THE STRUCTURE OF A MARKOV CHAIN

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1. Introduction

Let p be a standard transition function on the set I of integers, that is, a function from $(0, \infty) \times I \times I$ into $[0, 1]$ satisfying

$$\begin{aligned} &\sum_j p(t, i, j) = 1, \\ (1.1) \qquad &p(s + t, i, k) = \sum_j p(s, i, j)p(t, j, k), \end{aligned}$$

together with the continuity condition $\lim_{t \to 0} p(t, i, i) = 1$. Let f be an absolute probability function, that is, a function from $(0, \infty) \times I$ into $[0, 1]$, satisfying

$$(1.2) \qquad \sum_j f(t, j) = 1, \qquad f(s + t, j) = \sum_i f(s, i)p(t, i, j).$$

Let L be an arbitrary set containing I as a subset. There is then a Markov process $\{x(t), t > 0\}$ with state space L having the specified transition and absolute probability functions. The notation $x(t)$ will always refer to the tth random variable of such a process, and the process will be called "smooth" if L is topological and if almost every sample function is right continuous with left limits on $(0, \infty)$. Note that this condition does not require the existence of a right limit at 0. For each $t > 0$ the random variable $x(t)$ almost surely has its values in I, but it has been known since Ray's work [11] in 1959 that L and the process can be chosen to make the process and properly chosen extensions of the transition function have desirable smoothness properties. One can always choose L to be an entrance space in the sense of [4]; that is, one can choose L to satisfy the following conditions:

(a) L is a Borel subset of a compact metric space in which I is dense;

(b) for every absolute probability function f there is a smooth corresponding process with state space L;

(c) for every integer j, $p(\cdot, \cdot, j)$ has a continuous extension to $(0, \infty) \times L$ and (1.1) is satisfied for i allowed to be any point of L;

(d) if ξ is in L and if $\{x(t), t > 0\}$ is a smooth process with absolute probability function given by $f(t, i) = p(t, \xi, i)$, then $x(0+)$ exists (and is in L) almost surely.

In the following, i, j, k are integers and ξ, η are points of a specified entrance space. The probability measure determined by a smooth process with $f(t, i) =$

$p(t, \xi, i)$ will be denoted by P_ξ. If ξ is not a branch point, $P_\xi\{x(0+) = \xi\} = 1$ and we define $x(0) = \xi$. The process will then be called a smooth process with initial state ξ. There are many entrance spaces, and none can be described as best for all purposes. Any full analysis of Markov chains must include an analysis of the possible entrance spaces for a given transition function.

The present paper is devoted to showing how, after ramifying any given entrance space in order to get certain functions continuous, the resolvent of a process can be expressed in terms of the resolvent of the process killed at a certain type of terminal time together with certain extra operators. The case of interest is when all states are stable. The technique has been used for Hunt processes, but unfortunately under the hypothesis (too restrictive in the present context) that there are no branch points. The necessary theorems on additive functionals when the related Markov processes may have branch points, apparently, have not been stated in the literature, but the extension to the branch point case seems straightforward.

2. Stable case

We shall be most interested in the "stable case" by which we mean that for every i

$$-p'_{i,i}(0) = \sum_{j \neq i} p'_{i,j}(0) < \infty. \tag{2.1}$$

As usual, we write q_i for $-p'_{i,i}(0)$ and $q_{i,j}$ for $p'_{i,j}(0)$ when $j \neq i$.

In the stable case, any smooth process with initial state an integer proceeds at first by jumps. More precisely, if the initial state is i, there is an exponential holding time with expected value $1/q_i$; then there is a jump to j with probability $q_{i,j}/q_i$; then there is an exponential holding time with expected value $1/q_j$; and so on. If T_1 ($\leqq \infty$) is the first explosion time, that is, the supremum of these jump times, the process is integer valued to time T_1. Many papers have been devoted to the character of the paths after time T_1 when $T_1 < \infty$.

