ON THE SPAN IN L^p OF SEQUENCES OF INDEPENDENT RANDOM VARIABLES (II)

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1. Introduction

This work is motivated by the following problem: which Banach spaces are isomorphic (linearly homeomorphic) to a complemented subspace of $L^p(= L^p[0, 1])$ for $1 < p < \infty$, $p \neq 2$, and what are their linear topological properties? (A linear subspace A of a Banach space B is said to be complemented if there exists a bounded linear operator P on B with range A such that $P^2 = P$; such a P is called a projection onto A.) It is well known that ℓ^2, ℓ^p, $\ell^2 \oplus \ell^p$, $(\ell^2 \oplus \ell^2 \oplus \cdots)_p$, and of course L^p itself, are examples of such Banach spaces. As usual, ℓ^p denotes the Banach space of sequences of scalars (x_n) such that $(\Sigma|x_n|^p)^{1/p} = \|(x_n)\| < \infty$; given Banach spaces $B_1, B_2, \cdots$; we denote by $(B_1 \oplus B_2 \oplus \cdots)_p$ the Banach space of all sequences (b_n) such that $b_n \in B_n$ for all n and $\|(b_n)\| = (\Sigma\|b_n\|^p)^{1/p} < \infty$. Given a sequence (b_n) of elements of a Banach space B, we denote by $[b_n]$ the closed linear span of its terms; if $B = L^p$ of some probability space, then $[b_n]$ is denoted also by $[b_n]_p$.

To see that Hilbert space, that is, ℓ^2 is an example, one may consider a sequence (f_n) of two valued, symmetric, independent random variables; it follows from Khintchine's inequalities that $[f_n]_p$ is isomorphic to Hilbert space. Moreover, the 2 and p norms are equivalent on $[f_n]_p$, and orthogonal projection onto $[f_n]_2 = [f_n]_p$ shows that $[f_n]_p$ is complemented.

Now let $2 < p < \infty$, and let (f_n) be a sequence of independent, nonzero random variables belonging to L^p, each of mean zero. We proved in [11] that $[f_n]_p$ is isomorphic to a complemented subspace of L^p, and that $[f_n]_p$ is isomorphic to exactly one of four Banach spaces: ℓ^2, ℓ^p, $\ell^2 \oplus \ell^p$, or a new space which we denote as $\mathbf{X}_p$. We showed moreover that if the f_n are three valued, symmetric, then $[f_n]_p$ is complemented in L^p by means of orthogonal projection, with $[f_n]_q$ thus complemented and isomorphic to $([f_n]_p)^*$, the dual of $[f_n]_p$ (throughout, p and q used together, denote reals satisfying $1/p + 1/q = 1$). We thus obtained that $\mathbf{X}_q$ (defined as the dual of $\mathbf{X}_p$) is isomorphic to the span of a sequence of independent random variables in L^q, providing the starting point for the present

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paper. Since the isomorphism structure of the spans of sequences of independent random variables in L^p is completely determined for $p > 2$, by the results of [11], we investigate here the span of a sequence of independent random variables (g_n) in L^p for $1 < p < 2$.

The main result of the present paper, from the standpoint of Banach space theory, is that $[g_n]_p$ is isomorphic to a subspace of $\mathbf{X}_p$. (See Corollary 4.3.) Thus, in particular, ℓ^r is isomorphic to a subspace of $\mathbf{X}_p$ for all $p < r \leqq 2$. (The number of possible isomorphism types of $[g_n]_p$ is uncountable, since ℓ^r is such a type for all $p < r \leqq 2$; the classification of these spaces up to isomorphism appears difficult.) From the standpoint of probability theory, our main result is that for $0 < p \leqq 2$, every infinitely divisible, symmetric random variable with a finite absolute pth moment may be approximated in p mean by sums of independent three valued, symmetric random variables (Theorem 4.1). It is easily seen that the span in L^p of a sequence of independent random variables, each of mean zero, is isomorphic to the span of a sequence of symmetric, independent random variables (if $p \geqq 1$). We prove in Theorem 4.2 that given any sequence (X_n) of symmetric, independent random variables and $0 < p < 2$, there exists a sequence (Y_n) of infinitely divisible, symmetric, independent random variables and a sequence (f_n) (respectively, $(f_{n,p})$) of functions valued in the nonnegative reals, such that for any sequence (c_j) of scalars, $\Sigma c_j X_j$ converges a.e. (respectively, in L^p) if and only if $\Sigma c_j Y_j$ converges a.e. (respectively, in L^p) if and only if $\Sigma f_n(|tc_n|) < \infty$ (respectively, $\Sigma f_{n,p}(|tc_n|) < \infty$) for some $t > 0$.

Our main results are contained in Section 4; Section 3 is devoted to preliminaries concerned with distributional equivalents to convergence in L^p of sequences of random variables, and Section 2 to definitions and some standard facts. The remainder of the introduction is devoted to a summary, in greater detail, of some of the results of [11]; we shall also use the notation given below in the sequel.

The result of most interest to probability theory is probably the following.

THEOREM 1.1. *Let $2 < p < \infty$. Then there is a constant K_p so that for any sequence (X_n) of independent random variables belonging to L^p, each of mean zero, and for any integer n,*

$$\left(E\left|\sum_{j=1}^{n} X_j\right|^p\right)^{1/p} \leqq K_p \max\left\{\left(\sum_{j=1}^{n} E|X_j|^p\right)^{1/p}, \left(\sum_{j=1}^{n} E|X_j|^2\right)^{1/2}\right\} \tag{1.1}$$

and

$$E\left(\left|\sum_{j=1}^{n} X_j\right|^p\right)^{1/p} \geqq \frac{1}{2} \max\left\{\left(\sum_{j=1}^{n} E|X_j|^p\right)^{1/p}, \left(\sum_{j=1}^{n} E|X_j|^2\right)^{1/2}\right\}. \tag{1.2}$$

If moreover the variables are three valued and symmetric, then there is a linear projection P from L^p onto $[X_n]_p$ such that P^* is a projection from L^q onto $[X_n]_q$, with $\|P\| \leqq K_p$.

This has, incidentally, an immediate corollary.

COROLLARY 1.1. *Let $2 < p < \infty$ and let (X_n) be a sequence of independent random variables, each of mean zero. Then ΣX_n converges in L^p if and only if $\Sigma E|X_n|^2 < \infty$ and $\Sigma E|X_n|^p < \infty$.*

The proof of Theorem 1.1 uses standard inequalities as well as the following possibly new lemma.

LEMMA 1.1. *Let $1 < p < \infty$ and let $X_1, \cdots, X_n$ be nonnegative, independent random variables. Then*

$$(E(\Sigma X_i)^p)^{1/p} \leqq 2^p \max \{(\Sigma E X_i^p)^{1/p}, \Sigma E X_i\}. \tag{1.3}$$

For the proofs of these results, see Lemmas 1 and 2 and Theorems 3 and 4 of [11].

Now, given a sequence $w = (w_n)$ of positive scalars, $2 < p < \infty$, let $\mathbf{X}_{p,w}$ denote the Banach space of all sequences of scalars (α_n) such that

$$\|(\alpha_n)\| = \max \{(\Sigma|\alpha_n|^2 w_n^2)^{1/2}, (\Sigma|\alpha_n|^p)^{1/p}\} < \infty. \tag{1.4}$$

By the natural basis of $\mathbf{X}_{p,w}$, we refer to the sequence of elements (e^n) of $\mathbf{X}_{p,w}$, where $e_j^n = \delta_{n,j}$ for all j, n. It follows easily from Theorem 1.1 that if (f_n) is a sequence of independent random variables with $E|f_n|^p = 1$ and $Ef_n = 0$ for all n, then $[f_n]_p$ is isomorphic to $\mathbf{X}_{p,w}$, where

$$w_n = \frac{(E|f_n|^2)^{1/2}}{E(|f_n|^p)^{1/p}} \tag{1.5}$$

for all n. (In fact, there is an invertible bounded linear map T from $[f_n]_p$ onto $\mathbf{X}_{p,w}$ such that $Tf_n = e^n$ for all n.)

Let a sequence $w = (w_n)$ of positive reals be given and fix $2 < p < \infty$. If (w_n) satisfies

$$\inf_n w_n = 0, \qquad \sum_{w_n < \varepsilon} w_n^{2p/(p-2)} = \infty \quad \text{for all } \varepsilon > 0, \tag{1.6}$$

then we proved that $\mathbf{X}_{p,w}$ is isomorphic to $\mathbf{X}_p$. Precisely, we showed in Theorem 13 of [11] that if w and w' satisfy (1.6), then $\mathbf{X}_{p,w}$ is isomorphic to $\mathbf{X}_{p,w'}$. The symbol $\mathbf{X}_p$ denotes the one Banach space (up to isomorphism) thus represented; $\mathbf{X}_q$ or any of its representatives $\mathbf{X}_{q,w}$ are defined by duality; $\mathbf{X}_q = \mathbf{X}_p^*$, $\mathbf{X}_{q,w} = \mathbf{X}_{p,w}^*$. If (w_n) fails (1.6), there are exactly three possibilities; either $\inf w_n > 0$, or $\Sigma\, w_n^{2p/(p-2)} < \infty$, or the positive integers N split into two infinite sets N_1 and N_2 with $\Sigma_{n \in N_1} w_n^{2p/(p-2)} < \infty$ and $\inf_{n \in N_2} w_n > 0$. In these three cases, we have that $\mathbf{X}_{p,w}$ is isomorphic either to ℓ^2, ℓ^p, or $\ell^2 \oplus \ell^p$ respectively.

