GENERAL LATERAL CONDITIONS FOR SOME DIFFUSION PROCESSES

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1. Formulation of the problem and fundamental results

1.1. Let E be a plane domain bounded by a smooth contour L, and let $v(z)$ be a smoothly varying vector field on L. Let the point $\gamma \in L$ be called exclusive if the projection of the vector $v(z)$ on the inner normal to L changes sign at the point γ. Let us say that the function $u(z)$ satisfies the boundary condition $\mathfrak{A}$ if, at each nonexclusive point z of the contour L, the derivative of u in the direction $v(z)$ is zero. We are interested in solutions of the heat conduction equation $(\partial u_t(z)/\partial t) = \Delta u_t(z)$ in the domain E, which satisfy the initial condition $u_0(z) = f(z)$ and the boundary condition $\mathfrak{A}$. More accurately, our problem is to describe the general form of the lateral conditions at exclusive points, which will, together with the initial and boundary conditions, define a unique solution $u_t(z)$ of the heat conduction equation, wherein: (a) $u_t(z) \geq 0$ if $f(z) \geq 0$; (b) $\|u_t\| \leq \|f\|$ (we understand $\|f\|$ to be $\sup |f(z)|$ in the union E^* of the domain E and the set of all nonexclusive points of the contour L). (An analogous problem for the system of differential equations of Kolmogorov which describes Markov processes with countable phase space was studied by W. Feller [4]. However, Feller considered only a special class of supplementary conditions corresponding to "continuous exit" from the boundary. The supplementary conditions we found cover the most general case.)

In terms of probability theory the problem may be stated as follows. The heat conduction equation, together with the boundary condition $\mathfrak{A}$, prescribes a Brownian motion process in the domain E with reflection from the boundary in the domain E. The behavior of the trajectories after hitting an exclusive point of the boundary is not determined here. The problem is to describe all possible kinds of such behavior.

It is more convenient to pose and solve the problem in the terminology of semigroups of linear operators. Let $\mathcal{E}$ be some set and $\mathfrak{B}$ some σ-algebra of subsets of $\mathcal{E}$. Let $B = B(\mathcal{E})$ the space of all bounded $\mathfrak{B}$-measurable functions on $\mathcal{E}$ with the norm $\|f\| = \sup |f(z)|$. The family of linear operators T_t, $(t > 0)$, operating in the space B and satisfying the following conditions:

(1.1.A) $T_t f \geq 0$, if $f \geq 0$,
(1.1.B) $\|T_t f\| \leq \|f\|$,
(1.1.C) $T_s T_t = T_{s+t}$ for any $s, t > 0$,

is called a *Markov semigroup* in the space $\mathcal{E}$.

The semigroup T_t in the space E^* defined by the formula $T_tf(x) = u_t(x)$ corresponds to the boundary value problem described above for the heat conduction equation. (The σ-algebra of all Borel sets is always considered as the basic σ-algebra $\mathcal{B}$ in the space E^*.) Let $\mathfrak{A}$ be some linear operator defined in the subset $\mathfrak{D}_{\mathfrak{A}}$ of the space B. The Markov semigroup T_t is called an $\mathfrak{A}$-semigroup if the following conditions are satisfied.

(1.1.D) The infinitesimal generator A of the semigroup T_t is a contraction of the operator $\mathfrak{A}$.

(1.1.E) The set B_0 of all elements $f \in B$ for which $\lim_{t\to 0} \|T_tf - f\| = 0$, is everywhere dense in B in the sense of convergence w. We say that $f_n \xrightarrow{w} f$, if $f_n(x) \to f(x)$ for all $x \in \mathcal{E}$ and the sequence of norms $\|f_n\|$ is bounded.

Let us now define the operator $\mathfrak{A}$ as follows. Let $\mathfrak{D}$ be the set of all functions from $B(E^*)$ having Hölder-continuous first partial derivatives in E^* and Hölder-continuous second partial derivatives in E, and satisfying the boundary condition $\mathfrak{a}$. Let us consider the Laplace operator Δ in the domain $\mathfrak{D}$. It will be proved that a minimum w-closed extension exists for this operator. We denote this extension also by $\mathfrak{A}$. Our purpose is to describe all $\mathfrak{A}$-semigroups.

1.2. Let us move along the contour L passing the exclusive point γ in the direction of the vector $v(\gamma)$ and at the same time, observing the projection of the vector $v(z)$ on the inner normal to the contour L at the point z. Let us put $\gamma \in \Gamma_+$, if this projection changes sign from plus to minus, and $\gamma \in \Gamma_-$, if the sign changes from minus to plus. Let us set $\Gamma = \Gamma_+ \cup \Gamma_-$ (this is the set of all exclusive points).

It is expedient to "split" each point $\gamma \in \Gamma$ into two points γ^+ and γ^-. The union of all such pairs is denoted by Π. The decomposition of Π into Π_+ and Π_- corresponds to the decomposition of Γ into Γ_+ and Γ_-. If F is a function in E^*, then $F(\gamma^+)$, $F(\gamma^-)$ are its limits when z tends to γ along the contour L from the positive and negative sides, respectively. It is proved that if $F \in \mathfrak{D}_{\mathfrak{A}}$, then the limiting values $F(\gamma^+)$, $F(\gamma^-)$ exist for all $\gamma \in \Gamma$. To each $\alpha \in \Pi_-$ there corresponds just one bounded harmonic function $p_\alpha(z)$ satisfying the boundary condition $\mathfrak{a}$ and such that $p_\alpha(\alpha) = 1$ and $p_\alpha(\beta) = 0$ for $\beta \in \Pi_+$, $\beta \neq \alpha$. (If, say, $\alpha = \gamma^+$, then $p_\alpha(z)$ is the probability that the trajectory issuing from z will approach γ having touched L on the positive side of γ.)

1.3. Let us suppose the following are given.

(1) The partition of the set Π_+ into classes. The set of these classes is denoted by Ω.

(2) For each $\omega \in \Omega$ there is a set of nonnegative constants c_ω, σ_ω, $b_{\omega,\gamma}$, ($\gamma \in \Gamma_+$), and a measure ν_ω in the space $\mathcal{E} = E^* \cup \Pi_- \cup \Omega$.

Let E^*_ϵ denote the set of all points z of the set E^* for which $\rho(z, \Gamma_+) \geq \epsilon$ (the distance between the point z and the set M is denoted by $\rho(z, M)$); $\chi_{\gamma,\epsilon}$ is a function equal to $|z - \gamma|^2$ for $|z - \gamma| < \epsilon$ and zero for $|z - \gamma| \geq \epsilon$. Let $t = \rho(z, L)$. Let us put $\omega \in \Omega'$ if $\sigma_\omega = 0$, $b_{\omega,\gamma} = 0$ for all $\gamma \in \Gamma_+$ and $\nu_\omega(\mathcal{E}) < \infty$.

Let us assume that for every $\omega \in \Omega$ the following conditions are satisfied:

(1.3.A) $\nu_\omega(E^*_\epsilon) < \infty$ if $\epsilon > 0$,

(1.3.B) $(t, \nu_\omega) < \infty$ where $t = t(z) = \rho(z, L)$,
and the integral of the function f in measure ν (in the whole space $\mathcal{E}$) is denoted by (f, ν).

(1.3.C) For any $\gamma \in \Gamma_+$ and any sufficiently small $\epsilon > 0$, and

$$\int_{|z-\gamma|<\epsilon} |z - \gamma|^2 \nu_\omega(dz) < \infty, \tag{1.1}$$

(1.3.D) $(p_\alpha, \nu_\omega) < \infty$ for $\alpha \notin \omega$;

(1.3.E) $\nu_\omega(\omega) = 0$;

(1.3.F) $b_{\omega,\gamma} = 0$ if at least one of the points γ^+, γ^- does not belong to ω;

(1.3.G) $\nu_\omega(\Omega') = 0$ for $\omega \in \Omega'$;

(1.3.H) at least one of the numbers $\nu_\omega(\mathcal{E})$, $b_{\omega,\gamma}$, $(\gamma \in \Gamma_+)$, c_ω, σ_ω is positive.

To each set $\mathfrak{U} = \{c_\omega, \sigma_\omega, b_{\omega,\gamma}, \nu_\omega\}$ let us associate the manifold $\mathfrak{J}(\mathfrak{U})$ of all functions F defined on E^* and satisfying the following conditions:

(1.3.a) $F \in \mathfrak{D}_{\mathfrak{A}}$;

(1.3.b) to each $\omega \in \Omega$ there corresponds a number (we call it $F(\omega)$) such that $F(\alpha) = F(\omega)$ for all $\alpha \in \omega$;

(1.3.c) if $\sigma_\omega > 0$, there exists an analogously defined value $\mathfrak{A}F(\omega)$;

(1.3.d) if $b_{\omega,\gamma} > 0$, there exists

$$\frac{\partial F}{\partial n}(\gamma) = \lim_{t \downarrow 0} \frac{F[(1-t)\gamma] - F(\gamma)}{t}; \tag{1.2}$$

(1.3.e) for each $\omega \in \Omega$ the function $F - F(\omega)$ is summable in ν_ω-measure;

(1.3.f) for each $\omega \in \Omega$,

$$(F - F(\omega), \nu_\omega) + \sum_{\gamma \in \Gamma_+} b_{\omega,\gamma} \frac{\partial F}{\partial n}(\gamma) - c_\omega F(\omega) - \sigma_\omega \mathfrak{A}F(\omega) = 0. \tag{1.3}$$

We say that the $\mathfrak{A}$-semigroup T_t satisfies the lateral condition $\mathfrak{U}$, if $\mathfrak{D}_{\mathfrak{A}} \subseteq \mathfrak{J}(\mathfrak{U})$. The lateral condition $\mathfrak{U}$ is called a special one if $\sigma_\omega = 0$ for all $\omega \in \Omega$.

1.4. The fundamental results of this paper are formulated in theorems 1.1–1.3. (These results have been published without proof in [3].)

THEOREM 1.1. *Every $\mathfrak{A}$-semigroup satisfies some special lateral condition $\mathfrak{U}$. The arbitrary special lateral condition $\mathfrak{U}$ uniquely determines some $\mathfrak{A}$-semigroup.*

Theorem 1.1 solves the problem posed in section 1.1. However, the natural question arises of what is the sense of the conditions $\mathfrak{U}$ when some σ_ω are positive? In this case it is necessary to extend the phase space E^* by appending to it all points ω for which $\sigma_\omega > 0$. Let $\tilde{\Omega}$ denote the manifold of all such points, and let us put $\mathcal{E} = E^* \cup \tilde{\Omega}$. For any function F in the space $\mathcal{E}$ let F_0 denote its contraction in the space E^*. Let us set $F \in \tilde{B} = B(\mathcal{E})$ if $F_0 \in B(E^*)$. Let us define the operator $\mathfrak{A}$ in the space $\tilde{B}$ by the formulas:

$$\tilde{\mathfrak{A}}F(z) = \mathfrak{A}F_0(z) \qquad \text{for} \quad z \in E^*,$$
$$\tilde{\mathfrak{A}}F(\omega) = \frac{1}{\sigma_\omega}\left\{(F - F(\omega), \nu_\omega) + \sum_{\gamma \in \Gamma_+} b_{\omega,\gamma} \frac{\partial F}{\partial n}(\gamma) - c_\omega F(\omega)\right\} \text{for} \quad \omega \in \tilde{\Omega}. \tag{1.4}$$

The manifold $\mathfrak{D}_{\tilde{\mathfrak{A}}}$ of all functions $F \in \tilde{B}$ for which $F_0 \in \mathfrak{D}_{\mathfrak{A}}$ and the right side of (1.4) has meaning, is the domain of $\tilde{\mathfrak{A}}$. Let $\tilde{\mathfrak{J}}(\mathfrak{U})$ denote the set of all functions

$F \in \tilde{B}$ satisfying conditions (1.3.a), (1.3.b), (1.3.d), (1.3.e), and (1.3.f). It is necessary to replace $\mathfrak{A}$ by $\tilde{\mathfrak{A}}$ in (1.3.a) and (1.3.f) and $F(\omega)$ in (1.3.b) must be understood, for $\omega \in \tilde{\Omega}$, to be the value of the function F at the point ω. (Let us note that for $\omega \in \tilde{\Omega}$ condition (1.3.f) is automatically satisfied by virtue of the definition of the operator $\tilde{\mathfrak{A}}$.)

