METRIC MEASURE SPACES OF ECONOMIC AGENTS

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1. Introduction

An economic agent, who participates in a pure exchange economy is described by his *needs*, *tastes*, and *endowments*. If there are ℓ commodities, these economic concepts are formalized as follows:

the needs are described by a subset $X \subset R^{\ell}$; in choosing a commodity vector, the agent is restricted to the set X;

the tastes are described by a binary relation $\precsim$ on X; $x \precsim y$ means that the commodity vector y is at least as desired as the commodity vector x;

the endowments are described by a vector e in the commodity space R^{ℓ} (for more details see Chapters 2 and 4 of G. Debreu [6]).

A *pure exchange economy* is a finite family $\{(X_a, \precsim_a, e_a)\}_{a \in A}$ of economic agents. Since the endowments e_a typically are not a maximal element for $\precsim_a$ in X_a, there is an incentive to exchange commodities in order to improve the initial position.

The result of an exchange is a redistribution of the total endowments; it can be described by a function f of A into R^{ℓ} such that for every $a \in A$, $f_a \in X_a$ and $\Sigma_A f_a = \Sigma_A e_a$ or, if we assume free disposal of all commodities, $\Sigma_A f_a \leqq \Sigma_A e_a$.

The economic analysis of pure exchange economies consists of specifying a certain class of redistributions as possible outcomes of the exchange process.

Imagine a planner who cannot enforce his plan and who proposes a certain redistribution f. If there exists a subset B of agents and for every agent a in B a commodity vector $g_a \in X_a$ such that g_a is preferred to f_a by every member a in B and $\Sigma_B g_a \leqq \Sigma_B e_a$, then the coalition B has the desire and the power to block the proposed plan f. It seems reasonable to exclude as possible outcomes all redistributions which can be blocked by any coalition. The remaining redistributions are called the *core* of the exchange economy. The core, if not empty, contains in general many redistributions.

Suppose now that all agents agree to exchange commodities in fixed ratios. This agreement simplifies the exchange extremely. If $p \in R^{\ell}$ is the price system (that is, p_h/p_k is the amount of commodity k one has to give for one unit of commodity h), the agent $a = (X, \precsim, e)$ will only consider vectors in his budget set $\{x \in X | p \cdot x \leqq p \cdot e\}$ and will choose a most desired vector in this set. A redistribution f and a price vector p is called a *price equilibrium*, if the commodity vector f_a is for every agent a most desired vector in his budget set. The existence of price equilibria can be shown under quite general conditions (Debreu [6]).

One easily shows that a price equilibrium always belongs to the core and that the core is, in general, larger than the set of price equilibria. It is the purpose of this paper to elaborate the connection between the core and the set of price equilibria. It has been argued in economic theory (back to F. Y. Edgeworth [12] in 1881) that the "difference" between the core and the set of price equilibria is "small" if the economy is "large"; in the sense that the influence of every individual agent on the outcome of collective activities is negligible. And here is the connection with measure theory: in order to give a precise definition of "large economies" in the above sense, measure theory is the natural tool. The use of measure theory in order to describe large economies is due to R. J. Aumann [1]. For references of earlier work which led to the concept of a measure space of economic agents see the introduction of Aumann's paper.

Notation. If $x, y \in R^\ell$, we denote the scalar product by $x \cdot y$. The relation $x < y$ means $x \leqq y$ and $x \neq y$. The double inequality $x \ll y$ means $x_i < y_i$, $i = 1, \cdots, \ell$. By $x \leqq M$ is meant $x \leqq y$ for every $y \in M$. The set R^ℓ_+ is $\{x \in R^\ell | x \geqq 0\}$ and $R^\ell_- = -R^\ell_+$. A correspondence of T into S is a mapping of T into nonempty subsets of S. If φ denotes a correspondence of a probability space $(T, \mathscr{T}, \lambda)$ into R^ℓ, we call measurable selector for φ a measurable function f of T into R^ℓ such that $f(t) \in \varphi(t)$ almost everywhere (a.e.) in T. The set of all integrable selectors is denoted by $\mathscr{L}_\varphi$. The integral of a correspondence φ is defined by

$$\int \varphi \, dv = \left\{ \int f \, dv \,\middle|\, f \in \mathscr{L}_\varphi \right\}. \tag{1.1}$$

For details on the integration of correspondence see Aumann [3] and Debreu [8].

2. The space of agents' characteristics

We shall define the space of preference relations and give a precise meaning to the intuitive concept of "similar preferences."

Let $\mathbf{P}$ denote the set of all nonempty subsets P in $R^\ell \times R^\ell$ which have the following properties.

PROPERTY 2.1 (Continuity). *P is closed.*

PROPERTY 2.2 (Reflexivity). *$(x, y) \in P$ implies $(x, x) \in P$ and $(y, y) \in P$.*

PROPERTY 2.3 (Completeness). *$(x, x) \in P$ and $(y, y) \in P$ imply $(x, y) \in P$ or $(y, x) \in P$.*

PROPERTY 2.4 (Transitivity). *$(x, y) \in P$ and $(y, z) \in P$ imply $(x, z) \in P$.*

The generic element in $\mathbf{P}$ will be called a *preference relation* and is denoted by $\precsim$ (we write $x \precsim y$ or $(x, y) \in \precsim$, whichever is more convenient). The projection of $\precsim$ on R^ℓ, that is, the set $\{x \in R^\ell | (x, x) \in \precsim\}$, is called a *consumption set* and is denoted by $X_\precsim$. A consumption set is always closed.

We say that a sequence $(\precsim_n)$ of preference relations *converges* to a preference relation $\precsim$ in $\mathbf{P}$ if

$$\text{Lim Inf} \precsim_n \; = \; \precsim \; = \text{Lim Sup} \precsim_n, \tag{2.1}$$

where Lim Inf $\precsim_n$ denotes the set of points x in $ER^\ell \times R^\ell$ such that every neighborhood of x intersects all the $\precsim_n$ with sufficiently large n and Lim Sup $\precsim_n$ denotes the set of points x in $R^\ell \times R^\ell$ such that every neighborhood of x intersects infinitely many $\precsim_n$.

It is well known (see F. Hausdorff [14] and P. D. Watson [28]) that this concept of convergence for closed subsets of a locally compact separable metric space is metrizable. In fact, denote by $Y = R^\ell \times R^\ell \cup \{\infty\}$, the one point compactification of $R^\ell \times R^\ell$. Let δ_Y denote the Hausdorff distance ([14], p. 166) for subsets of the compact metric space Y, that is,

$$\delta_Y(F, F') = \inf\{\varepsilon > 0 \mid F \subset B_\varepsilon(F') \text{ and } F' \subset B_\varepsilon(F)\}, \tag{2.2}$$

where

$$B_\varepsilon(F) = \{x \in Y \mid \inf_{y \in F} \rho(x, y) < \varepsilon\} \tag{2.3}$$

denotes the open ε neighborhood of the set F with respect to a metric ρ in Y.