In terms of transition functions, the problem is the following. Define $\bar{p}$ by

$$\bar{p}(t, i, j) = P_i\{x(t) = j, T_1 > t\}. \tag{2.2}$$

Then $p \geqq \bar{p}$, with equality only under special restrictions on the matrix $(q_{i,j})$, and the problem is to find the class of possible transition functions p for a given $\bar{p}$, that is, for a given matrix $(q_{i,j})$. In earlier papers, it has always been assumed that the number of ways sample functions can "go to infinity" at T_1 is discrete (usually finite) in some suitable sense. See, for example, the recent literature [2], [5], [12]. If an entrance space has been chosen as state space, one can describe the process at time T_1: the distribution of $x(T_1)$ if $x(T_1 -) = \xi$ is the branching distribution $p(0+, \xi, \cdot)$, that is, the limiting distribution of $p(t, \xi, \cdot)$ when $t \to 0$. (See [4].) If ξ is not a branch point, and only then, this distribution is supported by the singleton $\{\xi\}$. If $p(0+, \xi, \cdot)$ is supported by I for every ξ, one can continue from T_1 using the matrix $(q_{i,j})$; and in fact, in this case $(q_{i,j})$ together with the branching distributions determine the process completely. If the branching distributions are not restricted as stated, we shall see below how the process can be continued—using only $(q_{i,j})$, the state space topology, and the branching distributions—to a time we designate as T_∞.

For an important class of stopping times T, including T_∞, we shall show that if a transition probability p^T is defined like $\bar{p}$ in equation (2.2) but using T instead of T_1, then any given entrance space can be ramified into one in which $p^T(t, \cdot, j)$ has a continuous extension to the space, and the corresponding resolvent operator will then take bounded functions into continuous functions. This property will be used in expressing the resolvent operator of the given process in terms of the resolvent operator of the process killed at T, together with certain other operators.

3. Terminal times and corresponding excessive functions

We assume the usual background of σ-algebras and so on for Hunt processes (using in the discussion of smooth processes, for example, the space of right continuous functions with left limits from $[0, \infty)$ into the entrance state space less the branch points (where branch points are allowed as left limits), identifying the value of the function at 0 with $x(0+)$. The obvious changes are to be made on the rare occasions when the absolute probability function is not chosen to ensure the existence of $x(0+)$. In particular, we shall use the translation operator θ_t. Note that our smooth processes are not necessarily quasi left continuous except in a slightly extended sense, but have the strong Markov property (see [3], [4]).

Let L be an entrance space. A terminal time will be called "admissible" if the following three conditions are satisfied:

(a) T is perfect (see [1]);

(b) for every nonbranch point ξ, $\lim_{n\to\infty} \theta_{s_n} T = 0$, P_ξ almost everywhere where $T = 0$, whenever $\{s_n, n \geqq 1\}$ is a sequence of positive numbers with limit 0;

(c) $P_\xi\{T > 0\} = 1$ if ξ is in I.

According to the Blumenthal zero-one law, the probability in (c) must be either 0 or 1 if ξ is not a branch point.

If ξ is not a branch point and $t > 0$, define

$$p^T(t, \xi, k) = P_\xi\{x(t) = k, T > t\}. \tag{3.1}$$

Then the Chapman-Kolmogorov equation system

$$\begin{aligned} p^T(s + t, \xi, k) &= \sum_j P_\xi\{x(s + t) = k, x(s) = j, T > s, \theta_s T > t\} \\ &= \sum_j E_\xi\{P\{x(s + t) = k, \theta_s T > t \mid \mathscr{F}(s)\}\, 1_{T > s, x(s) = j}\} \\ &= \sum_j p^T(s, \xi, j) p^T(t, j, k) \end{aligned} \tag{3.2}$$

is satisfied.

If ξ is a branch point, we make the obvious definition corresponding to starting the process at ξ and having it jump at once:

$$p^T(t, \xi, k) = \int_L p^T(t, \eta, k)p(0+, \xi, d\eta), \tag{3.3}$$

where we use the fact that $p(0+, \xi, \cdot)$ is supported by the set of nonbranch points. The Chapman-Kolmogorov equation system is satisfied by p^T for every initial state in L. If $s < t$,

$$\sum_j p(s, \xi, j)p^T(t - s, j, k) = P_\xi\{x(t) = k, \theta_s T > t - s\} \geqq p^T(t, \xi, k) \tag{3.4}$$

and the left side has limit $p^T(t, \xi, k)$ when $s \to 0$. Then the function $p(\cdot, \cdot, k) - p^T(\cdot, \cdot, k)$ is space time excessive (see [4]). Instead of $p(\cdot, \cdot, k)$, any other function on $(0, \infty) \times L$ which dominates $p^T(\cdot, \cdot, k)$ and is an exit law can be used, for example, the constant function 1.

