PROJECTIVE LIMITS OF MEASURE SPACES

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1. Introduction

This paper introduces analytic measures and proves several theorems on projective limits of analytic measures. The usual existence theorems assume either strong conditions on the conditional probabilities, or compactness of σ-measures and "continuity" of mappings (see Theorem 4.1 below). This is an attempt to assume something about measurable spaces only, and nothing about mappings. More explicitly, we want to find a theorem for general systems of measure spaces similar to the classical theorem that says the indirect product of perfect σ-measures always exists.

Throughout this paper, we consider a measure to be a finitely additive, finite, nonnegative measure and σ-additive measures are called σ-measures. It should be remarked that all results hold for σ-finite measures; the proof of that generalization is trivial.

In Section 2, the problem of the existence of the projective limit of σ-measure spaces is described and notation is introduced.

In Section 3, the "morphisms" between systems of measure spaces are introduced and applications to existence problems are mentioned. In Sections 2 and 3, the use of category theory may be helpful, however, no results of category theory are assumed—what is needed is explained without any sophistication.

In Section 4, the existence results of S. Bochner [3], J. Choksi [4] and M. Métivier [24] are recalled in Theorem 4.1.

In Section 5, the relevant properties of analytic spaces are recalled and the main results are proved, that is Theorems 2.2, 5.4, 5.5, 5.6, and 5.7. In conclusion, a short survey of analytic spaces is given.

2. Presheaves of measure spaces

2.1. Let $\langle I, \leqq \rangle$ be a directed set (that is, $\leqq$ is a reflexive order on I such that each finite subset of I has an upper bound), and let

$$(2.1) \qquad \langle \{\langle X_n, \mathscr{B}_n, \mu_n \rangle\}, \{f_{n,m} | n \geqq m\} \rangle$$

be a presheaf of measure spaces (over $\langle I, \leqq \rangle$). That means each

$$f_{n,m} : \langle X_n, \mathscr{B}_n, \mu_n \rangle \to \langle X_m, \mathscr{B}_m, \mu_m \rangle \tag{2.2}$$

is a measure morphism (that is, μ_m is the image of μ_n under $f_{n,m}$, or more specifically, $\mu_m B = \mu_n f_{n,m}^{-1}[B]$ for B in $\mathscr{B}_m$), each $f_{n,n}$ is an identity, and

$$f_{n,m} = f_{k,m} \circ f_{n,k} \tag{2.3}$$

whenever $n \geqq k \geqq m$.

We shall denote by $\langle X, \{f_n\} \rangle$ the usual realization of the projective limit of the underlying presheaf of sets; that means that X is a subset of the product $\Pi\{X_n\}$ consisting of all points $\{x_n\}$ such that $f_{k,m} x_k = x_m$ whenever $k \geqq m$ and $f_n : X \to X_n$ are restrictions of projections. Let

$$\mathscr{B}_n^* = f_n^{-1}[\mathscr{B}_n], \qquad \mathscr{B}^* = \bigcup \{\mathscr{B}_n^*\}. \tag{2.4}$$

Since I is directed, $\mathscr{B}^*$ is a field on X. If all mappings f_n are onto (or, more generally, if the inner measure of $X_n - f_n[X]$ is zero), then one can define a measure μ on $\mathscr{B}^*$ by setting

$$f_n^{-1}[B] = \mu_n B. \tag{2.5}$$

Obviously, each μ_n is the image of μ under f_n, and $\langle X, \mathscr{B}^*, \mu \rangle$ is the projective limit in the category of measure spaces.

2.2. Assume that all the $\mathscr{B}_n$ are σ-fields, and all the μ_n are σ-measures. Certainly we want the limit to be endowed with a σ-measure on a σ-field; in other words, we want to find the projective limit in the category of σ-measure spaces. It is easy to see that if a projective limit exists, then $\langle X, \mathscr{B}, \mu^* \rangle$ is a projective limit, where $\mathscr{B}$ is the smallest σ-field containing $\mathscr{B}^*$ and μ^* is the ***unique*** extension of μ from $\mathscr{B}^*$. Under our assumption that the f_n are onto and the μ_n are finite, the above is the case if and only if μ is σ-additive on $\mathscr{B}^*$. E. Andersen and B. Jessen [1] gave an example when μ is not σ-additive. The presheaf is very simple, namely, the n are finite subsets of integers, the X_n are finite products of the unit interval, the $f_{n,m}$ are projections, but the μ_n are *not* Borel measures. The projective limit of this presheaf is called the indirect product.