We now define a metric d on $\mathbf{P}$ by the formula

$$d(\precsim, \precsim') = \delta_Y(\precsim \cup \{\infty\}, \precsim' \cup \{\infty\}). \tag{2.4}$$

ASSERTION 2.1. *The metric space $(\mathbf{P}, d)$ is separable and a sequence $(\precsim_n)$ converges to $\precsim$ with respect to the metric d if and only if*

$$\text{Lim Inf} \precsim_n = \precsim = \text{Lim Sup} \precsim_n. \tag{2.5}$$

The metric d on the set of preferences $\mathbf{P}$ is justified economically by the fact that agents with similar preferences behave similarly in similar situations. This is made precise by the following result, due to Debreu [9]. Given a preference relation $\precsim$ in $\mathbf{P}$, an endowment vector e in the commodity space R^ℓ and a price vector $p \in R^\ell$, we denote by

$$\beta(\precsim, e, p) = \{x \in X_\precsim \mid p \cdot x \leqq p \cdot e\} \tag{2.6}$$

the *budget set* and by

$$\varphi(\precsim, e, p) = \{x \in \beta(\precsim, e, p) \mid x \succsim y \text{ for every } y \in \beta(\precsim, e, p)\} \tag{2.7}$$

the *demand set*.

ASSERTION 2.2. *Let M be a subset in $\mathbf{P} \times R^\ell \times R^\ell$ such that for every point $(\precsim, e, p)$ in M the budget set $\beta(\precsim, e, p)$ is convex and compact and $\inf p \cdot X_\precsim < p \cdot e$. Then the demand correspondence φ of M into R^ℓ is nonempty and compact valued and upper hemicontinuous (that is, for every point $(\precsim, e, p)$ in M and every open set G containing $\varphi(\precsim, e, p)$, there exists a neighborhood V of $(\precsim, e, p)$ such that $\varphi(\precsim', e', p') \subset G$ for every $(\precsim', e', p') \in V$).*

For a proof of Assertion 2.2 and related results see Debreu [9]. Actually, Debreu uses the Hausdorff distance on $\mathbf{P}$. One can, however, easily verify that the results also hold for the coarser metric d used in this paper (see [17], Appendix A). We use the metric d instead of the Hausdorff distance on $\mathbf{P}$ since, in general, $\mathbf{P}$ with the Hausdorff distance is not separable. The metric d extends on $\mathbf{P}$ the metric introduced by Y. Kannai [21], who considered only monotonic preferences.

A *pure exchange economy* is described by a finite family $\{a_i\}_{i\in I}$ of points in the space $\mathbf{P} \times R^\ell$ of agents' characteristics. Since we want to compare different economies, in particular, economies with different numbers $\#I$ of participants, it is convenient to consider the distribution ν of the family $\{a_i\}$ over $\mathbf{P} \times R^\ell$, that is, the probability measure

$$\nu = \frac{1}{\#I} \sum_{i\in I} \delta_{a_i}, \tag{2.8}$$

where δ_a denotes the probability measure concentrated at the point a.

Since an economy and its n fold replica have the same distribution, one may wish to introduce a set of "second names" in order to be able to distinguish agents with identical characteristics. Quite arbitrarily we choose the unit interval as the set of "second names." Then an economy can be described by a measure on $\mathbf{A} = \mathbf{P} \times R^\ell \times [0, 1]$ which has a finite support and assigns equal measure to every point in the support. For short, we call such measures *simple*.

More generally, every probability measure ν on the Borel field $\mathscr{B}_{\mathbf{A}}$ of the separable metric space $\mathbf{A}$ may be conceived as a distribution of agents' characteristics of a certain "economic system." The number $\nu(B)$ is interpreted as the fraction of the totality of agents whose characteristics belong to $B \in \mathscr{B}_{\mathbf{A}}$. If a denotes the generic element in $\mathbf{A}$, we denote by $\precsim_a$, $X(a)$, $e(a)$ the corresponding preference relation, consumption set, and initial endowments, respectively.

DEFINITION 2.1. *An* allocation *for the measure ν on $\mathbf{A}$ is a ν integrable function of $\mathbf{A}$ into R^ℓ such that $f(a) \in X(a)$ almost everywhere in $\mathbf{A}$ $(f(a)$ is a possible consumption plan for agent $a)$. An allocation f is called* attainable *for ν, if $\int f\,d\nu \leqq \int e\,d\nu$ (total demand does not exceed total supply).*

DEFINITION 2.2. *An allocation f for the measure ν is said to be* blocked *by the coalition $B \in B_{\mathbf{A}}$, if there exists a ν integrable function g of B into R^ℓ, where $g(a) \in X(a)$ such that*

(i) $g(a) \succ_a f(a)$ a.e. in B,

(ii) $\nu(B) > 0$ and $\int_B g\,d\nu \leqq \int_B e\,d\nu$.

The set of all unblocked attainable allocations for ν is called the core of ν.

DEFINITION 2.3. *A price vector $p \in R^\ell$ and an attainable allocation f for the measure ν are called a* price equilibrium *if $f(a) \in \varphi(a, p)$ almost everywhere in $\mathbf{A}$ $(f(a)$ is a greatest element for $\precsim_a$ in the budget set $\beta(a, p)$ with respect to the price vector $p)$ and $p\cdot\int f\,d\nu = p\cdot\int e\,d\nu$ (value of demand equals value of supply).*

It follows immediately from Definitions 2.2 and 2.3 that:

PROPOSITION 2.1. *For every measure ν on $\mathbf{A}$, a price equilibrium is unblocked.*

Let $\mathbf{E}$ denote the set of measures ν on $\mathbf{A}$ such that the following assumptions hold.

ASSUMPTION 2.1. *For ν* a.e. *in $\mathbf{A}$, the consumption set $X(a)$ is* convex *and uniformly* bounded *from below and the preference relation $\precsim_a$ is* locally non-saturated *(that is, for every $x \in X(a)$ and every neighborhood U of x, there is a vector $z \in X(a) \cap U$ such that $z \succ_a x$).*

ASSUMPTION 2.2. *For* ν a.e. *in* **A**, $e(a) \in X(a)$ *and* $\int e\, d\nu$ *belongs to the interior of* $\int X\, d\nu$.