It is easily seen that $\mathbf{X}_p$ is isomorphic to a subspace of $\ell^p \oplus \ell^2$; we proved in [11] that $\mathbf{X}_p$ is not a continuous linear image of $\ell^p \oplus \ell^2$. This enabled us to show that there is a subspace A of ℓ^p isomorphic to ℓ^p, but uncomplemented in ℓ^p for all $2 < p < \infty$ (see Theorem 6 of [11]). The existence of such an A is known for $1 < p < \frac{4}{3}$ and is open for $p = 1$ and for $\frac{4}{3} \leqq p < 2$. Actually, the fact that $\mathbf{X}_p$ is not a continuous linear image of $\ell^p \oplus \ell^2$ follows from the results of the present paper. For it is known that ℓ^r is not isomorphic to a subspace of

$\ell^q \oplus \ell^2$ for any $q < r < 2$, yet by Corollary 4.2, ℓ^r is isomorphic to a subspace of $\mathbf{X}_q$ for all $q < r < 2$. It seems surprising that $\mathbf{X}_q$ should be so rich in subspaces, when it is isomorphic to a quotient space of $\ell^q \oplus \ell^2$, a space seemingly poor in its variety of subspaces. It also follows from our results and the results of [2] that a great many of the Orlicz sequence spaces are isomorphic to subspaces of $\mathbf{X}_q$ and that every subspace of L^q with a symmetric basis is isomorphic to a subspace of $\mathbf{X}_q$, $1 < q < 2$ (see the first remark following Corollary 4.2).

For other applications of probability theory to the theory of the Banach spaces associated with L^p spaces, we refer the reader to [2] and [3]; for the theory of these spaces themselves, see [1] and [8]. We wish to thank S. Bochner and L. Le Cam for stimulating conversations connected with this paper.

2. Definitions, notation, and standard facts

Since this paper may be of interest to readers not completely familiar with the standard terminology in Banach space theory or in probability theory, we give here a rather thorough exposition of that terminology. For standard facts in Banach space theory, see [4] and [5]; for standard facts in probability theory, see [9].

Only real Banach spaces shall be considered ($\mathscr{R}$ denotes the set of real numbers). The assertions we make concerning isomorphic properties of Banach spaces carry over to complex Banach spaces as well (for example, that ℓ^r is isomorphic to a subspace of $\mathbf{X}_p$ for all $1 < p < r < 2$). Given a Banach space B, we denote by B^* its dual, the space of all bounded linear functionals on B. Let B be a Banach space and (b_n) a sequence of elements of B. We say that (b_n) is normalized if $\|b_n\| = 1$ for all n. It is said to be a basic sequence if for each $x \in [b_n]$, there exists a unique sequence of scalars (x_n) such that $x = \Sigma x_n b_n$, the series converging in norm. The sequence (b_n) is called a basis for B if it is a basic sequence with $[b_n] = B$. It is called an unconditional basic sequence if it is a basic sequence such that arbitrary subseries of $\Sigma x_n b_n$ converge whenever $\Sigma x_n b_n$ converges. (For various facts concerning unconditional bases and convergence, the reader may consult [4].) Two basic sequences (a_n) and (b_n) in Banach spaces A and B, respectively, are said to be equivalent if for all sequences of scalars (x_n), $\Sigma x_n a_n$ converges if and only if $\Sigma x_n b_n$ converges. It is well known and easily seen that (a_n) is equivalent to (b_n) if and only if there is a bounded bijective linear operator $T: [a_n] \to [b_n]$ with $Ta_n = b_n$ for all n.

Given a basic sequence (b_n), a sequence (y_j) is said to be a block basis of (b_n) if there exists a sequence (B_j) of disjoint finite subsets of the positive integers, with $a < b$ if $a \in B_i$ and $b \in B_j$ and $i < j$, and if there exist scalars λ_n for $n \in B_j$, such that $y_j = \Sigma_{n \in B_j} \lambda_n b_n$ for all j. Every block basis is a basic sequence in its own right.

It is a theorem of Banach that, if (b_n) is a basic sequence, then there is a constant K such that $\|\alpha_i x_i\| \leq K \|\Sigma_{j=1}^{\infty} \alpha_j x_j\|$ for all i and for all convergent series $\Sigma \alpha_j x_j$; let us call K the biorthogonal constant of (b_n). Suppose that (b_n) is a normalized basic sequence with biorthogonal constant equal to 1, and suppose that (a_n) is a sequence in B such that $\Sigma_{n=1}^{\infty}\|a_n - b_n\| < 1$. Then it is known (see [1]) that (a_n) is a basic sequence equivalent to (b_n). In fact, we have the following lemma.

LEMMA 2.1 (perturbation lemma). *The operator T defined on the linear span of the b_n and satisfying $Tb_n = a_n$ for all n extends (uniquely) to an invertible operator $T\colon [b_n] \to [a_n]$ satisfying $\|T\| \leqq 1 + \delta$ and $\|T^{-1}\| \leqq (1-\delta)^{-1}$, where $\delta = \Sigma\|a_n - b_n\|$.*

PROOF. All the assertions follow from the fact that if x is in the linear span of the b_n, then $\|Tx - x\| \leqq \delta\|x\|$. (For details and related results, see [1].)

By a probability space (Ω, P) we mean a set Ω and a probability measure P defined on some σ-algebra $\mathscr{S}$ of subsets of Ω; (Ω, P) may also be denoted $(\Omega, \mathscr{S}, P)$. When $\Omega = [0, 1]$, we take P to be Lebesgue measure with respect to the Lebesgue measurable subsets of $[0, 1]$. For the sake of definiteness, we usually state results on $[0, 1]$; all results stated as holding on the probability space $[0, 1]$ also hold on any atomless probability space (Ω, P). By a random variable, we mean a real valued function X defined on some probability space such that $X^{-1}(E)$ is measurable for all Borel sets E. *By a distribution, we mean a probability measure on the Borel subsets of $\mathscr{R}$.* Given n random variables $X_1, \cdots, X_n$; by dist $(X_1, \cdots, X_n)$, we mean the measure μ defined on the Borel subsets of $\mathscr{R}^n$ by $\mu(E) = P[(X_1, \cdots, X_n) \in E]$. Given a random variable X defined on a probability space (Ω, P) and $0 < r < \infty$, $E(X) = \int X(w)\, dP(w)$ and $\|X\|_r = (E|X|^r)^{1/r}$. *If X has μ as its distribution, we denote the characteristic function* (ch. f.) *of X (also called the* ch. f. of μ) *by $\hat{\mu}$*; thus, $\hat{\mu}(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itX}\, d\mu(x)$. The variable X is said to be symmetric if X and $-X$ have the same distribution. Given distributions μ and ν, we denote by $\mu * \nu$ the unique distribution with $(\mu * \nu)\hat{} = \hat{\mu}\hat{\nu}$.

By $L^r(\Omega)$, we mean the space of equivalence classes (under equality a.e.) of random variables X on Ω such that $\|X\|_r$ is finite; L^r denotes $L^r[0, 1]$. We follow the usual notation for the most part. For example, if X_n, X are random variables and $0 < r < \infty$, we write $X_n \xrightarrow{P} X$ if X_n converges to X in probability, $X_n \xrightarrow{\text{a.e.}} X$ if X_n converges to X with probability one, and $X_n \xrightarrow{r} X$ if $X_n \to X$ in r mean, that is, if $\|X_n - X\|_r \to 0$.

One final observation: as noted in [11] (p. 278, Lemma 2 and the remarks following it), it follows from [9], p. 263, that if (X_n) is a sequence of nonzero independent random variables, each of mean zero, all belonging to L^p for some fixed $1 \leqq p < \infty$, then (X_n) is an unconditional basic sequence in L^p. It also follows that the biorthogonal constant for (X_n) is one. (Thus, the perturbation lemma applies.)

3. Preliminaries

Our main general interest is convergence in L^r of sequences of random variables (for $1 \leqq r \leqq 2$). In this section, we first state some known results concerning this. We then give a proposition which shows, among other things, that a random variable X defined on $[0, 1]$ which may be approximated in distribution by simple random variables of a certain form, may be approximated a.e. by a sequence of such variables of the same form. (The point is that it is not necessary to consider another random variable Y with dist Y = dist X.)