There is one simple case when μ is σ-additive without any assumptions on $\langle \mathscr{B}_n, \mu_n \rangle$ or f_n (except for being onto). If every countable subset of I has an upper bound, then $\mathscr{B}^*$ is a σ-field, and μ is a σ-measure. Indeed, if $\{B_i^* | i \in N\}$ is any sequence in $\mathscr{B}^*$, then there exists an index $n \in I$ and a sequence $\{B_i\}$ in $\mathscr{B}_n$ such that $B_i^* = f_n^{-1}[B_i]$; since f_n^{-1} preserves unions, intersections and measure, the result follows.

This observation has an important consequence. By a subpresheaf, we mean the restriction of the presheaf to a directed subset J of I. If J is order isomorphic to N (integers), or to a subset of N, then the subpresheaf is called simple (finite or infinite). Denote by $\langle X_J, \mathscr{B}_J, \mu_M \rangle$, the projective limit of the subpresheaf, and by $f_n^J : X_J \to X_n$, $n \in J$, the projections. Clearly there is a mapping

(2.6) $$f_J : X \to X_J$$

such that

(2.7) $$f_n = f_n^J \circ f_J$$

for every $n \in J$.

DEFINITION 2.1. *The presheaf satisfies the Bochner condition if* $f_J: X \to X_J$ *is* onto *for every simple* J.

The argument used above yields the following useful observation.

THEOREM 2.1. *If the Bochner condition holds, and if every simple subpresheaf has a limit (that is, the limit measure is* σ*-additive), then* μ *is* σ*-additive.*

It should be remarked that it is enough to assume the existence of the limit for a sufficiently rich family of simple subpresheaves.

2.3. Since the example of Andersen and Jessen, many criteria for existence have been found. There have been two different approaches; one is based on the concept of "compactness" of measures, the other puts conditions on the conditional probabilities associated with $f_{n,m}$. The latter approach, introduced by C. Ionescu Tulcea [18], and developed by Choksi [4], J. Raoult [27], and N. Dinculeanu [11], is in spirit very natural for probability theory, but unfortunately it is not easy to verify its assumptions. We shall discuss the former approach which was introduced by Kolmogorov (in a very special case) and by Bochner, in our setting but with a topological underlying structure, and further developed by Choksi [4], Métivier [24], and others. It should be remarked that Choksi's and Métivier's analyses of the work of Bochner profited from papers on indirect products of compact, or more generally perfect, measures. The work of E. Marczewski [23], C. Ryll-Nardzewski [31], and M. Jiřina [19] should be mentioned.

The first positive result, which is very old and independent of the two approaches mentioned above, says that the direct product always exists. At first sight, there is no connection between direct products and projection limits of a *directed presheaf*. One obtains a directed presheaf by considering the finite subproducts with projections as presheaf maps. Then it is clear that we have a very special underlying presheaf of sets which will be called an *indirect product presheaf*, and very special $\{\mu_n\}$.

The next result, that the indirect product exists if all μ_n are perfect, has been proved by both approaches.

Here we want to prove the following result, the definitions for which are given below.

THEOREM 2.2. *If the Bochner condition is satisfied, if the presheaf* $\langle\{\langle X_n, \mathscr{B}_n\rangle\}, \{f_{n,m}\}\rangle$ *is productlike, and if all* σ*-measures* μ_n *are analytic (more generally, pseudoanalytic) then* μ *is* σ*-additive and pseudoanalytic.*

The author does not know whether the assumption that the presheaf is productlike is needed.

3. Morphisms of presheaves

One way of proving that μ is σ-additive is by considering the limits of closely related presheaves. One example was given above in Theorem 2.1. Another example follows.

3.1. Assume that $\langle\{\langle X'_n, \mathscr{B}'_n, \mu'_n\rangle\}, \{f'_{n,m}\}\rangle$ is another presheaf over the same $\langle I, \leqq\rangle$, and let $\langle\langle X', \mathscr{B}'^*, \mu'\rangle, \{f'_n\}\rangle$ have the obvious meaning. Assume that $\{\varphi_n | n \in I\}$ is a family of mappings, such that every

$$\varphi_n : \langle X_n, \mathscr{B}_n, \mu_n\rangle \rightarrow \langle X'_n, \mathscr{B}'_n, \mu'_n\rangle \tag{3.1}$$

is a measure morphism, and such that for every $n \geqq m$ the following diagram commutes

$$\begin{array}{ccc} \cdot & \xrightarrow{\;f_{n,m}\;} & \cdot \\ \varphi_n \big\downarrow & & \big\downarrow \varphi_m \\ \cdot & \xrightarrow[\;f'_{n,m}\;]{} & \cdot \end{array} \tag{3.2}$$

It is easy to see that there exists a unique map $\varphi : X \rightarrow X'$ such that

$$\varphi_n \circ f_n = f'_n \circ \varphi \tag{3.3}$$

for all n. It is also easy to check that

$$\varphi : \langle X, \mathscr{B}^*, \mu\rangle \rightarrow \langle X', \mathscr{B}'^*, \mu'\rangle \tag{3.4}$$

is a measure morphism. It follows that if μ is σ-additive then so is μ'.

