ON MARKOV GROUPS

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1. Introduction

Let $\{P_t\colon 0 \leq t < \infty\}$ be a strongly continuous one-parameter semigroup of transition operators on the Banach space ℓ_1 of absolutely convergent series, reducing to the identity at $t = 0$, with matrix representation

$$(P_t x)_j = \sum_i x_i p_{ij}(t), \qquad (j = 1, 2, \cdots), \tag{1}$$

so that in $\{p_{ij}(\cdot)\colon i, j = 1, 2, \cdots\}$ we have a standard family of Markov transition functions on $[0, \infty)$ in the terminology of K. L. Chung [1]. In this and in the succeeding papers [5], [6] we shall be interested in three loosely related questions:

(i) the analyticity or otherwise of the functions $p_{ij}(\cdot)$;

(ii) the identification of quasi-analytic classes of such functions;

(iii) the possibility of extending the Markov semigroup $\{P_t\colon 0 \leq t < \infty\}$ to a strongly continuous *group* $\{P_t\colon -\infty < t < \infty\}$ of bounded linear operators on ℓ_1.

The present paper is concerned with the last of these three topics, and consists largely of conjectures and scraps of evidence about them. Some further evidence will be found in the accompanying paper by Miss J. M. O. Speakman [7]. If our remarks lead others to solve the problems posed, we shall be delighted.

2. Property (*U*) and property (*G*)

It will be helpful to make the following definitions.

DEFINITION 1. *A Markov semigroup $\{P_t\colon 0 \leq t < \infty\}$ will be said to have property (U) when any one of the following equivalent conditions is satisfied:*

$(U1)$: $p_{ii}(t) \to 1$ *as* $t \to 0$, *uniformly with regard to* i;

$(U2)$: $\|P_t - I\| \to 0$ *as* $t \to 0$;

$(U3)$: $P_t = \exp(At)$, *where A is a bounded operator;*

$(U4)$: $\sup q_i < \infty$, *where* $q_i = -p'_{ii}(0)$.

The equivalence of $(U1)$ and $(U2)$ is due to the fact that

$$\|P_t - I\| = 2 \sup_i (1 - p_{ii}(t)). \tag{2}$$

The equivalence of $(U2)$ and $(U3)$ is a standard result in the theory of such semigroups ([4], theorem 9.6.1). The equivalence of $(U123)$ and $(U4)$ follows from the fact that

$$1 - p_{ii}(t) \leq 1 - \exp(-q_i t) \leq q_i t, \tag{3}$$

and from the consequence $q_i = -A_{ii}$ of ($U3$). Property (U) is so named because it is plainly a 'uniformity' property; Markov semigroups with property (U) are sometimes called '*q-bounded.*'

We note in passing that the q_i's always exist and can have values between 0 and $+\infty$ inclusive. The off-diagonal derivatives $q_{ij} = p'_{ij}(0)$ $(i \neq j)$ also exist and satisfy $0 \leq q_{ij} < \infty$. In the q-bounded case infinite values for the q_i's are of course excluded, and the matrix $Q = (q_{ij})$, with $q_{ii} = -q_i$, can then be identified with A at ($U3$).

It is clear that q-bounded Markov semigroups can always be extended to uniformly continuous groups $\{P_t\colon -\infty < t < \infty\}$ by taking the equation at ($U3$) to be the definition of P_t when $t < 0$, and in particular this is true when the number of states is finite.

It is worth remarking that the adjoined operators P_t with $t < 0$ will never (save in a trivial case) be transition operators. This follows from the fact (which I owe to P. Whittle) that *if P and Q are transition operators and if $PQ = QP = I$, then both P and Q must be permutations.* Thus if $P_{-\tau}$ is a transition operator for some positive τ, then P_τ must be a permutation. However it has a strictly positive diagonal, so it must be I. From this it follows that $P_{n\tau} = I$ for all integers n, and then it becomes evident that the whole semigroup must reduce to the identity.

