LIMIT THEOREMS FOR SUMS OF A RANDOM NUMBER OF POSITIVE INDEPENDENT RANDOM VARIABLES

B. V. GNEDENKO
MOSCOW UNIVERSITY

1. Statement of the problem

The work of A. Wald [11] and H. Robbins [10] played an important role in stimulating interest in the investigation of sums of a random number of independent random variables. Of late a large number of papers related to this topic have been published and their importance for numerous applied questions as well as theoretical mathematics has been shown.

Results of research, conducted over the last three years by myself and my students in connection with problems in the theory of reliability, theory of queuing, physics, and production organization, are presented in this paper. We shall begin the presentation with the consideration of one of these problems. Some of the formulations of problems and theorems which will be included here have been presented before when I delivered lectures at the Universities of London, Sheffield, Rome, Budapest and Warsaw where I was a guest in recent years. Several statements of problems and their solutions originated there.

Geiger-Müller counters are used in nuclear physics and also in the study of cosmic radiation. A particle having struck the counter and having been counted by it causes a breakdown in it lasting some time τ. Any particle striking the counter in the period of the breakdown is not registered by it. Usually one assumes that τ is a constant; however, a more realistic assumption is that τ is a random variable with some distribution $G(x)$. Our problem is to determine the distribution of the length of the time interval from the first registration of a particle to the first loss. Here we shall assume that the time intervals between successive entries of particles are independent and identically distributed with distribution function $F(x)$.

It is obvious that the counter does not lose a particle until the duration of the breakdown is as large as the interval between successive entries of particles. (See Figure 1.) This permits us to write the following equality:

$$\eta = \xi_1 + \xi_2 + \cdots + \xi_\nu, \tag{1.1}$$

where ξ_k is the time between the arrival of the kth particle and the $(k + 1)$st particle and $\nu = \min \{k, \xi_k < \tau_k\}$.

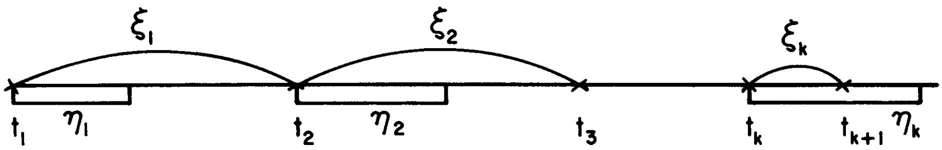

FIGURE 1
Entry of particles.

We introduce the notation $\alpha = P\{\xi \geqq \tau\}$. It is clear that the random variable ν has the geometric distribution

$$p_k = P\{\nu = k\} = \alpha^{k-1}(1 - \alpha), \qquad k = 1, 2, 3, \cdots . \tag{1.2}$$

Note that the number of terms in the sum (1.1) depends on the values which the summands ξ_k take. Since the event $\{\nu = k\}$ does not depend on the ξ_n, for $n > k$, one can use Wald's identity

$$E\eta = E\xi_1 \cdot E\nu, \tag{1.3}$$

for computing the mathematical expectation of η. Because $E\nu = 1/(1 - \alpha)$, then

$$E\eta = \frac{E\xi_1}{1 - \alpha}. \tag{1.4}$$

Computation of the distribution function of the variable η requires more complicated reasoning to which we now turn. Let $\Phi(x) = P\{\eta < x\}$, $\bar{\Phi}(x) = 1 - \Phi(x)$. It is not difficult to see that

$$\bar{\Phi}(t) = \bar{F}(t) + \int_0^t \bar{\Phi}(t - x)G(x)\, dF(x). \tag{1.5}$$

This equation is solved in terms of the Laplace transform:

$$\varphi(s) = \frac{f(s) - g(s)}{1 - g(s)}, \tag{1.6}$$

where

$$\varphi(s) = \int_0^\infty e^{-sx}\, d\Phi(x), \qquad f(s) = \int_0^\infty e^{-sx}\, dF(x), \tag{1.7}$$
$$g(s) = \int_0^\infty e^{-sx}\, G(x)\, dF(x).$$

Very often counters work under conditions where the duration of the breakdown is, as a rule, significantly smaller than the interval between entries of successive particles into the counter. In other words, there is interest in considering the case

$$\alpha = P\{\xi \geqq \tau\} \approx 1. \tag{1.8}$$

More precisely, we shall assume at the beginning that there is an integral parameter n on which the function $G(x) = G_n(x)$ depends in such a way that as $n \to \infty$

$$\alpha_n = \int_0^\infty G_n(x)\, dF(x) \to 1, \tag{1.9}$$

but $\alpha_n \neq 1$.