Let $\lambda, \lambda_1, \lambda_2, \cdots$ be finite positive Borel measures on $\mathscr{R}$. We say that $\lambda_n \xrightarrow{c} \lambda$ if $\int \phi d\lambda_n \to \int \phi d\lambda$ for all bounded continuous ϕ on $\mathscr{R}$. It is well known that if λ, λ_n are distributions (that is, probability measures), then $\lambda_n \xrightarrow{c} \lambda$ is equivalent to $\hat{\lambda}_n \to \hat{\lambda}$ uniformly on compact sets, which is equivalent to $\int \phi d\lambda_n \to \int \phi d\lambda$ for all continuous functions ϕ on $\mathscr{R}$ vanishing at infinity, and is also equivalent to the existence of random variables X, X_n on $[0, 1]$ with λ = dist X, λ_n = dist X_n for all n with $X_n \xrightarrow{\text{a.e.}} X$. Given $0 < r < \infty$, we say that $\lambda_n \xrightarrow{r} \lambda$ if $\lambda_n \xrightarrow{c} \lambda$,

$$\int |x|^r \, d\lambda_n(x) < \infty \quad \text{for all } n, \tag{3.1}$$

and

$$\int |x|^r \, d\lambda_n(x) \to \int |x|^r \, d\lambda(x) < \infty. \tag{3.2}$$

It is easily seen that $\lambda_n \xrightarrow{r} \lambda$ if and only if $\lambda_n \xrightarrow{c} \lambda$ and $\int_{|x| > K} |x|^r \, d\lambda_n(x) \to 0$ uniformly in n as $K \to \infty$. *We have that if* λ, λ_n *are distributions, then* $\lambda_n \xrightarrow{r} \lambda$ *if and only if there exist random variables* X, X_n *in* $L^r[0, 1]$ *with* λ = dist X, *and* λ_n = dist X_n *for all* n, *such that* $X_n \xrightarrow{r} X$. For, we may choose random variables with these distributions such that $X_n \to X$ a.e. Since $\lambda_n \xrightarrow{r} \lambda$,

$$\int_0^1 |X_n|^r \, dt \to \int_0^1 |X|^r \, dt, \tag{3.3}$$

whence, by a well-known consequence of Egorov's theorem, $\int_0^1 |X_n \to X|^r \, dt \to 0$.

In the sequel, we shall have need of the following proposition.

PROPOSITION 3.1. *Let* $X, X_1, \cdots, X_n$ *be simple (that is, finite ranged) random variables defined on some probability space, and let* Y *be a random variable on* $[0, 1]$ *such that* dist Y = dist X. *Then there exist simple random variables* $Y_1, \cdots, Y_n$ *on* $[0, 1]$ *such that* $\text{dist}(X, X_1, \cdots, X_n) = \text{dist}(Y, Y_1, \cdots, Y_n)$.

This proposition is, in turn, a simple consequence of the following lemma.

LEMMA 3.1. *Let* v *be a probability measure on a finite set* F, *with* $v\{f\} \neq 0$ *all* $f \in F$. *Let* $\mathscr{A}$ *be a Boolean subalgebra of the subsets of* F, *and let* $\tau: \mathscr{A} \to \mathscr{S}$ *be a measure preserving Boolean transformation, where* $\mathscr{S}$ *equals the Lebesgue measurable subsets of* $[0, 1]$. *Then there exists* $\tilde{\tau}: 2^F \to \mathscr{S}$ *such that* $\tilde{\tau}$ *is a measure preserving Boolean transformation with* $\tilde{\tau} \,|\, \mathscr{A} = \tau$.

Here, 2^F denotes the set of all subsets of F. Our hypotheses on τ mean simply that $\tau(\mathscr{A})$ is a subalgebra of $\mathscr{S}$, and τ is a measure isomorphism between the measure algebras $(\mathscr{A}, v)$ and $(\tau(\mathscr{A}), m)$ as defined in [6], page 67 (where m denotes Lebesgue measure).

PROOF. Since F is finite it suffices to prove that if E is a subset of F, then τ can be extended to a measure preserving transformation defined on the algebra of sets generated by $\mathscr{A}$ and E, which we shall denote by $\mathscr{A}'$. Let $E_1, \cdots, E_k$ be the atoms of $\mathscr{A}$. Since $\mathscr{A}$ is finite, $\mathscr{A}$ is generated by $E_1, \cdots, E_k$; but then $\mathscr{A}'$ is evidently generated by $E_1 \cap E, E_1 \cap E', \cdots, E_k \cap E, E_k \cap E'$ (where E' denotes the complement of E). Now by standard results (see [6]), we may choose for each i, a set $G_i \subset \tau(E_i)$ such that G_i is Lebesgue measurable and $m(G_i) = m(\tau(E_i) \cap E)$. (If $E_i \cap E$ is empty, let G_i be the empty set.) Then, defining $\tilde{\tau}(E_i \cap E) = G_i$, $\tilde{\tau}(E_i \cap E') = \tau(E_i) - G_i$, we have that $\tilde{\tau}$ extends uniquely to a well defined measure preserving Boolean transformation on $\mathscr{A}'$.

PROOF OF PROPOSITION 3.1. We may of course assume that the variables $X, X_1, \cdots, X_n$ are defined on a probability space (F, ν) such that F is a finite set, with $\nu\{f\} \neq 0$ for all $f \in F$. Let

$$\mathscr{A} = \{X^{-1}\{r\} : r \text{ is in the range of } X\}. \tag{3.4}$$

Since Y and X have the same distribution, it follows that there is a measure preserving Boolean transformation $\tau: \mathscr{A} \to \mathscr{S}$ such that $\tau^{\#}X = Y$ a.e., where $\tau^{\#}$ is defined by

$$\tau^{\#}\left(\sum_{i=1}^{k} r_i I_{E_i}\right) = \sum_{i=1}^{k} r_i I_{\tau(E_i)}, \tag{3.5}$$

where $E_1, \cdots, E_k$ are the distinct atoms of $\mathscr{A}$ and $r_1, \cdots, r_k$ are arbitrary real numbers. By Lemma 3.1, τ extends to a measure preserving Boolean transformation $\tilde{\tau}: 2^F \to \mathscr{S}$; since $\tilde{\tau}$ extends τ, we have that $\tilde{\tau}^{\#} X = Y$ a.e. also. We now merely define $Y_i = \tilde{\tau}^{\#}X_i$ for all $1 \leqq i \leqq n$; it is immediate that

$$\operatorname{dist}(X, X_1, \cdots, X_n) = \operatorname{dist}(\tilde{\tau}^{\#}X, \tilde{\tau}^{\#}X_1, \cdots, \tilde{\tau}^{\#}X_n) = \operatorname{dist}(Y, Y_1, \cdots, Y_n). \tag{3.6}$$

Q.E.D.

COROLLARY 3.1. *Let X and Y be random variables with* dist X = dist Y *with X defined on some probability space (Ω, P) and Y defined on $[0, 1]$. Assume that (X_n) is a sequence of simple random variables defined on Ω such that $X_n \xrightarrow{r} X$ for some $0 < r < \infty$ (respectively, $X_n \xrightarrow{P} X$). Then there exists a sequence of simple random variables defined on $[0, 1]$, such that* dist Y_n = dist X_n *for all n, and $Y_n \xrightarrow{r} Y$ (respectively, $Y_n \xrightarrow{P} Y$).*

PROOF. Throughout, let us use the notation "$Z_n \to Z$" to stand for "$Z_n \xrightarrow{r} Z$" for some fixed $0 < r < \infty$, or to stand for "$Z_n \xrightarrow{P} Z$". We may choose a sequence (g_k) of simple, real valued, Borel measurable functions, defined on the real numbers, such that $g_k(Y) \to Y$. Then since dist Y = dist X, we have $g_k(X) \to X$. Now by Proposition 3.1, for each k we may choose Y_k on $[0, 1]$ such that dist $(Y_k, g_k(Y))$ = dist $(X_k, g_k(X))$. Then of course dist Y_k = dist X_k for all k. Since $g_k(X) \to X$ and $X_k \to X$, we have $g_k(X) - X_k \to 0$; whence, $g_k(Y) - Y_k \to 0$, whence $Y_k \to Y$. *Q.E.D.*

REMARK 3.1. Proposition 3.1, Lemma 3.1, and Corollary 3.1 all hold for countably valued random variables rather than merely simple random variables. In fact, one obtains that *if (Ω, S, P) is an atomless probability space and if $X_1, \cdots, X_n, U_1, \cdots, U_m$ are countably valued random variables defined on some* *probability space, and* $W_1, \cdots, W_m$ *are variables defined on* Ω *with* dist $(U_1, \cdots, U_m) =$ dist $(W_1, \cdots, W_m)$, *then there exist variables* $Y_1, \cdots, Y_n$ *defined on* Ω *with*

$$\text{dist}(Y_1, \cdots, Y_n, W_1, \cdots, W_n) = \text{dist}(X_1, \cdots, X_n, U_1, \cdots, U_m). \tag{3.7}$$

Moreover, it follows from results of Maharam (Lemmas 1 and 2 of [10]) that this holds with no restriction at all on the cardinality of the range of the random variables $X_1, \cdots, X_n, U_1, \cdots, U_m$, provided that $(\Omega, \mathscr{S}, P)$ is a homogeneous nonseparable probability space (for example, the one obtained by taking the product measure on uncountably many copies of [0, 1]). For it can be deduced from her results that if $(S, \mathscr{B}, \nu)$ is a separable probability space (that is, $L^1(S)$ is separable), if $\mathscr{A}$ is a σ-subalgebra of $\mathscr{B}$, and if $\tau: \mathscr{A} \to \mathscr{S}$ is a measure preserving Boolean transformation, then there exists a measure preserving Boolean transformation $\tilde{\tau}: \mathscr{B} \to \mathscr{S}$ extending τ. (For an application of this, see Remark 4.2.)

