrary: $L \circ k_t < s$. Then from Definition 4.1 (ii) $L \circ k_s \circ k_t = L \circ k_t$, a contradiction since the right side is greater than 0, while the left side is equal to $L \circ k_s = 0$.

(iv) According to (ii), this "sup" is really a lim. We first have $L' \leqq L$ from (i). If $L < \infty$, take $s > L$ and apply Definition 4.1 (ii), we find $L' \geqq L \circ k_s = L$, so L and L' are equal on $\{L < \infty\}$. We have for finite t $L' \circ k_t = \lim_{s \to \infty} L \circ k_s \circ k_t = L \circ k_t$ (take $s > t$).

We must prove that L' is a coterminal time. We have seen that Definition 4.1 (iii) is equivalent to $L' \leqq \zeta$; since $L' \leqq L$ and L satisfies Definition 4.1 (iii), the property holds. Next $(L \circ k_u) \circ \theta_s = L \circ \theta_s \circ k_{u-s}$ if $u > s$, and this is equal to $(L \circ k_{u-s} - s)^+$ from Definition 4.1 (i). Letting $u \to \infty$, we get property Definition 4.1 (i) for L'. Let us finally prove Definition 4.1 (ii): assume $L' < s$, then we have $L \circ k_u < s$ if u is finite; hence, from Definition 4.1 (ii) $L \circ k_u = L \circ k_s \circ k_u = L \circ k_s$ if $u > s$. Letting $u \to \infty$, we get $L' = L' \circ k_s$.

DEFINITION 4.2. *The coterminal time L is exact if $L = \sup_{s < \infty} L \circ k_s$.*

If L is not exact, then the coterminal time L' we have just considered is exact, and differs from L only on $\{L = \infty\}$. Thus, exactness is not much of a restriction.

We are going now to construct the terminal time associated with the coterminal time L.

DEFINITION 4.3. *The time T_L (or simply T) is the function on Ω defined by*

$$T_L(\omega) = \inf \{t > 0: L(k_t \omega) > 0\}. \tag{4.1}$$

Note that $T_L = T_{L'}$.

All the results on T_L which may be needed below are collected in the following proposition.

PROPOSITION 4.2. *The function $T = T_L$ satisfies the following relations:*

(i) *T is an optional r.v.;*

(ii) *$T \leqq L$ on $\{L > 0\}$, $T = \infty$ on $\{L = 0\}$;*

(iii) *$T \circ \theta_t = \infty$ is equivalent to $L' \leqq t$; hence, $L' = \sup \{t : T \circ \theta_t < \infty\}$;*

(iv) *$T \circ k_s = \infty$ on $\{T > s\}$;*

(v) *T is a perfect, exact terminal time;*

(vi) *$a < L' \leqq b$ is equivalent to $T \circ \theta_a \leqq b - a$, $T \circ \theta_b = \infty$.*

PROOF. (i) The inequality $T < t$ holds if and only if there exists some rational $r < t$ such that $L \circ k_r > 0$. This implies that T is an optional r.v.

(ii) If $L = 0$, then $L \circ k_t \leqq L$ is 0 for all t, and hence $T = \infty$. The inequality $T \leqq L$ is obvious on $L = \infty$. If $0 < L < \infty$, take $t > L$; therefore $L = L \circ k_t$ from Definition 4.1 (ii), and since $L > 0$ we have $T \leqq t$, and finally $T \leqq L$.

(iii) Using Definition 4.1 (i),

$$
(4.2) \qquad \begin{aligned} T \circ \theta_t &= \inf \{u > 0 : L \circ k_u \circ \theta_t > 0\} = \inf \{u > 0 : L \circ \theta_t \circ k_{u+t} > 0\} \\ &= \inf \{u > 0 : L \circ k_{u+t} > t\}, \end{aligned}
$$

so $(T \circ \theta_t = \infty) \Leftrightarrow (L \circ k_{u+t} \leqq t \text{ for all } t) \Leftrightarrow (L' \leqq t)$.

(iv) is obvious, from the definition of T.

(v) A *terminal time* is an optional r.v. S such that for every t, $S = t + S \circ \theta_t$ a.s. on $\{S > t\}$. In the definition which is most commonly used, the exceptional set may depend on t; if it can be chosen independent of t, S is a *perfect* terminal time (in our case the exceptional set will turn out to be empty). Also S is *exact* if $t + S \circ \theta_t \to S$ a.s. as $t \to 0$.