Define R, R^T by

$$R_{\xi,j}(\lambda) = \int_0^\infty e^{-\lambda s}p(s, \xi, j)\, ds, \qquad R^T_{\xi,j}(\lambda) = \int_0^\infty e^{-\lambda s}p^T(s, \xi, j)\, ds \tag{3.5}$$

for $\lambda > 0$. Then $R_{\cdot,j}(\lambda)$ is λ excessive and what we have proved about $p - p^T$ implies that $R_{\cdot,j}(\lambda) - R^T_{\cdot,j}(\lambda)$ is also λ excessive.

A terminal state Δ can be introduced in the usual way to make p^T stochastic, and this function, with second and third arguments restricted to $I^\Delta = I \cup \{\Delta\}$ is then a standard transition function relative to I^Δ.

4. Terminal time examples

Throughout the rest of this paper we consider only the stable case. For a function which is right continuous with left limits from $[0, \infty)$ into an entrance space, we call a point of $[0, \infty)$ an "explosion point" of that function if it is a limit point of discontinuities. The set Γ of explosion points is closed. Because of the nature of the sample functions of smooth processes in the stable case, the set Γ is, stated roughly, determined by a sample function in such a way that Γ is independent of the state space. More precisely, consider a process $\{x(t), t \text{ rational} > 0\}$ with the given transition function and some absolute probability function. We take the state space as I (untopologized). Almost every sample function is identically constant in a neighborhood of each rational parameter value. Define an explosion time for a sample function as any real number with the property that in every neighborhood of the number there are infinitely many maximal constancy intervals. If L is any entrance space, the process can be extended into a smooth process with state space L by defining $x(t)$ for irrational $t > 0$ as $x(t+)$ when this limit exists in the L topology. For almost every sample function, the set Γ is then the same whether defined in terms of the original or in terms of the extended process.

Define a terminal time T_1 extending the definition in Section 2, as the infimum of points in $\Gamma - \{0\}$, (or ∞, if there is no such point). The qualification in parentheses will be omitted in further definitions. Then T_1 is admissible. Note that $x(T_1)$ may be integer valued and that $\theta_{T_1}T_1$ need not be 0.

Define $T_{1,1} = T_1$. If α is a countable ordinal for which $T_{1,\beta}$ is defined when $\beta < \alpha$, define $T_{1,\alpha} = \sup_{\beta<\alpha} T_{1,\beta}$ if α is a limit ordinal and $T_{1,\alpha} = T_{1,\alpha-1} + \theta_{T_{1,\alpha-1}}T_1$ otherwise. There is a first countable α, depending on the sample function, with $T_{1,\alpha} = T_{1,\alpha+1} = \cdots$; using this α, we define $T_{1,\infty} = T_{1,\alpha}$. Then $T_{1,\infty}$ is a terminal time, because this method applied to any terminal time yields a terminal time. The admissible terminal time $T_2 = \lim_{s\to 0}\theta_s T_{1,\infty}$ is the first limit point from the right of explosion times. The method of obtaining T_2 from T_1 is now applied to T_2 to obtain an admissible terminal time T_3, and so on. More precisely, if T_α is already defined for α a countable ordinal $T_{\alpha+1}$ is obtained from T_α as T_2 was from T_1. If T_β is defined for $\beta < \alpha$, where α is a (countable) limit ordinal, T_α is defined as $\lim_{s\to 0}\theta_s\,(\sup_{\beta<\alpha}T_\beta)$. A standard argument shows that for some countable ordinal α, depending on the sample function, $T_\alpha = T_{\alpha+1}$, so that $\theta_{T_\alpha}T_\alpha = 0$, and that, for any assignment of probability measures to the Markov process in question, there is an index α independent of the sample function, such that $T_\alpha = T_{\alpha+1}$ almost surely. Thus, if T_∞ is defined as T_α for α large enough to make $T_\alpha = T_{\alpha+1}$, we obtain a stopping time with $\theta_{T_\infty}T_\infty = 0$. Every terminal time we have obtained here is defined in terms of explosion times and is therefore meaningful for every entrance space, and the meanings are the same for all entrance spaces just as those of T_1 are.