One may ask if it is possible to prove some general regularities in the behavior of the sums $S_n = \xi_1 + \xi_2 + \cdots + \xi_{\nu_n}$ as $n \to \infty$.

2. Convergence to an exponential distribution

The purpose of this section is to prove the following two theorems.

THEOREM 1. *If the random variables ξ_k have a finite mathematical expectation and condition* (1.9) *is satisfied, then as $n \to \infty$*

$$P\left\{\frac{S_n}{ES_n} < x\right\} \to 1 - e^{-x}. \tag{2.1}$$

PROOF. Let $T_n = ES_n$. Then the Laplace transform of the distribution of S_n/ES_n is

$$\varphi\left(\frac{s}{T_n}\right) = \frac{f(s/T_n) - g(s/T_n)}{1 - g(s/T_n)}. \tag{2.2}$$

For brevity we introduce the notation

$$\beta_n(s) = f\left(\frac{s}{T_n}\right) - g\left(\frac{s}{T_n}\right). \tag{2.3}$$

It is obvious that for arbitrary $s \geqq 0$,

$$\beta_n(s) \leqq \beta_n(0) = 1 - g(0) = \int_0^\infty \big(1 - G_n(x)\big)\, dF(x) = 1 - \alpha_n. \tag{2.4}$$

Our immediate problem is to prove that, as $n \to \infty$

$$\frac{\beta_n(s)}{\beta_n(0)} \to 1 \tag{2.5}$$

uniformly in every finite interval of $s \geqq 0$. To this aim, we consider the difference

$$0 \leqq \beta_n(0) - \beta_n(s) = \int_0^\infty \big(1 - \exp\{-sx/T_n\}\big)\big(1 - G_n(x)\big)\, dF(x). \tag{2.6}$$

Since for any $z \geqq 0$, $1 - e^{-z} \leqq z$, one can write the following chain of inequalities:

$$\beta_n(0) - \beta_n(s) \leqq \frac{s}{T_n} \int_0^\infty x\big(1 - G_n(x)\big)\, dF(x) \tag{2.7}$$
$$\leqq \frac{s}{T_n} \left[\sqrt{T_n} \int_0^\infty [1 - G_n(x)]\, dF(x) + \int_{\sqrt{T_n}}^\infty x\, dF(x) \right].$$

Since $T_n = a/\beta_n(0)$, where $a = E\xi_k$, we arrive at the inequality

$$\beta_n(0) - \beta_n(s) \leqq \frac{s}{a} \beta_n(0) \left[\big(a\beta_n(0)\big)^{1/2} + \int^\infty x\, dF(x) \right]. \tag{2.8}$$

From the hypotheses of the theorem, both terms on the right side of the inequality tend to zero as $n \to \infty$. This completes the proof.

But

$$\varphi_n\left(\frac{s}{T_n}\right) = \frac{\big(\beta_n(s)/\beta_n(0)\big)}{\big(\beta_n(s)/\beta_n(0)\big) - \left[\left(f\left(\frac{s}{T_n}\right) - 1\right)\Big/\beta_n(0)\right]}. \tag{2.9}$$

Hence, from what has just been proved and the relation

$$\lim_{n\to\infty} \frac{f(s/T_n) - 1}{\beta_n(0)} = sa^{-1} \lim \frac{f(s/T_n) - 1}{s/T_n} = \frac{s}{a} f'(0) = -s, \tag{2.10}$$

we conclude that

$$\lim_{n\to\infty} \varphi_n\left(\frac{s}{T_n}\right) = \frac{1}{1+s}. \tag{2.11}$$

The theorem has been proved.