From the proof of (iii), we have

$$
(4.3) \qquad t + T \circ \theta_t = \inf \{w > t : L \circ k_w > t\} = \inf \{w > 0 : L \circ k_w > t\}
$$

(since $w \geqq L \circ k_w$, according to Definition 4.1 (iii)). Assume that $t < T$, and choose u such that $t < u < T$. Then $L \circ k_u = 0$, and from Proposition 4.1 (iii), $L \circ k_w > 0$ implies $L \circ k_w \geqq u > t$. Therefore, on $t < T$ we have

$$
(4.4) \qquad t + T \circ \theta_t = \inf \{w > 0 : L \circ k_w > 0\} = T,
$$

and this means that T is a perfect terminal time without exceptional set.

Let us prove that T is exact. We always have

$$
(4.5) \qquad \begin{aligned} t + T \circ \theta_t &= \inf \{w > 0 : L \circ k_w > t\} \\ &\geqq \inf \{w > 0 : L \circ k_w > 0\} = T. \end{aligned}
$$

There is equality on $\{T = \infty\}$. On $\{T < \infty\}$ we have $T < T + u$ for every $u > 0$. Therefore, $L \circ k_{T+u} > 0$, and for t small enough $t + T \circ \theta_t < T + u$; hence, $T = \lim_{t \to \infty} t + T \circ \theta_t$.

(vi) The relation $a < L' \leqq b$ is equivalent to $T\circ\theta_a < \infty$, $T\circ\theta_b = \infty$ according to (iii), but $(T\circ\theta_a \leqq b - a, T\circ\theta_b = \infty) \Rightarrow (T\circ\theta_a < \infty, T\circ\theta_b = \infty) \Rightarrow (T\circ\theta_a \leqq b - a, T\circ\theta_b = \infty)$ since $\infty > T\circ\theta_a > b - a$ implies $(b - a) + T\circ\theta_{b-a}\circ\theta_a = T\circ\theta_a < \infty$, according to (v), and therefore $T\circ\theta_b < \infty$.

We have now everything we need to prove that coterminal times are birth times. However, only one side of the duality between terminal and coterminal times appears in Propositions 4.1 and 4.2, and we must give the other side.

PROPOSITION 4.3. *Let T be a random variable on $(\Omega, \mathscr{F})$ which satisfies the following properties:*

(i) $T < s \Rightarrow T\circ k_s = T$;

(ii) $s < T \Rightarrow T\circ k_s = \infty$;

(iii) $s < T \Rightarrow T - s = T\circ\theta_s$.

Set L_T (or simply L) $= \sup\{t : T\circ\theta_t < \infty\}$. Then L is an exact coterminal time, and $T_L = T$ if and only if T is exact, that is $t + T\circ\theta_t \downarrow T$ as $t \downarrow 0$.

PROOF. Let us show first that $T < s \Leftrightarrow T\circ k_s < s$. The implication $\Rightarrow$ is obvious from (i). Conversely, if $T\circ k_s < r < s$, (i) implies that $T\circ k_r\circ k_s = T\circ k_r < r$, which excludes the possibility that $T > r$ according to (ii). Therefore, $T \leqq r < s$. As a consequence, we have that $\{T < s\} \in \mathscr{F}_s$; hence, T is optional and a terminal time according to (iii). Note that T is not an arbitrary terminal time, because of property (ii): ζ, for instance, is a terminal time and $s < \zeta$ implies $\zeta\circ k_s = s$, not ∞.

It follows at once from (iii) that $s + T\circ\theta_s \geqq T$, and then that $t + T\circ\theta_t$ increases in t. Let us set $T' = \lim_{t\to 0} t + T\circ\theta_t$. We have $T' \geqq T$, with $T' = T$ on $\{T > 0\}$. If $s < T'$, then $s < T\circ\theta_t$ for t small enough, and (iii) gives $s + T\circ\theta_{s+t} = T\circ\theta_t$; letting $t \to 0$, we get $s + T'\circ\theta_s = T'$. Hence, T' satisfies (iii). If $s < T'$, then $s < t + T\circ\theta_t$ for all t; hence $T\circ k_{s-t}\circ\theta_t = \infty$ from (ii), or $T\circ\theta_t\circ k_s = \infty$. Finally, $T'\circ k_s = \infty$. If $T' < s$, then $t + T\circ\theta_t < s$ for t small enough. Hence, $T\circ\theta_t = T\circ k_{s-t}\circ\theta_t$ from (i); and letting $t \to 0$, we find again that $T'\circ k_s = T'$, that is, T' satisfies (i). Thus, T' is an exact terminal time. Note that $L_T = L_{T'}$.