ASSUMPTION 2.3 (Irreducibility). *For every partition* (S, T) (*that is*, $S \cup T = \mathbf{A}$, $S \cap T \neq \varnothing$ *and* $0 < \nu(S) < 1$) *and every attainable allocation* f *for* ν, *there exists an allocation* h *for* ν *such that*

$$\int_T (e - h)\, d\nu + \int_S f\, d\nu \in \int_S \{x \in X(a) \mid x \succ_a f(a)\}\, d\nu. \tag{2.9}$$

Assumption 2.3 expresses the idea that the endowments of every coalition are desired.

To be more specific, let **M** denote the subset in the space **A** of agents' characteristics defined by the property: for every $a \in \mathbf{M}$, $X(a) = R^{\ell}_{+}$, $\precsim_a$ is monotonic (that is, $x < y$ implies $x \prec_a y$) and $e(a) > 0$. By $\mathbf{E}[\mathbf{M}]$, we denote the set of measures ν such that the support of ν belongs to **M** and $\int e\, d\nu \gg 0$. Clearly, $\mathbf{E}[\mathbf{M}] \subset \mathbf{E}$. If we add to the above properties that every preference relation is convex (that is, for every $z \in X(a)$, the set $\{x \in X(a) \mid x \succsim_a z\}$ is convex), we write $\mathbf{M}_c$ and $\mathbf{E}[\mathbf{M}_c]$, respectively.

3. The identity of the core and the set of price equilibria for atomless measures

The traditional economic concept of an economy where no individual agent can influence the outcome of a collective activity is described by an atomless measure on the space of agents' characteristics, that is, a measure ν on **A** such that $\nu(\{a\}) = 0$ for every $a \in \mathbf{A}$.

The concept of an atomless measure space of economic agents is due to Aumann who proved in [1] the identity of the core and the set of price equilibria for a pure exchange economy. Theorem 3.1 is a generalization of Aumann's result. An extension to an economy with production has been given in [15]. The proof given below differs from Aumann's proof and is a simplification of the proof in [15] suggested to me by D. Schmeidler.

THEOREM 3.1. *If* ν *is an atomless measure in* **E**, *then the core and the set of price equilibria coincide.*

PROOF. Let f be in the core of ν. We have to show that f is a price equilibrium. Consider for every agent $a \in \mathbf{A}$ the sets

$$\begin{aligned} \pi(a) &= \{x \in X(a) \mid x \succ_a f(a)\}, \\ \psi(a) &= \{\pi(a) - e(a)\} \cup \{0\}. \end{aligned} \tag{3.1}$$

Since by assumption $\nu(\{a\}) = 0$ for every agent $a \in \mathbf{A}$ and since **A** is a separable metric space, it follows that the measure space $(\mathbf{A}, \mathscr{B}_{\mathbf{A}}, \nu)$ has no atoms. It is well known (Richter [22]) that the integral of a correspondence with respect to an atomless measure space is a convex set. Hence,

(a) the set $\int \psi\, d\nu$ is a convex subset in R^{ℓ}.

Next, we show

(b) $R^{\ell}_{-} \cap \int \psi\, d\nu = \{0\}$.

Assume there is a selector $h \in \mathscr{L}_\psi$ with $\int h\, d\nu < 0$. Then the coalition $S = \{a \in \mathbf{A} | h(a) \neq 0\}$ can block the allocation f by assigning to every agent a in S the vector $g(a) = h(a) + e(a)$. In fact, by definition of g it follows that $g(a) \succ_a f(a)$ *on* S; furthermore, $\nu(S) > 0$ and $\int_S g\, d\nu \leqq \int_S e\, d\nu$.

From (a) and (b), it follows that there is a hyperplane separating R^ℓ_- and $\int \psi\, d\nu$; that is, there is a vector $p \in R^\ell$, $p > 0$, such that $p \cdot z \geqq 0$ for every $z \in \int \psi\, d\nu$.

It is not hard to show that the graph of the correspondence ψ belongs to $\mathscr{B}^\nu_{\mathbf{A}} \otimes \mathscr{B}(R^\ell)$, where $\mathscr{B}^\nu_{\mathbf{A}}$ denotes the completion of $\mathscr{B}_{\mathbf{A}}$ with respect to ν. Then (see Theorem C in the Appendix of [16]), it follows that

$$\inf_{h \in \mathscr{L}_\psi} p \cdot \int h\, d\nu = \int (\inf_{x \in \psi(\cdot)} p \cdot x)\, d\nu. \tag{3.2}$$

Therefore, $\int \inf p \cdot \psi\, d\nu \geqq 0$. Since by definition of $\psi(a)$, we have $\inf p \cdot \psi(a) \leqq 0$; it follows that $\inf p \cdot \psi(a) = 0$ a.e. in $\mathbf{A}$. Hence, together with Assumption 2.1, we obtain

(c) $p \cdot e(a) \leqq p \cdot x$ for every $x \succsim_a f(a)$ a.e. in $\mathbf{A}$.

Since $\int f\, d\nu \leqq \int e\, d\nu$. Property (c) clearly implies that $p \cdot f(a) = p \cdot e(a)$ a.e. in $\mathbf{A}$; that is, $f(a)$ belongs to the budget set $\beta(a, p)$ and $p \cdot \int f\, d\nu = p \cdot \int e\, d\nu$. By (c) we know that a.e. in $\mathbf{A}$, $z \in X(a)$ and $p \cdot z < p \cdot e(a)$ imply $z \precsim_a f(a)$. If $a \in \mathbf{A}$ is such that $\inf p \cdot X(a) < p \cdot e(a)$, then it follows from the convexity of the consumption set $X(a)$ that every point $z \in X(a)$ with $p \cdot z = p \cdot e(a)$ is the limit of a sequence of points $z_n \in X(a)$ with $z_n \precsim_a f(a)$; hence, $z \precsim_a f(a)$. Thus, we have

(d) a.e. in $\mathbf{A}$, $\inf p \cdot X(a) < p \cdot e(a)$ implies $f(a)$ is a greatest element in the budget set $\beta(a, p)$.

It remains to show that the set

$$T = \{a \in \mathbf{A} | \inf p \cdot X(a) = p \cdot e(a)\} \tag{3.3}$$

has measure zero. Assume $\nu(T) > 0$. Clearly,

$$p \cdot \int_T e\, d\nu = \int_T \inf p \cdot X\, d\nu = \inf p \cdot \int_T X\, d\nu. \tag{3.4}$$

Hence,

(e) $p \cdot \int_T (e - X)\, d\nu \leqq 0$.