Therefore, if we know that our presheaf is the image of a presheaf with σ-additive limit measure, then μ is σ-additive. This result is hardly useful, because there is no theory concerning liftings of presheaves to good ones.

3.2. If μ' is a σ-measure, then μ need not be σ-additive on $\varphi^{-1}[\mathscr{B}'^*]$; however, in some cases it follows that μ is a σ-measure. This situation gives rise to the following observation.

If φ is onto, then

$$\varphi^{-1} : X' \rightarrow X \tag{3.5}$$

preserves countable unions and measure. Hence, to prove that μ is σ-additive, it is enough to find, for every disjoint sequence $\{B_i\}$ in $\mathscr{B}^*$ such that the union B of $\{B_i\}$ belongs to $\mathscr{B}^*$, a morphism $\{\varphi_n\}$ into a presheaf with σ-additive limit measure such that φ is onto.

This observation is usually combined with Theorem 2.1 in an obvious way. This is the idea of the conceptual proof of the fact that the indirect product of perfect measures is σ-additive, and this is the way to prove Theorem 2.2.

DEFINITION 3.1. *Assume that $I = N$. Then the presheaf $\langle\{\langle X_n, \mathscr{B}_n\rangle\}, \{f_{n,m}\}\rangle$ of measurable spaces is called productlike if for every countable $\mathscr{D} \subset \mathscr{B}^*$ there* *exists a morphism* $\{\varphi_n\}$ *into a presheaf* $\langle\{\langle X'_n, \mathscr{B}'_n\rangle\}, \{f_{n,m}\}\rangle$ *such that* (*a*) *the map* φ *is* onto, (*b*) *all* $\mathscr{B}'_n$ *are countably generated, and* (*c*) $\mathscr{D} \subset \varphi^{-1}[\mathscr{B}'^*]$. *A general presheaf is called productlike if every infinite simple subpresheaf is productlike.*

It is obvious that every indirect product presheaf is productlike. Observe that "productlike" is a measurable property, independent of measure, but depending essentially on the *measurable* structure.

The result of Section 3.2 can be considerably improved. The same conclusion can be reached by replacing "φ is onto" by "the inner measure of $X - \varphi[X]$ is 0." One then gets the direct proof of the existence of the projective limit of indirect product presheaves of perfect measures.

EXAMPLE. Let X'_n be the Stone space of $\mathscr{B}_n$, let $\mathscr{B}'_n$ be the σ-field of Baire sets in X_n, let φ_n be the natural map from X_n into X'_n, and let μ'_n be the image of μ_n under φ_n. The maps $f_{n,m}$ induce maps $f'_{n,m}$ such that $\{\varphi_n\}$ is a morphism; the limit X' is a compact space by Theorem 3.2 of Bochner [3] (and measure μ' is σ-additive). Now it is clear that μ is σ-additive if and only if the inner measure of $X - \varphi[X]$ is 0. It is instructive to observe that φ is one to one.

Mallory [22] developed a similar idea. On the product $\Pi\{\langle X_n, \mathscr{B}_n\rangle\}$, one can construct something whose trace on X is μ whenever μ is σ-additive.

4. The theorem of Bochner, Choksi, and Métivier

4.1. Assume that $I = N$. To prove that μ is σ-additive it is enough to verify that if $\{B^*_i\}$ is a decreasing sequence in $\mathscr{B}^*$ such that $\mu B^*_i \geqq r > 0$ for each i, then the intersection of $\{B^*_i\}$ is nonvoid. One can easily check that there is no loss of generality in assuming that $B^*_i = f_i^{-1}[B_i]$ for $B_i \in \mathscr{B}_i$. For every k in N define

$$D_k = \bigcap\{f_{n,k}[B_n] \mid n \geqq k\}. \tag{4.1}$$

Now if, for each k in N,

$$f_{k+1,k}[D_{k+1}] \supset D_k \neq \varnothing, \tag{4.2}$$

then it is easy to construct (by induction) a point $X = \{x_n\}$ in X such that $x_n \in D_n \subset B_n$ for each n. Clearly relation (4.2) would hold if the $f_{n,k}$ were continuous, and the sets D_k were compact, while (4.2) would not hold in general, even if X_n were compact and $f_{n,k}$ were continuous. On the other hand, it is easily checked that it suffices to assume that (4.2) holds for enough sequences $\{B^*_i\}$ in $\mathscr{B}^*$. This is the proof of the existence part of the following theorem and corollary, the credit for which should be given to Kolmogorov, Bochner, Marczewski, Ryll-Nardzewski, Choksi, and Métivier. The definitions are given after the statement of the theorem.

