ALFRÉD RÉNYI, *IN MEMORIAM*

On their return from the United States at the end of June, 1969, Catherine and Alfréd Rényi visited me in Vienna, and we spent a beautiful evening together by the Danube-Oder canal. Kató Rényi told us that she thought her health was improving. But within weeks the news came of her death. On January 5, 1970, I received a letter from Alfréd Rényi in which he wrote of plans to visit Vienna again, and he incidentally added that he was not feeling well and had trouble with his glands. On February 2nd, Professor M. Csörgő of McGill University, who was then visiting at the Mathematic Institute of the University of Vienna, relayed the news that the Hungarian radio had announced that Alfréd Rényi had died the day before. Hungary, a country which had bred many mathematicians, had now lost one of its most remarkable.

Born on March 20, 1921 in Budapest, Alfréd Rényi had a heritage of learned men. His father had been an engineer and his maternal grandfather a professor of philosophy at the University of Budapest. When he married Catherine, he added another mathematician to his family.

When Rényi finished the gymnasium, the fascist administration of the Horty regime did not grant him permission to enroll at the university. Thus, he went to work in a factory, and it was only after he won a prize in a mathematics contest that he was allowed to enter the University of Budapest, where he was allowed to study from 1939 to 1944. Rényi almost fell victim to the last rage of fascism. He was sent to a labor camp, but he managed to escape. The next year he obtained his doctorate at the University of Szeged. The year 1946 was of decisive significance for Rényi's career, for it was then that he obtained a post-graduate fellowship at the University of Leningrad and came in contact with Yu. V. Linnik. On Rényi's return to Budapest, he was made assistant professor at the University of Budapest for 1947-1948. In 1948, he became privatdozent at Budapest, and later that year he was nominated professor at the University of Debrecen for two years. From 1950 through 1970, he was director of the Mathematical Institute of the Hungarian Academy of Sciences, Budapest. In 1952, he was nominated professor of mathematics at the Eötvös Loránd University, Budapest, and head of the chair of probability theory. These two full time jobs with their heavy administrative responsibilities did not seem to hinder his productive scientific work. On the contrary, his scientific work expanded while he took care of numerous students. This feat was possible because Rényi worked frequently through the night enveloped in tobacco smoke and the odor of coffee.

His great intellectual capacity and his love for mathematics insured his enormous success. Besides his many books, he authored and coauthored some 200 scientific papers. Nevertheless, that was not the most demanding part of Rényi's activities. He also travelled frequently, attended countless meetings, gave lectures across the world, and spent many years as a visiting professor. In 1960 he was at Stanford University, in 1961 at Michigan State University, in 1964 at the University of Michigan, in 1966 again at Stanford, in 1968 at Cambridge University and the University of Erlangen, and finally in 1969 at the University of North Carolina.

Rényi was editor or coeditor of many mathematical journals in addition to Hungarian journals. He was a member of the Hungarian Academy of Sciences, of the Presidium of the J. Bolyai Mathematical Society, vice president of the International Statistical Institute, fellow of the Institute of Mathematical Statistics, and overseas fellow of Churchill College, Cambridge.

Frequently I had the pleasure of meeting Rényi—in Hungary or Austria, or elsewhere, and I could not help admiring his many interests and thorough education. Rényi was concerned with art, politics, and the whole of culture. His English, German, and Russian were fluent, he knew some French, and he still remembered classical Greek from school. Rényi, with all his language ability, was capable of clever and caustic humor. He was an impressive man, full of genuine humanism. He always pleaded for peace and for friendship between different nations.

RÉNYI'S WORK

To many people Rényi was a specialist in probability theory. But actually he was a mathematician with diversified interests. He had an extensive knowledge of mathematics at his disposal which allowed him not only to work in many areas of mathematics, but also to see new relationships between different mathematical disciplines. Many of his papers demonstrate his ability to carry the methods of one field in mathematics over to another. His many skills become apparent from the variety of his papers which range from axiomatic investigations to the careful consideration of details. Certainly his main interest was concentrated on concrete problems. When Rényi made Archimedes say: "One has to be a dreamer of dreams to apply mathematics with real success," in [139], he certainly did this against the background of his own experiences.

Rényi's scientific work belongs to the following subjects: real analysis, number theory, probability theory, information theory and mathematical statistics, complex analysis, graph theory and combinatorial analysis, geometry of convex bodies, applied mathematics, and didactic of mathematics, the enumeration being more or less arbitrary.

We shall now give a short evaluation of the most important papers of Rényi and in doing so, try to see them in connection with related literature. His first publication [1], as well as his thesis which was only partially published later [27], is concerned with classical analysis. It deals with Tauberian conditions for the Abel summability of series and is related to some work of O. Szász (I). His thesis, as far as it was published, is concerned with the C^1–summability of Cauchy-Fourier series. Rényi often returned to related problems such as the theory of orthogonal series and the theory of summability [10], [32], [38], [39], [64], [114], [134]. Paper [114] illustrates in a simple way our remark about his ability to switch from one subject in mathematics to another. He starts with the simple observation that all summability procedures that are based on positive matrices with row sums 1 may be interpreted almost obviously in the framework of probability theory. Using this idea, he gets interesting results on the Hausdorff and Henrikson summability procedures, the first being related to the binomial distribution, the second to the Poisson distribution. These investigations were carried on by L. Schmetterer (II) and quite recently by J. G. Kemeny and J. L. Snell (III). Papers [38]. [39], which are concerned with a version of the Stone-Weierstrass theorem for measurable functions also deserve special attention. Let $\{f_n\}$, $0 \leqq f_n(x) \leqq 1$, be a sequence of measurable functions defined on [0, 1]. Suppose that the functions f_n separate points with probability 1 with respect to Lebesgue's measure L on [0, 1]. Then for every essentially bounded function f on [0, 1] and for every $\varepsilon > 0$ and $\delta > 0$ there exists an element g which belongs to the algebra generated by the set $\{f_n\}$ such that $L\{x: |f(x) - g(x)| < \varepsilon\} > 1 - \delta$. Obviously, this theorem follows from the Stone-Weierstrass theorem by an application of Lusin's theorem but in [38] an independent short proof is given which also provides a proof of the Stone-Weierstrass theorem (even for arbitrary compact spaces).