Let $S = \mathbf{A} \backslash T$. Since by Assumption 2.2 we have $\int e\, d\nu \in$ interior $\int X\, d\nu$, it follows that $\nu(S) > 0$. By Assumption 2.3 there exists a vector $z \in \int_T (e - X)\, d\nu$ such that $z + \int_S f\, d\nu \in \int_S \pi\, d\nu$. Thus, there is a function g of S into R^ℓ with $g(a) \succ_a f(a)$ such that $\int_S g\, d\nu = z + \int_S f\, d\nu$. It follows by (d) that $p \cdot g(a) > p \cdot f(a)$. Using (e) we, therefore, obtain

$$p \cdot \int_S f\, d\nu < p \cdot \int_S g\, d\nu = p \cdot z + p \cdot \int_S f\, d\nu \leqq p \cdot \int_S f\, d\nu, \tag{3.5}$$

a contradiction. *Q.E.D.*

REMARK 3.1. In an alternative model, where coalitions are taken as the primitive concept, and hence, where preferences are defined for coalitions, K. Vind [27] has proved the identity of the core and the set of price equilibria. Vind's result has been extended by R. Cornwall [5]. The equivalence between the model based on individual agents (Aumann) and the model based on coalitions (Vind) has been established by Debreu [7].

Conditions under which the identity of the core and the set of price equilibria still hold when the measure ν has atoms have been given by J. Gabszewicz and J. F. Mertens [13], and B. Shitovitz [25].

4. Existence of price equilibria

The existence of price equilibria for economies with finitely many participants has been studied extensively in the economic literature. A complete treatment of this problem can be found in Debreu [6]. An existence proof for an atomless measure space of economic agents was given first by Aumann [2]. Schmeidler [24] has given an alternative proof of Aumann's result and has shown that completeness of the preference relations is not needed. The existence of price equilibria for a private ownership economy with production and a measure space of consumers has been established in [16].

THEOREM 4.1. *Let ν be a measure in* **E** *such that for every agent a with $\nu(\{a\}) > 0$ the preference relation $\precsim_a$ is convex. Then there exists a price equilibrium.*

PROOF. We have to show that there is a price vector $p \in R^\ell$ such that the demand set $\varphi(a, p)$ is nonempty a.e. in **A** and

$$\int [\varphi(\cdot, p) - e]\, d\nu \cap R^\ell_- \neq \varnothing. \tag{4.1}$$

The proof of the existence of such a price vector in the price simplex

$$\Delta = \left\{p \in R^\ell_+ \,\middle|\, \sum_{i=1}^{\ell} p_i = 1\right\} \tag{4.2}$$

is based on the following result (see 5.6 (1) of Debreu [6]) which is a consequence of Kakutani's fixed point theorem:

> *Let K be a compact subset of R^ℓ. If Ξ is a correspondence of Δ into K such that the graph of Ξ is closed, for every $p \in \Delta$, the set $\Xi(p)$ is (nonempty) convex and satisfies $p \cdot \Xi(p) \leqq 0$, then there is a $p \in \Delta$ such that $\Xi(p) \cap R^\ell_- \neq \varnothing$.*

Since for a given $p \in \Delta$ the budget set

$$\beta(a, p) = \{x \in X(a) \,|\, p \cdot x \leqq p \cdot e(a)\} \tag{4.3}$$

may be unbounded, and hence, the demand set $\varphi(a, p)$ may be empty, we consider for every integer k the truncated consumption set

$$X^k(a) = \{x \in X(a) \mid x \leqq k[e(a) + 1]\}. \tag{4.4}$$

Define

$$\begin{aligned}
\beta^k(a, p) &= \{x \in X^k(a) \mid p \cdot x \leqq p \cdot e(a)\}, \\
\varphi^k(a, p) &= \{x \in \beta^k(a, p) \mid x \succsim_a y \text{ for every } y \in \beta^k(a, p)\}, \\
\Xi^k(p) &= \int [\varphi^k(\cdot, p) - e]\, dv.
\end{aligned} \tag{4.5}$$

We can show (for details see Theorem 3 of [16]), that the correspondence Ξ^k of Δ in R^ℓ has the following properties:

(a) there is a compact set K in R^ℓ such that $\Xi^k(p) \subset K$ for every $p \in \Delta$;

(b) the graph of the correspondence Ξ^k is closed;

(c) for every $p \in \Delta$, $p \cdot \Xi^k(p) \leqq 0$ and the set $\Xi^k(p)$ is nonempty and convex.

We now apply Kakutani's fixed point theorem to the correspondence Ξ^k. Hence, there exist a price vector $p^k \in \Delta$ and an integrable function f^k of $\mathbf{A}$ into R^ℓ such that

(d) $f^k(a) \in \varphi^k(a, p^k)$ and $\int f^k\, dv \leqq \int e\, dv$ a.e. in $\mathbf{A}$.

Without loss of generality, we can assume that the sequence (p^k) is convergent; say, $\lim p^k = p \in \Delta$.

Since every function f^k is a selector of the consumption set correspondence X, the sequence (f^k) is minorized. By (d), the sequence $(\int f^k\, dv)$ is bounded. Consequently, it follows from Fatou's lemma in ℓ dimension (see [23] or [19]) that there is an integrable function f of $\mathbf{A}$ into R^ℓ such that

(e) $\int f\, dv \leqq \int e\, dv$ and $f(a) \in \operatorname{Lim\,Sup} \{f^k(a)\}$ a.e. in $\mathbf{A}$.

Clearly, (d) and (e) imply that $f(a) \in \beta(a, p)$ a.e. in $\mathbf{A}$. Let $a \in \mathbf{A}$ be such that $\inf p \cdot X(a) < p \cdot e(a)$; that is, there is a vector $x \in X(a)$ such that $p \cdot x < p \cdot e(a)$. For k large enough, we obtain $p^k \cdot x < p^k \cdot e(a)$ and $x \in X^k(a)$. Hence, by (*d*) it follows that $x \precsim_a f^k(a)$. Since the graph of the preference relation is assumed to be closed, (e) implies $x \precsim_a f(a)$. Therefore, it follows that in the case $\inf p \cdot X(a) < p \cdot e(a)$, we have $f(a) \in \varphi(a, p)$ and $p \cdot f(a) = p \cdot e(a)$.

Finally, as in the proof of Theorem 3.1, we show that the set

$$T = \{a \in \operatorname{supp}(v) \mid \inf p \cdot X(a) = p \cdot e(a)\} \tag{4.6}$$

has measure zero. This completes the proof that (p, f) is a price equilibrium for v. *Q.E.D.*

5. Upper hemicontinuity of the equilibrium correspondence

In this section, we study the behavior of the set of price equilibria of an economy when the characteristic data of the economy, that is, the measure v, are changed. Let us mention that even under strong assumptions on the agents' characteristics there may be more than one price equilibrium.