Let us prove that L is an exact coterminal time. We have

$$L\circ\theta_s = \sup\{t : T\circ\theta_{t+s} < \infty\} = (L - s)^+. \tag{4.6}$$

Hence, L satisfies Definition 4.1 (i). Next, assume that T is not identically 0 (in which case everything would be trivial); denote by $[\Delta]$ the constant sample function equal to Δ, choose some ω such that $T(\omega) > 0$ and some $s < T(\omega)$. Then $T(k_s\omega) = \infty > s$ from (ii), and from (iii) $T(\theta_s(k_s\omega)) = \infty$; otherwise stated, $T([\Delta]) = \infty$.

Then we have

$$L\circ k_s = \sup\{t : T\circ\theta_t\circ k_s < \infty\} = \sup\{t : T\circ k_{(s-t)^+}\circ\theta_t < \infty\}. \tag{4.7}$$

If $t = s$, we have $T([\Delta]) = \infty$. Hence, $L\circ k_s \leqq s$, that is, Definition 4.1 (iii) is true. Let us remark that $T \leqq T\circ k_s$ for all s: if we had $T > r > T\circ k_s$ for some r, we would have from (i) $T\circ k_r = \infty$, from (ii) $T\circ k_r\circ k_s = T\circ k_s < \infty$. Hence,

$r > s$ and $T > s$; thus, $T \circ k_s = \infty$ from (ii), which is a contradiction. This implies that $L \circ k_s \leqq L$. On the other hand, if $s > L$, choose some s' such that $s > s' > L$. Then $T \circ \theta_{s'} = \infty$. Hence, $r < L$ implies $T \circ \theta_r < \infty$, which implies $T \circ \theta_r \leqq s' - r < s - r$ from (iii). Therefore from (i), we get $T \circ k_{s-r} \circ \theta_r = T \circ \theta_r$, or $T \circ \theta_r \circ k_s = T \circ \theta_r < \infty$; and finally, $r \leqq L \circ k_s$. Otherwise stated, $L = L \circ k_s$ and property (ii) of Definition 4.1 is proved.

If $L = \infty$, we have $T \circ \theta_t < \infty$ for all t. Given t, choose $w > t + T \circ \theta_t$. Then $w - t > T \circ \theta_t$ and from (i) $T \circ k_{w-t} \circ \theta_t = T \circ \theta_t < \infty$. Hence, $T \circ \theta_t \circ k_w < \infty$, and $L \circ k_w \geqq t$. Since t is arbitrary, we have also $L' = \infty$ and L is exact.

We finish with the proof that $T_L = T$. We have

$$(4.8) \quad \begin{aligned} T_L &= \inf\{t > 0 : L \circ k_t > 0\} = \inf\{t > 0 : \text{for some } s \quad T \circ \theta_s \circ k_t < \infty\} \\ &= \inf\{t > 0 : T' \circ k_t < \infty\} = \inf\{t > 0 : t > T'\} = T'. \end{aligned}$$

5. Birth of the process at a coterminal time

We consider a coterminal time L, exact, and its associated terminal time T. Let us define a new semigroup (K_t) by killing at time T,

$$(5.1) \qquad K_t(x, f) = E^x[f \circ X_t \cdot I_{\{t < T\}}].$$

It has Δ as an absorbing point (since $L([\Delta]) = 0$, $T([\Delta]) = +\infty$), but it does not satisfy $K_t 1 = 1$. That does not matter.