Later on Rényi [176] took a fancy to pointing out the effectiveness of probabilistic methods in real and complex analysis. He considered for instance probabilistic proofs for Wiman's theorem on the maximum modulus of entire functions, for the Cartan-Thullen theorem on domains of regularity of analytic functions of several complex variables, for the construction of power series which are uniformly but not absolutely convergent on the unit circle, and so on. Among the earliest publications of Rényi are his number theoretic investigations which are based on Linnik's large sieve [5], [7]. His results are concerned with the famous problem of Goldbach and can be considered a breakthrough in this field. He shows that there exists a fixed natural number ℓ (which may be very large) such that each integer is the sum of a prime number and an integer that has at most ℓ prime factors. Many mathematicians have tried to improve the statement about the nature of ℓ. After fundamental work by A. I. Vinogradov (IV), K. F. Roth (V) and E. Bombieri (VI), A. A. Buchštab (VII) and independently W. Jurhat, H. E. Richert, and H. Halberstam (VIII) obtained $\ell = 3$ as the best result so far. Rényi discovered very soon that the large sieve admits a probabilistic interpretation. Essentially the problem is the following. Let N be a natural number and n_i, $1 \leqq i \leqq k$, be integers satisfying $1 \leqq n_1 < n_2 < \cdots < n_k \leqq N$. Let p be a prime number and $Z(p, h)$ the number of such n_i which belong to the residual class h (mod p). Denote $\Sigma_{h=0}^{p-1} (Z(p, h) - k/p)^2$ by $D(p)$. The problem is to find an upper bound for $\Sigma_{p \leqq X}\, pD(p)$ (where X is for instance of the form $X = N^\alpha$ with $\frac{1}{3} \leqq \alpha \leqq \frac{1}{2}$).

It may be remarked that Rényi's method is powerful if α is near $\frac{1}{3}$, while Linnik's method gives good estimates if α is near $\frac{1}{2}$ and even equal to $\frac{1}{2}$.

But E. Bombieri (VI) has discovered that the important result of Rényi, $\Sigma_{p \leqq N^{1/3}} pD(p) = O(kN)$ is also correct for $\alpha \leqq \frac{1}{2}$.

As Rényi remarked, Linnik's approach [IX] to this problem can be described, roughly speaking, in the following way: for 'almost every' prime number $p \leqq \sqrt{N}$ 'almost every' residue class mod p is represented among the numbers n_i, $1 \leqq i \leqq k$ if k/N is 'not too small'. Rényi discovered that the distribution of the n_i in 'almost' all residue classes is 'almost' uniform for 'almost' all prime numbers $p \leqq N^{\alpha}$, $\alpha < \frac{1}{2}$. For a more precise formulation see [11], [20]. Regarding the fact that the distributions of the n_i in the residue class mod p and mod q with $p \neq q$ prime numbers, are 'almost' independent if p and q are small compared with N, the way was open to a probabilistic approach to the large sieve. He was developing this idea in his papers [22] and [98] and gave it its final form in [106] and [107]: let $\xi_1, \xi_2, \cdots$ be an arbitrary sequence of (real valued) random variables and let $M(\xi_i, \xi_j)$ be the Gebelein maximal correlation coefficient of ξ_i and ξ_j. Let η be any random variable and suppose that there exists a $C \geqq 1$ such that

$$\sum_{1 \leqq i, j < \infty} M(\xi_i \xi_j) x_i x_j \leqq C \sum_{i=1}^{\infty} x_i^2$$

for all real numbers $x_1, x_2, \cdots$. Then it follows that $\Sigma_{i=1}^{\infty} K_{\xi_i}^2(\eta) \leqq C$, where $K_{\xi_i}(\eta)$ is the correlation coefficient of η on ξ_i in the sense of Kolmogorov. This result also gives an interesting insight into the relation between some measures of dependence for random variables. This last subject has been treated by Rényi from an axiomatic point of view in paper [112] which inspired others in their further investigations in this field. Let us only mention papers by P. Csáki and J. Fischer (X), (XI), (XII). Let us finally mention that in one of his last papers together with P. Erdős [183], Rényi returned to the problem of the large sieve. Introducing random sets of natural numbers n_i, they show for instance, that some results of P. X. Gallagher (XIII) cannot be very much improved.

It is not astonishing that as early as 1948 Rényi [8] returned his interest to another topic which ranges between number theory and probability theory. It starts with É. Borel's theorem (XIIIa) on the uniform distribution of the decimal digits of almost all real numbers. Rényi [68] (see also [100]) generalized this result to Cantor series. Rényi's most important results in this direction are perhaps contained in paper [87] (see also [86]). He considers the f-expansions of real numbers which were studied for the first time by B. H. Bissinger (XIV) (and C. I. Everett (XV)) for a decreasing (increasing) homeomorphism f from $[1, \infty]$ ($[0, \infty]$) onto $[0, 1]$. Let x be any real number and define

$$\varepsilon_0(x) = [x], \qquad r_0(x) = x - [x],$$
$$\varepsilon_{n+1}(x) = [f^{-1}(r_n(x))], \qquad r_{n+1}(x) = f^{-1}(r_n(x)) - [f^{-1}(r_n(x))],$$