THEOREM 1.2. *To an arbitrary set* $\mathfrak{U} = \{c_\omega, \sigma_\omega, b_{\omega,\gamma}, \nu_\omega\}$ *satisfying the requirements* (1.3.A)–(1.3.H) *corresponds a uniquely defined Markov semigroup in the space* $\mathcal{E}$ *for which the infinitesimal generator* A *is a contraction of* $\tilde{\mathfrak{A}}$, *and* $\mathfrak{D}_A \subseteq \mathfrak{Z}(\mathfrak{U})$.

Let $w(z)$ be a function conformally mapping the domain E into the unit circle. Let P denote the manifold of all functions of the kind

$$F(z) = \sum_{\gamma \in \Gamma} k_\gamma \arg\left(1 - \frac{w(z)}{w(\gamma)}\right) + F_0(z) \tag{1.5}$$

where k_γ are constants, and $F_0(z)$ is a function continuous in the closed domain $E \cup L$. Each function $F \in P$ is extended naturally to the manifold $E^* \cup \Pi$. Let P_Ω denote the set of all functions $F \in P$, for which the value $F(\alpha)$ is constant in each class ω from Ω.

THEOREM 1.3. *For the Markov semigroup described in theorem* 1.2, *the space* $\tilde{B}_0 = B_0(\mathcal{E})$ *consists of all functions* $F \in P_\Omega$ *satisfying the conditions*

$$(F - F(\omega), \nu_\omega) - c_\omega F(\omega) = 0 \qquad \text{for all} \quad \omega \in \Omega'. \tag{1.6}$$

The domain of the infinitesimal generator consists of all functions $F \in \mathfrak{Z}(\mathfrak{U})$ *for which* $\mathfrak{A}F \in B_0(\mathcal{E})$.

1.5. Let us clarify the assumptions which have been made relative to the contour L and the vector field $v(z)$. It is assumed that the contour L is given by the equation $z = z(t)$, where the function $z(t)$ is differentiable and its derivative $z'(t)$ vanishes nowhere and is Hölder-continuous. It is furthermore assumed that the function $v[z(t)]$ has a Hölder-continuous derivative with respect to t. The function F satisfies the boundary condition $\mathfrak{a}$ if for any nonexclusive point z_0 of the contour L a neighborhood is found in which the partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$ exist and are Hölder-continuous, and

$$v_1(z_0) \frac{\partial F}{\partial x}(z_0) + v_2(z_0) \frac{\partial F}{\partial y}(z_0) = 0 \tag{1.7}$$

(v_1 and v_2 denote the coordinates of the vector v).

Under the assumptions we have made, a function $w(z)$ may be constructed which has a Hölder-continuous derivative in the closed domain $E \cup L$ and maps this domain conformally onto the unit circle $\{z\colon |z| \leq 1\}$ (see [5], p. 468, for instance). Hence, without any loss of generality, it may be assumed that $E = \{z\colon |z| < 1\}$, $L = \{z\colon |z| = 1\}$. (See [2] for details.)

2. Minimum principle. Green's operator

2.1. The minimum principle for the boundary value problem $\mathfrak{a}$ is formulated as follows.

THEOREM 2.1. *Let us assume that the function F is bounded from below, satisfies the boundary condition $\mathfrak{A}$ and the following conditions:*

(2.1.A) *for all $z \in E$, $\lambda F(z) - \Delta F(z) \geq 0$ where λ is a nonnegative constant;*

(2.1.B) $\liminf_{z \to \gamma} F(z) \geq 0$ *for all $\gamma \in \Gamma_+$.*

Then the function F is nonnegative.

When $\lambda = 0$, and condition (2.1.A) is satisfied with the equality sign, this theorem was proved in ([2], section 5). Additions to this proof, required in the general case, are presented in appendix A. Among the lemmas on which the proof of theorem 2.1 relies, is one which is of independent interest for the sequel. It is the following.

LEMMA 2.1. *Let a function F, not a constant, be given in the domain E bounded by the smooth contour L, and let $\lambda F(z) - \Delta F(z) \geq 0$ for all $z \in E$ and for a nonnegative constant λ. Let $z_0 \in L$ and*

$$F(z_0) \leq F(z) \qquad \text{for all} \quad z \in E. \tag{2.1}$$

If F has a derivative in the direction of some vector v making an acute angle with the inner normal at the point z_0, then this derivative is positive.

(Here and henceforth, the value $F(z_0)$ of the function F at the boundary point z_0 is the limit of $F(z)$ when z tends to z_0 along the set E.)

2.2. In appendix B, a function $g(z, w) = g_w(z)$, ($w \in E$, $z \in E \cup L$, $w \neq z$) is constructed under the assumption that the set Γ_+ is nonempty such that for each $w \in E$:

(2.2.A) $g_w(z) = -(\frac{1}{2}\pi) \ln |z - w| + h_w(z)$, where $h_w(z)$ is a harmonic function in E;

(2.2.B) $g_w(z)$ satisfies the boundary condition $\mathfrak{A}$;

(2.2.C) $g_w(z)$ is bounded in a neighborhood of each point $\gamma \in \Gamma$;

(2.2.D) $g_w(\gamma) = 0$ for $\gamma \in \Gamma_+$.

From the minimum principle (theorem 2.1) it follows at once that the conditions (2.2.A)–(2.2.D) define the function $g_w(z)$ uniquely. We shall call it the Green function.

It is proved in appendix B that the function $g_w(z)$ is nonnegative and has the following property:

(2.2.E) the function $|z - w| g_w(z)$ is bounded in the domain $z \in E \cup L$, $w \in E$.

2.3. The functions $p_\gamma(z)$, ($\gamma \in \Gamma_+$, $z \in E$), defined by the conditions:

(2.3.A) $p_\gamma(z)$ is a harmonic function in E;

(2.3.B) p_γ satisfies the boundary condition $\mathfrak{A}$;

(2.3.C) p_γ is bounded;

(2.3.D) $p_\gamma(\gamma) = 1$; $p_\gamma(\beta) = 0$ for $\beta \in \Gamma_+$, $\beta \neq \gamma$,

play an important part in the construction of the Green function.

From the minimum principle it follows that these conditions define the function p_γ uniquely, and that $0 \leq p_\gamma(z) \leq 1$. See appendix B for the construction of the function p_γ. The same appendix also gives a proof of the following property:

(2.3.E) $p_\gamma(z) = \sum_{\beta \in \Gamma_-} a_\beta^\gamma \varphi_\beta(z) + H_\gamma(z)$

where $\varphi_\beta(z) = (1/\pi) \arg (1 - (z/\beta))$ (the value of the argument is taken between $-\pi/2$ and $\pi/2$); a_β^γ are constants, and the function $H_\gamma(z)$ has Hölder-continuous partial derivatives in the closed circle $\{|z| \leq 1\}$ and Hölder-continuous second derivatives outside Γ_-.

2.4. Let us consider the integral equation

$$F(z) + \lambda \iint_E g(z, w)F(w)\,dw = f(z), \qquad (f \in B = B(E^*)). \tag{2.2}$$

By virtue of (2.2.E), Fredholm theory is applicable to equation (2.2) (see [6], no. 563, say). Therefore, if for some λ equation (2.2) has no nonzero solutions for $f = 0$, then (2.2) has a single solution defined by the formula

$$F(z) = f(z) - \lambda \iint_E g_\lambda(z, w)f(w)\,dw \tag{2.3}$$

where $g_\lambda(z, w)$ is a function satisfying the equation

$$g_\lambda(z, u) - \lambda \iint_E g(z, w)g_\lambda(w, u)\,dw = g(z, u). \tag{2.4}$$

The function $g_\lambda(z, w)$ is called the *resolvent* of equation (2.2). It is found as the ratio $D(z, w; \lambda)/D(\lambda)$ of two power series in λ, which converge in the whole complex plane. The solution F defined by (2.3) is continuous in $E \cup L$ if f is continuous in $E \cup L$. It is seen from (2.4) that $g_0(z, u) = g(z, u)$.

The operator

$$G_\lambda f(z) = \iint_E g_\lambda(z, w)f(w)\,dw, \qquad (f \in B) \tag{2.5}$$

is called the Green operator. Let us put $G_0 = G$. Substituting F from (2.3) into (2.2), we arrive at the identity

$$G_\lambda f = G[f - \lambda G_\lambda f]. \tag{2.6}$$

2.5. Let us list a number of properties of the Green operator. Let $F = G_\lambda f$, $(f \in B)$. Then

(2.5.A) $F \in B$; if $f_n \overset{w}{\to} f$, then $\|G_\lambda f_n - F\| \to 0$;

(2.5.B) $F(\gamma) = 0$ for $\gamma \in \Gamma_+$;

(2.5.C) $F(z) = \tilde{F}(z) + \sum_{\gamma \in \Gamma_-} c_\gamma \varphi_\gamma(z)$,

where c_γ are constants, φ_γ is defined by (2.1), and the function $\tilde{F}$ has Hölder-continuous derivatives in the closed circle $\{|z| \leq 1\}$;

(2.5.D) F satisfies the boundary condition $\mathfrak{A}$;

(2.5.E) the partial derivatives of the function F are given by the formulas

$$\begin{aligned} \frac{\partial F}{\partial x} &= \iint_E \frac{\partial g_w}{\partial x} f(w)\,dw; \\ \frac{\partial F}{\partial y} &= \iint_E \frac{\partial g_w}{\partial y} f(w)\,dw, \qquad (w \in E;\ z = x + iy \in E \cup L \backslash \Gamma_-). \end{aligned} \tag{2.7}$$

(2.5.F) The derivative of F at the point $\gamma \in \Gamma_+$ in the direction of the inner normal to L is given by

$$\frac{\partial F}{\partial n}(\gamma) = \iint_E B_\gamma^\lambda(w) f(w)\, dw \tag{2.8}$$

where

$$B_\gamma^\lambda(w) = \lim_{t \downarrow 0} \frac{g_\lambda((1-t)\gamma, w)}{t}. \tag{2.9}$$

(2.5.G) If f is Hölder-continuous in E, then F is twice Hölder-continuously differentiable in E and $\Delta F(z) = \lambda F(z) - f(z)$, $(z \in E)$.

The validity of conditions (2.5.A)–(2.5.G) in the $\lambda = 0$ case is proved in appendix B. Their validity for every λ for which the operator G_λ is defined, results from relationships (2.4) and (2.6).

Let C^0 denote the set of all Hölder-continuous functions in the set E^*. From (2.5.C), (2.5.D), and (2.5.G) follows the imbedding

$$G_\lambda(C^0) \subseteq \mathfrak{D}. \tag{2.10}$$

Now, let us show that the operator G_λ is defined for all $\lambda \geq 0$. As has already been remarked in section 2.4, to do this it is sufficient to verify that (2.2) has only a trivial solution for $\lambda \geq 0$ and $f = 0$. For this verification we shall use properties (2.5.A)–(2.5.G) for $\lambda = 0$. If $F + \lambda GF = 0$, then according to (2.5.C), $F \in C^0$. By virtue of (2.10), (2.5.G), and (2.5.B), $F \in \mathfrak{D}$, $\lambda F - \Delta F = 0$ and $F = 0$ on Γ_+. By the minimum principle, $F = 0$.

2.6. We shall now prove several new properties of the Green operator.

(2.6.A) For every $\lambda \geq 0$, the general form of the function $F \in \mathfrak{D}$ is given by the formula

$$F = G_\lambda f + h_\lambda, \qquad (f \in C^0,\ h_\lambda \in \mathfrak{D},\ \lambda h_\lambda - \Delta h_\lambda = 0), \tag{2.11}$$

where $\Delta F = \lambda F - f$;

(2.6.B) $0 \leq G_\lambda f \leq Gf$ for $f \geq 0$;

(2.6.C) $\lambda G_\lambda 1 \leq 1$;

(2.6.D) $\|\lambda G_\lambda f\| \leq \|f\|$.

