A LIMIT THEOREM FOR INDEPENDENT RANDOM VARIABLES

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The starting point of this paper is the question, what happens to the distribution of the sum of a large number of independent, identically distributed, integer valued, random variables or equivalently, what happens to a measure on the group of integers when convoluted by itself a large number of times? It is known that the probability of being at a fixed integer tends to a limit and that this limit is 0. Therefore, the finer information about what the distribution looks like is obtained by looking at the ratio of the probability of being at one fixed integer to the probability of being at another fixed integer. Such a theorem was proved by Chung and Erdös in [1].

There are two natural directions for generalizing this theorem. One generalizes to a Markov process and the other to convoluting measures on a more general group.

A generalization to Markov chains is given by Kingman and Orey in [3]. Another generalization is given by Jain in [2] for a fairly general Markov process, but the price for the generality is that the theorem is about the ratio of the expected number of visits to a set up to time n instead of the probability of being there at time n.

In this paper we generalize to convoluting on more general groups and prove a theorem in the case where the group is the line. The method used is a modification of the one used by Chung and Erdös. This method gives the same theorem for Euclidean space, and if we analyze the proof, we see that we use very little that is specific to the line, and hence we could get a theorem for a general locally compact abelian group. (Our assumption of mean 0 is used only in obtaining lemma 1, and hence in all cases when we have lemma 1, we have a general theorem.) We could do the same for the time ratio, thus generalizing theorem 4 of [1].

Charles Stone recently gave another proof of the main theorem of this paper in [4]. His method seems to give more information in the case of Euclidean space but does not seem to go over to more general groups.

THEOREM. *Let X_n be a sequence of independent, identically distributed, real-valued random variables with either mean 0 or with the integral of the positive and negative parts both infinite. Assume also that the values X takes are not all part of an arithmetic progression. Let $S_n = \sum_{i=1}^{n} X_i$. Let J_1 and J_2 be two intervals of the same length. Then*

$$\lim_{n\to\infty} \frac{\Pr\{S_n \subset J_1\}}{\Pr\{S_n \subset J_2\}} = 1. \tag{1}$$

The abbreviation Pr stands for probability of the event which follows in braces.

LEMMA 1. *Given ϵ and an interval J,* $\Pr\{S_n \subset J\} > (1-\epsilon)^n$ *for all sufficiently large n.*

PROOF. The proof is similar to the proof of theorem 2.2 in Erdös and Chung and will therefore be omitted.

LEMMA 2. *There are arbitrarily small numbers α such that there is an integer k and arbitrarily small pairs of intervals I and I' where* $\Pr\{S_k \subset I\} \neq 0$ *and* $\Pr\{S_k \subset I'\} \neq 0$ *and I and I' are α apart in the sense that there is a point in I that is at distance α from some point in I'.*

PROOF. This is straightforward and easy and will therefore be omitted.

DEFINITION. *Let I be an interval. We shall call the interval obtained by extending I in both directions by ϵ, I^{ϵ} and the one obtained by contracting by ϵ, $I^{-\epsilon}$.*

LEMMA 3. *Let J_1 and J_2 be disjoint intervals of the same length whose centers are α apart, and lemma 2 applies to α. Then given $\epsilon > 0$ and $\gamma > 0$, there is a K such that*

$$\frac{\Pr\{S_n \subset J_2^{\epsilon}\}}{\Pr\{S_n \subset J_1^{-\epsilon}\}} > 1 - \gamma \qquad \text{for all} \quad n > K. \tag{2}$$

PROOF. By lemma 2 there are intervals I and I' and an integer k such that there is a point in I and a point in I' α apart, the lengths of I and I' are both $< \epsilon$, and $\Pr\{S_k \subset I\} = p \neq 0$ and $\Pr\{S_k \subset I'\} = p' \neq 0$ [to fix ideas, assume J_2 is to the right of J_1 and that I' is to the right of I and that ϵ is small compared to α so that I and I' are disjoint. Take all intervals from now on to be half open].