$n \geqq 0$. Then the f expansion of x is of the form

$$x = \varepsilon_0(x) + f(\varepsilon_1(x) + f(\varepsilon_2(x) + f(\varepsilon_3(x) + \cdots))).$$

Let us consider for simplicity the case of a decreasing f only. Define by induction

$$f_1(y_1) = f(y_1), \qquad y_1 \geqq 1,$$
$$f_n(y_1, \cdots, y_n) = f_{n-1}(y_1, \cdots, y_{n-2}, y_{n-1} + f(y_n)), \qquad n \geqq 2.$$

Assume that $|f(y_2) - f(y_1)| \leqq |y_2 - y_1|$ for $1 \leqq y_1 < y_2$ and that $|f(y_2) - f(y_1)| \leqq \lambda|y_2 - y_1|$ for $1 + f(2) < y_1 < y_2$. The real number λ satisfies $0 < \lambda < 1$. Assume furthermore that

$$\frac{\sup_{0<t<1} \frac{d}{dt} f_n(\varepsilon_1(x), \cdots, \varepsilon_{n-1}(x), \varepsilon_n(x) + t)}{\inf_{0<t<1} \frac{d}{dt} f_n(\varepsilon_1(x), \cdots, \varepsilon_{n-1}(x), \varepsilon_n(x) + t)} \leqq C,$$

where C is independent of n and x. (Note that $t \to f_n(\varepsilon_1(x), \cdots, \varepsilon_{n-1}(x), \varepsilon_n(x) + t)$ has a derivative almost everywhere.) Rényi proves that under these assumptions $x \to f^{-1}(x) - [f^{-1}(x)]$ is an ergodic transformation T. This is also valid if N is an integer >2, f is a decreasing homeomorphism from $[1, N]$ onto $[0, 1]$, and $f(x) = 0$ for $x > N$, the other conditions remaining the same.

Analogous results have been obtained by Rényi for the case of an increasing homeomorphisms. Choosing $f(x) = x/q$ for $0 \leqq x \leqq q$, q an integer $\geqq 2$, $f(x) = 1$ for $x > q$, one gets the q-adic expansion of the real numbers and the well-known result that $Tx = xq - [xq]$ is an ergodic transformation. With $f(x) = 1/x$, $x \geqq 1$, one obtains the continued fraction representation of the real numbers which has been treated earlier in this context, for instance, by C. Ryll-Nardzewski (XVI) and S. Hartman (XVII). Rényi also considers in paper [87] a special transformation of a somewhat different type namely $f(x) = x/\beta$, $0 \leqq x \leqq \beta$, $f(x) = 1$, $x > \beta$, where β is a real number > 1 and not an integer. While he was able to show that the transformation $\beta x - [\beta x]$ is ergodic, he did not give an explicit formula for the (unique) normalized invariant measure. This has been done later on by W. Parry (XVIII) and by A. O. Gelfond (XIX) and along with a more general investigation by P. Roos (XX).

Rényi's interest in ergodic theory was not limited to number theory. Starting with a paper [28] whose results were generalized later by A. N. Kolmogorov (XXI), he finally introduced the following concept [92]. Let (R, S, μ) be an arbitrary measure space where μ is not necessarily a finite measure. A sequence $\{A_n\}$, $A_a \in S$, is called strongly mixing with density α, $0 < \alpha < 1$, if for any $B \in S$ with $\mu(B) < \infty$ the relation $\lim_{n\to\infty} \mu(A_n \cap B) = \alpha\mu(B)$ holds. It is easy to see that this concept is closely related to the definition of strongly mixing for a measure preserving transformation if $\mu(R) < \infty$. Using a lemma related to the fact that the closed unit sphere in a Hilbert space is weakly sequentially compact, Rényi proves that a sequence $\{A_n\}$ with $A_0 = R$, $\mu(A_0) = 1$, $\mu(A_n) > 0$, is strongly mixing with density α if and only if $\lim_{n\to\infty} \mu(A_n | A_k) = \alpha$, $k = 0, 1, 2, \cdots$. From the Radon-Nikodym theorem, it follows immediately that

$\lim_{n\to\infty} Q(A_n) = \alpha$, where Q is any probability measure which is absolutely continuous with respect to μ. Using the mixing property of sums of independent random variables, he concludes that the limit distribution (if it exists) of the normed sums under μ is left invariant when considered under Q. This is essentially the content of [28].

In [145], a slight generalization of the concept of strongly mixing events $\{A_n\}$ is introduced. Let P be a probability measure on (R, S) and suppose that $\lim_{n\to\infty} P(A_n \cap B) = Q(B)$ exists for every $B \in S$. Then $\{A_n\}$ is called a sequence of stable events by Rényi. The limit Q is a measure which is absolutely continuous with respect to P. Let us denote the corresponding density by f. Assuming that f is equal to α P-almost everywhere, one gets back the definition of a strongly mixing sequence. The sequence $\{A_n\}$ is stable if and only if $\lim_{n\to\infty} P(A_n | A_k)$ exists for $k \geqq 1$. It follows that any sequence of exchangeable events $\{A_k\}$ is stable. This has been used in [92] to deduce B. de Finetti's theorem (XXII) on exchangeable events and to give a generalization: namely, there exists a random variable f such that $P(A_{i_1} \cap \cdots \cap A_{i_j} | f) = f^j$ almost surely, $j \geqq 1$, $i_1 \neq i_k$, $1 \neq k$. This was proved later by D. G. Kendall (XXIIa) also by a martingale argument. Rényi's definition and theorems on strongly mixing events have been generalized by L. Sucheston (XXIII). An elementary and illuminating proof of these results has been given by J. Neveu (XXIV).