Whittle's lemma can be proved as follows. Let j be any state; then $(QP)_{jj} = \delta_{jj} > 0$, and hence $P_{ij} > 0$ for at least one state i. For such an i we must have $(PQ)_{ik} = \delta_{ik} = 0$, save when $k = i$, so that $Q_{jk} = 0$, except when $k = i$. From this it is clear that $P_{ij} > 0$ for exactly one i. Thus each row of Q and each column of P contains exactly one nonzero element. The result now follows from the symmetry of the data.

It is important to notice that (U) is a property of the system *as a whole*, and that it cannot be defined in terms of the irreducible classes of states, if such exist. This is obvious from ($U4$).

We now make a second definition.

DEFINITION 2. *A Markov semigroup will be said to have property* (G) *when it is possible to define bounded linear operators P_t for all $t < 0$ such that $\{P_t\colon -\infty < t < \infty\}$ is a strongly continuous group.*

Obviously (U) implies (G); the converse might be conjectured to be true, but whether this is so or not appears to be an open question.

Like (U), (G) is not a class property. This will be made clear in the following section of the paper, where we shall exhibit some direct sums of finite-state Markov semigroups which, in contrast to their summands, have neither the property (U) nor (G).

3. Remarks about the relation between (U) and (G), and some examples

To illustrate one important feature of the problem, it is useful to consider the simple birth process with unit birth rate. For this, $p_{ij}(t) = 0$ when $1 \leq j < i$, and

$$p_{ij}(t) = \binom{j-1}{i-1} e^{-it}(1 - e^{-t})^{j-i} \qquad \text{when} \quad 1 \le i \le j. \tag{4}$$

The matrix Q is given here by $q_{ii} = -i$, $q_{i,i+1} = +i$, $q_{ij} = 0$ otherwise, so that the system is certainly not q-bounded. If (G) is to hold, then P_t for $t = -\tau < 0$ must be P_τ^{-1}, and the upper triangular character of P_τ ensures that the elements of P_t are uniquely determined by the finite linear equations expressing the fact that $P_t P_\tau = I$. It follows in this way that the components of P_t are given by the same analytical formulae when $t < 0$ as when $t \ge 0$.

But now we observe that

$$\sum_j |p_{ij}(t)| = (2e^{-\tau} - 1)^{-i} \qquad \text{when} \quad 0 < \tau \le \log 2, \tag{5}$$

and that the series diverges when $\tau > \log 2$. Thus

$$\|P_t\| = \sup_i \sum_j |p_{ij}(t)| = \infty \qquad \text{for all} \quad t < 0, \tag{6}$$

and so the operators $P_t (t < 0)$ are not bounded. Thus (G) does not hold.

An elementary theorem about semigroups shows that in testing for (G), only one value of $t < 0$ need be examined.

THEOREM 1. *Property* (G) *holds if and only if* P_t *has a bounded inverse for some one* (*and then for all*) $t > 0$.

This is theorem 16.3.6 of [4]; for completeness we sketch a proof. In any Banach algebra, if T is boundedly invertible, then so is any R for which $RA = I = BR$; thus, if P_τ is invertible for some one $\tau > 0$, then so is P_t for $0 \le t \le \tau$, and therefore so is P_{nt} for all positive integers n, so that P_t is invertible for all $t \ge 0$. Putting $P_{-t} = (P_t)^{-1}$ for $t \ge 0$ yields a group $\{P_t \colon -\infty < t < \infty\}$, and strong continuity then follows from the inequalities

$$\begin{aligned} \|P_{-t}x - x\| &= \|P_{\alpha-t}(P_{-\alpha}x - P_{t-\alpha}x)\| \\ &\le \|P_{\alpha-t}\| \, \|P_t(P_{-\alpha}x) - (P_{-\alpha}x)\| \\ &\le \|P_t(P_{-\alpha}x) - (P_{-\alpha}x)\|, \end{aligned} \tag{7}$$

where $0 < t < \alpha$ (fixed).