Let us define also $g(x) = P^x\{L = 0\} = P^x\{T = \infty\}$ for all $x \in E$, including Δ (see Proposition 4.2 (iii)). The following computation shows that g is *invariant* for (K_t):

$$(5.2) \quad \begin{aligned} K_t(x, g) &= P^x\{T \circ \theta_t = \infty, t < T\} = P^x\{T - t = \infty, t < T\} \\ &= P^x\{T = \infty\} = g(x). \end{aligned}$$

We can therefore define the conditioned semigroup,

$$(5.3) \qquad K_t^g(x, dy) = \begin{cases} \dfrac{1}{g(x)} K_t(x, dy) g(y) & \text{if } g(x) \neq 0, \\ \varepsilon_\Delta(dy) & \text{if } g(x) = 0. \end{cases}$$

If $g(x) \neq 0$, this is a probability measure carried by $\{g > 0\}$.

We shall need the following elementary property of the semigroup (K_t^g). Let $(Y_s)_{s \geqq 0}$ be any right continuous Markov process with (K_s^g) as its transition semigroup, and let f be a bounded continuous function on E. Set $j = K_t^g f$. Then the process (Y_s) is strong Markov, and the process $(j \circ Y_s)_{s \geqq 0}$ is a.s. right continuous. These properties belong to the folklore of Markov processes, the second statement being a particular case of the so-called "Feller property of the semigroup in the fine topology." They are true for all right continuous strong Markov semigroups (see the bibliographical comments at the end of the paper).

We state and prove now our main result. The restriction to the open interval $(0, \infty)$ is essential.

THEOREM 5.1. *The process* $(X_{L+t})_{t>0}$ *is a strong Markov process with respect to the family* $(\mathscr{F}_{L+t})$, *with* (K_t^g) *as transition semigroup.*

PROOF. Implicit in the statement is the hypothesis that Ω has been provided with some law P^μ; we drop μ from our notation, however.

Let L_n be the dyadic approximation of L from above, that is, $L_n = k2^{-n}$ if and only if $(k-1)2^{-n} < L \leqq k2^{-n}$, $k \geqq 1$, and $L_n = 0$ on $\{L = 0\}$. We are going to prove that, for f continuous and bounded on E and $0 < s < t$,

$$E[f \circ X_{L_n+t} | \mathscr{F}_{L_n+s}] = K_{t-s}^g(X_{L_n+s}, f) \tag{5.4}$$

(the fields are defined as in Definition 3.1, L, L_n, $L_n + s$, $L_n + t$ being honest; X_{L_n+s} is $\mathscr{F}_{L_n+s}$ measurable according to Proposition 4.2). This relation will imply Theorem 5.1. Indeed, each process $(X_{L_n+u})_{u>0}$ is a right continuous Markov process with (K_u^g) as transition semigroup according to (5.4). Using the Feller property above, $K_t^g f$ is a.s. right continuous on the sample paths of every process (X_{L_n+u}), and therefore as $n \to \infty$, on the sample paths of $(X_{L+u})_{u>0}$. If $H \in \mathscr{F}_{L+s} \subset \mathscr{F}_{L_n+s}$, the relation

$$\int_H f \circ X_{L_n+t}\, dP = \int_H K_{t-s}^g f \circ X_{L_n+s}\, dP \tag{5.5}$$

passes nicely to the limit as $n \to \infty$, giving the result we seek.

Let us therefore prove (5.4). Take $H \in \mathscr{F}_{L_n+s}$. Then letting $J_k = \{k2^{-n} < L \leqq (k+1)2^{-n}\}$,

$$\int_H f \circ X_{L_n+t}\, dP = \int_{H, L=0} f \circ X_t\, dP + \sum_{k=0}^{\infty} \int_{H \cap J_k} f \circ X_{(k+1)2^{-n}}\, dP. \tag{5.6}$$

Since $H \in \mathscr{F}_{L_n} + s$, and we are on $\{L \leqq (k+1)2^{-n}\}$, therefore on $\{L_n + s \leqq (k+1)2^{-n} + s\}$, we may replace H by some $H' \in \mathscr{F}_{(k+1)2^{-n}+s}$. The kth term of the sum can be written

$$E[f \circ X_{(k+1)2^{-n}+t}, H', T \circ \theta_{k2^{-n}} \leqq 2^{-n}, T \circ \theta_{(k+1)2^{-n}} = \infty\}. \tag{5.7}$$