Since we want to compare the set of price equilibria for different economies in $\mathbf{E}$, it is convenient to describe an allocation f for v by the vector valued measure $f \cdot v$ which is defined by

$$(f\cdot\nu)(B) = \int_B f\,d\nu, \qquad B \in \mathscr{B}_{\mathbf{A}}. \tag{5.1}$$

Let $\mathscr{M}^\ell$ denote the set of R^ℓ valued finite measures on $\mathscr{B}_{\mathbf{A}}$. We define the *equilibrium correspondence* W of $\mathbf{E}$ into $\Delta \times \mathscr{M}^\ell$ by

$$W(\nu) = \{(p, f\cdot\nu)\,|\Delta \times \mathscr{M}^\ell|\,(p,f) \text{ is a price equilibrium for } \nu\}. \tag{5.2}$$

We endow the sets of measures $\mathbf{E}$ and $\mathscr{M}^\ell$ with the weak topology and ask for continuity properties of the correspondence W. This problem has been studied in [20]. In a different setup, Kannai [21] proved continuity properties of the equilibrium correspondence for atomless measures.

THEOREM 5.1. *Let (ν_n) be a sequence converging weakly to ν in $\mathbf{E}$ such that*

(i) *for ν a.e. in $\mathbf{A}$, the preference relation $\precsim_a$ is convex;*

(ii) *for every price equilibrium (p, f) for ν, total demand equals total supply, that is, $\int f\,d\nu = \int e\,d\nu$;*

(iii) $\lim \int e\,d\nu_n = \int e\,d\nu$;

(iv) *there is a vector $b \in R^\ell$ such that for every measure ν_n and ν, $b \leqq X(a)$ a.e. Let $(p_n, f_n\cdot\nu_n) \in W(\nu_n)$ for $n = 1, \cdots$. Then there exists $(p, f\cdot\nu) \in W(\nu)$ which is a limit in $\Delta \times \mathscr{M}^\ell$ of a subsequence of $(p_n, f_n\cdot\nu_n)$.*

PROOF. We shall only sketch the essential steps in order to show how the assumptions are used; a complete proof is given in [20].

Since every price vector p_n belongs to the compact price simplex Δ, we can assume that $\lim p_n = p \in \Delta$.

According to R. M. Dudley [11], there exist a probability space $(T, \mathscr{T}, \lambda)$ and measurable functions α_n and α of T into $\mathbf{A}$ such that

(a) $\nu_n = \lambda\circ\alpha_n^{-1}$, $\nu = \lambda\circ\alpha$ and $\lim \alpha_n(t) = \alpha(t)$ a.e. in T.

Consider the composed function $\tilde{f}_n = f_n\circ\alpha_n$ of T into R^ℓ. The sequence $(\tilde{f}_n)$ is bounded from below. Since

$$\int \tilde{f}_n\,d\lambda = \int f_n\,d\nu_n \leqq \int e\,d\nu_n, \tag{5.3}$$

it follows from assumption (iii) that the sequence $(\int \tilde{f}_n\,d\lambda)$ is bounded. Hence, we can assume that $\lim_n \int \tilde{f}_n\,d\lambda$ exists. According to Fatou's lemma in ℓ dimension ([23] or [19]), there exists a function $\tilde{f}$ of T into R^ℓ such that

(b) $\int \tilde{f}\,d\lambda \leqq \lim_n \int \tilde{f}_n\,d\lambda$ and $\tilde{f}(t) \in \operatorname{Lim\,Sup}\,\{\tilde{f}_n(t)\}$ a.e. in T.

Since $\tilde{f}_n(t) \in \varphi[\alpha_n(t), p_n]$ a.e. in T, it follows from (a) and (b) that $\tilde{f}(t) \in \varphi[\alpha(t), p]$ a.e. in T.

Consider the conditional expectation

$$E(\tilde{f}\,|\,\alpha^{-1}(\mathscr{B}_{\mathbf{A}})) = f^*. \tag{5.4}$$

It follows that $f^*(t) \in \varphi[\alpha(t), p]$ a.e. in T, since $\varphi[\alpha(t), p]$ is a closed convex set which does not contain a straight line. Since f^* is $\alpha^{-1}(\mathscr{B}_{\mathbf{A}})$ measurable, there exists a measurable function f of $\mathbf{A}$ into R^ℓ such that $f^* = f\circ\alpha$. Consequently, (p, f) is a price equilibrium for ν. Hence, by assumption (ii), it follows that $\int f\,d\nu = \int e\,d\nu$ and consequently, $\int \tilde{f}\,d\lambda = \lim \int \tilde{f}_n\,d\lambda$.

Now, we can apply the corollary in [19] which states that the sequence $(\tilde{f}_n)$ is $\sigma(L_1, L_\infty)$ relative compact and that every $\sigma(L_1, L_\infty)$ adherent point g has the property that $g(t) \in \text{co Lim Sup}\,\{\tilde{f}_n(t)\}$ a.e. in T, where co denotes the convex hull.

According to Eberlein's theorem, there exists a $\sigma(L_1, L_\infty)$ converging subsequence of $(\tilde{f}_n)$. Without loss of generality, we can therefore assume that the sequence $(\tilde{f}_n)$ is $\sigma(L_1, L_\infty)$ convergent to an integrable function $\tilde{f}$. Since $\text{Lim Sup}\,\{\tilde{f}_n(t)\} \subset \varphi[\alpha(t), p]$, which is a convex set, we have

(c) $\tilde{f}(t) \in \varphi[\alpha(t), p]$ a.e. in T.

As above, let f^* be the conditional expectation of $\tilde{f}$ with respect to $\alpha^{-1}(\mathscr{B}_{\mathbf{A}})$ and f a function of $\mathbf{A}$ into R^ℓ such that $f^* = f \circ \alpha$. It follows from (c) that $(p, f) \in W(\nu)$. It remains to verify that $(f_n \cdot \nu_n)$ converges in $\mathscr{M}^\ell$ to $f \cdot \nu$; that is, for every continuous and bounded function h of $\mathbf{A}$ into R it follows that $\lim \int f_n \cdot h\, d\nu_n = \int f \cdot h\, d\nu$.

Since the sequence $(\tilde{f}_n)$ is $\sigma(L_1, L_\infty)$ convergent to $\tilde{f}$ and since the sequence (α_n) converges a.e. to α, we verify easily that

$$\lim \int \tilde{f}_n \cdot (h \circ \alpha_n)\, d\lambda = \int \tilde{f} \cdot (h \circ \alpha)\, d\lambda. \tag{5.5}$$

Hence,

$$\begin{aligned} \lim \int f_n \cdot h\, d\nu_n &= \lim \int \tilde{f}_n \cdot (h \circ \alpha_n)\, d\lambda = \int \tilde{f} \cdot (h \circ \alpha)\, d\lambda \\ &= \int f^* \cdot (h \circ \alpha)\, d\lambda = \int f \cdot h\, d\nu. \end{aligned} \tag{5.6}$$

Q.E.D.