It is not difficult to find wide classes of Markov semigroups within which $(U) \equiv (G)$. For example, consider the Cohen semigroup $\{P_t^\odot \colon 0 \le t < \infty\}$ derived from a given Markov semigroup $\{P_t \colon 0 \le t < \infty\}$ by the following construction [2], [3]:

$$\begin{cases} P_0^\odot = I, \\ P_t^\odot x = \dfrac{1}{\Gamma(t)} \displaystyle\int_0^\infty e^{-u} u^{t-1} P_u x \, du, \qquad (0 < t < \infty). \end{cases} \tag{8}$$

It is easily verified that each $P_t^\odot$ is a transition operator, and the new operators form a semigroup in virtue of Dirichlet's integral formula. It is strongly continuous because

$$\|P_t^\odot x - x\| = \Gamma(t)^{-1} \left\| \int_0^\infty e^{-u} u^{t-1} \, du \int_0^u P_v A x \, dv \right\| \le t \|Ax\|, \tag{9}$$

if $t > 0$ and if x lies in the domain of the infinitesimal generator A. This shows that $||P_t^{\odot}x - x|| \to 0$ when $t \to 0$ for all such x, and then for every x by an appeal to the Banach-Steinhaus theorem.

Now $P_1^{\odot} = R_1 = (I - A)^{-1}$, where $\{R_\lambda: 0 < \lambda < \infty\}$ is the family of resolvent operators for the original semigroup. From this point of view, $\{P_t^{\odot}: 0 \leq t < \infty\}$ is a semigroup of fractional powers of R_1. In view of theorem 1, we know that the derived semigroup will have property (G) if and only if R^1 has a bounded inverse. This, however, is equivalent to requiring the original semigroup to have a bounded infinitesimal generator A, when by $(U3)$ it will be q-bounded. We should then have

$$(10) \qquad ||P_t^{\odot} - I|| \leq \Gamma(t)^{-1} \int_0^\infty e^{-u}u^{t-1}||P_u - I||\, du = o(1), \qquad (t \to 0),$$

and hence, arrive at proposition 1.

PROPOSITION 1. *A Cohen semigroup* $\{P_t^{\odot}: 0 \leq t < \infty\}$ *has property* (G) *if and only if it has property* (U).

We have remarked that (U) is not a class property, and that the same is true of (G). Thus, it is reasonable to look at Markov semigroups which are built up out of simpler ones by a direct sum construction. Let us write

$$(11) \qquad P_t^{\alpha,\beta,\rho} = \begin{pmatrix} \beta + \alpha e^{-\rho t} & \alpha(1 - e^{-\rho t}) \\ \beta(1 - e^{-\rho t}) & \alpha + \beta e^{-\rho t} \end{pmatrix},$$

where α, β, and ρ are nonnegative and $\alpha + \beta = 1$. These matrices represent the most general two-state Markov semigroup, which of course has both properties (U) and (G), and

$$(12) \qquad ||P_{-\tau}^{\alpha,\beta,\rho}|| = 1 + 2 \max(\alpha, \beta)(e^{\rho\tau} - 1), \qquad (\tau \geq 0).$$

If we now form the direct sum of a sequence of such semigroups with parameters $(\alpha_k, \beta_k, \rho_k)(k = 1, 2, \cdots,)$ it will be seen that the semigroup thus constructed will in any case be strongly continuous, and that it will have property (U) if and only if

$$(13) \qquad \sup_k \max(\alpha_k, \beta_k)\rho_k < \infty,$$

whereas it will have property (G) if and only if

$$(14) \qquad \sup_k \max(\alpha_k, \beta_k)(e^{\rho_k} - 1) < \infty.$$

Now $\frac{1}{2} \leq \max(\alpha_k, \beta_k) \leq 1$, hence the two boundedness conditions are equivalent. Thus we have proposition 2.

PROPOSITION 2. *A direct sum of two-state Markov semigroups has property* (G) *if and only if it has property* (U).

It would be desirable to extend this result to cover the direct sums of arbitrary finite-state semigroups, or more generally to direct sums of q-bounded semigroups. For a contribution to this problem see section 2 of [7].