Proof of (2.6.A). Let $f \in C^0$. Then by virtue of (2.10), $G_\lambda f \in \mathfrak{D}$. Hence, $F \in \mathfrak{D}$. On the contrary, if $F \in \mathfrak{D}$, then $f = \lambda F - \Delta F \in C^0$. According to (2.10), $\tilde{F} = G_\lambda f \in \mathfrak{D}$; by virtue of (2.5.G), $\lambda\tilde{F} - \Delta\tilde{F} = f$. Hence, $h_\lambda = F - \tilde{F} \in \mathfrak{D}$ and $\lambda h_\lambda - \Delta h_\lambda = 0$.

Proof of (2.6.B). If $f \in C^0$, then according to (2.10) and (2.5.G), the function $F = G_\lambda f$ belongs to $\mathfrak{D}$ and satisfies the equation $\lambda F - \Delta F = f \geq 0$. By the minimum principle, $F \geq 0$. Furthermore, it follows from (2.6) that $G_\lambda f \leq Gf$, and (2.6.B) results from the validity of the inequality $0 \leq G_\lambda f \leq Gf$ for all $f \in C^0$.

Proof of (2.6.C). According to (2.5.G), the function $F = 1 - \lambda G_\lambda 1$ satisfies the equation $\Delta F = \lambda F$. Furthermore, $F(\alpha) = 1$ for $\alpha \in \Gamma_+$ and $F \in \mathfrak{D}$. By the minimum principle, $F \geq 0$.

Proof of (2.6.D). By virtue of (2.6.B), the inequality $-\|f\| \leq f \leq \|f\|$ implies that $-\|f\|\lambda G_\lambda 1 \leq \lambda G_\lambda f \leq \|f\|\lambda G_\lambda 1$, that is, $|\lambda G_\lambda f| \leq \|f\|\lambda G_\lambda 1$. Hence, (2.6.D) follows by virtue of (2.6.C).

3. Structure of functions of class D

3.1. We shall start from proposition (2.6.A), according to which the general form of the function $F \in \mathfrak{D}$ is given by (2.11) (for any $\lambda \geq 0$). Let us first study the class $\mathfrak{D}_\lambda$ of all functions $h_\lambda \in \mathfrak{D}$ satisfying the equation $\lambda h_\lambda - \Delta h_\lambda = 0$.

LEMMA 3.1. *The formula*

$$h = \lambda G h_\lambda + h_\lambda \tag{3.1}$$

establishes a one-to-one correspondence between $h_\lambda \in \mathfrak{D}_\lambda$ *and* $h \in \mathfrak{D}_0$.

PROOF. According to section 2.4, each $h \in B$ is uniquely represented in the form (3.1) in terms of some $h_\lambda \in B$. If $h \in \mathfrak{D}$, then $h_\lambda \in C^0$ by virtue of (2.5.C), and $Gh_\lambda \in \mathfrak{D}$ by virtue of (2.10). Therefore, $h_\lambda = h - \lambda G h_\lambda \in \mathfrak{D}$. On the other hand, if $h_\lambda \in \mathfrak{D}$, then $h \in \mathfrak{D}$, according to (2.10). By virtue of (2.5.G), $\Delta h = -\lambda h_\lambda + \Delta h_\lambda$. Hence, $h \in \mathfrak{D}_0$ if and only if $h_\lambda \in \mathfrak{D}_\lambda$.

The set $\mathfrak{D}_0$ is studied in ([2], section 8). Namely, it has been shown in [2] that

(3.1.A) to each $\alpha \in \Pi_+$ corresponds a function $p_\alpha \in \mathfrak{D}_0$ such that $p_\alpha(\alpha) = 1$; $p_\alpha(\beta) = 0$ for $\beta \neq \alpha$, $\beta \in \Pi_+$.

(The definition of the set Π_+ is given in section 1.2.)

(3.1.B) Every function $h \in \mathfrak{D}_0$ is represented uniquely as a linear combination of functions p_γ, $(\gamma \in \Pi_+)$. In particular, the functions p_γ, $(\gamma \in \Gamma_+)$ introduced in section 2.3, are expressed in terms of p_α, $(\alpha \in \Pi_+)$ by means of the formula $p_\gamma = p_{\gamma+} + p_{\gamma-}$.

3.2. Let p_α^λ denote the solution of the integral equation (3.1) for $h = p_\alpha$. Let us list some properties of the function p_α^λ:

(3.2.A) $p_\alpha^\lambda(\alpha) = 1$; $p_\alpha^\lambda(\beta) = 0$ for $\beta \neq \alpha$, $\beta \in \Pi_+$;

(3.2.B) $\lambda p_\alpha^\lambda(z) - \Delta p_\alpha^\lambda(z) = 0$, $(z \in E)$;

(3.2.C) every function $h_\lambda \in \mathfrak{D}_\lambda$ is represented uniquely as a linear combination of the functions p_α^λ, $(\alpha \in \Pi_+)$;

(3.2.D) for every $\alpha \in \Pi_+$,

$$p_\alpha^\lambda = -\varphi_\alpha(z) + \sum_{\gamma \in \Gamma_-} A_{\alpha,\gamma}^\lambda \varphi_\gamma(z) + h_\gamma^\lambda(z) \tag{3.2}$$

where

$$\varphi_\alpha = \begin{cases} \varphi_\gamma & \text{for} \quad \alpha = \gamma^+, \\ -\varphi_\gamma & \text{for} \quad \alpha = \gamma^- \end{cases} \tag{3.3}$$

$A_{\alpha,\gamma}^\lambda$ are constants, and h_γ^λ are functions with Hölder-continuous first and second derivatives in the closed circle $E \cup L$.

For $\lambda = 0$, the statements (3.2.A)–(3.2.C) are valid according to section 3.1, and statement (3.2.D) is proved in [2] (see section 8.2). The case of arbitrary $\lambda \geq 0$ is reduced to the case $\lambda = 0$ with the aid of relationship (3.1).

LEMMA 3.2. *Every function* $F \in \mathfrak{D}$ *is represented as*

$$F = -G(\Delta F) + \sum_{\alpha \in \Pi} F(\alpha) p_\alpha. \tag{3.4}$$

If $F(\gamma^+) = \dot{F}(\gamma^-)$ *at some point* $\gamma \in \Gamma_+$, *then the function* F *is continuously differentiable in the neighborhood of this point.*

PROOF. According to (2.6.A), $F = Gf + h$, where $f = -\Delta F$ and $h \in \mathfrak{D}_0$. By virtue of (3.1.A)–(3.1.B), $h = \sum h(\alpha)p_\alpha$. There remains to note that $F(\alpha) = h(\alpha)$, because $Gf = 0$ on Π_+, and that the functions $G(\Delta F)$, $p\gamma = p_{\gamma+} + p_{\gamma-}$ and p_α (for α different from γ^+ and γ^-) are continuously differentiable in the neighborhood of $\gamma \in \Gamma_+$ (see (2.3.E), (2.5.C), and (3.2.D)).

3.3. Comparing (3.2.D) and lemma 3.2, we remark that each function $F \in \mathfrak{D}$ is represented as

$$F = \sum_{\gamma \in \Gamma} k_\gamma \varphi_\gamma + \tilde{F}, \tag{3.5}$$

where k_γ is a constant, and the function $\tilde{F}$ may be extended continuously on the domain $E \cup L$. Let P denote the class of all functions F in the set E^* which are representable as (3.5). Each function F has a natural extension to the set $E \cup L$. We shall denote the extended functions with the same letters as the originals.

The set P is a Banach space relative to the norm $\|F\| = \sup_{z \in E^*} |F(z)|$. The following lemma describes the general form of the linear functionals in this space.

LEMMA 3.3. *The arbitrary linear functional* ℓ *in the space* P *is written as*

$$\ell(F) = (F, \mu) \tag{3.6}$$

where μ *is a finite signed measure in the space* $E^* \cup \Pi$ *and* (F, μ) *is the integral of the function* F *with respect to the measure* μ. *If the functional* ℓ *is nonnegative, the measure* μ *is also nonnegative.*

PROOF. The functional ℓ induces a linear functional in the space $C(E \cup L)$, contained in P, of all continuous functions in the closed circle $E \cup L$. Hence, there exists a measure ν on $E \cup L$ such that for all $F \in C(E \cup L)$,

$$\ell(F) = \int_{E \cup L} F(z)\nu(dz). \tag{3.7}$$

If $F \in P$, then

$$F_1 = F - \sum_{\gamma \in \Gamma} \{F(\gamma - 0) - F(\gamma + 0)\}\varphi_\gamma \in C(E \cup L). \tag{3.8}$$

Hence,

$$\ell(F) = \int_{E^*} F(z)\nu(dz) + \sum_{\alpha \in \Pi} b_\alpha F(\alpha) \tag{3.9}$$

where

$$b_\alpha = \begin{cases} -\ell(\varphi_\gamma) + (\varphi_\gamma, \nu) + \frac{1}{2}\nu(\gamma) & \text{for } \alpha = \gamma^+, \\ \ell(\varphi_\gamma) - (\varphi_\gamma, \nu) + \frac{1}{2}\nu(\gamma) & \text{for } \alpha = \gamma^-. \end{cases} \tag{3.10}$$

Defining the measure μ by means of $\mu(M) = \nu(M \cap E^*) + \sum_{\alpha \in \Pi \cap M} b_\alpha$, one can rewrite (3.9) as (3.6).

If the functional ℓ is nonnegative, then the measure ν is also nonnegative. Let us prove that $b_\alpha \geq 0$. Let $\alpha = \gamma^+$ or $\alpha = \gamma^-$. Let us consider a continuous function $F_n(z)$ which is bounded by zero and one and is equal to $p_\alpha(z)$ for $|z - \gamma| \leq 1/n$, and equal to zero for $|z - \gamma| \geq 2/n$. From the relationship $\ell(F_n) \to b_\alpha$ there follows that $b_\alpha \geq 0$.

3.4. Let us now investigate the structure of the class $\mathfrak{D}$ near the set Γ_+ in more detail. Let us fix some point $\gamma \in \Gamma_+$. The vector $v(\gamma)$ is tangent to the contour L at the point γ. Without restricting the generality, it may be considered that its direction agrees with the positive direction of the contour L (otherwise, the field v could be multiplied by -1). Every point z sufficiently close to γ is represented uniquely in the form

$$z = \gamma e^{is}(1 - t), \qquad (|s| < \pi). \tag{3.11}$$

Hence, the numbers s, t may be considered as local coordinates in the neighborhood of γ. Evidently $t = 0$ for $z \in L$ and $t > 0$ for $z \in E$. We shall denote a point with coordinates s, t by $z(s, t)$. Let us put $z_s = z(s, 0) = \gamma e^{is}$. Let $\theta(s)$ denote an angle which the vector $v(z_s)$ forms with the positive direction of the contour L at the point s. Let us note that $\theta(0) = 0$ and θ changes sign from plus to minus at the point 0. Hence, $\kappa = -\theta'(0) \geq 0$. It is easy to see that

$$\frac{\partial F}{\partial v}(z_s) = \frac{\partial F}{\partial s}(z_s) \cos \theta(s) + \frac{\partial F}{\partial t}(z_s) \sin \theta(s). \tag{3.12}$$

If $F \in \mathfrak{D}$, then $(\partial F/\partial v)(z_s) = 0$ for $z_s \notin \Gamma$, and therefore

$$\frac{\partial F}{\partial s}(z_s) = -\tan \theta(s) \frac{\partial F}{\partial t}(z_s), \qquad (z_s \notin \Gamma). \tag{3.13}$$

Let $\gamma \in \Gamma_+$. If $F(\gamma^+) = F(\gamma^-)$, then by virtue of lemma 3.2, the equality (3.13) holds even for $s = 0$, and we shall have

$$\frac{\partial F}{\partial s}(\gamma) = 0. \tag{3.14}$$

Let us note that

$$\frac{\partial F}{\partial t}(\gamma) = \frac{\partial F}{\partial n}(\gamma). \tag{3.15}$$

If the function F is *twice continuously differentiable* in the neighborhood of γ, then differentiating (3.13) with respect to s we have

$$\frac{\partial^2 F}{\gamma s^2}(\gamma) = \kappa \frac{\partial F}{\partial n}(\gamma). \tag{3.16}$$

By the Taylor formula

$$F(z) = F(\gamma) + \frac{\partial F}{\partial s}(\gamma)s + \frac{\partial F}{\partial t}(\gamma)t + \frac{1}{2}\left[\frac{\partial^2 F}{\partial s^2}(\gamma)s^2 + 2\frac{\partial^2 F}{\partial s\, \partial t}(\gamma)st + \frac{\partial^2 F}{\partial t^2}(\gamma)t^2\right] + O(s^2 + t^2). \tag{3.17}$$

Taking into account (3.14)–(3.16), we have

$$F(z) = F(\gamma) + \frac{\partial F}{\partial n}(\gamma)\,[t + \tfrac{1}{2}\kappa s^2] + O(s^2 + t). \tag{3.18}$$

THEOREM 3.1. *If the function* $F \in \mathfrak{D}$ *is continuous at the point* $\gamma \in \Gamma_+$, *then the asymptotic formula* (3.18) *is valid as* $z \to \gamma$.