To simplify notation, we will prove lemma 3 only for n of the form $mk + r$ $(0 \leq r < k)$ $(0 \leq m < \infty)$ for just one r so we can keep r fixed throughout the proof. Call the random variable S_r, $\overline{X}_0$ and let $\overline{X}_i = \sum_{j=r+1+k(i-1)}^{j=r+ik} X_j$. Let $T_{I,m}(w)$ be the number of $\overline{X}_i(w)$ that are in the interval $I (1 \leq i \leq m)$. It is well known that given γ_1 (choose γ_1 so that $(1-\gamma_1)^2/(1+2\gamma_1) > 1-\gamma$, where γ is the γ in lemma 3), there is a $\delta < 1$ such that $\Pr\{|(T_{I,m})/(pm) - 1| > \gamma_1\} < \delta^m$ for all m. Hence, using lemma 1, we get

$$\Pr\left\{\left|\frac{T_{I,m}}{pm} - 1\right| > \gamma_1\right\} < \gamma_1 \Pr\{S_{mk+r} \subset J_1^{-2\epsilon}\} \qquad \text{for all } m \text{ large enough,} \tag{3}$$

$$\Pr\left\{\left|\frac{T_{I',m}}{p'm} - 1\right| > \gamma_1\right\} < p'\gamma_1 \Pr\{S_{mk+r} \subset J_2\} \qquad \text{for all } m \text{ large enough.} \tag{4}$$

Let m' be an m for which (3) and (4) work, and let $n' = m'k + r$. We will prove lemma 3 for n'.

Break up the real numbers into half open intervals, each of which has length $< \epsilon \cdot 1/n'$ and lies either entirely inside or outside I and I', and I and I' are broken up into the same number t of intervals. Denote those intervals in I by $g_1, \cdots, g_t$, those in I' by $g'_1, \cdots, g'_t$, and the rest by $h_1, \cdots, h_t, \cdots$.

For each w we get a sequence v of m intervals, the i-th interval being the interval that $\overline{X}_{i-1}(w)$ is in. Let Q_v be the set of w with the sequence v.

Let B be the union of all those Q_v that contain a w for which $S_{n'}(w) \subset J_1$, and B' the union of the Q_v that contain a w for which $S_{n'}(w) \subset J_2^{+2\epsilon}$.

Define a new measure space $\hat{B}$ as follows: for each Q_v in B take as many copies as there are g (that is, subintervals of I) in the second through last terms of the sequence v (that is, we will take $T_{I,m'}(w)$ copies of Q_v where $w \subset Q_v$). Let $Q_v^i(1 \leq i \leq T_{I,m'}(w))$ be the copies of Q_v. Let

$$\hat{B} = \bigcup_{Q_v \subset B} \left(\bigcup_{i=1}^{\substack{T_{I,m'}(w) \\ (w \subset Q_v)}} \right) Q_v^i. \tag{5}$$

The measure of a set in $\hat{B}$ will be the sum of the measure of the set intersected with each of the Q_v^i. Call the measure μ ($u(\hat{B})$ is much larger than the measure of B, for example).

Define $\hat{B}'$ in a similar way; namely,

$$\hat{B}' = \bigcup_{Q_v \subset B'} \left(\bigcup_{i=1}^{\substack{i = T_{I',m'}(w) \\ (w \subset Q_v)}} \right) Q_v^i \tag{6}$$

(we will call the measure on $\hat{B}'$ u and also the measure on our original probability space u).

To each Q_v^i in $\hat{B}$ there corresponds a $Q_{v'}$ in B' as follows: the index v' will be obtained by changing the i-th g in the second through last terms of v, call it g_j into g_j' and leaving the rest of the sequence alone. It is easy to check that $Q_{v'}$ is in B'.