The following two observations (which I owe to D. Williams) deny the property (G)-but-not-(U) to still further classes of Markov semigroups. First, suppose that $\{P_t\colon 0 \le t < \infty\}$ is measurable in the uniform sense. This requires that there be a sequence of measurable countably-valued operator functions of t which converge in the uniform sense to P_t for almost all t. From theorem 9.3.1 of [4], it then follows that P_t depends continuously on t, in the uniform sense, for $t > 0$. If (G) holds, the inequalities

$$(15) \qquad \|P_h - I\| \le \|P_{-t}\|\,\|P_{t+h} - P_t\| = o(1) \qquad \text{as} \quad h \to 0$$

(for fixed $t > 0$), then imply that (U) holds. We therefore have the next proposition.

Proposition 3 (Williams). *A uniformly measurable Markov semigroup has property (G) if and only if it has property (U). In particular this is true of Markov semigroups uniformly continuous for $t > 0$.*

It will be noticed that uniform continuity at any one $t > 0$ is enough to force the conclusion. Uniform continuity often arises as a result of compactness, but the following result is best established directly, although it could be exhibited as a corollary to proposition 3.

Proposition 4 (Williams). *If the Markov semigroup $\{P_t\colon 0 \le t < \infty\}$ has a resolvent operator R_λ which is compact for some one, and then for all, $\lambda > 0$, then it has property (G) if and only if it has property (U), and the state space must then be finite.*

This depends on a surprising theorem [8] of Williams according to which the compactness of R_λ implies that of P_t for every $t > 0$. If (G) holds, therefore, we shall have a compact identity $I = P_{-t}P_t$, and so the state space must be finite and (U) must hold.

These propositions help to delimit the region within which one should look for counter examples to the conjecture that $(U) \equiv (G)$.

4. The (Γ, γ)-diagram

Theorem 1 shows that a Markov semigroup will have the property (G) if it has the property defined below.

Definition 3. *A Markov semigroup will be said to have property (F) when there is a positive value of t for which $f(t) \equiv \|P_t - I\| < 1$.*

Thus $(U) \Rightarrow (F) \Rightarrow (G)$. Let class A consist of those Markov semigroups which have property (F) but not property (U), and let class B consist of those Markov semigroups which have property (G) but not property (F). If it is true that $(U) \equiv (G)$, then the classes A and B will both be vacuous.

It is useful to write

$$(16) \qquad g(t) \equiv \inf_i p_{ii}(t), \qquad (0 \le t < \infty),$$

because we have

$$(17) \qquad f(t) = 2(1 - g(t)), \qquad g(t) = 1 - f(t)/2.$$

Thus the (F) semigroups are those for which $g(t) > \frac{1}{2}$ for some $t > 0$. (We must exclude $t = 0$ because in all cases $g(0) = 1$.) Miss Speakman [7] has shown that a direct sum of (U) semigroups which does not itself have property (U) must be such that $g(t) \leq \frac{1}{2}$ for all $t > 0$. Thus, if class B could be shown to be empty, we could strengthen proposition 2 to include all direct sums of (U) semigroups.

Let us put

$$\Gamma \equiv \limsup_{t \to 0} g(t), \qquad \gamma \equiv \liminf_{t \to 0} g(t). \tag{18}$$

Then the (U) semigroups are precisely those for which $(\Gamma, \gamma) = (1, 1)$. For the direct sum of two-state Markov semigroups considered in proposition 2, with $\alpha_k = \alpha$, $\beta_k = \beta$, and $\rho_k = k$ (an example suggested by J. F. C. Kingman), we have $\Gamma = \gamma = \min(\alpha, \beta)$, and so we can have $(\Gamma, \gamma) = (c, c)$ where c has any