THEOREM 4.1. *Assume that* $I = N$, *and that for each n there is an* $\aleph_0$ *compact class* $\mathscr{C}_n \subset \mathscr{B}_n$ *such that the following three conditions are fulfilled:*

(i) $\mathscr{C}_n$ *is an approximating class for* $\mathscr{B}_n$;

(ii) $f_{n,m}[\mathscr{C}_n] \subset \mathscr{C}_m$ *for* $n \geqq m$; *and*

(iii) $f_{n,m}^{-1}[(x_m)] \cap [\mathscr{C}_n]$ *is an* $\aleph_0$ *compact class for any* $n \geqq m$, *and* $x_m \in X_m$. *Then* μ *is* σ*-additive and* $\aleph_0$ *compact.*

COROLLARY 4.1. *Assume that the Bochner condition holds and that every infinite simple subpresheaf satisfies the assumptions in Theorem 4.1. Then* μ *is* σ*-additive and perfect.*

It should be remarked that $\aleph_0$ compactness implies σ-additivity. For very detailed proofs we refer to Métivier [24].

DEFINITION 4.1. *We say that* $\mathscr{C} \subset \mathscr{B}$ *is an approximating (inner or from below) class for* μ *on* $\mathscr{B}$ *if for each* B *in* $\mathscr{B}$ *and each* $r > 0$, *there exists a* C *in* $\mathscr{C}$ *such that* $C \subset B$ *and*

$$\mu C \geqq \mu B - r. \tag{4.3}$$

A collection $\mathscr{C}$ *of sets is said to be* $\aleph_0$ *compact if the intersection of every countable subcollection of* $\mathscr{C}$ *with the finite intersection property is nonvoid. A measure* μ *on* $\mathscr{B}$ *is said to be* $\aleph_0$ *compact is there exists an* $\aleph_0$ *compact approximating class* $\mathscr{C} \subset B$ *for* μ. *A measure* μ *on* $\mathscr{B}$ *is said to be perfect if* μ *is compact in any countably generated* σ*-field contained in* $\mathscr{B}$.

For a list of various characterizations of perfect measures we refer to Ryll-Nardzewski [31], or Frolík [17].

4.2. *Capacity* $\aleph_0$ *compact measures.* It is not necessary to assume that $\mathscr{C}_n \subset \mathscr{B}_n$. Of course one should assume something which relates $\mathscr{C}_n$ to $\mathscr{B}_n$. A condition which is needed for Theorem 4.1 reads as follows (omitting the subscript): if $C_i \in \mathscr{C}$, and

$$\bigcap \{C_i | i \leqq k\} = B \in \mathscr{B} \tag{4.4}$$

then

$$\bigcap \{B_i | i \leqq k\} \subset B \tag{4.5}$$

for some $B_i \in \mathscr{B}$, $B_i \supset C_i$.

The measure admitting such an $\aleph_0$ compact approximating class is called *capacity compact.* The analytic measures (introduced in Section 5.2) give, perhaps, the most important example of *capacity compact* measures. The theory of capacity compact measures is developed in Frolík [17]. The theory contains a very general theorem on extension of measures. It is a mixture of the capacity approach, and of the Caratheodory outer measure approach to measure theory. In the proofs of our theorems on projective limits of analytic measures it is sufficient to use Theorem 4.1 if one takes Theorem 5.3 into account.

4.3. *Regular Borel measures.* Theorem 4.1 is usually applied in a topological setting, namely when the $f_{n,m}$ are continuous, and the μ_n are regular Borel measures (that means, the X_n are endowed with topologies). The details are obvious, and we shall use the topological setting (the original theorem by Bochner) without any further comment.

5. Analytic measures

5.1. *Baire measures.* In the topological context the measures are considered either on the σ-field of all Borel sets (the smallest σ-field containing the closed sets), or on the σ-field of all Baire sets (the smallest σ-field such that every continuous real valued function is measurable). The former measures are called Borel measures, the latter are called the Baire measures. In the literature there is a strong preference for Borel measures, in particular for regular Borel measures, because then all compact sets (at least in the Hausdorff spaces) are Borel sets, and not many compact sets are Baire sets in general. There is one very important property of the Baire σ-field which *is not* enjoyed by the Borel σ-field. The Baire field reflects important properties of the underlying topological space. For example, if X is a compact space, and if the Baire σ-field is countably generated, then X is metrizable; if we know that the Borel field is countably generated, then we may not conclude that X is metrizable, and in addition, it seems that nothing interesting can be said in this situation. The advantage of regular Borel measures is weakened by the fact that, in many important cases, every σ-measure on the Baire σ-field extends uniquely to a regular Borel measure. This is the case of Baire measures on analytic spaces (see Section 5.3). The point is that considering measures on the Baire σ-field, we have a choice of good topology. Perhaps the most striking example using this fact is Theorem 5.4.