PROOF. According to lemma 3.2, the function F is continuously differentiable in the neighborhood of γ. Let us first assume that $(\partial F/\partial n)(\gamma) = 0$. By the theorem of Lagrange,

$$F(z) - F(\gamma) = F(s, t) - F(0, 0) = F(s, 0) - F(0, 0) + F'_t(s, \hat{t})t, \quad (0 \le \hat{t} \le t). \tag{3.19}$$

For $z \to \gamma$,

$$F'_t(s, \hat{t}) \to F'_t(0, 0) = \frac{\partial F}{\partial n}(\gamma) = 0. \tag{3.20}$$

Furthermore,

$$F(s, 0) - F(0, 0) = F'_s(\hat{s}, 0)s \tag{3.21}$$

where $\hat{s}$ lies between 0 and s. According to (3.13),

$$F'_s(s, 0) = -\tan\theta(s) F'_t(s, 0). \tag{3.22}$$

For $s \to 0$, $F'_t(s, 0) \to F'_t(0, 0) = 0$, and therefore,

$$F'_s(s, 0) = o[\tan\theta(s)] = o[\theta(s)] = o(s). \tag{3.23}$$

By virtue of (3.21) and (3.23), $F(s, 0) - F(0, 0) = o(s^2)$ and (3.18) results from (3.19) and (3.20).

Let us now note that the function p_γ belongs to $\mathfrak{D}$ and is twice continuously differentiable in the neighborhood of γ (see (2.10) and (2.3.E)). Hence, (3.18) is valid for p_γ. Since $\Delta p_\gamma = 0$, then by lemma 2.1, $(\partial p_\gamma/\partial n)(\gamma) = c > 0$.

Finally, let F be an arbitrary function from $\mathfrak{D}$ continuous at γ. Then the function

$$F_1(z) = F(z) - \frac{1}{c}\frac{\partial F}{\partial n}(\gamma)\, p_\gamma(z) \tag{3.24}$$

satisfies the condition $(\partial F_1/\partial n)(\gamma) = 0$. According to the above, (3.18) is satisfied for F_1. Since it is also satisfied for p_γ, it is then satisfied also for F.

3.5. The following theorem holds.

THEOREM 3.2. *Let* $\gamma \in \Gamma_+$ *and* $\alpha = \gamma^-$ *or* $\alpha = \gamma^+$. *Then*

$$\lim_{z \to \gamma} \frac{p_\alpha(z)}{t + \frac{1}{2}\kappa s^2} = +\infty. \tag{3.25}$$

PROOF 1. For definiteness, let $\alpha = \gamma^-$. According to (3.2.D),

$$p_\alpha(z) = \varphi(z) + h(z) \tag{3.26}$$

where h is twice continuously differentiable in the neighborhood of γ and

$$\varphi(z) = \varphi_\gamma(z) = \frac{1}{\pi}\arg\left(1 - \frac{z}{\gamma}\right) = -\frac{1}{\pi}\arctan\frac{(1 - t)\sin s}{1 - (1 - t)\cos s}. \tag{3.27}$$

Let us note that

$$(3.28)\qquad \varphi(z_s) = \frac{1}{\pi}\arg(1 - e^{is}) = \begin{cases} \dfrac{s}{2\pi} - \frac{1}{2} & \text{for } s > 0, \\ \dfrac{s}{2\pi} + \frac{1}{2} & \text{for } s < 0. \end{cases}$$

For $s \downarrow 0$, $p_\alpha(z_s) \to p_\alpha(\gamma^+) = 0$; $\varphi(z_s) \to -\frac{1}{2}$. Hence,

$$(3.29)\qquad h(\gamma) = \tfrac{1}{2}.$$

Furthermore, let us note that

$$(3.30)\qquad \begin{cases} \varphi_s'(z_s) = \dfrac{1}{2\pi}; \\ \varphi_t'(z) = \dfrac{\sin^2[\pi\varphi(z)]}{\pi(1-t)^2 \sin s}; \qquad \varphi_t'(z_s) = \dfrac{\sin^2\left[\dfrac{s}{2} - \dfrac{\pi}{2}\right]}{\pi \sin s}. \end{cases}$$

For $s \neq 0$,

$$(3.31)\qquad 0 = \frac{\partial p_\alpha}{\partial \nu}(z_s) = \frac{\partial p_\alpha}{\partial s}(z_s)\cos\theta(s) + \frac{\partial p_\alpha}{\partial t}(z_s)\sin\theta(s).$$

By virtue of (3.21) and (3.27),

$$(3.32)\qquad \frac{\partial p_\alpha}{\partial s}(z_s) = \frac{1}{2\pi} + h_s'(z_s); \qquad \frac{\partial p_\alpha}{\partial t}(z_s) = \frac{\sin^2\left[\dfrac{s}{2} - \dfrac{\pi}{2}\right]}{\pi \sin s} + h_t'(z_s).$$

Substituting these values in (3.31), and letting $s \to 0$, we have $0 = (1/2\pi) + h_s'(\gamma) - (\kappa/\pi)$. This means

$$(3.33)\qquad h_s'(\gamma) = \frac{\kappa}{\pi} - \frac{1}{2\pi}.$$

PROOF 2. If the statement of the theorem is false, a sequence $z_n = z(s_n, t_n) \to \gamma$ will be found for which

$$(3.34)\qquad p_\alpha(z_n) \le c(t_n + \tfrac{1}{2}\kappa s_n^2)$$

where c is some constant. Hence, $p_\alpha(z_n) \to 0$, and by virtue of (3.26) and (3.29), $\varphi(z_n) \to -\frac{1}{2}$. Hence, it follows that $(s_n/t_n) \to +\infty$. This means $(t_n/s_n) \to 0$ and $s_n > 0$ starting with some n.

By the Lagrange theorem

$$(3.35)\qquad p_\alpha(s, t) = p_\alpha(s, 0) + \frac{\partial p_\alpha(s, \hat{t})}{\partial t}\, t$$

where $0 \le \hat{t} \le t$. By virtue of (3.26), (3.28), (3.29), and (3.33),

$$(3.36)\qquad p_\alpha(s, 0) = \frac{s}{2\pi} - \tfrac{1}{2} + h(s, 0) = h(s, 0) - h(0, 0) - h_s'(0, 0)s + \frac{\kappa}{\pi} = \frac{\kappa}{\pi}s + O(s^2).$$

Hence, it follows from (3.34) and (3.35) that

$$c(t_n + \tfrac{1}{2}\kappa s_n^2) \geq p_\alpha(s_n, t_n) \geq p_\alpha(s_n, 0) = \frac{\kappa s_n}{\pi} + O(s_n^2). \tag{3.37}$$

Dividing both sides by $s_n > 0$, and taking into account that $(t_n/s_n) \to 0$, we have $0 \geq \kappa/\pi$. This means $\kappa = 0$. But for $\kappa = 0$, we have from (3.34) and (3.35) that

$$c \geq \frac{p_\alpha(z_n)}{t_n} \geq \frac{\partial p_\alpha(s_n, \hat{t}_n)}{\partial t}. \tag{3.38}$$

However, it is seen from (3.30) that $\varphi_t'(s_n, \hat{t}_n) \to +\infty$. At the same time, $h_t'(s_n, \hat{t}_n) \to h_t'(0, 0) < \infty$. Hence, the relation (3.38) may not be satisfied. The obtained contradiction proves the theorem.

3.6. We shall now examine the constant κ and the local coordinate system introduced in section 3.4, for different points $\gamma \in \Gamma_+$ simultaneously. The point γ will hence be given in the form of a subscript (let us note that t_γ does not actually depend on γ, and coordinates s_γ differ only by a constant factor). In $E \cup L$, for each $\gamma \in \Gamma_+$ let us construct a continuous nonnegative function τ_γ which coincides with $t + \frac{1}{2}\kappa_\gamma s_\gamma^2$ in the neighborhood of γ and is equal to zero in the neighborhood of all the rest of the points of the set Γ. Furthermore, let us construct a function η, continuous in $E \cup L$, coinciding with $t + s_\gamma^2$ in some neighborhood of γ for any $\gamma \in \Gamma_+$, positive everywhere in $E \cup L \backslash \Gamma_+$ and equal to 1 in some neighborhood of Γ_-.

Let $\hat{P}$ denote the set of all functions $F \in P$ which vanish on the set Γ_+, and let us put $F \in P_1$ if

$$F = \sum_{\gamma \in \Gamma_+} k_\gamma \tau_\gamma + \eta F_1 \tag{3.39}$$

where k_γ are constants and $F_1 \in \hat{P}$. It is easy to verify that every function $F \in P_1$ has a normal derivative at the points $\gamma \in \Gamma_+$ and

$$k_\gamma = \frac{\partial F}{\partial n}(\gamma), \qquad (\gamma \in \Gamma_+). \tag{3.40}$$

The imbedding

$$\mathfrak{D} \cap \hat{P} \subseteq P_1 \tag{3.41}$$

results from theorem 3.1. Furthermore, evidently $P_1 \subseteq \hat{P} \subseteq P$.

Let us introduce a norm into P_1 by putting $\|F\| = \max_{z \in E^*} |F(z)/\eta(z)|$. Every linear functional in the space P induces some linear functional in P_1. In fact, if $F \in P_1$, then

$$\ell(F) \leq k \max |F(z)| \leq k \max |\eta(z)| \|F\| \tag{3.42}$$

(here the maximum is taken in $E \cup L$; k is a positive constant).

3.7. Linear functionals on P_1 can be characterized as follows.

LEMMA 3.4. *An arbitrary linear nonnegative functional ℓ in the space P_1 has the form*

$$\ell(F) = (F, \mu) + \sum_{\gamma \in \Gamma_+} b_\gamma \frac{\partial F}{\partial n}(\gamma) \tag{3.43}$$

where b_γ are nonnegative constants, μ is a measure on $E^* \cup \Pi$ such that $(\eta, \mu) < \infty$ and $\mu(\Pi_+) = 0$.

PROOF. The formula $\hat{\ell}(F_1) = \ell(\eta F_1)$, $(F_1 \in \hat{P})$ defines a linear functional in the space $\hat{P}$. This means that there is a finite measure ν on the space $E^* \cup \Pi$ such that $\hat{\ell}(F_1) = (F_1, \nu)$, $(F_1 \in \hat{P})$. Without loss of generality, it may be considered that $\nu(\Pi_+) = 0$. Let us put $\mu(dz) = (1/\eta)\nu(dz)$. Then $(\eta, \mu) < \infty$, $\mu(\Pi_+) = 0$, and $\ell(\eta F_1) = \hat{\ell}(F_1) = (F_1, \nu) = (\eta F_1, \mu)$. If $F \in P_1$, then by virtue of (3.39) and (3.40),

$$F - \sum_{\gamma \in \Gamma_+} \frac{\partial F}{\partial n}(\gamma)\tau_\gamma = \eta F_1, \qquad (F_1 \in \hat{P}). \tag{3.44}$$

Hence,

$$\ell\left\{F - \sum \frac{\partial F}{\partial n}(\gamma)\tau_\gamma\right\} = \ell(\eta F_1) = (\eta F_1, \mu). \tag{3.45}$$

Putting $b_\gamma = \ell(\tau_\gamma) - (\tau_\gamma, \mu)$, we have (3.41). The proof of the nonnegativity of b_γ is carried out exactly as in section 3.3.