I do not doubt that one of the most important achievements of Rényi in probability theory, namely, the axiomatic approach to conditional probability [56], [75], was motivated by Rényi's interest in the application of probability (see, for instance, [78]) and by number theoretic ideas. If $\mathscr{N}$ is the set of all natural numbers, $\mathscr{A}$ the power set of $\mathscr{N}$, $\mathscr{B} \subseteqq \mathscr{A}$ the set of all finite nonempty sets and ν the counting measure on $\mathscr{A}$ then

$$P(A | B) = \frac{\nu(A \cap B)}{\nu(B)}, \qquad A \in \mathscr{A}, B \in \mathscr{B},$$

defines a conditional probability on $(\mathscr{N}, \mathscr{A}, \mathscr{B})$ in the sense of Rényi

It is true that similar attempts to establish an axiomatic theory of conditional probability have been made earlier. (See, for instance, H. Jeffreys (XXV), F. I. Good (XXVI), G. A. Barnard (XXVII), and B. O. Koopman (XXVIII).) Furthermore, Rényi himself reports in [56] (*Acta Math. Acad. Sci. Hungar.*, Vol 6 (1955)) that he was informed in June 1954 by Gnedenko "that Kolmogorov has put forward the idea to develop his theory in such a manner that conditional probability should be taken as the fundamental concept but he never published his ideas regarding this question." But it is obvious that Rényi's considerations are the only ones which give a satisfactory approach in the framework of measure theory.

Let R be a nonempty set and let $\mathscr{A}$ be a σ-algebra of subsets of R. Let $\mathscr{B}$ be a nonempty subset of $\mathscr{A}$. Let $(A, B) \to P(A | B)$ be a mapping from $\mathscr{A} \times \mathscr{B}$ in the nonnegative real numbers satisfying the following conditions:

(i) $P(B|B) = 1, B \in \mathscr{B}$;

(ii) $A \to P(A|B)$ is a σ-additive measure on $\mathscr{A}$ for every $B \in \mathscr{B}$;

(iii) whenever $A \in \mathscr{A}, B \in \mathscr{A}, C \in \mathscr{B}$, and $B \cap C \in \mathscr{B}$, then $P(A|B \cap C)P(B|C) = P(A \cap B|C)$.

Such a mapping is called a conditional probability on $(R, \mathscr{A}, \mathscr{B})$. It follows easily from axioms (i)–(iii) that the empty set cannot belong to $\mathscr{B}$. If P is a probability measure on $(R, \mathscr{A})$ in the usual sense, then clearly the mapping $(A, B) \to P(A \cap B)/P(B)$, $A \in \mathscr{A}$, $B \in \mathscr{B}$, satisfies axioms (i)–(iii), where $\mathscr{B} = \{B : B \in \mathscr{A}, P(B) > 0\}$. Of course, essentially new results were to be expected only when conditional probability, in the sense of Rényi, could not be represented by such a quotient with a finite measure P.

The interesting question—under which conditions can a conditional probability on $(R, \mathscr{B}, \mathscr{A})$ be represented as such a quotient with a fixed measure—has been partly answered by Rényi. A more complete result in this direction has been obtained by Á. Császár (XXIX). For every conditional probability space in the sense of Rényi there exists a set J of indices α so that this space can be realized by a set of measures $\{\mu_\alpha\}_{\alpha \in J}$: that is, to every $B \in \mathscr{B}$ there exists at least one $\alpha \in J$ with $0 < \mu_\alpha(B) < \infty$ so that $P(A|B) = \mu_\alpha(A \cap B)/\mu_\alpha(B)$, $A \in \mathscr{A}$. The set J may have the same power as the set $\mathscr{B}$. Now, the problem is under which assumptions does J contain only one element. Császár shows that this is the case if and only if the following conditions for the conditional probability space are satisfied: (1) whenever $n \geqq 1$, $A_i \in \mathscr{A}$, $B \in \mathscr{B}$, $A_i \subseteqq B_i \cap B_{i+1}$, $1 \leqq i \leqq n_i$, $B_{n+1} = B_1$, then

$$\prod_{i=1}^{n} P(A_i|B_i) = \prod_{i=1}^{n} P(A_i|B_{i+1});$$

(2) $P(B \cap B^*|B)$ and $P(B \cap B^*|B^*)$, $B, B^* \in \mathscr{B}$ are always both 0 or >0.

Rényi also introduced the concept of a Cavalieri space anticipating some aspects of the modern theory of the disintegration of measures. Furthermore, he considers in paper [56] (*Acta Math. Acad. Sci. Hungar.*, Vol. 6 (1955)) conditional ergodicity of Markov chains. Let $(p_{i,j}^{(n)})$, i, j, integers, $n \geqq 1$, be the n step transition probabilities of a homogeneous Markov chain whose state space is, say, the set of all integers. If there exist positive real numbers q_i, $i = 0, \pm 1, \cdots$, such that

$$\lim_{n \to \infty} \frac{p_i^{(n)}}{p_{j,k}^{(n)}} = \frac{q_i}{q_k} \qquad \text{for all} \quad h, i, j, k = 0, \pm 1, \pm 2, \cdots,$$

then the Markov chain is called conditionally ergodic. Clearly, every ergodic Markov chain is conditionally ergodic. P. Erdős and K. L. Chung (XXIXa) have shown that under some weak conditions a class of Markov chains of the random walk type is conditionally ergodic. Later F. J. Dyson and K. L. Chung showed that not every Markov chain is conditionally ergodic (XXX). Using the generalization of Kolmogorov's inequality given by J. Hájek and Rényi [65], which is interesting in itself, a conditional law of large numbers is presented in paper [56] (*Acta Math. Acad. Sci. Hungar.*, Vol. 6 (1955)). Subsequently, the

Hájek-Rényi inequality was generalized to semimartingales by Y. S. Chow (XXXI). It is apparent that the conditional probability in the sense of Rényi is of importance also for the number theoretic equidistribution in locally compact spaces (which are not compact). That is dealt with in more detail in (XXXII) (see also E. Schnell (XXXIII)). Rényi's work on conditional probability spaces has also been transferred to measures on Boolean algebras by P. H. Krauss (XXXIV).