It is natural to ask oneself about the set of all those $F(x)$ for which, with suitably chosen normalizing factors $B_n > 0$, the distribution functions of the sums

$$\frac{S_n}{B_n} = \frac{\xi_1 + \cdots + \xi_{\nu_n}}{B_n} \tag{2.12}$$

approach the exponential distribution $\Phi(x) = 1 - e^{-x}$. The answer to this question is given by the following theorem.

THEOREM 2. *In order that the distribution functions of the sums* $(\xi_1 + \cdots + \xi_{\nu_n}/B_n)$ *with suitably chosen nonrandom* $B_n > 0$ *converge as* $n \to \infty$ *to the exponential distribution, it is necessary and sufficient that the condition*

$$\lim_{x\to\infty} \frac{x[1 - F(x)]}{\int_0^x [1 - F(z)]\, dz} = 0 \tag{2.13}$$

is satisfied.

We refer to [4] for the proof of this theorem. However, we note that condition (2.13) was mentioned as long ago as 1935 by A. Ya. Khinchin [7]. In this work he introduced the concept of relative stability of sums which was a generalization of the law of large numbers for nonnegative random variables $\xi_1, \xi_2, \cdots$. The sums $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ are called relatively stable if there exist normalizing factors $B_n > 0$ such that for arbitrary $\varepsilon > 0$

$$P\left\{\left|\frac{S_n}{B_n} - 1\right| \leqq \varepsilon\right\} \to 1 \tag{2.14}$$

as $n \to \infty$.

For identically distributed summands, condition (2.13) is necessary and sufficient for the relative stability of sums. This theorem was proved in [7].

3. The class of limit distributions

The limit theorems which were proved in the preceding paragraph permit an important generalization. We ask ourselves about the class of limit distributions for the normalized sums

$$S_n = \frac{\xi_1 + \xi_2 + \cdots + \xi_{\nu_n}}{B_n} \tag{3.1}$$

of independent, identically distributed, nonnegative random variables $\xi_1, \xi_2, \cdots$, with suitably chosen constants B_n. The random variables ν_n are defined as before and satisfy condition (1.9). In addition, we assume that as n increases, the mathematical expectations of the variables ν_n do not increase too rapidly. More precisely, we assume that as $n \to \infty$

$$\frac{E\nu_{n+1}}{E\nu_n} = \frac{1 - \alpha_n}{1 - \alpha_{n+1}} \to 1. \tag{3.2}$$

This section is devoted to the proof of the following theorem.

THEOREM 3. *Under the conditions just formulated, the distribution function* $\Psi(x)$ *can be a limit of the distributions of sums* (3.1) *if and only if its Laplace transform has the form*

$$\psi(s) = \frac{1}{1 + Cs^{\gamma}}, \tag{3.3}$$

where C *and* γ *are nonnegative constants,* $0 < \gamma \leqq 1$.

We shall preface the proof of this theorem with several lemmas.

LEMMA 1. *If nonnegative functions* $\varphi_1(x)$ *and* $\varphi_2(x)$ *are respectively nondecreasing and nonincreasing,* $F(x)$ *is a distribution function and all integrals under discussion exist, then*

$$\int_0^\infty \varphi_1(x)\varphi_2(x)\, dF(x) \leqq \int_0^\infty \varphi_1(x)\, dF(x) \cdot \int_0^\infty \varphi_2(x)\, dF(x). \tag{3.4}$$

PROOF. Indeed, let $a = \int_0^\infty \varphi_2(x)\, dF(x)$ and $z = \sup\{x, \varphi_2(x) \geqq a\}$. Then, for $x > z$ the inequality $\varphi_2(x) < a$ is satisfied. Now it is obvious that

$$(3.5)\qquad \int_0^\infty \varphi_1(x)\varphi_2(x)\,dF(x) - \int_0^\infty \varphi_1(x)\,dF(x)\cdot\int_0^\infty \varphi_2(x)\,dF(x)$$

$$= \int_0^\infty \varphi_1(x)[\varphi_2(x) - a]\,dF(x)$$

$$= \int_0^z \varphi_1(x)[\varphi_2(x) - a]\,dF(x) + \int_z^\infty \varphi_1(x)[\varphi_2(x) - a]\,dF(x)$$

$$\leqq \varphi_1(z)\int_0^z [\varphi_2(x) - a]\,dF(x) + \varphi_1(z)\int_z^\infty [\varphi_2(x) - a]\,dF(x) = 0.$$

This inequality proves the lemma.