Obviously, p^{T_1} is determined completely by the matrix $(q_{i,j})$ if the initial state is an integer. If the initial state is neither an integer nor a branch point, the continuity properties of space time excessive functions [4] imply that

$$p^{T_1}(t, \xi, j) = \limsup_{i\to\xi} p^{T_1}(t, i, j). \tag{4.1}$$

If the initial state is a branch point, $p^{T_1}(t, \xi, \cdot)$ is determined by (3.3). Thus, p^{T_1} is determined by $(q_{i,j})$, the branching distributions, and the entrance space topology. Evidently these same elements also determine $p^{T_2}, \cdots, p^{T_\infty}$.

5. Ramification of an entrance space

Let K_0 be an entrance space, a Borel subset of the compact metric space K, with metric ρ. Let T be an admissible terminal time defined in terms of the explosion points of sample functions. Then T has a well-defined meaning for every entrance space, in fact, a meaning independent of the entrance space (see Section 4). Enlarge K to K^Δ by the adjunction of an isolated state Δ, setting $\rho(\xi, \Delta) = 1$ and $p(t, \Delta, \Delta) = 1$. Let ρ^* be a metric on $I^\Delta = I \cup \{\Delta\}$, chosen in such a way that if K^* is the completion of I^Δ under ρ^* there is a Borel subset K_0^* of K^* which is an entrance space for the restriction of p^T to $(0, \infty) \times I^\Delta \times I^\Delta$. Define R, R^T as in Section 3. It is known (see [3], for example) that ρ^* can be chosen so that $\{i_n, n \geqq 1\}$ is a Cauchy sequence if and only if $\{R^T_{i_n,j}(\lambda), n \geqq 1\}$ is a Cauchy sequence for all j and all $\lambda > 0$, equivalently for λ in a countable dense subset of $(0, \infty)$. We assume that such a choice has been made.

Now let $\hat{K}$ be the completion of I^Δ in the metric $\rho + \rho^*$. Then $\hat{K}$ is a compact metric space in which I^Δ is dense and there is a unique continuous map α $[\alpha^*]$ from $\hat{K}$ onto K^Δ $[K^*]$ leaving I^Δ invariant. Since $\hat{K}$ can also be considered the completion of K^Δ $[K^*]$ in the metric $\rho + \rho^*$, K^Δ and K^* can be thought of as subsets of $\hat{K}$.

Define

$$
(5.1)\qquad \begin{aligned} &\hat{K}_0 = \alpha^{-1}(K_0^\Delta) \cap \alpha^{*-1}(K_0^*) \\ &\hat{p}(t, \xi, j) = p(t, \alpha(\xi), j), \qquad \hat{p}_T(t, \xi, j) = p^T(t, \alpha^*(\xi), j), \qquad \xi \in \hat{K}_0, \end{aligned}
$$

so that $\hat{p} = p$ and $\hat{p}_T = p^T$ on $(0, \infty) \times I^\Delta \times I^\Delta$. For each $j, \hat{p}(\cdot, \cdot, j)$ and $\hat{p}_T(\cdot, \cdot, j)$ are continuous on $(0, \infty) \times \hat{K}_0$. (In discussing these functions, Δ is considered an honorary integer.) Moreover, $\hat{p}$ and $\hat{p}_T$ are stochastic transition functions satisfying the Chapman-Kolmogorov equation system.

Let $\{x(t), t \text{ rational} > 0\}$ be a process with state space I and an integer initial state. Almost every sample function of this process has right and left limits at all real positive ($\geqq 0$) times in the K topology. Furthermore, the fact that $R_{\cdot,j}(\lambda)$ and $R_{\cdot,j}(\lambda) - R^T_{\cdot,j}(\lambda)$ are both λ excessive on K (see Section 3) implies that almost every sample function of the $R^T_{x(t)j}(\lambda)$ process has right and left limits at all real positive times in the K^* topology. We conclude that almost every $x(\cdot)$ process sample function has right and left limits at all real positive times in the K^* topology and therefore also in the $\hat{K}$ topology.