I have mentioned that Rényi was greatly interested in the application of probability theory to other fields such as biology [76], [187], [194]; chemistry [52], [61]; operations research [42], [46], [54], [71], [122], [157]; and physics [45], [78], [89]. Therefore, it is not surprising that he often was concerned with the Poisson process. In this field Rényi came to a quite remarkable result in paper [177]. Let $\mathscr{I}$ be the semiring consisting of the real intervals of the form $[a, b)$ and let $\mathscr{R}$ be the ring generated by $\mathscr{I}$. Assume that ζ is a random additive set function defined on $\mathscr{R}$ with the following property: for every $E \in \mathscr{R}$ and $n = 0, 1, 2, \cdots$, the relation

$$(*) \qquad P\big(\zeta(E) = n\big) = e^{-\lambda(E)} \frac{\big(\lambda(E)\big)^n}{n!}$$

holds, where λ is a nonatomic Radon measure defined on the Borel sets of Euclidean R_1. Then ζ is a Poisson process. This proves that assumption $(*)$ already implies the independence of the random variables $\zeta(E_i)$, $E_i \in \mathscr{R}$, $E_i \cap E_j = \emptyset$, $i \neq j$, $j = 1, \cdots, k$, $k \geqq 2$. In this paper, Rényi questioned whether the theorem stated above remains valid if $\mathscr{R}$ is replaced by $\mathscr{I}$. P. A. P. Moran (XXXV) and L. Shepp (see J. R. Goldman (XXXVI)) and P. M. Lee (XXXVIa) have shown that the answer to this question is no.

In [72], Rényi gave a characterization of the Poisson process. Consider a renewal process, that is, a sequence of events which occur in the random points $0 = t_0 < t_1 < t_2 < \cdots$ so that the random variables $t_i - t_{i-1}$, $i \geqq 1$, are independentally and identically distributed with distribution function F. Let $\lambda = 1/\int_0^\infty x dF(x)$ be the (positive) intensity of the process. Now one applies a transformation T_q to the renewal process which has the following significance. Replace t_i by qt_i, $0 < q < 1$ and cancel the events independently of each other with probability $1 - q$. Rényi shows that only the Poisson process is invariant if T_q is applied and that an arbitrary renewal process becomes a Poisson process with intensity λ as $q \to 0$. This result has attracted great attention. Soon K. Nawrotzki (XXXVII) proved even the following theorem. If the set of the renewal processes is replaced by the set of all homogeneous point processes, then the statement remains valid with the compound Poisson processes substituted for the Poisson processes. Let me add that (XXXVII) is one of the first papers of a whole series of the probability theoretical school in the German Democratic Republic. Moreover, I shall only mention a paper by D. Szász (XXXVIII) to outline the further development of this subject.

We shall describe another characterization of the Poisson process by Rényi using information theoretical tools later on. Finally, we mention a limit theorem by Rényi [80] on the asymptotic behavior of the sum of a random number of independent random variables which is frequently quoted in the literature. Let $\xi_1, \xi_2, \cdots$ be a sequence of independent random variables with $E(\xi_i) = 0$ and $E(\xi_i^2) = 1$. Define $s_n = \xi_1 + \cdots + \xi_n$, $n \geqq 1$. Let $\nu(t)$ be a positive integer valued random variable for every $t > 0$ which converges in probability to ∞ as $t \to \infty$. The asymptotic behavior of $s_{\nu(t)}$ as $t \to \infty$ has been quite thoroughly investigated by R. L. Dobrušin (XXXVIIIa) already in 1955, but with the restriction that $\nu(t)$ is independent of all ξ_n, $n \geqq 1$. A result without this restriction has been given by F. J. Anscombe (XXXIX). Rényi generalized this investigation and gave the following result which will be presented in a form given later by J. R. Blum, D. L. Hanson, J. I. Rosenblatt (XL) and independently by J. Mogyoródi (XLI). Suppose that $\nu(t)/t$ converges in probability to a positive random variable as $t \to \infty$. Then $s_\nu(t)/(\nu(t))^{1/2}$ has an asymptotic normal distribution. Subsequently, H. Wittenberg (XLII) studied the corresponding problem for the Kolmogorov-Smirnov distance and obtained the above mentioned result as a special case.

We proceed to describe the many and important contributions of Rényi to the theory of information. In paper [77] Rényi and J. Balatoni consider the definition of entropy for arbitrary random variables (on the real line). Let ξ be such a random variable and define $\xi^{(n)} = [n\xi]/n$. If $H_0(\xi^{(n)})$, the Shannon entropy of $\xi^{(n)}$ exists, then for every n

$$\limsup_{n\to\infty} \frac{H_0(\xi^{(n)})}{\log_2 n} \leqq 1.$$

If

$$\lim_{n\to\infty} \frac{H_0(\xi^{(n)})}{\log_2 n} = d(\xi)$$

exists, then $d(\xi)$ is called the dimension of ξ. Furthermore,

$$H_d(\xi) = \lim_{n\to\infty} [H_0(\xi^{(n)}) - d \log_2 n]$$

is called the d dimensional entropy of ξ (if it exists). If ξ is a random variable of the discrete type then it has dimension 0 and its 0 dimensional entropy coincides with the classical Shannon entropy. If ξ has a density (with respect to Lebesgue's measure), then ξ has dimension 1 and $H_1(\xi)$ coincides with the usual definition of entropy. (Of course, the last two statements only make sense if the entropies are well defined.) More precisely, the following statement holds (see I. Csiszar (XLIII)). The limit

$$\lim_{n\to\infty} \left(H_0(\xi^{(n)}) - \log_2 n\right) \geqq -\infty$$

exists for every random variable ξ with $H_0([\xi]) < \infty$. The distribution of ξ is absolutely continuous (with density f with respect to Lebesgue's measure) if $H_0([\xi]) < \infty$ and if

$$\lim_{n\to\infty}\left(H_0(\xi^{(n)}) - \log_2 n\right) = \int_{-\infty}^{+\infty} f(x)\log\frac{1}{f(x)}\,dx > -\infty.$$