LEMMA 2. $(\beta_n(s))/(\beta_n(0)) \to 1$ *uniformly in every finite interval of* s *as* $n \to \infty$.

PROOF. It is clear that

$$(3.6)\qquad 0 \leqq 1 - \frac{\beta_n(s)}{\beta_n(0)}$$

$$= \frac{\beta_n(0) - \beta_n(s)}{\beta_n(0)}$$

$$= \frac{1}{\beta_n(0)}\left[\int_0^\infty [1 - G_n(x)]\,dF(x) - \int_0^\infty e^{-sx/B_n}[1 - G_n(x)]\,dF(x)\right]$$

$$= \frac{1}{\beta_n(0)}\int_0^\infty (1 - e^{-sx/B_n})(1 - G_n(x))\,dF(x).$$

According to Lemma 1,

$$(3.7)\qquad \int_0^\infty (1 - e^{-sx/B_n})(1 - G_n(x))\,dF(x)$$

$$\leqq \int_0^\infty (1 - e^{-sx/B_n})\,dF(x)\int_0^\infty (1 - G_n(x))\,dF(x)$$

$$= \beta_n(0)\left[1 - f\left(\frac{s}{B_n}\right)\right].$$

Since $B_n \to \infty$ as $n \to \infty$, the lemma is proved.

Lemmas 1 and 2 were proved by A. D. Soloviev. The first lemma is essentially nothing more than the assertion of the negativeness of the correlation coefficient of two random variables, one of which decreases when the other increases.

We note that by Lemma 2, if the limit distribution of the sums (3.1) exists and its Laplace transform is $\psi(s)$, then the limit

$$(3.8)\qquad \lim_{n\to\infty} \frac{1 - f(s/B_n)}{B_n(0)} = \alpha(s)$$

exists and

$$\psi(s) = \frac{1}{1 + \alpha(s)}. \tag{3.9}$$

Our problem now is to find the class of possible functions $\alpha(s)$. With this goal, we note that for any $s > 0$ and $s' > 0$ one can find an integer $m = m_n(s'/s)$ such that as $n \to \infty$

$$\frac{B_n\, s/s'}{B_m} \to 1. \tag{3.10}$$

One can write the following identity:

$$\frac{1 - f\left(\dfrac{s'}{B_n}\right)}{\beta_n(0)} = \frac{1 - f\left(\dfrac{s}{B_n\, s/s'}\right)}{\beta_m(0)} \cdot \frac{\beta_m(0)}{\beta_n(0)}. \tag{3.11}$$

Since the limit as $n \to \infty$ of the left side of this equality exists as well as the limit of the first factor on the right side, the limit of the second factor must exist also, that is,

$$\alpha(s') = \alpha(s)\, k\left(\frac{s'}{s}\right). \tag{3.12}$$

But since $\alpha(s)$ for $\operatorname{Re} s > 0$ is an analytic function, we have the decomposition

$$\alpha(s') = \alpha(s) + \alpha'(s)(s' - s) + o(s' - s). \tag{3.13}$$

Using this decomposition, the preceding equality takes the form

$$\alpha'(s)(s' - s) = \alpha(s)\left[k\left(\frac{s'}{s}\right) - 1\right] + o(s' - s). \tag{3.14}$$

Note that $k(1) = 1$. We divide the last equality by $\alpha(s)(s' - s)$,

$$\frac{\alpha'(s)}{\alpha(s)} = \frac{k(s'/s) - k(1)}{(s'/s) - 1} \cdot \frac{1}{s} + o(1), \tag{3.15}$$

and let $s' \to s$. We obtain, as a result, the equation

$$\frac{\alpha'(s)}{\alpha(s)} = \frac{\gamma}{s}, \tag{3.16}$$

where $\gamma = [k'(s)]_{s=1}$. The solution of this equation has the form $\alpha(s) = C^{s\gamma}$, where C is an arbitrary constant.