Each $Q_{v'}$ in B' has at most $T_{I',m'}(w)$, $(w \subset Q_{v'})$ inverse images (this is easy to check). Hence, we can get a one-to-one mapping $\hat{B}$ into $\hat{B}'$ such that if $Q_v^i \to Q_{v'}^j$ then v' will be obtained from v by the process described in the previous paragraph.

Partition $\hat{B}$ into disjoint pieces D_j in the following way: the set Q_v^i will be in the same group as $Q_{\bar{v}}^j$ if $i = j$ and v and $\bar{v}$ differ in only one coordinate, the i-th g of the second through last terms of v (or equivalently $\bar{v}$). Call D_j full if it contains a Q_v^i for every possible choice of g for the i-th g (that is, if there are t, Q_v^i in D_j). (The set D_j may not be full because some of the necessary Q_v may not be in B.)

It is easy to see that every Q_v^i such that there is a w in Q_v with $S_{n'}(w) \subset J_1^{-2\epsilon}$ is in a full D_j. Call the union of the Q_v that contain a w with $S_{n'}(w) \subset J_1^{-2\epsilon}$, $B_{-2\epsilon}$.

If D_j is full and D_j' is its image, then

$$u(D_j) = \frac{p'}{p} u(D_j). \tag{7}$$

This comes from the independence of the $\overline{X}_i$.

By (3) the measure of the part of $B_{-2\epsilon}$ for which $T_{I,m'} > pm'(1 - \gamma_1)$ is greater than $(1 - \gamma_1)u(B_{-2\epsilon})$. Hence, the measure of the full D_j in $\hat{B}$ is greater than

$$pm'(1 - \gamma_1)(1 - \gamma_1)u(B_{-2\epsilon}). \tag{8}$$

Combining (7) and (8) we get

$$u(\hat{B}') > \frac{p'}{p} \cdot pm'(1 - \gamma_1)(1 - \gamma_1)u(B_{-2\epsilon}). \tag{9}$$

By (4), the measure of the part of B' for which $T_{I',m'} > p'm'(1 + \gamma_1)$ is less than $p'\gamma_1 u(B')$. Hence, $u(\hat{B}') < u(B')p'm'(1 + \gamma_1) + m'p'\gamma_1 u(B')$, and

$$u(\hat{B}') < u(B')p'm'(1 + 2\gamma_1). \tag{10}$$

Putting (9) and (10) together we get $u(B')p'm'(1 + 2\gamma_1) > p'm'(1 - \gamma_1)(1 - \gamma_1) \times u(B_{-2\epsilon})$ and

$$u(B') > \frac{(1 - \gamma_1)^2}{(1 + 2\gamma_1)} u(B_{-2\epsilon}), \tag{11}$$

$$\Pr\{S_{n'} \subset J_2^{+3\epsilon}\} > u(B'), \tag{12}$$

and

$$u(B_{-2\epsilon}) > \Pr\{S_{n'} \subset J_1^{-2\epsilon}\} > \Pr\{S_{n'} \subset J_1^{-3\epsilon}\}. \tag{13}$$

Inequality (11) together with (12) and (13) proves the lemma for 3ϵ, and hence for ϵ.

LEMMA 4. *Let I and $\hat{I}$ be intervals such that the length of I is greater than k times the length of $\hat{I}$ (let $\epsilon = \frac{1}{4}$ length of $\hat{I}$). Then* $\Pr\{S_n \subset I^{-\epsilon}\} > \frac{1}{6}k \Pr\{S_n \subset I^{-\epsilon}\}$ *for all sufficiently large n.*

PROOF. We can fit more than $\frac{1}{3}k$ intervals, I_j, of length 4ϵ into $I^{-\epsilon}$ such that I_j^{ϵ} is disjoint from I_i^{ϵ} if $i \neq j$. Now apply lemma 3 ($\frac{1}{3}k$ times) with $\gamma < \frac{1}{2}\hat{I} = J_1$ and $I_j = J_2$.

We get our theorem by combining lemmas 3 and 4.

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