We write $\{T \circ \theta_{(k+1)2^{-n}} = \infty\}$ as $\{T \circ \theta_{(k+1)2^{-n}} > t\} \cap \{T \circ \theta_{(k+1)2^{-n}+t} = \infty\}$ and take conditional expectations with respect to $\mathscr{F}_{(k+1)2^{-n}+t}$. We get

$$\begin{aligned} &E[f \circ X_{(k+1)2^{-n}+t}, H', T \circ \theta_{k2^{-n}} \leqq 2^{-n}, T \circ \theta_{(k+1)2^{-n}} > t, g \circ X_{(k+1)2^{-n}+t}] \\ &\quad = E[H', T \circ \theta_{k2^{-n}} \leqq 2^{-n}, T \circ \theta_{(k+1)2^{-n}} > s, \\ &\qquad\qquad T \circ \theta_{(k+1)2^{-n}+s} > t - s, (fg) \circ X_{(k+1)2^{-n}+t}]. \end{aligned} \tag{5.8}$$

Taking now conditional expectations with respect to $\mathscr{F}_{(k+1)2^{-n}+s}$, equation (5.8) is equal to

$$E[H', T \circ \theta_{k2^{-n}} \leqq 2^{-n}, T \circ \theta_{(k+1)2^{-n}} > s, K_{t-s}(X_{(k+1)2^{-n}+s}, fg)]. \tag{5.9}$$

We replace $K_{t-s}(fg)$ by $gK^g_{t-s}(f)$, a legitimate step since $K_{t-s}(fg) = 0$ on $\{g = 0\}$, and we get

$$(5.10)\qquad E[H', T\circ\theta_{k2^{-n}} \leqq 2^{-n}, T\circ\theta_{(k+1)2^{-n}} > s, g\circ X_{(k+1)2^{-n}+s}, \\ K^g_{t-s}f\circ X_{(k+1)2^{-n}+s}]$$

which is equal to

$$(5.11)\qquad E[H', k2^{-n} < L \leqq (k+1)2^{-n}, K^g_{t-s}f\circ X_{(k+1)2^{-n}+s}],$$

according to a reasoning similar to the first transformation of this proof. We now replace H' by H, sum over k (the first term on the right side of (5.6) needs a similar, but slightly different treatment), and we get

$$(5.12)\qquad \int_H f\circ X_{L_n+t}\,dP = \int_H K^g_{t-s}f\circ X_{L_n+s}\,dP,$$

which is equivalent to (5.4).

REMARK. An easy computation shows that

$$(5.13)\qquad K^g_t(x, f) = \frac{1}{g(x)}\,Q_t(x, fg) = \frac{E^x[f\circ X_t, T < \infty]}{P^x\{T < \infty\}}.$$

Thus, (K^g_t) appears as the transition semigroup conditioned by the fact that $T = \infty$. For instance, if L is the last hitting time of A, (K^g_t) is the semigroup of the original process conditioned not to hit A. This is reasonably intuitive.

REFERENCES

[1] J. AZEMA, M. DUFLO, and D. REVUZ, "Propriétés relatives des processus de Markov recurrents," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 13 (1969), pp. 286–314.

[2] R. M. BLUMENTHAL and R. K. GETOOR, *Markov Processes and Potential Theory*, New York and London, Academic Press, 1968.

[3] K. L. CHUNG and J. L. DOOB, "Fields, optionality and measurability," *Amer. J. Math.*, Vol. 87 (1965) pp. 397–424.

[4] K. L. CHUNG and J. B. WALSH, "To reverse a Markov process," *Acta Math.*, Vol. 123 (1970), pp. 225–251.

[5] E. B. DYNKIN, *Markov Processes I, II* (translated by J. Fabius, V. Greenberg, A. Maitra, and G. Majone), New York, Academic Press; Berlin–Göttingen–Heidelberg, Springer-Verlag, 1965.

[6] P. A. MEYER, "Le retournement du temps d'après Chung et Walsh," *Séminaire de probabilités V*, Université de Strasbourg Lecture Notes in Mathematics, Heidelberg, Springer-Verlag, 1971.

[7] M. NAGASAWA, "Time reversions of Markov processes," *Nagoya Math. J.*, Vol. 24 (1964), pp. 177–204.

[8] D. REVUZ, oral communication.

[9] J. B. WALSH, "Some remarks on the Feller Property," *Ann. Math. Statist.*, Vol. 41 (1970), pp. 1672–1683.