Define $\hat{x}(t) = x(t+)$ (limit in the $\hat{K}$ topology). The process $\{\hat{x}(t), t \geqq 0\}$ is a smooth process with state space $\hat{K}_0$ and the image process $\{\alpha[\hat{x}(t)], t \geqq 0\}$ is a smooth process with state space K_0 and the same initial state. The three processes $\{\alpha[\hat{x}(t)], t \geqq 0\}$, $\{\alpha^*[\hat{x}(t)], t \geqq 0\}$, $\{\hat{x}(t), t \geqq 0\}$, when made identically Δ at times $\geqq T$ have identical finite dimensional distributions. More generally, the corresponding discussion goes through for any absolute probability function, for example by starting processes at time $1/n$ and then making $n \to \infty$. Thus, we have the situation where K_0 satisfies conditions (a), (b), (c) of an entrance space relative to the restrictions to $(0, \infty) \times I^\Delta \times I^\Delta$ of both p and p^T. That is, in the terminology of [4], $\hat{K}_0$ is an entrance adapted space for these two restrictions. According to [4], there is then a Borel subset L of $\hat{K}_0$ which is an entrance space for the first restriction and therefore for the second restriction (because if $\hat{x}(0+)$ exists this right limit also exists when the process is made identically Δ at the times $\geqq T$). We observe that we have now identified $\hat{p}_T$ with $\hat{p}^T$.

We have thus found an entrance space L on which the function $\hat{p}^T(\cdot, \cdot, j)$ is continuous as desired. A trivial modification of this discussion would yield the corresponding result simultaneously for countably many admissible terminal times T. Thus, for example, there is a space which is simultaneously an entrance space for every T_j in Section 4. Moreover, the corresponding reversed processes can be handled at the same time, for example, to make the final state space also an entrance-exit space.

6. Continuation of Markov chains

Analyses of the continuation of a Hunt type Markov process from the first time a set is hit have been made by many authors (for example recently by Dynkin [7], Motoo [9], and Okabe [10]). The corresponding analysis of Markov chains in the stable case has been limited to the analysis of what happens after the first explosion time, with strong restrictions on the number of ways sample functions can "go to ∞" at this time (see, for example, Chung [2] and the more recent Shih [12]). The techniques used in the analysis of Hunt process continuation have not been applied to the stable chain case. We shall now make such an application, following Dynkin [7] (who treated Hunt processes) with appropriate modifications. A new difficulty is the fact that no attack on chains can avoid the possibility of branch points, whereas the no-branch-point hypothesis has not been thought improperly restrictive in the Hunt theory and the associated theory of additive functionals.

Let L be an entrance space for our transition function in the stable case and let T be an admissible terminal time depending on the explosion points with the additional property that $\theta_T T = 0$. For example, T_∞, as defined in Section 4, is such a terminal time. Then p^T is well defined and we suppose that L has been ramified if necessary to make $p^T(\cdot, \cdot, j)$ continuous for every choice of j including the terminal state. The function $R^T_{\cdot,j}(\lambda)$ is then continuous on L. More generally, if f is a bounded function on L (or even merely on I) and if the obvious operational notation is used, the function $R^T(\lambda)f$ is continuous on L. This follows, for example, from the particular case just mentioned and the fact that the function

$$\sum_j R^T_{\cdot,j}(\lambda) = \frac{1}{\lambda} - R_{\cdot,\Delta}(\lambda) \tag{6.1}$$

is continuous, so that the sum on the left converges uniformly on compacta.

Let F be the closed set of points ξ at which $P_\xi\{T = 0\} = 1$, that is, at which $\Sigma_j p^T(t, \xi, j) = 0$ for all $t > 0$, or equivalently at which $\lambda R^T_{\xi,\Delta}(\lambda) = 1$ for every $\lambda > 0$. Let F_b be the set of branch points in F. Since $\theta_T T = 0$, for any smooth process $\{x(t), t > 0\}$ the random variable $x(T)$ has its values almost surely in F where $0 < T < \infty$, including 0 if $x(0+)$ exists and $x(0)$ is defined as this right limit. In fact, the distribution of $x(T)$ is supported by $F - F_b$, since almost no sample path hits a branch point.

If S is the hitting time of F, then $T = S + \theta_S T$ where $T > S$, whereas $\theta_S T = 0$ by definition of F. Thus, $T = S$ is the hitting time of F, and the condition $\theta_T T = 0$, combined with condition (b) for an admissible stopping time, implies that every point ξ of F is regular for F. Then $p(0+, \xi, F) = 1$ for

ξ in F. For a smooth process, if T is the limit of an increasing sequence $\{S_n, n \geqq 1\}$ of stopping times then $p[0+, x(T-), F] = 1$ almost everywhere where $S_n < T < \infty$ for all n. In particular if $T = T_\infty$ as defined in Section 4, then $x(T-) \in F$ almost everywhere where $0 < T < \infty$.