Recall that by $\mathbf{E}[\mathbf{M}_c]$ we denoted the set of measures in $\mathbf{E}$ which are concentrated on $\mathbf{M}_c$ (monotonic and convex preferences). If τ is a strictly positive vector in R^ℓ, then we denote

$$\mathbf{E}_\tau = \{\nu \in \mathbf{E}[\mathbf{M}_c] \mid \nu \text{ a.e., } e(a) \leqq \tau\}. \tag{5.7}$$

COROLLARY 5.1. *The equilibrium correspondence W of $\mathbf{E}_\tau$ into $\Delta \times \mathscr{M}^\ell$ is (nonempty and) compact valued and upper hemicontinuous.*

PROOF. The space $\mathscr{M}^\ell_+$ of R^ℓ_+ valued finite measures on $\mathbf{A}$ endowed with the weak topology is metrizable, since the underlying space $\mathbf{A}$ is metrizable. Hence, W is a correspondence of the metrizable space $\mathbf{E}_\tau$ into the metrizable space $\Delta \times \mathscr{M}^\ell_+$. Therefore, W is compact valued and upper hemicontinuous if and only if for every sequence (ν_n) converging weakly to ν in $\mathbf{E}_\tau$ and every $(p_n, f_n \cdot \nu_n) \in W(\nu_n)$ there exists a converging subsequence of $(p_n, f_n \cdot \nu_n)$ whose limit belongs to $W(\nu)$. But the existence of such a subsequence follows from Theorem 5.1 since assumptions (i) to (iv) are clearly fulfilled for measures in $\mathbf{E}_\tau$.

REMARK 5.1. Every measure ν on $\mathbf{A}$ is a limit of a sequence (ν_n) of measures, where every ν_n has a finite support. Thus, the result of this section and the well known existence theorem for finite economies imply the existence of price equilibria for measures.

6. A limit theorem on the core

Consider the *core correspondence* C of $\mathbf{E}$ into $\mathcal{M}^\ell$ defined by

$$C(\nu) = \{f \cdot \nu \in \mathcal{M}^\ell \mid f \text{ belongs to the core of } \nu\}. \tag{6.1}$$

As in the previous section, we endow the sets $\mathbf{E}$ and $\mathcal{M}^\ell$ with the weak topology and ask for continuity properties of the correspondence C. This problem was first studied by Kannai [21] in a different setup. There are examples which show that one cannot expect a general result similar to Corollary 5.1. Consider a sequence of economies $(\mathscr{E}_n)$, where the economy $\mathscr{E}_n$ is an n fold replica of the first economy $\mathscr{E}_1$. Debreu and H. Scarf [10] have shown that in this case Edgeworth's assertion [12], which states that the core "shrinks" to the set of price equilibria, can be made precise and proved. Every economy $\mathscr{E}_n$ can be represented by a simple measure ν_n on $\mathbf{A}$ and if the "second names" are properly selected, the sequence (ν_n) converges to an atomless measure ν. The result of Debreu and Scarf then implies that every neighborhood U in $\mathcal{M}^\ell$ of $C(\nu) = W(\nu)$ contains the core $C(\nu_n)$ for n large enough. Theorem 6.1, below, generalizes this result. Recent results of T. Bewley [4] indicate that Theorem 6.1 probably holds for a finer topology on the space of allocations.

THEOREM 6.1. *Let (ν_n) be a sequence of simple measures converging weakly to an atomless measure ν in $\mathbf{E}[\mathbf{M}_c]$ with compact support such that* supp $(\nu_n) \subset$ supp (ν). *Then every neighborhood U of $C(\nu)$ in $\mathcal{M}^\ell$ contains $C(\nu_n)$ for n sufficiently large.*

Theorem 6.1 follows from Propositions 6.1 and 6.2 stated below.

For every $a \in \mathbf{M}$ and $x \in R^\ell_+$ define

$$u(a, x) = \max\{\xi \in R_+ \mid (\xi, \cdots, \xi) \precsim_a x\}. \tag{6.2}$$

We verify easily (for details, see Kannai [21] or [17], Appendix B) that the function u of $\mathbf{M} \times R^\ell$ into R is continuous and for every $a \in \mathbf{M}$, $u(a, x) \leqq u(a, y)$ if and only if $x \precsim_a y$.

Let (ν_n) be a sequence of measures on $\mathbf{M}$ converging weakly to ν and let f_n be in the core of ν_n. We say that for the sequence (f_n) *similar treatment in utility* prevails if there exists for every n a subset $E_n \subset \mathbf{M}$ such that $\lim_n \nu_n(E_n) = 0$, and the limit $u[a_n, f_n(a_n)]$ exists for every converging sequence (a_n) with $a_n \in \text{supp}(\nu_n) \backslash E_n$.

These conditions express the idea that (with the possible exception of relatively few) similar agents are treated similarly in utility.

PROPOSITION 6.1. *Let (ν_n) be a sequence of simple measures converging weakly to an atomless measure ν in $\mathbf{E}[\mathbf{M}_c]$ with compact support such that* supp $(\nu_n) \subset$ supp (ν). *Let f_n be an allocation in the core of ν_n. Then there exists a subsequence of (f_n) for which similar treatment in utility prevails.*

A proof of this result has been given by Bewley [4]. The proof is too long to be included here. An easy proof can be found in [18] for the special case where the projection of supp (ν) of $\mathbf{A} = \mathbf{P} \times R^\ell \times [0, 1]$ onto $\mathbf{P} \times R^\ell$ is a finite set (economies with a finite set of types).

PROPOSITION 6.2. *Let* (ν_n) *be a sequence of measures converging weakly to a measure* ν *in* $\mathbf{E}[\mathbf{M}_c]$ *such that* supp (ν_n) *and* supp (ν) *belong to a compact subset* $\mathbf{K}$ *in* $\mathbf{M}_c$. *Let* f_n *be an allocation in the core of* ν_n *and assume that for the sequence* (f_n) *similar treatment in utility prevails. Then there exist an allocation* f *in the core of* ν *and a subsequence of* (f_n) *converging in* $\mathscr{M}^\ell$ *to* f.