4. The main results

DEFINITION 4.1. *Let* $\mathscr{S}$ *denote the set of all distributions* μ, *such that there exist* k *and* k*-independent three valued, symmetric random variables* $X_1, \cdots, X_k$ *such that* $\mu = \text{dist}(X_1 + \cdots + X_k)$.

We note that $\mu \in \mathscr{S}$ if and only if there is a k and positive real numbers $r_1, \cdots, r_k, \beta_1, \cdots, \beta_k$ with $\beta_i < 1$ for all i such that

$$\hat{\mu}(x) = \prod_{i=1}^{k} [1 - \beta_i(1 - \cos r_i x)]. \tag{4.1}$$

Let $0 < r \leqq 2$ and let F_r (respectively, F) denote the set of all distributions μ such that there exists a sequence $\mu_n \in \mathscr{S}$ with $\mu_n \xrightarrow{r} \mu$ (respectively, with $\mu_n \xrightarrow{c} \mu$). The next proposition follows from the results of Section 3.

PROPOSITION 4.1. *If* $\mu \in F_r$ (*respectively*, F), *then for any random variable* X *defined on* [0, 1] *with* dist $X = \mu$, *there exist three valued, symmetric random variables* $X_{n,k}$ *for* $1 \leqq k \leqq k_n$, $n = 1, 2, \cdots$ *such that for each* n, $X_{n,1}, X_{n,2}, \cdots, X_{n,k_n}$ *are independent with* $\sum_{k=1}^{k_n} X_{n,k} \xrightarrow{r} X$ (*respectively*, $\sum_{k=1}^{k_n} X_{n,k} \xrightarrow{\text{a.e.}} X$). To see this, we have by Corollary 3.1 that if X is on [0, 1] and dist $X = \mu$, then there exist random variables X_n on [0, 1] such that dist $X_n \in S$ for all n and $X_n \xrightarrow{r} X$ (respectively, $X_n \xrightarrow{\text{a.e.}} X$). But by Proposition 3.1, it follows that for each n, we may choose $X_{n,1}, \cdots, X_{n,k_n}$ independent, three valued, symmetric random variables such that $X_n = \sum_{k=1}^{k_n} X_{n,k}$. (Of course it is trivial that if X satisfies the conclusion of Proposition 4.1, then dist $X \in F_r$ (respectively, dist $X \in F$).)

W come now to our main result from the standpoint of probability theory.

THEOREM 4.1. *Let* μ *be a symmetric, infinitely divisible distribution. Then* $\mu \in F$. *If, moreover,* $\int |x|^r \, d\mu(x) < \infty$ *for some* $0 < r \leqq 2$, *then* $\mu \in F_r$.

We are indebted to S. Bochner for showing us that every symmetric, infinitely divisible distribution belongs to F.

To prove Theorem 4.1, we first require two lemmas, whose proofs use known techniques. (For the definition and standard facts concerning the infinitely divisible laws, see [9]; the main result we use here is the explicit representation of their characteristic functions given on p. 309 of [9].) The first lemma shows that the classes F_r and F are closed under the obvious operations.

LEMMA 4.1. *Let* $0 < r \leqq 2$. *Then if* $\mu \in F_r$ (*respectively*, $\in F$) *and* $\lambda > 0$, $\mu_\lambda \in F_r$ (*respectively*, $\in F$), *where* $\mu_\lambda(E) = \mu(\lambda E)$ *for all Borel* E. *If* $\mu, \nu \in F_r$ (*respectively* F), *then* $\mu * \nu \in F_r$ (*respectively*, F). *Finally, if* $\mu_n \in F_r$ (*respectively*, F) *and* $\mu_n \xrightarrow{r} \mu$ (*respectively*, $\mu_n \xrightarrow{c} \mu$), *then* $\mu \in F_r$ (*respectively*, $\mu \in F$).

PROOF. The first assertion is evident; for if we choose $\mu_n \in \mathscr{S}$ such that $\mu_n \xrightarrow{r} \mu$ (respectively, $\mu_n \xrightarrow{c} \mu$), then $(\mu_n)_\lambda \xrightarrow{r} \mu_\lambda$ (respectively, $(\mu_n)_\lambda \xrightarrow{c} (\mu)_\lambda$), and of course $\nu \in \mathscr{S} \Rightarrow \nu_\lambda \in \mathscr{S}$. To see the second assertion, we may choose independent random variables X and Y on $[0, 1]$ such that $\mu = \text{dist}\, X$, $\nu = \text{dist}\, Y$, and sequences of random variables (X_n), (Y_n) such that for all n, X_n and Y_n are independent, and dist X_n, dist $Y_n \in \mathscr{S}$, with $X_n \xrightarrow{r} X$ and $Y_n \xrightarrow{r} Y$ (respectively, $X_n \xrightarrow{\text{a.e.}} X$ and $Y_n \xrightarrow{\text{a.e.}} Y$). But then $X_n + Y_n \xrightarrow{r} X + Y$ (respectively, $X_n + Y_n \xrightarrow{\text{a.e.}} X + Y$). Of course, dist $(X + Y) = \mu * \nu$ and dist $(X_n + Y_n) \in S$ for all n. The final assertion follows from our initial remarks. For we may choose random variables X_n and X on $[0, 1]$ such that $\mu_n = \text{dist}\, X_n$ for all n, $\mu = \text{dist}\, X$, and $X_n \xrightarrow{r} X$ (respectively, $X_n \xrightarrow{P} X$). By Proposition 4.1, we may choose for each n, a random variable Y_n with dist $Y_n \in S$, such that $\|Y_n - X_n\|_r < 1/n$ (respectively, such that $d(Y_n, X_n) < 1/n$, where d denotes the usual convergence in probability metric, $d(Y, Z) = E(|Y - Z|/(1 + |Y - Z|))$ for any random variables Y and Z). Then $Y_n \xrightarrow{r} X$ (respectively, $Y_n \xrightarrow{P} X$), so of course $\mu \in F_r$ (respectively, $\mu \in F$). *Q.E.D.*

Parts of the next lemma are certainly known, but we are unaware of suitable references for all of its assertions.

LEMMA 4.2. *Let* μ, μ_n *be symmetric, infinitely divisible distributions, and let* λ, λ_n *be the unique finite symmetric measures on the reals such that*

$$\hat{\mu}(x) = \exp\left\{-\int_{-\infty}^{\infty} (1 - \cos yx)\frac{1 + y^2}{y^2}\, d\lambda(y)\right\},$$

$$\hat{\mu}_n(x) = \exp\left\{-\int_{-\infty}^{\infty} (1 - \cos yx)\frac{1 + y^2}{y^2}\, d\lambda_n(y)\right\}. \tag{4.2}$$

Then $\mu_n \xrightarrow{c} \mu$ *if (and only if)* $\lambda_n \xrightarrow{c} \lambda$. *Furthermore, if* $0 < r \leqq 2$, *then* $\int |x|^r\, d\mu(x) < \infty$ *if and only if* $\int |x|^r\, d\lambda(x) < \infty$, *and* $\mu_n \xrightarrow{r} \mu$ *if (and only if)* $\lambda_n \xrightarrow{r} \lambda$.

PROOF. We delete the proofs of the two parenthetical "only if's", since we have no need of these assertions in the sequel. If $\lambda_n \xrightarrow{c} \lambda$, then trivially $\hat{\mu}_n \to \hat{\mu}$ pointwise; whence, $\mu_n \xrightarrow{c} \mu$, since the μ_m and μ are distributions.

If $\int x^2\, d\mu(x) < \infty$, then

$$\begin{aligned}\int x^2\, d\mu(x) &= -\left(\frac{d^2\hat{\mu}}{dx^2}\right)(0) = \lim_{y\to 0} \frac{2 - \hat{\mu}(y) - \hat{\mu}(-y)}{y^2} \\ &= \lim_{y\to 0} 2\int \frac{1 - \cos yx}{y^2}\frac{1 + x^2}{x^2}\, d\lambda(x) \\ &\geqq \int (1 + x^2)\, d\lambda(x)\end{aligned} \tag{4.3}$$

by Fatou's lemma; whence $\int x^2\, d\lambda(x) < \infty$. If conversely $\int x^2\, d\lambda(x) < \infty$, then $\hat{\mu}$ is twice continuously differentiable and so μ has a finite second moment (see page 199 of [9]). We thus have that

$$\int x^2\, d\mu(x) = \int (1 + x^2)\, d\lambda(x), \tag{4.4}$$

which shows immediately that if $\lambda_n \xrightarrow{2} \lambda$ then $\mu_n \xrightarrow{2} \mu$.