Since $\Sigma_j R^T_{\xi,j}(\cdot)$ is the Laplace transform of a monotone decreasing function, the function $\lambda \rightsquigarrow \lambda\Sigma_j R^T_{\xi,j}(\lambda)$ is an increasing function. Then if α is a strictly positive number, to be retained thoughout the following, and if $K_\lambda f = R^T(\lambda)f/R^T(\alpha)1$ off F, then off F

$$|K_\lambda f| \leqq \begin{cases} \sup|f| & \text{if } \lambda \geqq \alpha, \\ \frac{\alpha}{\lambda} \sup|f| & \text{if } \lambda \leqq \alpha. \end{cases} \tag{6.2}$$

Let $\{x(t), t \geqq 0\}$ be a smooth process with $x(0) = \xi$, where we allow ξ to be arbitrary; smoothness at 0 in this context means that $x(0+)$ exists, but is almost certainly ξ if and only if ξ is not a branch point. Then, since almost every sample function is right continuous on $(0, \infty)$ (and we ignore the exceptional sample functions below), the parameter set for which a sample path lies in the open set $L - F$ is the union of disjoint (maximal) intervals open on the right. We denote the endpoints of the intervals generically by γ, δ. Here $\gamma + \theta_\gamma T = \delta$. Since almost every sample function is integer valued for Lebesgue almost every parameter value, these intervals cover Lebesgue almost every point of $(0, \infty)$. In this property, the context is simpler than that in [7], [9], [10]. As Dynkin pointed out in [6], for a smooth process and bounded f, under certain hypotheses satisfied in the present study, $K_\lambda f$ has a right limit at every left endpoint γ, for almost every sample function. We shall use this fact.

Let f be a bounded function on L (only its values on I are relevant). Let h be a bounded continuous function on $L - F$ with the property that for almost every sample function of a smooth process $\lim_{s\to 0} h[x(\gamma + s)]$ exists for all γ. We denote this limit by $h[x(\gamma)]^+$. In a similar context [7] Dynkin proved that

$$\begin{aligned} E_\xi\left\{\sum \exp\{-\lambda\gamma\}\, h[x(\gamma)]^+ \int_\gamma^\delta f[x(t)] \exp\{-\mu(t-\gamma)\}\, dt\right\} \\ = E_\xi\left\{\sum \exp\{-\lambda\gamma\}\, (hK_\mu f)[x(\gamma)]^+ \int_\gamma^\delta \exp\{-\alpha(t-\gamma)\}\, dt\right\}. \end{aligned} \tag{6.3}$$

Here λ and μ are strictly positive and the sum here and below is over all strictly positive γ. (Dynkin's proof yields (6.3), although his stated hypotheses are more special than ours.) Define ϕ^λ_h by

$$\phi^\lambda_h(\xi) = E_\xi\left\{\sum \exp\{-\lambda\gamma\}\, h[x(\gamma)]^+ \int_\gamma^\delta \exp\{-\alpha(t-\gamma)\}\, dt\right\}. \tag{6.4}$$

It is easy to see that this function is λ excessive and Dynkin's proof [7] that ϕ_h^λ is a regular λ potential is valid here. Thus, applying the Šur-Meyer theorem, we find as Dynkin did, that there is a continuous additive functional A_h^λ such that

(6.5) $$\phi_h^\lambda(\xi) = E_\xi\left\{\int_0^\infty \exp\{-\lambda t\}\, dA_h^\lambda(t)\right\}.$$

Dynkin's proof in [7] that A_h^λ does not depend on λ is valid in our context and we omit the superscript from now on. We shall write A instead of A_1. When $\lambda = \alpha$ and $h = 1$, we find

(6.6) $$\phi_1^\alpha(\xi) = E_\xi\{e^{-\alpha T}\}\alpha^{-1} = E_\xi\left\{\int_0^\infty e^{-\alpha t}\, dA(t)\right\}.$$

Hence,

(6.7) $$E_\xi\{e^{-\alpha T}\phi_1^\alpha[x(T)]\} = E_\xi\{e^{-\alpha T}\}\alpha^{-1} = E_\xi\left\{\int_{(T,\infty)} e^{-\alpha t}\, dA(t)\right\},$$

where we have used the fact that $\theta_T T = 0$ and must make the obvious conventions if T is not finite. Comparing (6.7) with (6.6), we see that $A(T) = 0$, P_ξ almost surely for all ξ. Thus (see [1]), A is supported by F; more precisely, the measure $dA(t)$ is P_ξ almost surely supported by the set of t with $x(t)$ in F.