Thus, if the limit distribution exists, its Laplace transform must have the form $\psi(s) = 1/(1 + Cs^\gamma)$.

We need to determine for which C and γ these functions can be Laplace transforms. It is easy to verify that this is possible only for $C > 0$ and for $0 < \gamma \leqq 1$, and also that every such function is a Laplace transform. The theorem has been proved.

The proof presented here was proposed for a closely related problem by I. N. Kovalenko [8].

4. Properties of the limit distributions

We shall now formulate a series of properties of the distributions of the class which we found. For brevity we shall denote this class by the symbol K.

PROPERTY 1. *All distributions of class K are infinitely divisible. This property can be proved by many methods. We shall prove it as a corollary to Theorem 6.*

PROPERTY 2. *Distributions of class K have densities; for $\gamma \neq 1$, these densities are unbounded.*

PROPERTY 3. *All distributions of class K are unimodal with the mode at the point $a = 0$.*

PROPERTY 4. *For $C = 1$ and $\gamma = 0.5$, the density of the distribution has the following form: for $x > 0$*

$$f(x) = \frac{1}{\sqrt{\pi x}} - \frac{2e^x}{\sqrt{\pi}} \int_{\sqrt{x}}^{\infty} e^{-z^2}\, dz. \tag{4.1}$$

The proof of all these statements can be found in [4].

We need to introduce a certain concept for the formulation and proof of the following property.

Let us consider a sequence of nonnegative random variables $\xi_1, \xi_2, \cdots$, independently distributed according to a law $\Psi(x)$. We mark on the axis OX the points $0, \xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \xi_3, \cdots$. Now let p be any number in the interval $0 \leqq p \leqq 1$. We shall perform the following operation: we begin to eliminate with probability p each of the marked points, excluding the point $x = 0$. We will call this operation screening with probability p. We denote by $\Phi_p(x)$ the probability distribution of the abscissa of the first point retained.

We will call the distribution $\Psi(x)$ stationary with respect to screening if for every p there exists $a_p > 0$ such that for all x, the equality

$$\Phi_p(x) = \Psi\left(\frac{x}{a_p}\right), \tag{4.2}$$

holds, or in terms of the Laplace transform $\varphi_p(s) = \psi(a_p s)$.

Property 5 is characteristic of the laws of the class K; therefore we shall formulate it in the form of the following theorem.

THEOREM 4. *The distribution $\Psi(x)$ belongs to class K if and only if it is stationary with respect to the operation of screening.*

PROOF. Since the probability that the first retained point will be the kth point in order is equal to $p^{k-1}q$, $k = 1, 2, \cdots$, we have

$$\Phi_p(x) = \sum_{k=1}^{\infty} p^{k-1} q \Psi^{(k)}(x). \tag{4.3}$$

In terms of the Laplace transform this equality takes the simple and elegant form

$$\varphi_p(s) = \frac{q\psi(s)}{1 - p\psi(s)}. \tag{4.4}$$

In one direction this theorem is proved immediately by substitution in the last equality. Actually, if $\psi(s) = 1/(Cs^\gamma + 1)$, then

$$\varphi_p(s) = \frac{q}{q + Cs^\gamma} = \frac{1}{1 + C(q^{-1/\gamma}s)^\gamma} = \psi(q^{-1/\gamma}s). \tag{4.5}$$

We have convinced ourselves that the operation of screening does not change the type of a distribution of the class K. The constants a_p are associated with the screening parameter p in the transform by the simple relation $a_p = q^{-1/\gamma}$, $q = 1 - p$. In particular, for $p = 0$, $a_p = 1$. This is obvious and immediate since for $p = 0$, the distribution $\Psi(x)$ is not affected by screening.