4. The operator $\mathfrak{A}$. Resolvents of $\mathfrak{A}$-semigroups

4.1. Let us put $F \in \mathfrak{D}_{\mathfrak{A}_\lambda}$ and $\mathfrak{A}_\lambda F = \lambda F - f$ if

$$F = G_\lambda f + h, \qquad \text{where } f \in B,\ h \in \mathfrak{D}_\lambda \tag{4.1}$$

(let us recall that according to section 3.1, $\mathfrak{D}_\lambda$ denotes the set of all $h \in \mathfrak{D}$, for which $\lambda h - \Delta h = 0$).

THEOREM 4.1. *For every $\lambda \geq 0$ the operator $\mathfrak{A}_\lambda$ is a w-closure of the operator Δ, defined in the domain $\mathfrak{D}$.*

To prove this theorem, we shall rely upon the following lemma.

LEMMA 4.1. *If $h_n \in \mathfrak{D}_\lambda$ and $h_n \xrightarrow{w} h$, then $h \in \mathfrak{D}_\lambda$.*

For $\lambda = 0$, this lemma was proved in [2] (see section 4.7). The $\lambda > 0$ case reduces to the $\lambda = 0$ case by the use of lemma 3.1 and (2.5.A).

PROOF OF THEOREM 4.1. According to (2.6.A), the operator $\mathfrak{A}_\lambda$ is an extension of the operator Δ considered in the domain $\mathfrak{D}$.

Let us prove that the operator $\mathfrak{A}_\lambda$ is w-closed. In fact, if $F_n \in \mathfrak{D}_{\mathfrak{A}_\lambda}$, $F_n \xrightarrow{w} F$, $\mathfrak{A}_\lambda F_n \xrightarrow{w} \varphi$, then $F_n = G_\lambda f_n + h_n$, $(f_n \in B,\ h_n \in \mathfrak{D}_\lambda)$ where $f_n = \lambda F_n - \mathfrak{A}_\lambda F_n$. It is clear that $f_n \xrightarrow{w} \lambda F - \varphi$. According to (2.5.A), $G_\lambda f_n \xrightarrow{w} G_\lambda(\lambda F - \varphi)$. Therefore, $h_n \xrightarrow{w} h = F + G_\lambda(\varphi - \lambda F)$. By lemma 4.1, $h \in \mathfrak{D}_\lambda$, and by definition of $\mathfrak{A}_\lambda$, $F = G_\lambda(\lambda F - \varphi) + h \in \mathfrak{D}_{\mathfrak{A}_\lambda}$, and $\mathfrak{A}_\lambda F = \varphi$.

Finally, let us consider an arbitrary w-closed extension $\mathfrak{A}'$ of the operator Δ. Let us put $f \in Q$ if $F = G_\lambda f + h \in \mathfrak{D}_{\mathfrak{A}'} \cap \mathfrak{D}_{\mathfrak{A}_\lambda}$ for any $h \in \mathfrak{D}_\lambda$, and if $\mathfrak{A}_\lambda F = \mathfrak{A}'F = \lambda F - f$. By virtue of (2.10) and (2.5.G), $C^0 \subseteq Q$. Furthermore, let $f_n \in Q$ and $f_n \xrightarrow{w} f$. Then $F_n = G_\lambda f_n + h \xrightarrow{w} G_\lambda f + h = F$, $\mathfrak{A}'F_n = \mathfrak{A}_\lambda F_n = \lambda F_n - f_n \xrightarrow{w} \lambda F - f$, and by virtue of the closedness of $\mathfrak{A}'$ and $\mathfrak{A}_\lambda$, $F \in \mathfrak{D}_{\mathfrak{A}'} \cap \mathfrak{D}_{\mathfrak{A}_\lambda}$, and $\mathfrak{A}'F = \mathfrak{A}_\lambda F = \lambda F - f$. This means Q is w-closed. Since the w-closure C^0 coincides with B, then $Q \supseteq B$ and $\mathfrak{A}' \supseteq \mathfrak{A}_\lambda$.

It results from theorem 4.1 that: (a) the w-closure $\mathfrak{A}$ of the operator Δ defined in the domain $\mathfrak{D}$ has been determined; (b) $\mathfrak{A}_\lambda = \mathfrak{A}$ for any $\lambda \geq 0$.

4.2. The operator

$$R_\lambda f(z) = \int_0^\infty e^{-\lambda t} T_t f(z)\, dt \tag{4.2}$$

is called the resolvent of the semigroup T_t. This operator has the following properties:

(4.2.A) if $f \geq 0$, then $R_\lambda f \geq 0$;

(4.2.B) $\|R_\lambda f\| \leq \frac{1}{\lambda} \|f\|$;

(4.2.C) for every $\lambda > 0$, R_λ maps B_0 in a one-to-one way on $\mathfrak{D}_A$. The inverse mapping is determined by the operator $\lambda \mathcal{I} - A$ (where $\mathcal{I}$ denotes the identity operator);

(4.2.D) if $f_n \xrightarrow{w} f$, then $R_\lambda f_n \xrightarrow{w} R_\lambda f$;

(4.2.E) $R_\lambda(B) \subseteq B_0$.

The properties (4.2.A), (4.2.B), and (4.2.D) are obvious. The property (4.2.C) has been proved in [1] (see section 1.4). The property (4.2.E) is verified by a simple computation.

LEMMA 4.2. *If R_λ is the resolvent of some $\mathfrak{A}$-semigroup, then $F = R_\lambda f \in \mathfrak{D}_\mathfrak{A}$ and $\lambda F - \mathfrak{A}F = f$ for every $f \in B$.*

PROOF. Let $\mathcal{K}$ denote the set of all functions f for which the statement of the lemma is satisfied. Let $f_n \in \mathcal{K}$, $f_n \xrightarrow{w} f$. According to (4.2.D), $F_n = R_\lambda f_n \xrightarrow{w} R_\lambda f = F$. We have $\mathfrak{A}F_n = \lambda F_n - f_n \xrightarrow{w} \lambda F - f$. Since the operator $\mathfrak{A}$ is w-closed, then $F \in \mathfrak{D}_\mathfrak{A}$ and $\mathfrak{A}F = \lambda F - f$. Therefore, the set $\mathcal{K}$ is w-closed. According to (4.2.C), $\mathcal{K} \supseteq B_0$ and by virtue of (4.2.D), $\mathcal{K} \supseteq B$.

4.3. Let R_λ be the resolvent of some $\mathfrak{A}$-semigroup. According to lemma 4.2, for any $f \in B$, $R_\lambda f \in \mathfrak{D}_\mathfrak{A}$, and by virtue of section 4.1, $R_\lambda f = G_\lambda f + h$ where $h \in \mathfrak{D}_\lambda$. By virtue of (3.2.C), this formula may be rewritten as

$$R_\lambda f = G_\lambda f + \sum_{\alpha \in \Pi_+} Q_\alpha^\lambda p_\alpha^\lambda \tag{4.3}$$

(Q_α^λ are constants dependent on f). From (4.3), (2.5.C), and (3.2.D) there follows that the function $R_\lambda f$ belongs to the space P described in section 3.3 for every $f \in B$. Taking into account (4.2.C), we have

$$\mathfrak{D}_A \subseteq R_\lambda(B_0) \subseteq R_\lambda(B) \subseteq P. \tag{4.4}$$

As is known (see [2], (1.3.B) say) the set $\mathfrak{D}_\alpha$ is everywhere dense in B_0 in the sense of convergence in the norm. Hence $B_0 \subseteq P$ and

$$\mathfrak{D}_A \subseteq R_\lambda(P). \tag{4.5}$$

By virtue of (3.2.A), there results from (4.3) that

$$Q_\alpha^\lambda(f) = R_\lambda f(\alpha), \qquad (\alpha \in \Pi_+). \tag{4.6}$$

4.4. Let $\alpha, \beta \in \Pi_+$. Let us put $\alpha \sim \beta$, if $F(\alpha) = F(\beta)$ for all $F \in \mathfrak{D}_A$. Hence, the validity of the equality $F(\alpha) = F(\beta)$ for all $F \in B_0$ follows. Let us note that

$\alpha \sim \beta$ if for some $\lambda > 0$, $Q^{\lambda}_{\alpha}(f) = Q^{\lambda}_{\beta}(f)$ for all $f \in B$. In fact, according to (4.6), the equality $R_{\lambda}f(\alpha) = R_{\lambda}f(\beta)$ results from the equality $Q^{\lambda}_{\alpha}(f) = Q^{\lambda}_{\beta}(f)$, and, therefore (see (4.4)), so does the equality $F(\alpha) = F(\beta)$ for all $F \in \mathfrak{D}_A$.

Let Ω denote the set obtained from Π_+ by identification of equivalent points. The elements of the set Ω (that is, the classes of equivalent points of the set Π_+) will be denoted by the letters ω, ζ, ξ. Let us put

$$p^{\lambda}_{\omega} = \sum_{\alpha \in \omega} p^{\lambda}_{\alpha}; \qquad Q^{\lambda}_{\omega} = Q^{\lambda}_{\alpha}, \ (\alpha \in \omega). \tag{4.7}$$

Formulas (4.3) and (4.6) may be rewritten as

$$R_{\lambda}f = G_{\lambda}f + \sum_{\omega \in \Omega} Q^{\lambda}_{\omega} p^{\lambda}_{\omega}, \tag{4.8}$$

$$Q^{\lambda}_{\omega}(f) = R_{\lambda}f(\omega), \qquad (\omega \in \Omega). \tag{4.9}$$

Let P_{Ω} denote the set of all functions $F \in P$ for which $F(\alpha) = F(\beta)$ for $\alpha \sim \beta$. There results from (4.4) and (4.8) that

$$\mathfrak{D}_A \subseteq R_{\lambda}(B) \subseteq P_{\Omega}. \tag{4.10}$$

Let us prove that

$$1 - \lambda G_{\lambda} 1 - \sum_{\alpha \in \Pi_+} p^{\lambda}_{\alpha} = 0. \tag{4.11}$$

Let us denote by u the function in the left side of (4.11). By virtue of (2.5.D) and (2.5.G) this function satisfies the boundary condition A and the equation $\lambda u - \Delta u = 0$. By virtue of (3.2.A), $u(\alpha) = 0$ for all $\alpha \in \Gamma_+$. By the minimum principle (see theorem 2.1), $u \geq 0$ and $-u \geq 0$; therefore $u = 0$.

4.5. The resolvents satisfy the following lemma.

LEMMA 4.3. *In order that the operator* R_{λ} *defined by* (4.8) *satisfy condition* (4.2.A), *it is necessary and sufficient that the functionals* $Q^{\lambda}_{\omega} (\omega \in \Omega)$ *satisfy the condition*

(4.5.A) $Q^{\lambda}_{\omega}(f) \geq 0$ *for* $f \geq 0$.

Under these circumstances, condition (4.2.B) *is equivalent to the condition*

(4.5.B) $\lambda Q^{\lambda}_{\omega}(1) \leq 1$.

PROOF. Since $G_{\lambda}f \geq 0$ for $f \geq 0$, the equivalence of (4.2.A) and (4.5.A) follows from (4.8) and (4.9). Furthermore, it is easy to see that under condition (4.2.A) the condition (4.2.B) is equivalent to the inequality

$$\lambda R_{\lambda} 1 \leq 1. \tag{4.12}$$

According to (4.9), the value of the function $\lambda R_{\lambda} 1$ at the point ω is $\lambda Q^{\lambda}_{\omega}(1)$. Hence, (4.12) implies (4.5.B). On the other hand, if (4.5.B) is satisfied, then, by virtue of (4.8) and (4.11),

$$\lambda R_{\lambda} 1 = \lambda G_{\lambda} 1 + \lambda \sum_{\omega} Q^{\lambda}_{\omega}(1) p^{\lambda}_{\omega} \leq 1. \tag{4.13}$$

4.6. Let us show, in conclusion, that the infinitesimal operator $\mathfrak{A}$ is the closure of the Laplace operator Δ if the latter is considered on a suitable class of functions.