Then and only then $H_1(\xi)$ exists.

These definitions can be transferred in a trivial way to multidimensional variables and, as Rényi pointed out in [104], also to random elements in a precompact metric space. This last generalization is of course closely related to Kolmogorov's (XLIV) work on the ε-entropy. Based on the results of Rényi, M. Rudemo (XLV) has developed these ideas for some stochastic processes. His arguments exemplified on purely discontinuous processes are as follows.

Let $X(t, w)$, $t \in [0, \infty)$, $w \in R$, be such a process. Let be $T > 0$ and denote by $0 < t_1(w) < t_2(w) < \cdots < t_{N(T)}(w)$ the positions of the jumps in the interval $(0, T)$. Furthermore, suppose that the sample functions of the process are almost surely continuous from the right. Then $X(t, w)$ is determined almost surely on $(0, T)$ by the process

$$\zeta(T) = \left(t_1, \cdots, t_{N(T)}, X(+0), X(t_1 + 0), \cdots, X(t_{N(T)} + 0)\right).$$

Define $q_n(T) = P\left(N(T) = n\right)$ and let

$$\xi_n(T) = \left(t_1, \cdots, t_n, X(+0), X(t_1 + 0), \cdots, X(t_n + 0)\right),$$

$n \geqq 0$, be the $(2n + 1)$ dimensional random variable whose distribution is the conditional distribution of $\zeta(T)$ given $N(T) = n$. The dimension of $\zeta(T)$ is then defined by $\Sigma_{n=0}^{\infty} q_n(T)\, d\left(\xi_n(T)\right)$ and the entropy by $\Sigma_{n=0}^{\infty} q_n(T) H_d\left(\xi T)\right)$ (if they exist). These have been used by Rényi [155] to prove essentially the following result. Among all homogeneous point processes with given intensity $\lambda > 0$, the Poisson process has the greatest (λT dimensional) entropy in every interval $(0, T)$.

A. Ja. Khintchin (XLVI) and D. K. Fadeev (XLVII) have given well-known characterizations of Shannon's entropy (for discrete probability distributions). Axiomatic studies of the concept of entropy play an important part in Rényi's scientific work. [118], [119]. We make use of a formulation which goes back to Z. Daróczy (XLVIII). (See, also, J. Aczél (XLIX).) Let $\{p_1, \cdots, p_n\}$, $n \geqq 1$, be an arbitrary finite set of positive numbers with $\Sigma_{i=1}^{n} p_i \leqq 1$. Define

$$H_\alpha(p_1, \cdots, p_n) = \frac{1}{1-\alpha}\log_2\frac{\sum_{k=1}^{n} p_k^\alpha}{\sum_{k=1}^{n} p_k}$$

for any real $\alpha \neq 1$ and $H_1(p_1, \cdots, p_n) = -\Sigma_{k=1}^{n} p_k \log p_k (\Sigma_{k=1}^{n} p_k)^{-1}$ if $\alpha = 1$. Let H be a function defined on all sets $\{p_1, \cdots, p_n\}$ with the following properties. The map $p_1 \to H(p_1)$ is continuous in $0 < p_1 \leqq 1$. Furthermore, $H(\frac{1}{2}) = 1$ and

$$H(p_1q_1, \cdots, p_nq_1, \cdots, p_1q_n, \cdots, p_nq_n) = H(p_1, \cdots, p_n) + H(q_1, \cdots, q_n),$$

$q_i > 0$, $\Sigma_{i=1}^n q_i \leqq 1$. Finally, it is supposed that there exists a real homeomorphism g so that

$$H(p_1, \cdots, p_n, q_1, \cdots, q_n)$$
$$= g^{-1}\left[\frac{\sum_{i=1}^n p_i g(H(p_1, \cdots, p_n)) + \sum_{i=1}^n q_i g(Hq_1, \cdots, q_n))}{\sum_{i=1}^n p_i + \sum_{i=1}^n q_i}\right]$$

for all $\{p_1, \cdots, p_n\}$ and $\{q_1, \cdots, q_n\}$ with $\Sigma_{i=1}^n p_i + \Sigma_{i=1}^n q_i \leqq 1$. If $g(x) = ax + b$ then $H = H_1$, if $g(x) = a2^{(1-\alpha)x} + b$ then $H = H_\alpha$, $\alpha \neq 1$. Daróczy also gives an affirmative answer to the question raised by Rényi whether the axioms cited above imply that g can only be one of the mentioned functions. Rényi's paper [118] had a great impact and it is impossible to enumerate even partly the literature based on this work. We only mention I. Csiszár (L), D. G. Kendall (LI), and J. Aczél and P. Nath (LII).