PROOF. Since (ν_n) converges weakly to ν and $\lim \nu_n(E_n) = 0$, it follows that supp $(\nu) \subset \limsup (\text{supp}\,(\nu_n)\backslash E_n) = S$. We easily verify that there is a continuous function v of S into R such that

(a) $(a_n) \to a$, where $a_n \in \text{supp}\,(\nu_n)\backslash E_n$, implies $v_n(a_n) = u[a_n, f_n(a_n)] \to v(a)$. (For details see Appendix E of [17].)

We shall prove that:

ASSERTION 6.1. *There exists an attainable allocation* f *for* ν *such that a.e. in* $\mathbf{A}$, $v(a) \leqq u[a, f(a)]$.

According to A. V. Skorokhod's lemma (p. 10 in [26]), there exist measurable mappings α_n, $n = 1, \cdots$, and α of the unit interval $T = [0, 1]$ into $\mathbf{K}$ such that

(b) $\nu_n = \lambda \circ \alpha_n^{-1}$ and $\nu = \lambda \circ \alpha_n$ (λ denotes the Lebesgue measure),

(c) $\lim_n \alpha_n(t) = \alpha(t)$ a.e. in T.

For every n, we define the function $\tilde{f}_n = f_n \circ \alpha_n$ of T into R^ℓ. By (b) we have $\int \tilde{f}_n\, d\lambda = \int f_n\, d\nu_n$. Since $\int f_n\, d\nu_n \leqq \int e\, d\nu_n$ and $\lim \int e\, d\nu_n = \int e\, d\nu$, there is a subsequence, which again we denote by (f_n), such that

(d) $\lim_n \int \tilde{f}_n\, d\lambda \leqq e\, d\nu$.

According to Fatou's lemma in ℓ dimension [23], [19], there exists an integrable function $\tilde{f}$ of T into R^ℓ_+ such that

(e) $\int \tilde{f}\, d\lambda \leqq \lim \int \tilde{f}_n\, d\lambda$ and $\tilde{f}(t) \in \text{Lim Sup}\, \{\tilde{f}_n(t)\}$ a.e. in T.

Since the sequence $(v_n \circ \alpha_n)$ of T into R converges in probability, a consequence of (a) and (b), and since the utility function u of $\mathbf{K} \times R^\ell_+$ into R is continuous, the relation

$$v_n[\alpha_n(t)] = u[\alpha_n(t), \tilde{f}_n(t)] \tag{6.3}$$

implies that $v[\alpha(t)] = u[\alpha(t), \tilde{f}(t)]$ a.e. in T.

The function $\tilde{f}$ can be written in the form $f \circ \alpha$, where f is a measurable function of $\mathbf{K}$ into R^ℓ, if and only if $\tilde{f}$ is $\alpha^{-1}(\mathscr{B}_{\mathbf{K}})$ measurable. The conditional expectation $E[\tilde{f} | \alpha^{-1}(\mathscr{B}_{\mathbf{K}})] = f^*$ has the properties:

$$\int f^*\, d\lambda \leqq \int e\, d\nu, \qquad f^*(t) \geqq v[\alpha(t)] \qquad \text{a.e. in } T. \tag{6.4}$$

In fact, the first property follows from (d) and (e). To prove the second property, we consider the correspondence π of supp (ν) into R^ℓ_+ defined by

$$\pi(a) = \{x \in R^\ell_+ \,|\, u(a, x) \geqq v(a)\}. \tag{6.5}$$

Clearly, $\tilde{f}$ is a selector of the correspondence $\tilde{\pi} = \pi \circ \alpha$. Since the set $\tilde{\pi}(t)$ is convex, closed, and contains no straight line, we can show that the conditional expectation f^* is also a selector of $\tilde{\pi}$, which establishes the second property.

Since f^* is $\alpha^{-1}(\mathscr{B}_{\mathbf{K}})$ measurable there is a measurable function f of $\mathbf{K}$ into R_+^ℓ such that $f^* = f \circ \alpha$. Clearly, $\int f\, d\nu \leqq \int e\, d\nu$ and a.e. in supp (ν), $u[a, f(a)] \geqq v(a)$. This completes the proof of Assertion 6.1.

Next, we shall prove:

ASSERTION 6.2. *Every allocation f for ν with $u[a, f(a)] \geqq v(a)$ for ν a.e. $a \in \mathbf{K}$ is unblocked.*

Assume the allocation f is blocked, that is, there is a coalition $B \in \mathscr{B}_{\mathbf{K}}$ and a measurable function g of B into R_+^ℓ such that

(i) $g(a) \succ_a f(a)$ on B

and

(ii) $\int_B g\, d\nu \leqq \int_B e\, d\nu$.

It is not hard to show that in this case there exists even a function g of B into R_+^ℓ such that (i) holds and that property (ii) can be strengthened to

(iii) $\int_B g_k\, d\nu < \int_B e_k\, d\nu$ for every coordinate k for which $\int_B e_k\, d\nu > 0$.

Consequently, Lusin's theorem implies that there is a compact coalition B and a continuous function g of B into R_+^ℓ such that (i) and (iii) hold. It follows from property (i) that on the coalition B we have $u[a, g(a)] > v(a)$.

We now extend the function g to a continuous function on $\mathbf{K}$ in such a way that $g_k = 0$ if k is a coordinate for which $\int_B e_k\, d\nu = 0$. Since B is compact and since the functions u, g and v are continuous, there exist an $\varepsilon > 0$ and an open set $\mathcal{O}$ containing B such that on $\mathcal{O}$ we have $u[a, g(a)] > v(a) + \varepsilon$. For every $\eta > 0$, there is a set Q whose boundary has ν measure zero such that $B \subset Q \subset \mathcal{O}$ and $\int_Q g_k\, d\nu < \int_Q e_k\, d\nu$ for every coordinate k for which $\int_Q e_k\, d\nu > 0$. Hence, the weak convergence of the sequence (ν_n) to ν implies that for n sufficiently large we have $\int_Q g\, d\nu_n \leqq \int_Q e\, d\nu_n$.

Since by (a) the sequence v_n converges uniformly to v, for n large enough, the coalition Q can block the allocation f_n which contradicts the assumption that $f_n \in C(\nu_n)$. This completes the proof of Assertion 6.2.

Since we assumed that the preference relations are monotonic, it is clear that for every unblocked and attainable allocation f, total demand equals total supply, that is, $\int f\, d\nu = \int e\, d\nu$. Hence, according to Assertions 6.1 and 6.2 we have shown that, given the sequence $(\tilde{f}_n)$ of integrable functions of T into R_+^ℓ such that $\lim \int \tilde{f}_n\, d\lambda$ exists, for every function $\tilde{f}$ with the properties $\int \tilde{f}\, d\lambda \leqq \lim \int \tilde{f}_n\, d\lambda$ and $\tilde{f}(t) \in \text{Lim Sup}\, \{\tilde{f}_n(t)\}$ a.e. in T, it follows that $\int \tilde{f}\, d\lambda = \lim \int \tilde{f}_n\, d\lambda$.