LEMMA 4.4. *The strong closure of the operator* Δ *considered on the set* $\mathfrak{D}_{\mathfrak{A}} \cap \mathfrak{D}$ *coincides with* A.

PROOF. If $F \in \mathfrak{D}_A \cap \mathfrak{D}$, then $AF = \mathfrak{A}F = \Delta F$. The operator A is closed. Hence, it is sufficient to prove that for each $F \in \mathfrak{D}_A$ there is a sequence $F_n \in \mathfrak{D}_A \cap \mathfrak{D}$ such that $\|F_n - F\| \to 0$ and $\|AF_n - AF\| \to 0$. According to (4.2.C), $f = \lambda F - AF \in B_0$. Hence, there exist functions $f_n \in \mathfrak{D}_A$ such that $\|f_n - f\| \to 0$. According to (4.2.C), $F_n = R_\lambda f_n \in \mathfrak{D}_A$ and $AF_n = \lambda F_n - f_n$. By virtue of (4.2.B), $\|F_n - F\| \to 0$. This means that

$$(4.14) \qquad \|AF_n - AF\| = \|\lambda(F_n - F) + f_n - f\| \to 0.$$

By virtue of (4.4), $f_n \in R_\lambda(B)$, and from (4.8) and (2.5.C), it follows that $f_n \in C^0$. According to (2.10), $G_\lambda(C^0) \subseteq \mathfrak{D}$. Hence, $F_n = R_\lambda f_n \in \mathfrak{D}$.

5. Lateral conditions for smooth functions

5.1. It will be shown herein that for any $\mathfrak{A}$-semigroup T_t a set $\mathfrak{U}$ is found which satisfies conditions (1.3.A)–(1.3.H) and such that $\mathfrak{D}_A \cap \mathfrak{D} \subseteq \mathfrak{Z}(\mathfrak{U})$. (The set $\mathfrak{Z}(\mathfrak{U})$ has been defined in section 1.3.)

LEMMA 5.1. *Every function* $F \in \mathfrak{D}_A \cap \mathfrak{D}$ *satisfies for every* $\lambda > 0$ *the following conditions:*

$$(5.1) \qquad F(\omega) = \ell_\omega^\lambda\left(F - \frac{1}{\lambda}\Delta F\right), \qquad (\omega \in \Omega)$$

where ℓ_ω^λ *is a linear nonnegative functional on the space* P_Ω *such that* $\ell_\omega^\lambda(1) \leq 1$.

PROOF. According to (4.8), every function $F \in \mathfrak{D}_A$ is representable as $F = R_\lambda f$, where $f \in P$. By virtue of (4.12), $F(\omega) = Q_\omega^\lambda(f)$. But $f = \lambda F - \Delta F$, and hence, $F(\omega) = \lambda Q_\omega^\lambda[F - (1/\lambda)\Delta F]$ so that relation (5.1) is satisfied for the functional $\ell_\omega^\lambda = \lambda Q_\omega^\lambda$. The properties of this functional mentioned in the formulation of the lemma follow from lemma 4.2.

REMARK. According to lemma 3.3, an arbitrary nonnegative linear functional ℓ on the space P is defined by (3.6) in terms of some finite measure μ on the space $E^* \cup \Pi$. It follows that every nonnegative linear functional on the space P_Ω is described by the same formula in terms of some measure μ on the space $\mathcal{E} = E^* \cup \Pi_- \cup \Omega$.

5.2. The space P_Ω is separable. Hence (see, for example, [7], section 24), a convergent subsequence may be selected from every sequence of linear functionals which is bounded in norm. It is easy to see that the norms of all the functionals ℓ_ω^λ do not exceed 1. Therefore, one can find linear functionals ℓ_ω and a sequence $\lambda_n \to \infty$ such that $\ell_\omega^{\lambda_n}(f) \to \ell_\omega(f)$ for every $f \in P_\Omega$ and any $\omega \in \Omega$. For $\lambda \to \infty$, $|\ell_\omega^\lambda((1/\lambda)\Delta F)| \leq \|(1/\lambda)\Delta F| \to 0$. Hence, from equality (5.1) we obtain in the limit

$$(5.2) \qquad F(\omega) = \ell_\omega(F).$$

According to the remark at the end of section 5.1,

$$(5.3) \qquad \ell_\omega(F) = (F, \mu_\omega)$$

where μ_ω is a finite measure on the space $\mathcal{E}$. We have

$$(1, \mu_\omega) = \ell_\omega(1) \leq 1. \tag{5.4}$$

From (5.2) and (5.3) we have

$$F(\omega) = (F, \mu_\omega). \tag{5.5}$$

5.3. Let us put $\omega \in \Omega_1$ if μ_ω is a unit measure concentrated at the point ω, and let $\Omega_0 = \Omega \backslash \Omega_1$. For $\omega \in \Omega_1$, equation (5.5) becomes an identity which all the functions F satisfy. In this case, another passage to the limit is necessary.

Let us note that for $\omega \in \Omega_1$,

$$\lim \ell_\omega^\lambda(f) = \ell_\omega(f) = (f, \mu_\omega) = f(\omega), \qquad (f \in P) \tag{5.6}$$

(the limit is taken over some sequence of values of λ which tend to $+\infty$). Let us put

$$\begin{cases} F_0 = F - \sum_{\zeta \in \Omega} F(\zeta) p_\zeta, \\ \tilde{F} = F_0 + \sum_{\zeta \in \Omega} \Delta F(\zeta) G p_\zeta. \end{cases} \tag{5.7}$$

Evidently,

$$\Delta \tilde{F} = \Delta F - \sum_{\zeta \in \Omega} \Delta F(\zeta) p_\zeta. \tag{5.8}$$

From (5.1), (5.7), and (5.8), we have

$$\begin{aligned} \ell_\omega^\lambda(F_0) + \sum_{\zeta \in \Omega} \ell_\omega^\lambda(p_\zeta)[F(\zeta) - F(\omega)] - [1 - \ell_\omega^\lambda(1)]F(\omega) - \frac{1}{\lambda} \ell_\omega^\lambda(\Delta \tilde{F}) \\ - \frac{1}{\lambda} \sum_{\zeta \in \Omega} \ell_\omega^\lambda(p_\zeta)[\Delta F(\zeta) - \Delta F(\omega)] - \frac{\ell_\omega(1)}{\lambda} \Delta F(\omega) = 0. \end{aligned} \tag{5.9}$$

When $\lambda \to +\infty$ along the sequence selected earlier, then according to (5.6),

$$\ell_\omega^\lambda(F_0) \to 0, \qquad \ell_\omega^\lambda(\Delta \tilde{F}) \to 0, \qquad \ell_\omega^\lambda(p_\zeta) \to 0, \qquad \text{for} \quad \zeta \neq \omega, \ell_\omega^\lambda(1) \to 1. \tag{5.10}$$

The function F_0 belongs to the space $\hat{P}$ defined in 3.6. By virtue of (3.41), $F_0 \in P_1$. According to the remark at the end of section 3.6, the functional l_ω^λ induces some linear functional on the space P_1. Let n_ω^λ denote the norm of this induced functional. Let us put $\Omega_\omega = \Omega \backslash \{\omega\}$,

$$\delta_\omega^\lambda = n_\omega^\lambda + \sum_{\zeta \in \Omega} \ell_\omega^\lambda(p_\zeta) + \frac{1}{\lambda} + 1 - \ell_\omega^\lambda(1). \tag{5.11}$$

For any $f \in P_1$, $|\ell_\omega^\lambda(f)| \leq n_\omega^\lambda \|f\|_{P_1}$. The space P_1 is separable. Hence, linear functionals ℓ_ω may be constructed in P_1, and a sequence of values λ may be selected which converges to $+\infty$ such that $\ell_\omega^\lambda(f) \to \ell_\omega(f)$ for all $f \in P_1$. Passing to a subsequence, if necessary, one can satisfy relations (5.6) and (5.10), and at the same time insure the existence of the limits

$$\lim \frac{\ell_\omega^\lambda(p_\zeta)}{\delta_\omega^\lambda} = q_{\omega,\zeta}(\zeta \neq \omega); \qquad \lim \frac{1 - \ell_\omega^\lambda(1)}{\delta_\omega^\lambda} = c_\omega; \qquad \lim \frac{1}{\lambda \delta_\omega^\lambda} = \sigma_\omega. \tag{5.12}$$

Passing to the limit in (5.9), we have

$$\ell_\omega(F_0) + \sum_{\zeta\in\Omega} q_{\omega,\zeta}[F(\zeta) - F(\omega)] - c_\omega F(\omega) - \sigma_\omega \Delta F(\omega) = 0. \tag{5.13}$$

Here ℓ_ω are nonnegative functionals in P_1, $q_{\omega,\zeta}$, c_ω, σ_ω are nonnegative constants, and $\|\ell_\omega\| + \sum_{\zeta\in\Omega_\omega} q_{\omega,\zeta} + c_\omega + \sigma_\omega = 1$.

LEMMA 5.2. *The functional ℓ_ω in* (5.13) *has the form*

$$\ell_\omega(f) = (f, \tilde{\nu}_\omega) + \sum_{\gamma\in\Gamma_+} b_{\omega,\gamma} \frac{\partial f}{\partial n}(\gamma) \tag{5.14}$$

where $\tilde{\nu}_\omega$ is the measure in $E^* \cup \Pi_-$ such that $(\eta, \tilde{\nu}_\omega) < \infty$; $b_{\omega,\gamma}$ are nonnegative constants satisfying condition (1.3.F); *if $b_{\omega,\gamma} > 0$, then all the functions $p_\zeta^\lambda(z)$, ($\zeta \in \Omega$, $\lambda \geq 0$) are continuously differentiable in the neighborhood of the point γ and for $\zeta \neq \omega$,*

$$\tilde{q}_{\omega,\zeta} = q_{\omega,\zeta} - (p_\zeta, \tilde{\nu}_\omega) - \sum_{\gamma\in\Gamma_+} b_{\omega,\gamma} \frac{\partial p_\zeta(\gamma)}{\partial n} \geq 0. \tag{5.15}$$

PROOF 1. The representation (5.14) of the functional ℓ_ω results from lemma 3.4. Let us show that (1.3.F) is satisfied. Let α be that one of the two points γ^+, γ^- which does not belong to ω, and let ζ be a class from Ω, containing α. By virtue of (5.12) there exists a constant c such that for all λ of the sequence under consideration

$$\ell_\omega^\lambda(p_\zeta) \leq c\delta_\omega^\lambda. \tag{5.16}$$

According to theorem 3.2, for every $N > 0$ there exists $\epsilon > 0$ such that

$$p_\alpha(z) > N\tau_\gamma(z) \qquad \text{for} \quad |z - \gamma| \leq \epsilon. \tag{5.17}$$

Let us consider a function $\psi(z)$ given in $E \cup L$ which satisfies the inequalities $0 \leq \psi \leq 1$ everywhere, is zero for $|z - \gamma| \geq 2\epsilon$, and one for $|z - \gamma| \leq \epsilon$. Evidently for all $z \in E \cup L$, $p_\zeta(z) \geq p_\alpha(z) \geq N_{\tau_\gamma}(z)\psi(z)$, and hence

$$\ell_\omega^\lambda(p_\zeta) \geq N\ell_\omega^\lambda(\tau_\gamma\psi). \tag{5.18}$$

Let us note that $\tau_\gamma\psi \in P_1$. Hence

$$\lim \frac{\ell_\omega^\lambda(\tau_\gamma\psi)}{\delta_\omega^\lambda} = \ell_\omega(\tau_\gamma\psi) = (\tau_\gamma\psi, \tilde{\nu}_\omega) + b_{\omega,\gamma}. \tag{5.19}$$

From (5.16), (5.18), and (5.19), we have $b_{\omega,\gamma} \leq c/N$, and $b_{\omega,\gamma} = 0$ because of the arbitrariness of N.