Now fix $0 < r < 2$. We have that

$$|x|^r = C_r \int_{-\infty}^{\infty} \frac{1 - \cos xy}{|y|^{1+r}}\, dy, \quad \text{where } C_r^{-1} = \int_{-\infty}^{\infty} \frac{1 - \cos y}{|y|^{1+r}}\, dy. \tag{4.5}$$

Now assume that $\int |x|^r\, d\mu(x) < \infty$. Since μ is symmetric, $\hat{\mu}(y) = \int_{-\infty}^{\infty} \cos xy\, dy$; thus, substituting for $|x|^r$ by using (4.5) and changing the order of integration, we have that

$$C_r^{-1} \int |x|^r\, d\mu(x) = \int_{-\infty}^{\infty} \frac{1 - \hat{\mu}(y)}{|y|^{1+r}}\, dy < \infty. \tag{4.6}$$

Now choose δ such that $1 - \hat{\mu}(y) \geqq -\frac{1}{2} \log \hat{\mu}(y)$ for all $|y| \leqq \delta$. Then

$$\begin{aligned} \infty &> 2 \int_{-\delta}^{\delta} \frac{1 - \hat{\mu}(y)}{|y|^{1+r}}\, dy \\ &\geqq \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} \frac{1 - \cos yx}{|y|^{1+r}} \frac{1 + x^2}{x^2}\, d\lambda(x)\, dy \\ &\geqq \int_{|x|>1} \left(\int_{-\delta}^{\delta} \frac{1 - \cos yx}{|y|^{1+r}}\, dy \right) \frac{1 + x^2}{x^2}\, d\lambda(x) \\ &\geqq K \int_{|x|>1} |x|^r \frac{1 + x^2}{x^2}\, d\lambda(x), \end{aligned} \tag{4.7}$$

where

$$K = \int_{-\delta}^{\delta} \frac{1 - \cos y}{|y|^{1+r}}\, dy > 0. \tag{4.8}$$

Thus, $\int |x|^r\, d\lambda(x) < \infty$. If conversely $\int |x|^r\, d\lambda(x) < \infty$, then since $1 - \hat{\mu}(y) \leqq -\log \hat{\mu}(y)$ for all y,

$$\begin{aligned} \int_{-1}^{1} \frac{1 - \hat{\mu}(y)}{|y|^{1+r}}\, dy &\leqq \int_{-1}^{1} \int_{-1}^{1} \frac{1 - \cos yx}{|y|^{1+r}} \frac{1 + x^2}{x^2}\, d\lambda(x)\, dy \\ &\quad + \int_{-1}^{1} \int_{|x|>1} \frac{1 - \cos yx}{|y|^{1+r}} \frac{1 + x^2}{x^2}\, d\lambda(x)\, dy \\ &\leqq \int_{-1}^{1} \left(\int_{-1}^{1} \frac{1 - \cos yx}{y^2 x^2} (1 + x^2)\, d\lambda(x) \right) |y|^{1-r}\, dy \\ &\quad + C_r^{-1} \int_{|x|>1} |x|^r \frac{1 + x^2}{x^2}\, d\lambda(x) < \infty; \end{aligned} \tag{4.9}$$

whence by (4.6), $\int |x|^r\, d\mu(x) < \infty$.

Now suppose $\lambda_n \xrightarrow{r} \lambda$, $0 < r < 2$. Then $\mu_n \xrightarrow{c} \mu$, since $\lambda_n \xrightarrow{c} \lambda$; hence for all y, $\hat{\mu}_n(y) \to \hat{\mu}(y)$. To show that $\mu_n \xrightarrow{r} \mu$, we need only show (by (4.6)) that

$$\int_{-\infty}^{\infty} \frac{1 - \hat{\mu}_n(y)}{|y|^{1+r}}\, dy \to \int_{-\infty}^{\infty} \frac{1 - \hat{\mu}(y)}{|y|^{1+r}}\, dy. \tag{4.10}$$

This in turn will follow if we show that the sequence of functions $[1 - \hat{\mu}_n(y)]/|y|^{1+r}$ is uniformly integrable near 0. Let $\varepsilon > 0$. We shall prove that there is a $\delta > 0$ such that

$$\int_{-\delta}^{\delta} \frac{1 - \hat{\mu}_n(y)}{|y|^{1+r}}\, dy < \varepsilon \quad \text{for all } n, \tag{4.11}$$

and thus, be done with this lemma.

Fix δ, n, and N with $N > 1$. Then

$$\begin{aligned} \int_{-\delta}^{\delta} & \frac{1 - \hat{\mu}_n(y)}{|y|^{1+r}}\, dy \\ &\leqq \int_{-\delta}^{\delta}\int_{-\infty}^{\infty} \frac{1 - \cos yx}{|y|^{1+r}} \frac{1 + x^2}{x^2}\, d\lambda_n(x)\, dy \\ &= \int_{-\delta}^{\delta}\left(\int_{-N}^{N} \frac{1 - \cos yx}{y^2x^2}(1 + x^2)\, d\lambda_n(x)\right)|y|^{1-r}\, dy \\ &\quad + \int_{|x|>N}\left(\int_{-\delta}^{\delta} \frac{1 - \cos yx}{|y|^{1+r}}\, dy\right)\frac{1 + x^2}{x^2}\, d\lambda_n(x) \\ &\leqq \sup_k \lambda_k(\mathscr{R})N^2 \int_{-\delta}^{\delta} |y|^{1-r}\, dy \\ &\quad + \sup_k \int_{|x|>N} C_r^{-1}|x|^r \frac{1 + x^2}{x^2}\, d\lambda_k(x). \end{aligned} \tag{4.12}$$

Now choose N so that

$$\sup_k \int_{|x|>N} C_r^{-1}|x|^r \frac{1 + x^2}{x^2}\, d\lambda_k(x) < \tfrac{1}{2}\varepsilon; \tag{4.13}$$

then choose δ so that

$$\sup_k \lambda_k(\mathscr{R})N^2 \int_{-\delta}^{\delta} |y|^{1-r}\, dy < \tfrac{1}{2}\varepsilon. \tag{4.14}$$

Q.E.D.

PROOF OF THEOREM 4.1. We first show that the "symmetric Poisson" distributions belong to F_2; that is, if there is a real x_0 and a real λ with $\lambda > 0$ such that $\hat{\mu}(t) = \exp\{-(1 - \cos x_0 t)\lambda\}$, then $\mu \in F_2$. By Lemma 4.1, we may assume $x_0 = 1$. Let μ_n be the distribution such that $\hat{\mu}_n(t) = [1 - \lambda(1 - \cos t)/n]^n$. If n is such that $n > \lambda$, then $\mu_n \in \mathscr{S}$, and of course $\mu_n \xrightarrow{c} \mu$. Moreover, $\int x^2\, d\mu_n \to \int x^2\, d\mu$; in fact

$$\int x^2\, d\mu_n = -\left(\frac{d^2\mu_n}{dx^2}\right)(0) = \lambda = \int x^2\, d\mu \quad \text{for all } n. \tag{4.15}$$

Next we have that the Gaussian distributions belong to F_2. Of course by Lemma 4.1, it suffices to prove that $\mu \in F_2$ if $\hat{\mu}(t) = \exp\{-\frac{1}{2}t^2\}$. But if μ_n is the distribution

$$\hat{\mu}_n(t) = \exp\left\{-\left[1 - \cos\left(\frac{t}{n}\right)\right]n^2\right\}, \tag{4.16}$$

then $\mu_n \xrightarrow{c} \mu$, and again, $\int x^2\, d\mu_n(t) = 1 = \int x^2\, d\mu(t)$ for all n, so $\mu_n \xrightarrow{2} \mu$, and thus, $\mu \in F_2$ by Lemma 4.1.

Now let μ be a symmetric, infinitely divisible distribution, and let λ be the unique symmetric measure on the reals related to μ as in Lemma 4.2. We may assume, without loss of generality, that μ has no Gaussian part, that is, that $\lambda\{0\} = 0$. For if μ had a Gaussian part, we could write $\mu = \mu_1 * \mu_2$ with μ_1 a Gaussian distribution and μ_2 Gaussian free. Then of course, we would have automatically that μ_2 is symmetric and infinitely divisible with $\int |x|^r\, d\mu_2 < \infty$ if $\int |x|^r\, d\mu < \infty$, by Lemma 4.2. Thus, since $\mu_1 \in F_2$, we would have that $\mu \in F$ if $\mu_2 \in F$ and $\mu \in F_r$ if $\mu_2 \in F_r$, by Lemma 4.1.