It is known [19] that in this case the sequence $(\tilde{f}_n)$ is $\sigma(L_1, L_\infty)$ relative compact and every $\sigma(L_1, L_\infty)$ adherent point $\tilde{f}$ has the property:

(f) $\tilde{f}(t)$ belongs to the convex hull of Lim Sup $\{\tilde{f}_n(t)\}$ a.e. in T.

Therefore, according to Eberlein's theorem, there is a $\sigma(L_1, L_\infty)$ converging subsequence. Without loss of generality, we can thus assume that the sequence $(\tilde{f}_n)$ is $\sigma(L_1, L_\infty)$ convergent to an integrable function $\tilde{f}$. Since the preference relations are assumed to be convex, it follows by (f) that $u[\alpha(t), \tilde{f}(t)] \geqq v[\alpha(t)]$ a.e. in T. As in the proof of Assertion 6.1 we show that the conditional expectation $E[\tilde{f} \mid a^{-1}(\mathscr{B}_{\mathbf{K}})] = f^*$ has the property: $u[\alpha(t), f^*(t)] \geqq v[\alpha(t)]$ a.e. in T.

Since we can write $f^* = f \circ \alpha$, where f is a measurable function of supp (ν) into R^{ℓ}_+, it follows from Assertion 6.2 that $f \in C(\nu)$.

It remains to verify that for every continuous and bounded function h of supp (ν) into R it follows that $\lim \int f_n \cdot h \, d\nu_n = \int f \cdot h \, d\nu$. Since the sequence $(\tilde{f}_n)$ is $\sigma(L_1, L_\infty)$ converging to $\tilde{f}$ and since the sequence (α_n) converges a.e. to α, we verify easily that

$$\lim \int \tilde{f}_n \cdot (h \circ \alpha_n) \, d\lambda = \int \tilde{f} \cdot (h \circ \alpha) \, d\lambda. \tag{6.6}$$

Hence,

$$\begin{aligned}\lim \int f_n \cdot h \, d\nu_n &= \lim \int \tilde{f}_n \cdot (h \circ \alpha_n) \, d\lambda = \int \tilde{f} \cdot (h \circ \alpha) \, d\lambda \\ &= \int f^* \cdot (h \circ \alpha) \, d\lambda = \int f \cdot h \, d\nu.\end{aligned} \tag{6.7}$$

Q.E.D.

REFERENCES

[1] R. J. AUMANN, "Markets with a continuum of traders," *Econometrica*, Vol. 32 (1964), pp. 39-50.

[2] ———, "Existence of competitive equilibria in markets with a continuum of traders," *Econometrica*, Vol. 34 (1966), pp. 1-17.

[3] ———, "Integrals of set-valued functions," *J. Math. Anal. Appl.*, Vol. 12 (1965), pp. 1-12.

[4] T. BEWLEY, "Edgeworth's conjecture," Working Paper No. 311, Center for Research in Management Science, University of California, Berkeley, 1970.

[5] R. CORNWALL, "The use of prices to characterize the core of an economy," *J. Econ. Theory*, Vol. 1 (1969), pp. 353-373.

[6] G. DEBREU, *Theory of Value*, New York, Wiley, 1959.

[7] ———, "Preference functions on measure spaces of economic agents," *Econometrica*, Vol. 35 (1967), pp. 111-122.

[8] ———, "Integration of correspondences," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1967, Vol. 2, Part 1, pp. 351-372.

[9] ———, "Neighboring economic agents," *La Décision*, Editions du Centre National de la Recherche Scientifique, Paris, 1968, pp. 85-90.

[10] G. DEBREU and H. SCARF, "A limit theorem on the core of an economy," *Internat. Econ. Review*, Vol. 4 (1963), pp. 235-246.

[11] R. M. DUDLEY, "Distances of probability measures and random variables," *Ann. Math. Statist.*, Vol. 39 (1968), pp. 1563-1572.

[12] F. Y. EDGEWORTH, *Mathematical Psychics*, London, Kegan, 1881.

[13] J. GABSZEWICZ and J. F. MERTENS, "An equivalence theorem for the core of an economy whose atoms are not 'too' big," *Econometrica*, Vol. 39 (1971).

[14] F. HAUSDORFF, *Set Theory*, New York, Chelsea, 1962.

[15] W. HILDENBRAND, "The core of an economy with a measure space of economic agents," *Rev. Econ. Studies*, Vol. 35, (1968), pp. 443-452.

[16] ———, "Existence of equilibria for economies with production and a measure space of consumers," *Econometrica*, Vol. 38 (1970), pp. 608-623.

[17] ———, "On economies with many agents," *J. Econ. Theory*, Vol. 2 (1970), pp. 161-188.

[18] W. Hildenbrand and A. P. Kirman, "Size removes inequality," *CORE Discussion Paper*, No. 7129, Center for Operations Research and Econometrics, Catholic University of Louvain, 1970.

[19] W. Hildenbrand and J. F. Mertens, "On Fatou's Lemma in several dimensions," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 17 (1971), pp. 151–155.

[20] ———, "Upper hemi-continuity of the equilibrium-set correspondence," *Econometrica*, Vol. 39 (1971).

[21] Y. Kannai, "Continuity properties of the core of a market," *Econometrica*, Vol. 39 (1971), pp. 791–815.

[22] H. Richter, "Verallgemeinerung eines in der Statistik benötigten Satzes der Masstheorie," *Math. Ann.*, Vol. 150 (1963), pp. 85–90.

[23] D. Schmeidler, "Fatou's Lemma in several dimensions," *Proc. Amer. Math. Soc.*, Vol. 24 (1970), pp. 300–306.

[24] ———, "Competitive equilibria in markets with a continuum of traders and incomplete preferences," *Econometrica*, Vol. 37 (1969), pp. 578–585.

[25] B. Shitovitz, "Oligopoly in markets with a continuum of traders," *Research Memorandum*, No. 47, The Hebrew University of Jerusalem, Department of Mathematics, 1969.

[26] A. V. Skorokhod, *Studies in the Theory of Random Processes*, Reading, Addison-Wesley, 1965.

[27] K. Vind, "Edgeworth-allocations in an exchange economy with many traders," *Internat. Econ. Review*, Vol. 5 (1964), pp. 165–177.

[28] P. D. Watson, "On the limits of sequences of sets," *Quart. J. Math. Oxford Ser.*, Vol. 4 (1953), pp. 1–3.