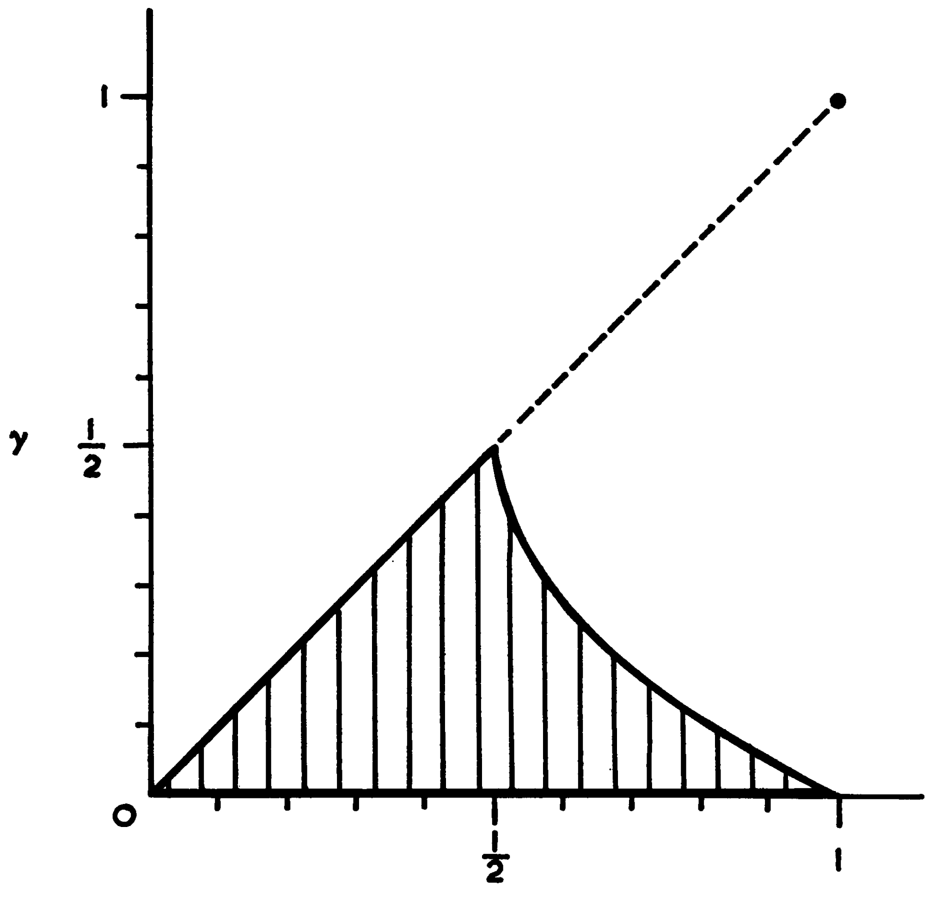

FIGURE 1

The (Γ, γ) diagram

value in the closed interval $[0, \frac{1}{2}]$. When $\Gamma = \gamma$ *there are no other possibilities.* This is a consequence of the following unpublished result of G. E. H. Reuter.

THEOREM 2 (Reuter). *If* $\gamma > \frac{1}{2}$, *then property* (U) *holds.*

Instead of giving Reuter's proof of theorem 2, we shall modify it slightly so as to give rather more information. We shall prove the following theorem.

THEOREM 3. *For every Markov semigroup, the pair* (Γ, γ) *must be* $(1, 1)$, *or must correspond to a point lying in the closed shaded region in figure 1; that is, the only possibilities are*

(a) $\Gamma = \gamma = 1$;

(b) $\Gamma > \frac{1}{2}$ *and* $0 \leq \gamma \leq \{1 - \sqrt{(2\Gamma - 1)}\}/2$;

(c) $0 \leq \Gamma \leq \frac{1}{2}$ *and* $0 \leq \gamma \leq \Gamma$.

We do *not* assert that all such pairs (Γ, γ) can in fact be realized, although we have seen that this is true so far as pairs of the form $(\Gamma, \gamma) = (c, c)$ are concerned. In an early stage of this investigation it seemed possible that Γ and γ would always be equal, but Miss Speakman [7] has shown that this is not so; in fact, it is now clear from her work that the region actually occupied by the pairs (Γ, γ) is two-dimensional, and is definitely smaller than that permitted by theorem 3. In an attempt to delimit it more exactly we have called in automatic computing aids, and a provisional account of this work will be found in [6] and [7].

It will be noticed that Reuter's theorem follows from theorem 3 by projection onto the γ-axis. Although the maximum allowable region in the (Γ, γ)-plane projects onto the whole interval $[0, 1]$ on the Γ-axis, we do not know whether all of this is actually attained; in fact, we know of no example with $\frac{1}{2} < \Gamma < 1$. (After this was written, Miss Speakman [7] proved that parts of the interval $[0, 1]$ are inaccessible to Γ.) If such an example could be found, then we should have discovered a class A semigroup, and we should know that property (F), and so also property (G), was not equivalent to (U).