Finally, it is worth remarking that Rényi gave in [118] the first information theoretic proof for the ergodicity of Markov chains. Rényi uses in his proof a theorem of matrix theory. That can be avoided as has been shown by Csiszár (LIII). Kendall (LIIIa) has adapted Rényi's method to the case of Markov chains with countably infinite states. The idea of using information theory to prove limit theorems goes back to Linnik (LIV). Many papers [175], [178] by Rényi are concerned with the information theoretic point of view on statistics. Inspired by a paper of D. V. Lindley (LV), he considers an n dimensional random variable ζ, 'the sample' and a discrete random variable θ 'a parameter' assuming only the values $\theta_1, \cdots, \theta_r$, $r \geqq 2$. He defines the standard decision $\Delta(\zeta)$ by $\Delta(\zeta) = \theta_k$ if $P(\theta = \theta_k \mid \zeta) = \max_{1 \leqq j \leqq r} P(\theta = \theta_j \mid \zeta)$. If $D(\zeta)$ is any other Borel measurable function from Euclidean R_n in the set $\{\theta_1, \cdots, \theta_r\}$, then $\min_D P(D(\zeta) \neq \theta) = P(\Delta(\zeta) \neq \theta)$. Moreover $P(\Delta(\zeta) \neq \theta) \leqq 1 - 2^{-E(H(\theta \mid \zeta))}$, where $H(\theta \mid \zeta)$ is the conditional entropy of θ given ζ. If $\zeta = (\xi_1, \cdots, \xi_n)$, where $\xi_1, \cdots, \xi_n$ are independently and identically distributed, then there exist an $A > 0$ and a q with $0 < q < 1$ (which do not depend on the *a priori* distribution of θ) such that $E(H(\theta \mid \zeta)) \leqq Aq^n$. I. Vincze (LVI) has extended this information theoretic Bayes approach to a continuous parameter θ.

Rényi was also interested in other aspects of mathematical statistics such as nonparametric tests [55]. In particular, he was working on the theory of order statistics [51], [53], [58]. He reduced the problems to the theory of Markov chains which are defined by sums of independent random variables. For this purpose the essential fact used is that a random variable ξ has an exponential distribution if and only if the equation $P(\xi > x + y \mid \xi > y) = P(\xi > x)$ is satisfied for arbitrary positive numbers x, y. These relationships have been observed simultaneously by B. Epstein and M. Sobel (LVII), too, but a systematic investigation is given only by Rényi. Rényi's method has frequently been used, very recently by M. Csörgő and V. Seshadri (LVIII). A very well-known result is the central limit theorem of Lindeberg type for samples from a finite population [109]. Later this topic has been treated thoroughly by Hájek (LIX).

I would like to mention Rényi's investigations on search theory [165], [190]. He based his theory on the following concept. Let S be a finite set with n elements, $n \geqq 2$. The problem is to find an unknown element x of S. A system $\mathscr{F}$ of 'known' functions defined on S is given. One makes a successive choice of functions $f_i \in \mathscr{F}$ in order to determine x. It is supposed that the f_i separate the points of S. Introducing an equidistribution in S, the functions $f \in \mathscr{F}$ become random variables whose entropy $H(f)$ is well defined. Then it is easy to show that $\Sigma_{f \in \mathscr{F}} H(f) \geqq \log n$. If $\mathscr{F}$ is a finite set, then an equidistribution is defined for the elements of $\mathscr{F}$. Functions $f_1, f_2, \cdots, f_N$ are chosen independently and in every case the value $f_i(x)$ is determined. The probability that the sequence $f_1(x), \cdots, f_N(x)$ determines x uniquely is decisive in connection with the duration of this random search procedure. The importance of this approach is demonstrated by many examples (see also [129]).

In the last 10 years of his scientific work Rényi was also concerned with another field of mathematics, namely, with graph theory which has developed very quickly in the last decades. Rényi's work on graph theory can be split up roughly into two directions. On the one hand he applied probabilistic methods; on the other hand he wrote several papers on enumerative problems in graph theory. "The area of probability arose from a theorem of Ramsey which may be simply explained as follows. Among any six people at a gathering there will always be three mutual acquaintances or three mutual non acquaintances" (quoted from Erdős (LX); for Ramsey's paper see (LXI)). Rényi's papers on probabilistic graph theory are mostly written together with P. Erdős. In [117] a simple counting measure has been introduced in the set of all undirected finite graphs (without multiple edges and loops) with N edges and n labeled vertices such that every graph of this set has probability inversely proportional to the number of ways of selecting N objects out of $\binom{n}{2}$. The probability of certain properties (connectedness, number of components, being a tree) of a random graph is studied if $N = N(n)$ and $n \to \infty$. In spite of the simplicity of this idea, it proved very useful. Let us illustrate some of the results which in many cases were substantially better than previously known ones.

A random graph almost surely consists of trees if $N(n) = o(n)$. The above mentioned set of graphs contains trees of order k if $N(n) \sim n^{(k-2)/(k-1)}$. Interesting results on the structure of the components of a random graph are also obtained. Paper [101] contains the following result. Let $P_k(n, N)$ be the probability that a random graph consists of a connected component and $k \leqq n$ isolated points. If $N(n) = [\frac{1}{2}n \log n + cn]$ and c is a real number, then $\lim_{n \to \infty} P_k(n, N(n))$ is a Poisson distribution with mean value e^{-2c}. (See also [127].)