Now it is easily seen (for example, by passing to Riemann sums) that there exists a sequence of symmetric measures (λ_n) on $\mathscr{R}$ such that for all n, the support of λ_n is a finite set not containing 0 with $\lambda_n \xrightarrow{c} \lambda$, with the additional property that $\int |x|^r\, d\lambda_n \to \int |x|^r\, d\lambda$ if $\int |x|^r\, d\lambda < \infty$. Fix n, and let μ_n be the distribution related to λ_n as in Lemma 4.2. Then for each n, μ_n is a convolution of a finite number of symmetric Poisson distributions; hence, $\mu_n \in F_2$ by Lemma 4.1. Thus, $\mu_n \xrightarrow{c} \mu$ by Lemma 4.2, so $\mu \in F$ by Lemma 4.1. If $\int |x|^r\, d\mu < \infty$, then $\mu_n \xrightarrow{r} \mu$ by Lemma 4.2, so again $\mu \in F_r$ by Lemma 4.1. *Q.E.D.*

Our next result shows that for $1 \leqq r < 2$, the span of a sequence of independent, infinitely divisible, symmetric random variables in L^r, is arbitrarily close to a subspace of the span of a certain sequence of three valued, symmetric random variables. (The result as stated is trivial for $r = 2$.)

COROLLARY 4.1. *Let $1 \leqq r < 2$ and let $X_1, X_2, \cdots$ be a sequence of nonzero, independent, symmetric, infinitely divisible random variables such that $E|X_i|^r < \infty$ for all i. Then given $\varepsilon > 0$, there exists a sequence $Z_1, Z_2, \cdots$ of independent, symmetric, three valued random variables, and a block basis (Y_n) of (Z_n) such that there is a unique invertible linear operator $T\colon [Z_n]_r \to [X_n]_r$ with $\|T\|\,\|T^{-1}\| \leqq 1 + \varepsilon$ and $T(Y_n) = X_n$ for all n.*

PROOF. We may assume, without loss of generality, that $E|X_n|^r = 1$ for all n. We may also assume that $X_1, X_2, \cdots$ are defined on $[0, 1]^\infty$, the countable infinite product of the unit interval with itself (endowed with the product measure), so that X_n depends only on the nth factor, for all n. That is, there exist random variables $\tilde{X}_1, \tilde{X}_2, \cdots$ defined on $[0, 1]$ so that $X_n(t) = \tilde{X}_n(t_n)$ for all n and $t = (t_n) \in [0, 1]^\infty$. Now choose $\varepsilon' < 1$ such that $(1 + \varepsilon')/(1 - \varepsilon') < 1 + \varepsilon$. By Theorem 4.1 and Proposition 4.1, we may choose for each n, three valued, symmetric, independent random variables $\tilde{X}_{n,1}, \cdots, \tilde{X}_{n,k_n}$ defined on $[0, 1]$ such that

$$\left\| \sum_{j=1}^{k_n} \tilde{X}_{n,j} - \tilde{X}_n \right\|_r < \frac{\varepsilon'}{2^n}. \tag{4.17}$$

Then for each n and j with $1 \leqq j \leqq k_n$, define $X_{n,j}$ by $X_{n,j}(t) = \tilde{X}_{n,j}(t_n)$ for all $t \in [0, 1]^\infty$; let τ be the unique one to one correspondence between $\{(n, j): 1 \leqq j \leqq k_n, n = 1, 2, \cdots\}$ and the positive integers such that $\tau(n, j) < \tau(n', j')$ if $n < n'$ or if $n = n'$ and $j < j'$, and define Z_n and Y_n for all n by $Z_n = X_{\tau^{-1}(n)}$ and $Y_n = \Sigma_{j=1}^{k_n} Z_{\tau(n,j)}$. Then the variables $Z_1, Z_2, \cdots$ are independent, (Y_n) is a block basis of (Z_n), and $\| Z_n - X_n \|_r < \varepsilon'/2^n$ for all n. The final assertion follows from the perturbation lemma, Lemma 2.1., *Q.E.D.*

COROLLARY 4.2. *The space ℓ^r is isomorphic to a subspace of $\mathbf{X}_q$ for all $1 < q \leqq r \leqq 2$.*

PROOF. Let $q < r < 2$, and let $X_1, X_2, \cdots$ be independent, identically distributed, stable, symmetric random variables of exponent r, such that $E|X_1|^q = 1$. Thus, letting μ be the distribution of X_1, $\hat{\mu}(x) = \exp\{-c|x|^r\}$, where c is the fixed constant (depending on q and r) determined by $\int |x|^q \, d\mu(x) = 1$. (As is well known and also a consequence of Lemma 4.2, such random variables have a finite absolute qth moment.) Then $[X_n]_q$ is isometric to ℓ^r. Indeed, given n real numbers $c_1, \cdots, c_n$ not all zero, we have that

$$[\text{ch.f. } (c_1 X_1 + \cdots + c_n X_n)](x) = \exp\{-c|(\Sigma|c_i|^r)^{1/r} x|^r\}, \tag{4.18}$$

whence the distribution of $\Sigma c_i X_i$ equals μ_λ where $\lambda^{-1} = (\Sigma|c_i|^r)^{1/r}$; whence,

$$E|\Sigma c_i X_i|^q = \int |\lambda x|^q \, d\mu(x) = \lambda^q; \tag{4.19}$$

and thus, $\|\Sigma c_i X_i\|_q = \lambda = (\Sigma|c_i|^r)^{1/r}$. Therefore the linear span of the X_n is linearly isometric to a dense linear subspace of ℓ^r; this isometry uniquely extends to an isometry of all of $[X_n]_q$ with ℓ^r.

By Corollary 4.2, $[X_n]_q$ is isomorphic to a subspace of $[Z_n]_q$ for a certain sequence (Z_n) of symmetric, three valued, independent random variables, but by the results of [11], $[Z_n]_q$ is isomorphic to a complemented subspace of $\mathbf{X}_q$. (Actually, $[Z_n]_q$ will be isomorphic to $\mathbf{X}_q$ in this case, since the other possible isomorphism types of $[Z_n]_q$ cannot contain an isomorph of ℓ^r.) Finally, $\ell^q \oplus \ell^2$ is isomorphic to a subspace of $\mathbf{X}_q$, so the cases $r = q$, $r = 2$ are taken care of also. *Q.E.D.*

REMARK 4.1. A basis (b_n) in a Banach space is said to be symmetric if for all permutations σ of the positive integers and sequences (x_n) of scalars, $\Sigma x_{\sigma(n)} b_n$ converges whenever $\Sigma x_n b_n$ converges. *Let $1 < q < 2$ and let B be a closed linear subspace of L^q with a symmetric basis. Then by our results and the theorems of* [2], *B is isomorphic to a subspace of* $\mathbf{X}_q$. Indeed, it follows from the results in [2] that either B is isomorphic to ℓ^q or there exists a sequence (X_n) of i.i.d., infinitely divisible, symmetric random variables belonging to L^q such that B is isomorphic to $[X_n]_q$. In either case B is thus isomorphic to a subspace of $\mathbf{X}_q$ by Corollary 4.1 and the results of [11]. It is also proved in [2] that there exists a convex function f with domain and range the nonnegative reals with $f(x)/x^2$ equivalent (near zero) to a decreasing function and $f(x)/x^q$ equivalent to an increasing function, such that B is isomorphic to the Orlicz sequence space ℓ_f; and conversely given any function f satisfying these conditions, ℓ_f is isomorphic to a subspace of L^q (and consequently to a subspace of $\mathbf{X}_q$ by our results). Finally, it is also proved in [2] that the above spaces include (isomorphically) all reflexive subspaces of L^1 with a symmetric basis.

The definitions of the terms used above are as follows. If g and h are two functions with domain and range the nonnegative reals, g and h are said to be equivalent near zero if there exist positive constants a, b, c, d and δ such that $h(ax) \leqq b\, g(x)$ and $g(cx) \leqq d\, h(x)$ for all $0 \leqq x \leqq \delta$. If g is convex, vanishing at zero, ℓ_g refers to the space of all sequences (x_n) of scalars such that $\Sigma g(|tx_n|) < \infty$ for some $t > 0$, under any norm equivalent to the norm

$$\|(x_n)\|_g = \inf\left\{\frac{1}{t}: t > 0 \text{ and } \Sigma g(|tx_n|) \leqq 1\right\}. \tag{4.20}$$

REMARK 4.2. The first part of our argument for Corollary 4.2 (valid for $r = 2$ also) shows that ℓ^r is isometric to a subspace of L^p for all $1 \leqq p < r \leqq 2$. Actually, by the results of [3], L^r is isometric to a subspace of L^p for all $1 \leqq p < r \leqq 2$. An explicit proof of this may be obtained by applying the last remark of Section 3 as follows. Let (Ω, P) be the probability space obtained by letting P be the product measure on Ω, an uncountable product of unit intervals. Let $X_{0,1}$ be a stable, symmetric random variable of exponent r, defined on Ω, with $E|X_{0,1}|^p = 1$. Let $n \geqq 0$, and assume that 2^n independent, identically distributed random variables $X_{n,1}, \cdots, X_{n,2^n}$ have been defined, such that $X_{0,1} = \Sigma_{j=1}^{2^n} X_{n,j}$. By the last remark of Section 3 since $X_{n,1}$ is thus symmetric and stable of exponent r, we may choose i.i.d. random variables $X_{n+1,1}, \cdots, X_{n+1,2^{n+1}}$ such that for each j with $1 \leqq j \leqq 2^n$, $X_{n,j} = X_{n+1,2j-1} + X_{n+1,2j}$.