PROOF OF THEOREM 3. Let $\{P_t\colon 0 \leq t < \infty\}$ be any Markov semigroup. From the identities

$$P_{t+s} - I = P_s(P_t - I) + (P_s - I) \tag{19}$$

and

$$2(P_t - I) = (P_{2t} - I) - (P_t - I)^2, \tag{20}$$

we find that

$$f(t + s) \leq f(t) + f(s) \tag{21}$$

and

$$2f(t) \leq f(2t) + \{f(t)\}^2, \tag{22}$$

so that

$$g(t) + g(s) \leq 1 + g(t + s) \tag{23}$$

and

$$g(2t) \leq 1 - 2g(t) + 2\{g(t)\}^2. \tag{24}$$

Note that the first g-inequality also gives $g(2t) \geq 2g(t) - 1$. We must have $0 \leq g(t) \leq 1$ for all t. Suppose that $g(\tau) = c > \frac{1}{2}$ for some $\tau > 0$. Put $t_n = \tau/2^n$. From (24) we find that *either*

$$g(t_1) \geq \{1 + \sqrt{(2c - 1)}\}/2, \tag{25}$$

or

$$g(t_1) \leq \{1 - \sqrt{(2c - 1)}\}/2. \tag{26}$$

If the first alternative holds, then $g(t_1) > \frac{1}{2}$ and we can continue the argument. Thus *either*

$$g(t_n) \geq \{1 + (2c - 1)^{2^{-n}}\}/2 \qquad \text{for } all \quad n = 1, 2, \cdots, \tag{27}$$

or

$$g(t_n) \leq \{1 - (2c - 1)^{2^{-n}}\}/2 \qquad \text{for } some \quad n \geq 1. \tag{28}$$

Suppose the first alternative holds. Then if we put

$$\alpha = (2\tau)^{-1} \log \frac{1}{2c - 1}, \tag{29}$$

so that $\alpha \geq 0$, and is finite, we shall have $g(t_n) \geq 1 - \alpha t_n$ for all $n = 1, 2, \cdots$; hence, $f(t_n) \leq 2\alpha t_n$ for all n. But $f(\cdot)$ is subadditive; therefore, $f(t) \leq 2\alpha t$ whenever $t = \rho\tau$ where ρ is a rational with binary denominator. For such values of t we therefore must have $g(t) \geq 1 - \alpha t$, and $p_{ii}(t) \geq 1 - \alpha t$ for all states i. But $p_{ii}(\cdot)$ is continuous for each i, and so the last inequality must hold for all $t \geq 0$, whence every $q_i \leq \alpha < \infty$, and the semigroup is q-bounded. Thus the first alternative holds only if the semigroup has property (U).

If the second alternative holds, then we cannot identify the value of n for which (28) is true, but at least for that value of n we must have

$$g(t_n) \leq \{1 - \sqrt{(2c - 1)}\}/2; \tag{30}$$

hence, we arrive at the following lemma.

LEMMA. *If a Markov semigroup does not have property* (U), *and if* $g(\tau) > \frac{1}{2}$ *for some* $\tau > 0$, *then*

$$g(t) \leq \{1 - \sqrt{(2g(\tau) - 1)}\}/2 \tag{31}$$

for some $t = \tau/2^n$, $(n = 1, 2, \cdots)$.

We can now prove theorem 3. We simply have to show that, if $\frac{1}{2} < \Gamma \leq 1$ for some non-(U) semigroup, then γ cannot exceed the upper bound stated at (b). In these circumstances we can find a sequence $\tau_k \downarrow 0$ at which $g(\tau_k) = c_k \to \Gamma$, with every $c_k > \frac{1}{2}$. We can therefore, by the lemma, find another sequence $t_k \to 0$ such that

$$g(t_k) \leq \{1 - \sqrt{(2c_k - 1)}\}/2 \to \{1 - \sqrt{(2\Gamma - 1)}\}/2, \tag{32}$$

and so γ cannot exceed the limit on the right-hand side.

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