NOTATION. If X is a topological space we denote by Baire (X) the σ-field of the Baire sets in X, as well as the set X endowed with the σ-field of all Baire sets.

CONVENTION. In this section a topological space, means a completely regular Hausdorff space. This is a very natural convention, because we are interested in Baire sets, and completely regular spaces are precisely the spaces with enough continuous real valued functions.

5.2. A short survey of analytic (K analytic, in the terminology of Choquet [5]) spaces is given in Section 6. Here we recall one possible definition: *a topological space X is called analytic if there exists a continuous mapping of a $K_{\sigma,\delta}$ onto* X where by $K_{\sigma,\delta}$ we mean a countable intersection of σ-compact subspaces of some space.

It is important to bear in mind that every compact space is analytic and every analytic space is Lindelöf (in particular, every metrizable analytic space is separable).

DEFINITION 5.1. *A measurable space $\langle X, \mathscr{B}\rangle$ is said to be analytic if there exists an analytic topology t for X such that $\mathscr{B}$ is the σ-field of all Baire sets in $\langle X, t\rangle$; the topology t is then called an analytic topologization of $\langle X, \mathscr{B}\rangle$. A measurable space is said to be pseudoanalytic if every countably generated measurable image is analytic. An analytic (pseudoanalytic) measure is a σ-measure on an analytic (pseudoanalytic) measurable space.*

5.3. We proceed to state three very important theorems on analytic spaces and measures. The first two theorems are taken from Frolík [16], the third one is new (for the proof we refer to Frolík [17]). The second theorem is a generalization of the classical theorem of Lusin (the classical theorem assumes A separable metrizable, and M separable).

THEOREM 5.1. *Let f be a continuous mapping of an analytic space X onto a space Y. Then Y is analytic* (this is obvious), *and*

$$f : \text{Baire}\,(X) \rightarrow \text{Baire}\,(Y) \tag{5.1}$$

is a measurable quotient map (that is, $M \subset Y$ is a Baire set in Y if and only if $f^{-1}[M]$ is a Baire set in X). In particular, if f is one to one then f is an isomorphism.

This theorem has the following consequence.

COROLLARY 5.1. *Let f be a map of space X onto a space Y such that the graph ρ of f is an analytic subspace of $X \times Y$. Then X and Y are analytic, and the projection of ρ onto X is a Baire isomorphism.*

Corollary 5.1 is applied as follows. If the graph of f is analytic, we can, without changing the Baire σ-field, replace the topology of X by an analytic topology such that f is continuous with respect to the new topology.

THEOREM 5.2. *Let f be a Baire measurable map of an analytic topological space X onto a metrizable space Y. Then the graph of f is analytic, Y is analytic, and f: Baire $(X) \rightarrow$ Baire (Y) is a measurable quotient map.*

COROLLARY 5.2. *If f is a measurable mapping of an analytic measurable space onto a countably generated measurable space Y, then Y is analytic and f is a measurable quotient map. In particular, every analytic measurable space is pseudoanalytic.*

Theorem 5.2 is applied as follows. Suppose that $\{h_n\}$ is a countable family of Baire measurable maps of an analytic measurable space into metrizable spaces. There exists an analytic topologization of the domain such that all maps h_n are continuous.

THEOREM 5.3. *Let μ be an analytic measure on $\langle X, \mathscr{B}\rangle$. For every analytic topologization t of $\langle X, \mathscr{B}\rangle$ the measure μ extends (uniquely) to a regular Borel measure ν on $\langle X, t\rangle$. In addition, ν coincides with the outer measure $\bar{\mu}$ (generated by μ) on compact sets in $\langle X, t\rangle$, $\bar{\mu}$ is a regular capacity with respect to the compact sets, and for every analytic subspace A of $\langle X, t\rangle$ (in particular, for every Baire set in $\langle X, t\rangle$),*

$$\bar{\mu}A = \sup\,\{\nu C \mid C \text{ compact, } C \subset A\}. \tag{5.2}$$

The standard proofs apply throughout. The above theorem has one important corollary.

COROLLARY 5.3 (Sion [34], Blackwell [2]). *Every analytic measure is perfect.*

The proof follows immediately from either Theorem 5.3 (every regular Borel measure is perfect), or from Theorem 5.2 and the last statement of Theorem 5.3.

REMARK. Every atomic measure is perfect. Hence if every σ-measure on $\langle X, \mathscr{B}\rangle$ is atomic, then every σ-measure on $\langle X, \mathscr{B}\rangle$ is perfect. R. Darst and R. Zink [9] gave an example of an uncountable subset of the line such that every Baire σ-measure is atomic; such a set cannot be analytic because every metrizable analytic space contains a copy of the irrational numbers. The set is a universal uncountable negligible set.