PROOF 2. Now, let $\zeta \in \Omega$ and $b_{\omega,\gamma} > 0$. According to (4.6), $p_\zeta^\lambda(z)$ is represented as the sum of functions $p_\alpha^\lambda(z)$, ($\alpha \in \zeta$). Since ζ either does not contain any of the points γ^-, γ^+ or contains both, and since the functions p_α^λ, ($\alpha \in \Pi_+$, $\alpha \neq \gamma^-$, $\alpha \neq \gamma^+$) and $p_\gamma = p_{\gamma-} + p_{\gamma+}$ are continuously differentiable in the neighborhood of γ; this is also true for the function ρ_ζ^λ.

Let us put $\gamma \in \Gamma_\omega$ if $b_{\omega,\gamma} > 0$, and let us consider the continuous function $f_n(z)$ in E^*, which equals $p_\zeta(z)$ for $\rho(z, \Gamma_\omega) \leq 1/n$, equals zero for $\rho(z, \Gamma_\omega) \geq 2/n$ and is everywhere between zero and one. The function f_n coincides with p_ζ near Γ_ω and equals zero near $\Gamma_+\backslash\Gamma_\omega$ (for sufficiently large n). Hence, $f_n \in P_1$ and

$$\frac{\ell_\omega^\lambda(f_n)}{\delta_\omega^\lambda} \to \ell_\omega(f_n) = (f_n, \tilde{\nu}_\omega) + \sum_\gamma b_{\omega,\gamma} \frac{\partial p_\zeta}{\partial n}(\gamma). \tag{5.20}$$

But $\ell_\omega^\lambda(p_\zeta) \geq \ell_\omega^\lambda(f_n)$. Therefore,

$$q_{\omega,\zeta} = \lim \frac{\ell_\omega^\lambda(p_\zeta)}{\delta_\omega^\lambda} \geq \ell_\omega(f_n) = (f_n, \tilde{\nu}_\omega) + \sum_\gamma b_{\omega,\gamma} \frac{\partial p_\zeta}{\partial n}(\gamma). \tag{5.21}$$

Since $f_n \to p_\zeta$ for $n \to \infty$, then (5.15) results from (5.21).

5.5. The expression (5.7) for the function F_0 may be rewritten as follows:

$$F_0 = F - F(\omega) - \sum_{\zeta \in \Omega} [F(\zeta) - F(\omega)] p_\zeta. \tag{5.22}$$

Substituting this expression into (5.13) and taking into account lemma 5.2, we have

$$\begin{aligned}(F - F(\omega), \tilde{\nu}_\omega) + \sum_{\gamma \in \Gamma_+} b_{\omega,\gamma} \frac{\partial F}{\partial n}(\gamma) + \sum_{\zeta \in \Omega} \tilde{q}_{\omega,\zeta}[F(\zeta) - F(\omega)] \\ - c_\omega F(\omega) - \sigma_\omega \Delta F(\omega) = 0.\end{aligned} \tag{5.23}$$

Let ν_ω denote the measure in the space $\mathcal{E} = E^* \cup \Pi_+ \cup \Omega$ which coincides with $\tilde{\nu}_\alpha$ in $E^* \cup \Pi_-$, equals $\tilde{q}_{\omega,\zeta}$ at the point $\zeta \in \Omega_\omega$, and equals zero at the point ω. Then the relation (5.23) may be rewritten as

$$(F - F(\omega), \tilde{\nu}_\omega) + \sum_{\gamma \in \Gamma_+} b_{\omega,\gamma} \frac{\partial F}{\partial n}(\gamma) - c_\omega F(\omega) - \sigma_\omega \Delta F(\omega) = 0. \tag{5.24}$$

THEOREM 5.1. *For any $\mathfrak{A}$-semigroup, there exists a set* $\mathfrak{U} = (c_\omega, \sigma_\omega, b_{\omega,\gamma}, \nu_\omega)$ *satisfying conditions* (1.3.A)–(1.3.H), *such that* $\mathfrak{D} \cap \mathfrak{D}_A \subset \mathfrak{J}(\mathfrak{U})$.

PROOF. For $\omega \in \Omega_1$ the set $c_\omega, \sigma_\omega, b_{\omega,\gamma}, \nu_\omega$ has been constructed in section 5.3. In the $\omega \in \Omega_0$ case we put

$$c_\omega = 1 - (1, \mu_\omega); \qquad b_{\omega,\gamma} = \sigma_\omega = 0; \qquad \nu_\omega(M) = \mu_\omega[M \cap \{\mathcal{E} \backslash \omega\}]. \tag{5.25}$$

Let $F \in \mathfrak{D}_A \cap \mathfrak{D}$; then $\mathfrak{A}F = \Delta F$. Evidently F satisfies conditions (1.3.a)–(1.3.f). Therefore, $F \in \mathfrak{J}(\mathfrak{U})$.

It is necessary to be convinced of the validity of properties (1.3.A)–(1.3.H). All these properties, except (1.3.G), are evident for $\omega \in \Omega_0$ and follow easily from lemma 5.2, for $\omega \in \Omega_1$. The condition (1.3.G) may not be satisfied, but we show that the system of relations constructed here can be replaced by an equivalent system satisfying all the conditions (1.3.A)–(1.3.H).

Let us note first that according to the definition of Ω (see section 4.4), for any two points $\omega_1 \neq \omega_2$ from Ω there exists a function $F \in \mathfrak{D}_A$ such that $F(\omega_1) \neq F(\omega_2)$. According to lemma 4.4, any function from $\mathfrak{D}_A$ may be approximated uniformly by functions from $\mathfrak{D}_A \cap \mathfrak{D}$. Hence $F(\omega_1) \neq F(\omega_2)$ for some function $F \in \mathfrak{D}_A \cap \mathfrak{D}$.

For $\omega \in \Omega'$ the relation (1.1) takes the form

$$(F - F(\omega), \nu_\omega) - c_\omega F(\omega) = 0, \tag{5.26}$$

or equivalently,

$$F(\omega) - \sum_{\omega \in \Omega'} p(\omega, \zeta) F(\zeta) = (F, \nu_\omega^*) \tag{5.27}$$

where

$$p(\omega, \zeta) = \frac{\nu_\omega(\zeta)}{c_\omega + \nu_\omega(\mathcal{E})}, \qquad \nu_\omega^*(M) = \frac{1}{c_\omega + \nu_\omega(\mathcal{E})} \nu_\omega\{M \cap (\mathcal{E}\backslash\Omega')\}. \tag{5.28}$$

Evidently $\sum_{\zeta \in \Omega'} p(\omega, \zeta) \leq 1$. Hence, lemma 1 of appendix C is applicable. Let us assume that the set K defined in this lemma is not empty. We know that if $\omega \in K$, then $p(\omega, \zeta) = 0$ for $\zeta \notin K$. This means $\nu_\omega(\Omega'\backslash K) = 0$ for $\omega \in K$. From the condition $\sum_{\zeta \in \Omega'} p(\omega, \zeta) = 1$ it follows that $c_\omega = \nu_\omega(\mathcal{E}\backslash\Omega') = 0$. This means $\nu_\omega(\mathcal{E}\backslash K) = 0$. From (5.27) we have

$$F(\omega) - \sum_{\zeta \in K} p(\omega, \zeta)F(\zeta) = 0, \qquad (\omega \in K). \tag{5.29}$$

Let $F_1, \cdots, F_r$ be a fundamental system of solutions of (5.29). Let us put $K_0 = K$ and let K_m be the set of points $\omega \in K_{m-1}$ at which F_m achieves its greatest value on K_{m-1}, $m = 1, 2, \cdots, r$. By induction we confirm that if $\omega \in K_m$ and $\zeta \notin K_m$, then $p(\omega, \zeta) = 0$. Since $p(\omega, \omega) = 0$, each set K_m consists of not less than two points. All the functions $F_1, \cdots, F_r$ are constants in the set K_r; therefore, all the solutions of (5.29) are constants in K_r. Since this contradicts the previous paragraph, the set K should be empty. According to lemma 1 of appendix C, the matrix $Q = (I - P)^{-1} = \sum_{n=0}^\infty P^n$ has nonnegative elements. Hence, the system (5.27) is equivalent to the system

$$(F - F(\omega), \tilde{\nu}_\omega) - \tilde{c}_\omega F(\omega) = 0 \tag{5.30}$$

where

$$\tilde{\nu}_\omega = \sum_{\zeta \in \Omega'} q(\omega, \zeta)\nu_\zeta^*, \qquad \tilde{c}_\omega = 1 - \tilde{\nu}(\mathcal{E}). \tag{5.31}$$

It is easy to verify that

$$\tilde{c}_\omega - \sum_{\zeta \in \Omega'} p(\omega, \zeta)\tilde{c}_\zeta = \frac{c_\omega}{c_\omega + \nu_\omega(\mathcal{E})} \geq 0. \tag{5.32}$$

Hence,

$$\tilde{c}_\omega = \sum_{\zeta \in \Omega'} q(\omega, \zeta) \frac{c_\zeta}{c_\zeta + \nu_\zeta(\mathcal{E})} \geq 0. \tag{5.33}$$

Replacing (5.26) by the equivalent relation (5.30) we obtain the lateral condition satisfying all the requirements (1.3.A)–(1.3.H).

6. Investigation of the class $\mathfrak{J}(\mathfrak{U})$

6.1. To each set $\mathfrak{U}$ satisfying the conditions (1.3.A)–(1.3.H) and each $\lambda > 0$ there corresponds a matrix $(a_{\omega,\zeta}^\lambda)$, $(\omega, \zeta \in \Omega)$ which is defined by the following formulas:

$$\begin{cases} a_{\omega,\zeta}^\lambda = -(p_\zeta^\lambda, \nu_\omega) - \sum_\gamma b_{\omega,\gamma} m_{\gamma,\zeta}^\lambda \quad \text{for} \quad \omega \neq \zeta, \\ a_{\omega,\omega}^\lambda = \sum_\gamma b_{\omega,\gamma}\lambda\{B_\gamma^\lambda, 1\} + (\lambda G_\lambda 1, \nu_\omega) + c_\omega + \lambda\sigma_\omega - \sum_{\zeta \in \Omega_\omega} a_{\omega,\zeta}. \end{cases} \tag{6.1}$$

Here B^λ_γ is defined by (2.9); the integral of the product f_1f_2 over the circle E is denoted by $\{f_1, f_2\}$, and

$$m^\lambda_{\zeta,\gamma} = \lim_{t \downarrow 0} \frac{p^\lambda_\omega[(1-t)\gamma]}{t} = \frac{\partial p^\lambda_\omega}{\partial n}(\gamma). \tag{6.2}$$

It is clear that B^λ_γ and $m^\lambda_{\zeta,\gamma}$ are nonnegative. From lemma 2.1 it follows that $m^\lambda_{\zeta,\gamma} > 0$. Indeed, p^λ_ζ satisfies the equation $\lambda f - \Delta f = 0$ and $p^\lambda_\zeta(z) \geq p^\lambda_\zeta(\gamma) = 0$ for all $z \in E$. From these remarks there results

$$a^\lambda_{\omega,\omega} > 0, \qquad a_{\omega,\zeta} \leq 0 \qquad \text{for} \quad \omega \neq \zeta, \tag{6.3}$$

$$\sum_{\zeta \in \Omega} a_{\omega,\zeta} \geq 0. \tag{6.4}$$

If the equality sign holds in (6.4), then

$$b_{\omega,\gamma} = \sigma_\omega = c_\omega = \nu_\omega(\mathcal{E}\backslash\Omega) = 0. \tag{6.5}$$

It follows from (6.5) that $\omega \in \Omega'$.

LEMMA 6.1. *For the matrix* $(a^\lambda_{\omega,\zeta})$ *there exists an inverse matrix* $(r^\lambda_{\omega,\zeta})$. *Here* $r^\lambda_{\omega,\zeta} \geq 0$.