Define for g a bounded Borel measurable function on L,

(6.8) $$v^\lambda(\xi, g) = E_\xi\left\{\int_0^\infty e^{-\lambda t} g[x(t)]\, dA(t)\right\}.$$

Then $v^\lambda(\xi, \cdot)$ defines a measure, and $v^\lambda(\xi, g)$ is the integral of g with respect to this measure. Below, "null set" is a set of $v^\lambda(\xi, \cdot)$ measure 0 for all ξ. This condition is independent of $\lambda > 0$. For example, $L - F$ is a null set.

By hypothesis, the space L is a Borel subset of a compact metric space L'; it is convenient to introduce L' at this point in order to follow Dynkin in [7]. By a linearity argument, he shows (translating his result into our context) that if ξ is not in some null set then there is a measure $b(\xi, \cdot)$ of Borel subsets of L' with $b(\xi, L') \leqq 1$ such that if h is continuous on L', $b(\cdot, h)$ is Borel measurable and $dA_h(t) = b[x(t)]\, dA(t)$. Thus, for such a choice of h, and hence for every bounded Borel measurable h on L',

(6.9) $$E_\xi\left\{\sum \exp\{-\lambda\gamma\}\, h[x(\gamma)] \int_\gamma^\delta \exp\{-\alpha(t-\gamma)\}\, dt\right\} = E_\xi\left\{\int_0^\infty \exp\{-\lambda t\}\, b[x(t), h]\, dA(t)\right\}.$$

If h is the indicator function of L and $\lambda = \alpha$, (6.7) together with (6.9) imply that $b(\xi, L) = 1$ for ξ not in a null set. In other words for ξ not in a null set, $b(\xi, \cdot)$ is a probability measure supported by L, and we can drop L' again. If f is bounded on L, (6.3) yields

$$\begin{aligned} (6.10)\qquad & E_\xi\left\{\int_T^\infty \exp\{-\lambda t\} f[x(t)]\,dt\right\} \\ &= E_\xi\left\{\sum \exp\{-\lambda\gamma\} \int_\gamma^\delta \exp\{-\lambda(t-\gamma)\} f[x(t)]\,dt\right\} \\ &= E_\xi\left\{\sum \exp\{-\lambda\gamma\}\,(K_\lambda f)[x(\gamma)]^+ \int_\gamma^\delta \exp\{-\alpha(t-\gamma)\}\,dt\right\}. \end{aligned}$$

If $K_\lambda f$ were defined bounded and Borel measurable on L, with $(K_\lambda f)[x(\gamma)] = (K_\lambda f)[x(\gamma)]^+$, or if $h[x(\gamma)]$ in (6.9) could be replaced by $h[x(\gamma)]^+$, then we could identify $K_\lambda f$ with h in (6.9) to get

$$\begin{aligned} (6.11)\quad E_\xi\left\{\int_T^\infty \exp\{-\lambda t\} f[x(t)]\,dt\right\} &= E_\xi\left\{\int_0^\infty \exp\{-\lambda t\}\, b[x(t), K_\lambda f]\,dA(t)\right\} \\ &= v^\lambda[\xi, b(\cdot, K_\lambda f)]. \end{aligned}$$

In order to get some version of (6.11), one can ramify $L - F$ to a space on which $K_\lambda f$ has a continuous extension [7]. The measure $v^\lambda(\xi, \cdot)$ is then a measure on this new space. With this interpretation of (6.11), the representation of the resolvent R in terms of R^T, v, K_λ is now trivial:

$$\begin{aligned} (6.12)\qquad [R(\lambda)f](\xi) &= E_\xi\left\{\int_0^\infty \exp\{-\lambda t\} f[x(t)]\,dt\right\} \\ &= [R^T(\lambda)f](\xi) + v^\lambda[\xi, b(\cdot, K_\lambda f)]. \end{aligned}$$

The continuous additive functional A determines the "boundary" process on F as usual.

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