5.4. *Limit theorems.* Using the notation introduced in Section 2, recall that all $f_n : X \to X_n$ are onto, and that all μ_n are σ-measures.

THEOREM 5.4. *Assume $I = N$. Let all the μ_n be analytic measures. If there exist analytic topologizations t_n of $\langle X_n, \mathscr{B}_n \rangle$ such that all the*

$$f_{n,m} : \langle X_n, t_n \rangle \to \langle X_m, t_m \rangle \tag{5.3}$$

are continuous, then μ is σ-additive, and the extension of μ to $\mathscr{B}$ is analytic.

PROOF. The σ-additivity follows from Theorem 5.3 and Section 4.2. The space X is a closed subspace of the product Z of $\{\langle X_n, t_n \rangle\}$, because the t_n are Hausdorff, and hence X is analytic because all t_n are analytic. It remains to be shown that $\mathscr{B}$ is the field of all Baire sets in X. This follows from two facts. The first is that $\mathscr{B}^*$ consists of the intersections with X of cylinders in Z over Baire sets in the factors, and these sets generate the σ-field of all Baire sets in Z. The second is that if X is an analytic set in a space Z, then every Baire set in X is a trace on X of a Baire set in Z. (The latter fact is not completely obvious. It follows from the Separation Theorem 6.1.)

THEOREM 5.5. *Assume that $I = N$, and that there exist analytic topologizations t_n of $\langle X_n, \mathscr{B}_n \rangle$ such that the graph of each $f_{n,m}$ is an analytic subspace of $\langle X_n, t_n \rangle \times \langle X_m, t_m \rangle$. Then μ is σ-additive, and the extension of μ to $\mathscr{B}$ is analytic.*

PROOF. For each n let

$$Y_n = \prod \{\langle X_k, t_k \rangle \mid k \leqq n\}, \tag{5.4}$$

and let X'_n be the subspace of Y_n consisting of all points $\{x_k \mid k \leqq n\}$ such that $f_{m,k} x_m = x_k$ for $n \geqq m \geqq k$. Thus,

$$X'_n = \varprojlim \langle \{\langle X_k, t_k \rangle \mid k \leqq n\}, \{f_{m,k}\} \rangle \tag{5.5}$$

and

$$\varphi_n = \{x \to \{f_{n,k} x \mid k \leqq n\}\} : X_n \to X'_n \tag{5.6}$$

is a one to one mapping onto. If we define $f'_{n,m} : X'_n \to X'_m$ by $f'_{n,m} \circ \varphi_n = \varphi_m \circ f_{n,m}$, then $f'_{n,m}$ is easily seen to be the restriction of the projection of Y_n onto its subproduct Y_m. Hence, each $f'_{n,m}$ is continuous. The inverse of each φ_n is the restriction of the projection of Y_n onto the nth coordinate space, and hence φ_n^{-1} is continuous. Assume that each X'_n is analytic. By Theorem 5.1 each

$$\varphi_n : X_n \to X'_n \tag{5.7}$$

is a Baire isomorphism. Define μ'_n to be the image of μ_n under φ_n. Evidently,

$$\varphi : \langle X, \mathscr{B}^*, \mu \rangle \to \langle X', \mathscr{B}'^*, \mu' \rangle \tag{5.8}$$

is a measure isomorphism. Theorem 5.4 applies to $\{X'_n\}$; thus μ' is σ-additive, and $\mathscr{B}'$ is the Baire σ-field of X. Hence μ is σ-additive, and to prove that $\mathscr{B}$ is the set of all Baire sets in X it is sufficient to observe that $\varphi^{-1} : X' \to X$ is continuous, which follows from the fact that all $\varphi_n^{-1} : X'_n \to X_n$ are continuous.

It remains to be shown that each X_n is analytic. Fix n. For each $m \leqq n$, $m > 1$ let F_m be the set of all $x = \{x_i\}$ in Y_n such that $f_m x_m = x_{m-1}$. Clearly (up to a formal distinction between two different ways of looking on finite products)

$$F_m = X_1 \times \cdots \times X_{m-2} \times (\text{graph } f_{m,m-1}) \times X_{m+1} \times \cdots \times X_n, \tag{5.9}$$

and evidently

$$X'_n = \bigcap \{F_m | m \leqq n\}. \tag{5.10}$$

Hence X_n is analytic, because countable products and countable intersections of analytic sets are analytic.

Corollary 5.4. *Let $I = N$, and let all $\mathscr{B}_n$ be countably generated and analytic. Then μ is σ-additive and $\mathscr{B}$ is analytic.*

Proof. For each n let t_n be a *metrizable* analytic topologization of $\langle X_n, \mathscr{B}_n\rangle$. By Theorem 5.2, the graph of each $f_{n,m}$ is analytic in $\langle X_n, t_n\rangle \times \langle X_m, t_m\rangle$, and hence Theorem 5.5 applies.