One of the earliest results in enumerative graph theory was the Cayley formula (LXII) on the number $T(n)$ of labeled trees with n points stating that $T(n) = n^{n-2}$. Rényi was very interested in the theory of trees. In one of his first papers on graph theory [108], he gave a new proof of this formula. Cayley has stated in (LXII) that the number of graphs on n labeled points consisting of $k \leqq n$ disjoint trees so that the first k points all belong to different trees is kn^{n-k-1}. Rényi gave a proof of this statement also. He used in paper [108] the method of H. Prüfer (LXIII). This method was originally created in order to show that there exist n^{n-2} possibilities to generate the permutations of n elements by $n - 1$ transpositions. Following a suggestion by I. Schur, Prüfer has already interpreted this problem in terms of graph theory. In [203] Rényi, together with C. Rényi, studied Prüfer's method thoroughly and expanded it. It is worth mentioning paper [142] in which the degree of asymmetry $A[G]$ of an undirected graph G (without loops and multiple edges) is introduced. Consider an asymmetric graph G (that is, the automorphism group of G is the identity). Delete r edges and adjoin s new ones such that the new graph has a nontrivial automorphism. Define $A[G]$ by $\min(r + s)$. Further, let $A(n) = \max A[G]$, where the maximum extends over all graphs with n vertices. It is shown that $\lim_{n\to\infty} A(n)/n = \frac{1}{2}$.

In [171], a problem was considered which according to G. Katona and E. Szemerédi (LXIV) permits the following interpretation. "There are n airports. Any ordered pair A, B of these airports is connected by at most one directed flight from A to B. How many directed connections have to be established to assure the possibility to fly from every airport to any other by changing planes at most once?" Rényi thought highly of these investigations in combinatorial analysis. He reported some of these investigations in [173] for the first time and pointed out the many applications (for example, the Ising model of ferromagnetism). He also had planned to compose another comprehensive presentation, yet he was not to carry out this plan.

A description of Rényi's work would be certainly incomplete without a short consideration of his papers in classical theory of complex functions. In his first papers [19], [91], he was concerned with the famous conjecture of Bieberbach on the coefficients a_n of schlicht functions. For functions close to convex of type β, $0 \leqq \beta \leqq \frac{1}{2}\pi$, he proves the following variant of this conjecture: $|a_n| \leqq 1 + (2\beta/\pi)(n - 1)$, $a_1 = 1$. The same result has been found by O. M. Raede (LXV). Let us mention that W. K. Hayman (LXVI) has proved $\limsup_{n\to\infty} |a_n|/n \leqq 1$ without any additional assumption.

Paper [103] is also of interest. Suppose that $f(z) = \Sigma_{k=1}^{\infty} c_k z^{n_k}$ is analytic and unbounded in $|z| < 1$. The point $e^{i\theta}$ is called B-singular for f if f is unbounded in $|z| < 1$, $\theta - \varepsilon < \arg z\, \theta + \varepsilon$ for every $\varepsilon > 0$. If f has Hadamard gaps, then all points of the circumference of the unit circle are singular (see D. Gaier and W. Meyer-König (LXVII)). Erdős (LXVIII) has constructed an f with $n_{k+1} - n_k \to \infty$ such that $z = 1$ is the only B-singular point for f. By a simple probability theoretical device, the existence of a class of such functions is established in [103]. Of course, these considerations are related to summability problems for series.

The other papers of Rényi in complex theory of functions are mostly concerned with entire functions. In [161], the following problem is considered. Let f, g

be entire nonconstant functions. Under which conditions is $f \circ g$ periodic? If g is a polynomial of degree $\geqq 3$, then $f \circ g$ cannot be periodic. The special case $g(z) = z^k$, $k \geqq 3$, was treated earlier by C. Rényi (LXIX). If g is not periodic and f any polynomial of degree $\geqq 1$, then $f \circ g$ is not periodic. The simple proof of this statement is based on clever application of the maximum principle. This last result has been generalized later by I. N. Baker (LXX): if f is an entire function of order $< \frac{1}{2}$ (or of order $\frac{1}{2}$ and minimal type) and g is not periodic then $f \circ g$ is not periodic.

As far as the papers [70], [81] are concerned, let us only mention the following interesting result which is related to the Whittaker constant w_1 (see below). If f is an entire function of order $\alpha \geqq 1$, $g(r) = \log \max_{|z|=r} |f(z)|$, then

$$\lim_{k \to \infty} \frac{N_k(f(z))g^{-1}(k)}{k} \leqq \exp\left\{2 - \frac{1}{\alpha}\right\},$$

where $N_k(f)$ is the number of zeros of $f^{(k)}$ in the unit circle. (This result has been improved later by Ju. K. Suetin (LXXI).) It follows for functions of finite exponential type that $w_p \geqq p/e$, where w_p is defined as follows: if f is an entire function of exponential type $\tau < \infty$, then $w_p = \max_w \{w > \tau$: if $f^{(i)}$, $i = 0, 1, 2 \cdots$ has at least p zeros in the unit circle then $f \equiv 0\}$. Later H. Wilf (LXXII) sharpened this lower bound for w_p.

Of Rényi's publications in book form, *Probability Theory* [57], which comprises all properties of his mathematical creativeness, is perhaps the most outstanding. "This book has its origin in a series of lectures which the author gave at the University of Budapest, beginning in 1948. The present form of Rényi's book reflects his particular mathematical interests. It appears that the author is not only an expert in probability theory but also in many fields of analysis and in the theory of numbers. Applications of probability theory, especially in physics and chemistry, also belong to the sphere of interests of the author. This wide scope of interests is reflected in the numerous problems that are added to each chapter." (Quoted from L. Schmetterer (LXXIII).)

Rényi also wrote a series of witty essays in form of dialogues [139], [191]. The dialogue between Mrs. Niccolini, Galileo and Torricelli was influenced to some extent by a drama of Ladislaus Németh as Rényi himself indicated. Undoubtedly, his acquaintance with the Hungarian mathematician-historian A. Szabó has stimulated his Socratic dialogue. The dialogues show Rényi's deep understanding of the position of mathematics in general. In these, the extent to which Rényi's didactic skill and powerful imagination extended beyond the field of mathematics is at once clear.

L. SCHMETTERER

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