We now define a linear operator T from the linear combinations of the indicators of the dyadic intervals on $[0, 1]$ into $L^p(\Omega)$ as follows: given $n \geqq 0$ and scalars $c_1, \cdots, c_{2^n}$, put

$$T\left(\sum_{j=1}^{2^n} c_j I_{[(j-1)/2^n, j/2^n]}\right) = \sum_{j=1}^{2^n} c_j X_{n,j}. \tag{4.21}$$

Then the definition of the $X_{n,j}$ shows that T is a well defined map; the argument of Corollary 4.2 shows that T is an isometry (since $E|X_{n,j}|^q = 2^{-nq/r}$); whence, T extends uniquely to an isometry from $L^r[0, 1]$ onto B, the closed linear span of $\{X_{n,j}: 1 \leqq j \leqq 2^n; n = 0, 1, 2, \cdots\}$ in $L^q(\Omega)$. Finally, since B is a separable subspace of $L^q(\Omega)$, it follows easily that B is isometric to a subspace of $L^q[0, 1]$.

Our final theorem reduces the study of the span of sequences of independent random variables to the span of sequences of symmetric, infinitely divisible random variables.

THEOREM 4.2. *Let (X_n) be a sequence of independent random variables, each possessing a nonnegative characteristic function. Let (Y_n) be a sequence of independent random variables such that for all n the* ch. f. *of Y_n equals $\exp\{-\int_{-\infty}^{\infty}(1 - \cos yx)\, d\mu_n(x)\}$, where μ_n is the distribution of X_n, and let $0 < r < 2$. Then ΣX_n converges* a.e. *if and only if ΣY_n converges* a.e., *and ΣX_n converges in r mean if and only if ΣY_n converges in r mean. Moreover, there exists* *a sequence of functions* (f_n) *(respectively,* $(f_{n,r})$*) with domain and range the non-negative real numbers such that for all* n, f_n *respectively,* $f_{n,r}$*) depends only on the distribution of* X_n *(respectively, on the distributions of* X_n *and on* r*), and such that for any sequence of scalars* (c_j), $\Sigma c_j X_j$ *converges a.e. (respectively, in* r *mean) if and only if there is a* $t > 0$ *such that* $\Sigma f_j(|tc_j|) < \infty$ *(respectively, such that* $\Sigma f_{j,r}(|tc_j|) < \infty$*).*

PROOF. Given a symmetric random variable X with characteristic function h, define

$$g_X^1(t) = \begin{cases} \log 1/h(t) & \text{if } h(t) > 0 \text{ and } \log 1/h(t) < 2, \\ 2 & \text{otherwise,} \end{cases} \tag{4.22}$$
$$g_X^2(t) = 1 - h(t).$$

Now define

$$f_X^i(y) = \int_0^1 g_X^i(yt)\,dt, \qquad f_{X,r}^i(y) = \int_0^1 \frac{g_X^i(yt)}{t^{1+r}}\,dt \tag{4.23}$$

for $y \geqq 0, i = 1, 2$. (The definition of these functions was suggested by Lemma 4.1 of [12], where the integral defining f_X^1 is given for X symmetric and infinitely divisible.) We shall prove that for $i = 1$ or $i = 2$, ΣX_n converges a.e. (respectively, in r mean) if and only if there is a $\delta > 0$ such that $\Sigma f_{X_n}^i(\delta) < \infty$ (respectively, $\Sigma f_{X_n,r}^i(\delta) < \infty$). This suffices to prove all the assertions of Theorem 4.2; in view of the obverations that $g_{Y_n}^1 = g_{X_n}^2$ (and thus $f_{Y_n}^1 = f_{X_n}^2$ and $f_{Y_n,r}^1 = f_{X_n,r}^2$) for all n, and $f_{cX}^i(y) = f_X^i(|c|y)$ and $f_{cX,r}^i(y) = f_{X,r}^i(|c|y)$ for any random variable X, scalar c, $i = 1, 2$.

To simplify the notation, let us put $g_{X_n}^i = g_n^i$, $f_{X_n}^i = f_n^i$, $f_{X_n,r}^i = f_{n,r}^i$, and let $h_n = \text{ch. f. } X_n$ for all n. In what follows, we make constant use of the elementary identities

$$1 - t \leqq \log \frac{1}{t} \qquad \text{for all } t \geqq 0, \tag{4.24}$$
$$\log \frac{1}{t} \leqq 2(1 - t) \qquad \text{for all } t, \quad \frac{1}{2} \leqq t \leqq 1,$$

as well as the result that a series of independent random variables converges a.e., if it converges in probability (see p. 249 of [9]).

Now assume that ΣX_j converges a.e. Then $\Pi_{j=1}^\infty h_j(t)$ converges uniformly on compact subsets of the real line. In particular, we may choose a δ such that $1 - \Pi_{j=1}^N h_j(x) < \frac{1}{2}$ for all $0 \leqq x \leqq \delta$. Then $\Sigma \log 1/h_j(t)$ converges uniformly for $0 \leqq t \leqq \delta$. Since then $g_n^2(t) \leqq g_n^1(t) = \log 1/h_n(t)$ for all n sufficiently large and all such t, it follows that $\Sigma_{j=1}^\infty \int_0^\delta g_j^i(x)\,dx < \infty$, and, after changing variables, we have that $\Sigma f_n^i(\delta) < \infty$ for $i = 1, 2$.

Conversely, fix $i = 1$ or 2, and assume that $\Sigma_{n=1}^\infty f_n^i(\delta) < \infty$ for some $\delta > 0$. Again, after changing variables, we have that $\Sigma_{n=1}^\infty \int_0^\delta g_n^i(t)\,dt < \infty$; whence by the Beppo-Levi theorem, $\Sigma g_n^i(t) < \infty$ a.e., $0 \leqq t \leqq \delta$. But then by Egorov's theorem, there exists a closed subset E of $[0, \delta]$ of positive measure such that Σg_n^i converges uniformly on E; the continuity of the g_n^i implies then that there exists an N such that $g_n^i(x) \leqq \frac{1}{2}$ for all $n \geqq N$. Thus by (4.24),

$$\sum_{n=N}^{\infty} \log \frac{1}{h_n(x)} \leqq 2 \sum_{n=N}^{\infty} g_n^i(x) < \infty \quad \text{for all } x \in E, \tag{4.25}$$

and so $\Pi_{n=1}^{\infty} h_n(x)$ converges to a nonzero limit on a set of positive measure. Then (by 4°, pp. 197–198 and also B, p. 251 of [9]) $\Sigma_{j=N}^{\infty} X_j$, and hence, $\Sigma_{j=1}^{\infty} X_j$ converges in probability, and thus a.e.

Now suppose that ΣX_j converges in L^r. Then applying (4.6) of Lemma 4.2,

$$\lim_{n\to\infty} E\left|\sum_{j=1}^{n} X_j\right|^r = \lim_{n\to\infty} C_r \int_{-\infty}^{\infty} \frac{1 - \prod_{j=1}^{n} h_j(t)}{|t|^{1+r}}\, dt < \infty. \tag{4.26}$$

Now since ΣX_j converges in probability, Πh_j converges uniformly on compacta; whence, we may choose a $\delta > 0$ so that $1 - \Pi_{j=1}^{n} h_j(x) \leqq \frac{1}{2}$ for all $|x| \leqq \delta$, all n. Now we have by (4.26) that

$$\lim_{n\to\infty} \int_{-\delta}^{\delta} \frac{1 - \prod_{j=1}^{n} h_j(t)}{|t|^{1+r}}\, dt < \infty; \tag{4.27}$$

whence by (4.25),

$$\lim_{n\to\infty} \int_{0}^{\delta} \frac{\sum_{j=1}^{n} \log 1/h_j(t)}{t^{1+r}}\, dt < \infty; \tag{4.28}$$

so by the monotone convergence theorem,

$$\sum_{j=1}^{\infty} \int_{0}^{\delta} \frac{\log 1/h_j(t)}{t^{1+r}}\, dt < \infty, \tag{4.29}$$

and hence $\Sigma_{j=1}^{\infty} f_{n,r}^i(\delta) < \infty$ for $i = 1, 2$.