Remark. In the proof we only use the classical part of Theorem 5.2. In addition, we need not use Theorem 5.5 because the classical part of Theorem 5.2 reduces Corollary 5.4 to Theorem 5.4. For a particular case of indirect product presheaves Corollary 5.4 is found in Blackwell [2].

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Assume that $I = N$, that the presheaf is productlike, and that all $\mathscr{B}_n$ are analytic. Let $\mathscr{D}$ be a countable subset of $\mathscr{B}^*$. Since the presheaf is productlike, there exists a morphism $\{\varphi_n\}$ onto a presheaf

$$\langle\{\langle X'_n, \mathscr{B}'_n, \mu'_n\rangle\}, \{f'_{n,m}\}\rangle \tag{5.11}$$

such that

$$\varphi: X \to X' \tag{5.12}$$

is onto, all $\mathscr{B}'_n$ are countably generated, and $\varphi^{-1}[\mathscr{B}'^*] \supset \mathscr{D}$. Since φ is onto, each

$$\varphi_n: X_n \to X'_n \tag{5.13}$$

must be onto, and by Theorem 5.2 (nonclassical case), each $\langle X'_n, \mathscr{B}'_n\rangle$ is analytic, and it follows now by Corollary 5.4 that μ' is σ-additive. By Section 2, μ is σ-additive on $\mathscr{B}^*$, and μ is pseudoanalytic by Theorem 5.2.

Remark. I do not know if the assumption that the presheaf is productlike is needed.

5.5. Let f be a Baire measurable mapping of an analytic space A onto a space A'. The space A' need not be analytic even if f is a Baire isomorphism. For example, let A be the unit interval and let A' be the unit interval with the topology having for an open base the intervals $[x, y)$.

Assume that A' is also analytic. The graph of f need not be analytic even if A and A' are compact. For example, consider a Baire isomorphism g of the unit interval J onto the Cantor space $2^{\aleph_0}$, and let $f = g^m$, thus

(5.14) $$f : J^m \to (2^{\aleph_0})^m,$$

with m uncountable. This shows that the assumptions in Theorem 5.5 are very restrictive, and moreover, it shows that the concept of "analytic graph" depends on the topologizations (it is not a measurable property).

If $\langle X, \mathscr{B}\rangle$ is an analytic measurable space, and if t is any analytic topologization of $\langle X, \mathscr{B}\rangle$, then every B in $\mathscr{B}$ is an analytic subspace of $\langle X, t\rangle$; and more generally, every Souslin set derived from $\mathscr{B}$ is an analytic subspace of $\langle X, t\rangle$. In other words,

(5.15) $$\text{Souslin }(\mathscr{B}) \subset \bigcap \{\text{anal }(t) \mid t \text{ analytic topologization of } \langle X, \mathscr{B}\rangle\},$$

where anal (t) denotes the collection of all analytic subspaces of $\langle X, t\rangle$. It follows that if f is a measurable mapping of an analytic measurable space $\langle A, \mathscr{B}\rangle$ onto an analytic measurable space $\langle A', \mathscr{B}'\rangle$, and if the graph of f is a Souslin set derived from measurable sets in $\langle A, \mathscr{B}\rangle x \langle A', \mathscr{B}'\rangle$, then the graph of f is analytic in $\langle A, t\rangle \times \langle A', t'\rangle$ for every analytic topologization t and t' of $\mathscr{B}$ and $\mathscr{B}'$, respectively. Thus, Theorem 5.5 implies the following theorem with no topological assumptions on $f_{n,m}$.

If all μ_n are analytic, and if all $f_{n,m}$ are Souslin sets derived from the measurable sets, then μ is σ-additive and $\mathscr{B}$ is analytic.

It is easy to see via the nontrivial Theorem 5.2 that this result is equivalent to Corollary 5.4 (if the graph is a Souslin set derived from the measurable sets, then the range σ-field is countably generated).

If $f : \langle A, \mathscr{B}\rangle \to \langle A', B'\rangle$ is measurable, if $B \in \mathscr{B}$, and if $\mathscr{B}$ and $\mathscr{B}'$ are analytic, then $f[B]$ need not be a Souslin set derived from measurable sets (unless $\mathscr{B}'$ is countably generated), and, moreover, given an analytic topologization of $\langle A', \mathscr{B}'\rangle, f[B]$ need not be analytic. Using a more complicated version of the proof of Theorem 4.1 one can prove the following result (which is not worthy of the work needed for the proof).