We shall now prove the converse of the theorem—that is, every distribution which does not change its type with screening belongs to the class K. For this we shall assume that, for arbitrary p, $(0 \leqq p \leqq 1)$ we have the identity

$$\varphi_p(s) = \frac{q\psi(s)}{1 - p\psi(s)} = \psi(a_p s), \tag{4.6}$$

where a_p depends on p, but does not depend on s. Hence

$$\psi(a_p s) - \psi(s) = p\,\frac{\psi(s)\big(\psi(s) - 1\big)}{1 - p\psi(s)} \tag{4.7}$$

or

$$\frac{\psi(a_p s) - \psi(s)}{(a_p - 1)s} = \frac{p}{a_p - 1}\cdot\frac{\psi(s)\big(\psi(s) - 1\big)}{s[1 - p\psi(s)]}. \tag{4.8}$$

Let $s > 0$. We pass to the limit in the last equality, assuming that $p \to 0$. The limit of the left side exists and is equal to $\psi'(s)$. The second factor on the right side also tends to a limit which is $(1/s)\psi(s)\big(\psi(s) - 1\big)$. Consequently the limit of the first factor on the right side exists; we will call this γ. By our assumption γ does not depend on s. As a result we arrive at the equation

$$\psi'(s) = \frac{\gamma}{s}\psi(s)[\psi(s) - 1]. \tag{4.9}$$

It is easy to verify that the general solution of this equation has the form $\psi(s) = 1/(1 + Cs^\gamma)$. We have earlier convinced ourselves that the only functions among these that are Laplace transforms of distribution functions are those with $C \geqq 0$ and $0 < \gamma \leqq 1$. We have arrived at the class K of distributions. The theorem has been proved.

The operation of screening was introduced by A. Rényi [9], and subsequently studied by Yu. K. Belyayev [1] and I. N. Kovalenko [8].

5. Domain of attraction of distributions of class K for $\gamma \neq 1$

Now we shall consider a problem that is in a sense the reverse of the above. We shall determine the conditions which the distribution $F(x)$ must satisfy in order that it lead to the given distribution of class K as a result of summation of a random number of random variables with suitable normalizing constants B_n. For brevity of speech with respect to such distributions, we will say that they belong to the domain of attraction of this limit law. Theorem 2 of this paper gives us necessary and sufficient conditions for the distribution $F(x)$ to belong to the domain of attraction of an exponential distribution. Our immediate problem is to determine the domains of attraction of all the remaining distributions of class K. The answer is contained in the following theorem.

THEOREM 5. *The function $F(x)$ belongs to the domain of attraction of the distribution $\Psi(x)$ with Laplace transform $\psi(s) = 1/(1 + Cs^{\gamma})$, if and only if, for every $c > 0$*

$$\lim_{x \to \infty} \frac{1 - F(cx)}{1 - F(x)} = c^{-\gamma}. \tag{5.1}$$

The proof of this theorem is contained in a joint paper by two authors [4] and is not reproduced here. I will note only that it, as well as the proof of Theorem 2, is based on a Tauberian theorem (see [2], Chapter XIII).

The following fact merits attention: the formulation of Theorem 5 literally coincides with the theorem on the domain of attraction of stable laws in the classical theorem on limit distributions (see [6], Section 35, Theorem 2).

Coincidences with classical theorems already noted here twice, as well as the fact that in class K, the parameter γ can only take values in the interval $0 < \gamma \leqq 1$ (we recall that stable distributions of nonnegative random variables are also possible only for values of the characteristic exponent in the interval zero to one) lead to a thought about a deep connection existing between the problem which we have been considering and the classical limit distributions for sums. Our task now is to demonstrate this connection.

6. A general limit theorem

We shall consider sequences of independent and identically distributed (for every n) random variables

$$\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \cdots \tag{6.1}$$

We introduce the notation

$$F_n(x) = P\{\xi_{n,k} < x\}, \qquad f_n(s) = \int_0^{\infty} e^{-sx}\, dF_n(x). \tag{6.2}$$

We shall assume further that the random variables of the sequence $\{\nu_n\}$ take on only nonnegative, integral values and for every n, the variable ν_n is independent of all variables $\xi_{n,k}$, $1 \leqq k < \infty$.