Suppose, conversely, that fixing $i = 1$ or 2, $\Sigma_{j=1}^{\infty} f_{n,r}^i(\delta) < \infty$. Then

$$\sum_{j=1}^{\infty} \int_{0}^{\delta} \frac{g_n^i(x)}{x^{1+r}}\, dx < \infty; \tag{4.30}$$

whence, $\Sigma g_n^i(x) < \infty$ a.e., $0 \leqq x \leqq \delta$. As we proved above, this implies that ΣX_n converges a.e.; consequently, Πh_n converges uniformly on compacta and we can choose δ', $0 < \delta' < \delta$ so that $h_n \geqq \frac{1}{2}$ on $[0, \delta']$ for all n. Thus by (4.24)

$$\int_{0}^{\delta'} \sum_{n=1}^{\infty} \frac{\log 1/h_n(x)}{x^{1+r}}\, dx < \infty \tag{4.31}$$

and hence

$$\lim_{n\to\infty} \int_{0}^{\delta'} \frac{1 - \prod_{j=1}^{n} h_j(x)}{x^{1+r}}\, dx < \infty. \tag{4.32}$$

Then, taking into account the symmetry of the h_j and the monotone convergence theorem,

$$\lim_{n\to\infty}\int_{-\delta'}^{\delta'}\frac{1-\prod_{j=1}^{n}h_j(x)}{|x|^{1+r}}\,dx=\int_{-\delta'}^{\delta'}\frac{1-\prod_{j=1}^{\infty}h_j(x)}{|x|^{1+r}}\,dx. \tag{4.33}$$

But

$$\lim_{n\to\infty}\int_{|x|>\delta'}\frac{1-\prod_{j=1}^{n}h_j(x)}{|x|^{1+r}}\,dx=\int_{|x|>\delta'}\frac{1-\prod_{j=1}^{\infty}h_j(x)}{|x|^{1+r}}\,dx, \tag{4.34}$$

by the dominated convergence theorem. Thus letting $X = \Sigma X_j$ and observing that the ch. f. of X equals Πh_j, we have by (4.6) that

$$\begin{aligned} E\left|\sum_{j=1}^{n} X_j\right|^r &= C_r\int_{-\infty}^{\infty}\frac{1-\prod_{j=1}^{n}h_j(x)}{|x|^{1+r}}\,dx \\ &\to C_r\int_{-\infty}^{\infty}\frac{1-\prod_{j=1}^{\infty}h_j(x)}{|x|^{1+r}}\,dx \\ &= E|X|^r \quad \text{as } n\to\infty; \end{aligned} \tag{4.35}$$

whence, $\Sigma_{j=1}^{n} X_j \xrightarrow{r} X$. *Q.E.D.*

Let $2 < p < \infty$, $1/p + 1/q = 1$. Let us call a basis in $\mathbf{X}_p$ a standard basis, provided it is equivalent to the natural basis of $\mathbf{X}_{p,w}$ for some w satisfying (1.6). (By the results of [11], given any w satisfying (1.6), then the natural basis of $\mathbf{X}_{p,w}$ is equivalent to some basis of $\mathbf{X}_p$.) Then let us call a basis (b_n) in $\mathbf{X}_q$ a standard basis, provided its dual (b_n^*) is a standard basis in $\mathbf{X}_p$ (where $b_n^*(b_j) = \delta_{n,j}$ for all j). Again, it follows from the results of [11] that a basis (b_n) in $\mathbf{X}_q$ is a standard basis if and only if there exists a sequence (X_n) of three valued, symmetric random variables (seminormalized in L^q) such that (b_n) is equivalent to (X_n) (in L^q). Actually, any standard basis (of either $\mathbf{X}_p$ or $\mathbf{X}_q$), is equivalent to a block basis of any other standard basis. For if w and w' are given satisfying (1.6), then by Lemma 7, p. 288 of [11], the natural basis of $\mathbf{X}_{p,w}$ is equivalent to a block basis (y_j) of the natural basis of $\mathbf{X}_{p,w'}$, and there is an "orthogonal" projection P from $\mathbf{X}_{p,w'}$ onto $[y_j]$. The range of P^* is then the closed linear span of a block basis of the natural basis of $\mathbf{X}_{q,w'}$, and this block basis is also equivalent to the natural basis of $\mathbf{X}_{q,w}$.

Our final result shows that all the spaces considered here are isomorphic to the spaces spanned by block bases of any standard basis of $\mathbf{X}_q$, and also reveals the lack of gain in generality (for our purposes) in considering nonsymmetric random variables.

COROLLARY 4.3. *Let $1 \leqq r < 2$, and let (X_n) be a sequence of independent, nonzero random variables belonging to L^r, each of mean zero. Then there exists a sequence (Z_n) of independent, three valued, symmetric random variables such that (X_n) is equivalent to a block basis of (Z_n) (in L^r). If $1 < r$, then (X_n) is equivalent to a block basis of any standard basis of $\mathbf{X}_r$.*

PROOF. Let $(\tilde{X}_n)$ be a sequence of independent random variables symmetrizing the X_n. Precisely, let $X_1', X_1'', X_2', X_2'', \cdots$ be a sequence of independent random variables such that for all n, $\text{dist } X_n = \text{dist } X_n' = \text{dist } X_n''$, and put $\tilde{X}_n = X_n' - X_n''$ for all n. Then fixing $1 \leqq r < 2$, (X_n) is equivalent to $(\tilde{X}_n)$ (in L^r). Indeed, when scalars (c_i) are given, if $\Sigma c_i X_i$ converges in L^r, then $\Sigma c_i X_i'$ and $\Sigma c_i X_i''$ converge, hence, $\Sigma c_i(X_i' - X_i'')$ converges. Conversely, if $\Sigma c_i(X_i' - X_i'')$ converges, then since $(X_1', X_1'', X_2', X_2'', \cdots)$ is an unconditional basic sequence (see Section 2), $\Sigma c_i X_i'$ converges; whence, $\Sigma c_i X_i$ converges.

Now, of course, if $h_n = \text{ch.f. } X_n$, then $|h_n|^2 = \text{ch.f. } \tilde{X}_n$ for all n. Hence, by Theorem 4.2, there exists a sequence (Y_n) of independent, symmetric, infinitely divisible random variables such that $(\tilde{X}_n)$ is equivalent to (Y_n). (The association $X_n \to Y_n$ of Theorem 4.2 is homogeneous; that is, if $X_n \to Y_n$, then $tX_n \to tY_n$). By Corollary 4.1, (Y_n) (and consequently (X_n)) is equivalent to a block basis of some sequence (Z_n) of symmetric, three valued independent random variables, and thence, if $r > 1$, to a block basis of any standard basis of $\mathbf{X}_r$, by our remarks above. *Q.E.D.*

REMARK 4.3. It is well known (and a consequence of Lemma 4.2) that random variables which are stable of exponent r, $0 < r < 2$, fail to have absolute rth moments. The preceding results show that if (X_n) is a sequence of i.i.d. random variables possessing absolute rth moments, then the closed linear span of the X_n in L^r cannot be isomorphic to ℓ^r for $1 \leqq r < 2$. Let $\tilde{X}_n$ be the "symmetrization" of the random variable $X_n - EX_n$ for all n (as defined in the preceding proof). Then $[\tilde{X}_n]_r$ is isomorphic to $[X_n]_r$ (provided the latter is of infinite dimension). Now if f is the characteristic function of $\tilde{X}_1$, then letting $f_{n,r} = f^2_{\tilde{X}_n, r}$ as defined in the proof of Theorem 4.2, we have that

$$f_{n,r}(y) = y^r \int_0^y \frac{1 - f(t)}{t^{1+r}}\, dt = o(y^r) \quad \text{as } y \to 0 \tag{4.36}$$

and $\Sigma c_j X_j$ converges in r mean if and only if $\Sigma f_{j,r}(|tc_j|) < \infty$ for some $t > 0$. This shows that the basic sequence (X_n) is not equivalent to the usual basis of ℓ^r, while every subsequence of (X_n) is equivalent to the whole sequence. Hence, $[X_n]_r$ is not isomorphic to ℓ^r, since ℓ^r has only one "subsequence equivalent" basis (up to equivalence).

In a sense we have studied here the richness of the spaces spanned by sequences of independent, three valued, symmetric random variables by working with infinitely divisible distributions. In light of our final corollary, it is now possible (in theory, at any rate) to turn this around and study the spaces spanned by arbitrary sequences of independent random variables, by studying block bases of sequences of independent, three valued, symmetric random variables, that is, of any standard basis of X_q.

Fix $1 < q < 2$, and let (X_n) be any sequence of three valued, symmetric, independent random variables such that w satisfies (1.6), where $w_n = \|X_n\|_q/\|X_n\|_2$ for all n. (Since X_n is three valued, $w_n = \|X_n\|_2/\|X_n\|_p$.) Also, assume that (X_n) is normalized in L^q. Thus, (X_n) is equivalent to an arbitrary standard basis of $\mathbf{X}_q$. As we mentioned in [11], it can be shown that for any sequence of scalars (c_j), $\Sigma c_j X_j$ converges in L^q if and only if

$$\sum_n \min\left\{c_n^2 w_n^{-2}, \frac{2}{q}|c_n|^q\right\} < \infty. \tag{4.37}$$

It follows that given any sequence (Y_n) of independent random variables in L^q, each of mean zero, and given any sequence $w = (w_n)$ satisfying (1.6), there exists a sequence $B_1, B_2, \cdots$ of disjoint finite subsets of the integers, and there exist, for each j, scalars λ_n for $n \in B_j$, such that for any sequence of scalars (c_j), the series $\Sigma c_j Y_j$ converges in L^q if and only if

$$\sum_{j=1}^{\infty} \sum_{n \in B_j} \min\left\{c_j^2 \lambda_n^2 w_n^{-2}, \frac{2}{q}|c_j|^q|\lambda_n|^q\right\} < \infty. \tag{4.38}$$

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