THEOREM 5.6. *If $I = N$, and if there exist analytic topologizations t_n such that the preimages of points under the $f_{n,m}$ are closed, the images under the $f_{n,m}$ of measurable sets are Souslin sets derived from measurable sets, and if the compact elements of each $\mathscr{B}_n$ form an approximating class for μ_n, then μ is σ-additive, and the extension of μ to $\mathscr{B}$ is compact.*

5.6. Assume that our presheaf satisfies the Bochner condition, and that for every simple subpresheaf the limit measure is σ-additive, and the limit σ-field is pseudoanalytic. From Section 3.2, μ is σ-additive; and since every countably generated σ-field contained in $\mathscr{B}$ is contained in the preimage of the limit σ-field of a simple presheaf, say over J, under the canonical map f_J, $\mathscr{B}$ is pseudoanalytic. This concludes the proof of Theorem 2.2, and proves the following general theorem.

THEOREM 5.7. *Assume the Bochner condition. If every infinite simple subpresheaf satisfies the assumptions in Theorems 5.4, 5.5 or 5.6 or Corollary 5.4, then μ is σ-additive, and $\mathscr{B}$ is pseudoanalytic.*

The concept of analytic measurable space depends on topology. It was noticed by G. Mackey [21] (and others) that analytic countably generated space can be defined without any topology. Indeed, every countably generated analytic measurable space is a measurable quotient of the Baire σ-field of the Cantor space $2^{\aleph_0}$, and Baire $(2^{\aleph_0})$ can be defined as the smallest σ-field containing the sets B_i^n, $n \in N$, $i = 0, 1$, where B_i^n consists of all points with the nth coordinate equal to i.

It follows that "pseudoanalytic" is purely measurable concept. No topological characterization of pseudoanalytic measurable spaces is known. It is very important that the class of all pseudoanalytic spaces is closed under the formation of arbitrary products (it is obvious that the class of analytic measurable spaces is closed under countable products, because the class of topological analytic spaces has this property, and the behavior of uncountable products is not known).

6. Short survey of analytic spaces

The theory of analytic sets in nonmetrizable spaces has been developed by Choquet, Rogers, Sion, the present author, and others. A survey of the theory of analytic sets and related topics is given in Frolik [13]; for the theory in general spaces we refer to Frolík [15], and for the abstract theory of analytic spaces to Frolík [14]; the last two papers have the advantage that the proofs and definitions are clearer and more economical. Here we recall the basic facts which have been used in Sections 5.3 and 5.4 without any reference, and we add a few new facts which may help to clarify the relevance of analytic sets for some questions in topological measure theory.

Each of the following conditions is necessary and sufficient for a space X to be analytic:

(a) X is a Souslin set in every $Y \supset X$;

(b) X is a Souslin set in some compact $Y \supset X$;

(c) there exists an upper semicontinuous compact valued map (usco compact) correspondence from the space Σ of irrational numbers onto X;

(d) there exists an usco compact correspondence from a separable completely metrizable space onto X; and

(e) X is a continuous image of a proper (that is, perfect) preimage of a closed subspace of Σ.

It follows from description (c) of analytic spaces that the class of all analytic spaces is closed under usco compact correspondences (in particular, under continuous maps), countable products, countable sums, and the formation of closed subspaces. Each of the conditions implies that every compact space is analytic, and every analytic space is Lindelöf.

For any space X the collection anal (X) of all analytic subspaces of X is closed under the Souslin operation (that is, operation A), and if X is analytic then anal (X) coincides with the collection of all Souslin sets. It follows that in every analytic space X every Baire set is analytic, and using the separation theorem given below, we get

$$\text{(6.1)} \qquad \text{Baire}\,(X) = E\{Y \mid Y \in \text{anal}\,(X),\, X - Y \in \text{anal}\,(X)\}.$$

The following result is the main fact needed for the proof of Theorem 5.1.

THEOREM 6.1 (Separation theorem of Frolík [12]). *If A_1 and A_2 are disjoint analytic subspaces of a space Z, then there exists a Baire set B in Z such that $A_1 \subset B \subset Z - A_2$.*

This is a generalization of Lusin's first separation principle.

If A is analytic then the topology generated by compact subspaces of A is analytic. It follows from Theorem 5.1 that every analytic measurable space can be topologized so that the compact sets generate the topology (such a space is called a k space). This is a useful property. Topological measures on k spaces have been studied by Scheffer [32].

A space X is said to be Borelian if there exists a *disjoint* usco compact correspondence from Σ onto X. One can show that X is Borelian if and only if X is a one to one continuous image of a proper pre-image of Σ. This implies (Theorem 5.1) that every "Borelian σ-measure" is compact with respect to compact sets in $\langle X, t\rangle$ for *some* Borelian topologization of the measurable sets.

If we are interested in the regular capacities rather than in Baire σ-measures, then the proper class of topological spaces for developing the theory of analytic sets is the class of all Hausdorff spaces. The main theorems are due to Choquet.

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