THEOREM 6. *If there exists a sequence of integers* k_n *such that as* $n \to \infty$

(i) $k_n \to \infty$,

(ii) $P\left\{\sum_{k=1}^{k_n} \xi_{n,k} < x\right\} \to \Phi(x)$,

(iii) $P\{\nu_n < xk_n\} \to A(x)$,

where $\Phi(x)$ *and* $A(x)$ *are distribution functions, then*

(iv) *the distributions of the sums*

$$S_{\nu_n} = \xi_{n,1} + \xi_{n,2} + \cdots + \xi_{n,\nu_n} \tag{6.3}$$

converge to the limit distribution function $\Psi(x)$ *whose Laplace transform is defined by the formula*

$$\psi(s) = \int_0^\infty \varphi^x(s)\, dA(x), \tag{6.4}$$

where $\varphi(s)$ *is the Laplace transform of the function* $\Phi(x)$.

PROOF. The Laplace transform of the sum S_{ν_n} is

$$\psi_n(s) = \sum_{j=0}^{\infty} p_{n,j}(f_n(s))^j, \tag{6.5}$$

where $p_{n,j} = P\{\nu_n = j\}$.

We put

$$A_n(x) = P\{\nu_n < x\}, \tag{6.6}$$

and then obviously

$$\psi_n(s) = \int_0^\infty f_n{}^x(s)\, dA_n(x). \tag{6.7}$$

Now let

$$\bar{A}_n(z) = P\left\{\frac{\nu_n}{k_n} < z\right\} = A_n(k_n z). \tag{6.8}$$

In our notation

$$\psi_n(s) = \int_0^\infty (f_n^{k_n}(s))^z\, dA_n(z). \tag{6.9}$$

From the assumptions of this theorem and known theorems on convergence of integrals

$$\psi_n(s) \to \psi(s) = \int_0^\infty \varphi^z(s)\, dA(z) \tag{6.10}$$

as $n \to \infty$.

The theorem has been proved. It is a special case of a result in [5].

According to the classical result of A. Ya. Khinchin, the function $\varphi(s)$ must be infinitely divisible. Thus $\varphi(s) = \exp\{\omega(s)\}$,

$$\omega(s) = \int_0^\infty \frac{1 - e^{-sx}}{x} \, dM(x), \tag{6.11}$$

$M(u)$ is monotonic, and $\int_1^\infty (1/x)\, dM(x) < \infty$.

As my students, D. Saas and B. Freier, have shown in the case considered here, one can prove two theorems that are in a sense converses of Theorem 6. Namely, if requirements (i), (ii), and (iv) are satisfied, then (iii) holds. If (i), (iii), and (iv) hold, then (ii) is satisfied.

In the problems studied here, the ν_n are distributed geometrically. Thus conditions (i) and (iii) are satisfied. By Theorem 3, condition (iv) is satisfied and the ν_n increase regularly. From the results of D. Saas and B. Freier just cited, condition (ii) must be satisfied. But this limit distribution must be stable. Let $\varphi(s) = e^{-Cs^\gamma}$ where $C > 0$ and $0 < \gamma \leqq 1$. And because $A(x) = P\{(\nu_n/n) < x\} = 1 - e^{-x}$, then all limit distributions of the sums S_{ν_n} have the form $\psi(s) = 1/(1 + C^{s\gamma})$.

We have obtained the result of Theorem 3.

In a similar way we can obtain the results of Theorems 2 and 4.

We must make only a small comment relative to what has been said. In Theorem 6 the variables ν_n are assumed to be independent of $\xi_{n,k}$. In the problem we considered there was not this independence but only asymptotic independence. But it is easily shown that with asymptotic independence, Theorem 6 is still valid.

From a result of V. M. Zolotarev, if in formula (6.4) the distribution $A(x)$ is infinitely divisible, then the distribution $\Psi(x)$ is infinitely divisible also. Hence, the infinite divisibility of distributions of the class K follows automatically.

A generalization of the results presented here for the case of summands of arbitrary sign is given by B. Freier ([3], Theorem 6).

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