PROOF. According to (6.3)–(6.4), lemma 2 of appendix C is applicable to the matrix $(a^\lambda_{\omega,\zeta})$. In order to prove lemma 6.1, it is sufficient to verify that the set K described in lemma 2 is empty. We know that if $\omega \in K$, the equality sign holds in (6.4) and $a_{\omega,\zeta} = 0$ for $\omega \in K$, $\zeta \notin K$. Hence, the equality $\nu_\omega(\Omega\backslash K) = 0$ follows, as does (6.5). It is clear that $K \subseteq \Omega'$. But according to (1.3.G), $\nu_\omega(\Omega') = 0$. Hence, $\nu_\omega(K) = 0$. However, $\nu_\omega(K) = \nu_\omega(\Omega\backslash K) = 0$ together with (6.5) contradict (1.3.H).

6.2. According to section 1.4, we put $\tilde{\mathcal{E}} = E^* \cup \tilde{\Omega}$, where $\tilde{\Omega}$ is the set of all $\omega \in \Omega$, for which $\sigma_\omega > 0$. We shall also use the notation $\tilde{B}$, $\tilde{\mathfrak{A}}$, $\mathfrak{Z}(\mathfrak{U})$ introduced in section 1.4. We shall write $f_n \xrightarrow{w} f$ if $f_n(z) \to f(z)$ for all $z \in \mathcal{E}$ and the sequence $\|f_n\|$ is bounded.

For each $f \in \tilde{B}$ we put

$$H^\lambda_\omega(f) = (G_\lambda f, \nu_\omega) + \sigma_\omega f(\omega) + \sum_\gamma b_{\omega,\gamma}\{B^\lambda_\gamma, f\}, \tag{6.6}$$

$$Q^\lambda_\omega(f) = \sum_{\zeta \in \Omega} r^\lambda_{\omega,\zeta} H^\lambda_\zeta(f), \tag{6.7}$$

where $r^\lambda_{\omega,\zeta}$ are defined in lemma 6.1. In the space $\tilde{B}$ let us consider the operators R_λ defined by the formula

$$R_\lambda f = G_\lambda f + \sum_{\omega \in \Omega} Q^\lambda_\omega(f) p^\lambda_\omega. \tag{6.8}$$

THEOREM 6.1. *For any* $\lambda > 0$ *the operator* R_λ *maps* $\tilde{B}$ *in a one-to-one way onto* $\mathfrak{Z}(\mathfrak{U})$. *The inverse mapping is given by the operator* $\lambda\mathfrak{Z} - \tilde{\mathfrak{A}}$.

PROOF. According to sections 4.1 and (3.2.C), the general form of the functions satisfying conditions (1.3.a)–(1.3.b) is given by

(6.9) $$F = G_\lambda f + \sum_{\omega \in \Omega} Q_\omega^\lambda p_\omega^\lambda.$$

All these functions automatically satisfy conditions (1.3.d) and (1.3.e). Let $F \in \mathfrak{J}(\mathfrak{U})$. According to the above, F has the form (6.9). According to sections 1.4 and 4.1,

(6.10) $$\lambda F(z) - \tilde{\mathfrak{A}}F(z) = \lambda F(z) - \mathfrak{A}F(z) = f(z) \qquad \text{for} \quad z \in E^*.$$

The values of $f(\omega)$ remain undetermined as yet for $\omega \in \tilde{\Omega}$. Let us put $f(\omega) = \lambda F(\omega) - \tilde{\mathfrak{A}}F(\omega)$. Let us recall that $\tilde{\mathfrak{A}}F(\omega)$ is defined by (1.4). Let us now note that the function F defined by (6.9) satisfies condition (1.3.f) if and only if the constants Q_ω^λ satisfy the system of equations

(6.11) $$\sum_{\zeta \in \Omega} a_{\omega,\zeta}^\lambda Q_\omega^\lambda = H_\omega^\lambda(f).$$

By virtue of lemma 6.1, (6.11) is equivalent to (6.7). Hence, the condition $F \in \mathfrak{J}(\mathfrak{U})$ is equivalent to the condition $F = R_\lambda f$, $(f \in \tilde{B})$. From the relation $(\lambda\mathfrak{J} - \tilde{\mathfrak{A}})R_\lambda f = f$ already proved, the remaining statements of the theorem result.

6.3. Condition (1.3) takes the form (1.6) for $\omega \in \Omega'$. We may rewrite it as

(6.12) $$F(\omega) = (F, \tilde{\nu}_\omega)$$

where $\tilde{\nu}_\omega = ((\nu_\omega)/(1, \nu_\omega) + c)$. Evidently, $(\tilde{\nu}_\omega, 1) \leq 1$.

Let $P(\mathfrak{U})$ denote the set of all functions $F \in P_\Omega$ satisfying the conditions (6.12) for all $\omega \in \Omega'$. Let us put $F \in \mathfrak{D}(\mathfrak{U})$ if $F \in \mathfrak{J}(\mathfrak{U})$, and $\mathfrak{A}F \in P(\mathfrak{U})$. It is clear that $\mathfrak{D}(\mathfrak{U}) \subseteq P(\mathfrak{U})$. There results from theorem 6.1 that for any $\lambda > 0$

(6.13) $$\mathfrak{D}(\mathfrak{U}) = R_\lambda[P(\mathfrak{U})].$$

Our purpose is to prove the following theorem.

THEOREM 6.2. *The set $\mathfrak{D}(\mathfrak{U})$ is everywhere dense in $P(\mathfrak{U})$ (in the sense of uniform convergence).*

Let us first prove some auxiliary propositions:

(6.3.A) $P(\mathfrak{U})$ is everywhere dense in $\tilde{B}$ (in the sense of w-convergence);

(6.3.B) if $f_n \xrightarrow{w} f$, then $\|R_\lambda f_n - R_\lambda f\| \to 0$;

(6.3.C) the strong closures of the sets $\mathfrak{D}(\mathfrak{U})$ and $R_\lambda(\tilde{B})$ coincide.

PROOF OF (6.3.A). Let φ^n be a continuous function in $E \cup L$ satisfying the inequalities $0 \leq \varphi^n \leq 1$, which equals 1 for $\rho(z, \Gamma_+) \leq (1/n)$ and zero for $\rho(z, \Gamma_+) \geq (2/n)$. Evidently, $\varphi_\zeta^n = \varphi_n p_\omega \in P_\Omega$. Let $f \in P_\Omega$. In order for the function

(6.14) $$f_n = f + \sum_{\zeta \in \Omega'} x_\zeta \varphi_\zeta^n$$

to belong to $P(\mathfrak{U})$, it is necessary and sufficient that the numbers x_ζ satisfy the system of equations

(6.15) $$x_\omega - \sum_{\zeta \in \Omega'} \Pi_{\omega,\zeta}^n x_\zeta = (f, \tilde{\nu}_\omega) - f(\omega), \qquad (\omega \in \Omega')$$

where $\Pi_{\omega,\zeta}^n = (\varphi^n, \tilde{\nu}_\omega)$. But $\Pi_{\omega,\zeta}^n \to 0$ (because $\varphi_\zeta^n(z) \to 0$ for $z \notin \Omega'$ and $\nu_\omega(\Omega') = 0$ by virtue of (1.3.G). According to lemma 1 of appendix C, the system (6.15) has a unique solution for sufficiently large n. Evidently it is bounded for $n \to \infty$, and according to (6.14), $f_n \xrightarrow{w} f$. Thus, $P(\mathfrak{U})$ is everywhere dense in P_Ω. But as is easy to see, the w-closure of P_Ω coincides with $\tilde{B}$. Therefore the w-closure of $P(\mathfrak{U})$ is also equal to $\tilde{B}$.

In order to prove (6.3.B), it is sufficient to compare (6.6)–(6.8) with (2.5.A). The statement of (6.3.C) results from (6.3.A) and (6.3.B).

6.4. Let Γ_c denote the set of all points of the contour L at which the vector field $v(z)$ is tangent to L. Evidently $\Gamma_c \supseteq \Gamma$.

LEMMA 6.2. *For any thrice continuously differentiable function $a(z)$ on the contour L which is zero in the neighborhood of Γ_c, there exists a function $A(z)$ coinciding with $a(z)$ on L.*

For each $\gamma \in \Gamma_+$ and any sufficiently small $\epsilon > 0$ a function $B_\epsilon(z)$ may be constructed which is continuously differentiable in $E \cup L$, equal to 1 for $\rho(z, \gamma) \leq \epsilon$, equal to zero for $\rho(z, \gamma) \geq 2\epsilon$, and satisfying the inequalities $0 \leq B \leq 1$ everywhere.

For any point γ from Γ_c a function $C_\gamma(z)$ may be constructed such that $C_\gamma(\gamma) = 1$ and $C_\gamma(z) = 0$ at all points of Γ_c except γ.

PROOF. Let $\theta(s)$ denote the angle between $v(e^{is})$ and the positive direction of L at the point e^{is}. On the segment $[0, 1]$ let us construct a twice continuously differentiable function $b(r)$ equal to 1 near 1, equal to zero near zero, and such that $0 \leq b(r) \leq 1$ for all r. A function $A(z)$ may be given by the formula

$$A(re^{is}) = a(e^{is}) - \frac{(1 - r)b(r)}{\tan \theta(s)} \frac{da(e^{is})}{ds}. \tag{6.16}$$

The functions B_ϵ and C_γ are obtained by means of the same formula. In order to obtain B_ϵ, it is possible to start from the function a, which equals 1 for $|z - \gamma| \leq \frac{1}{2}\epsilon$, equals zero for $|z - \gamma| \geq \epsilon$, and satisfies the inequality $0 \leq a \leq 1$ at all the rest of the points of the contour L. The function $b(r)$ must be selected so that it equals zero for $r < 1 - \frac{1}{2}\epsilon$. In order to determine C_γ, it is sufficient to construct the function $a(z)$ on the contour L so that it equals zero in the neighborhood of the set $\Gamma_c \backslash \{\gamma\}$ and satisfies the equality

$$a(e^{is}) = 1 + \int_{s_0}^{s} \tan \theta(s)\, ds \tag{6.17}$$

for $s_0 - \epsilon < s < s_0 + \epsilon$ (if $\gamma = \exp(is_0)$).

LEMMA 6.3. *If for all Hölder-continuous functions f*

$$\sum_{\gamma \in \Gamma_-} k_\gamma [Gf(\gamma^+) - Gf(\gamma^-)] = 0, \tag{6.18}$$

then all the constants k_γ are zero.

PROOF. Let $f_n(z)$ be Hölder-continuous functions in $E \cup L$ such that: $f_n(z) = 0$ for $|z - w| \geq (1/n)$, and $\{f_n, 1\} = 1$. Relying on the minimum principle, it is easy to show that the functions Gf_n converge to $g(z, w)$ uniformly in the neighborhood of Γ_-. Hence, from (6.18) there results

$$\sum_{\gamma \in \Gamma_-} m_\gamma [g(\gamma^+, w) - g(\gamma^-, w)] = 0, \qquad (w \in E). \tag{6.19}$$

To conclude, apply theorem 1 of appendix B.

PROOF OF THEOREM 6.2. By virtue of the Hahn-Banach theorem and (6.3.C), it is sufficient to prove that every linear functional ℓ on the space P_Ω which vanishes on $R_\lambda(\tilde{B})$, will vanish also on $P(\mathfrak{U})$. According to lemma 3.3 and the remark of section 5.1, $\ell(F) = (F, \xi)$, where ξ is a signed measure on the space $\mathcal{E}$. Thus, let

$$(6.20) \qquad (R_\lambda f, \xi) = 0 \qquad \text{for all } f \in \tilde{B}.$$

It is necessary to prove that $(F, \xi) = 0$ for all $F \in P(\mathfrak{U})$.

1. Let us put

$$(6.21) \qquad q_\omega = (p_\omega^\lambda, \xi); \qquad r_\zeta = \sum_{\omega \in \Omega} r_{\omega,\zeta}^\lambda q_\omega.$$

By virtue of (6.6)–(6.8), the relation (6.20) is equivalent to the relation

$$(6.22) \qquad (F, \nu) + \sum_{\zeta \in \Omega} r_\zeta \sigma_\zeta f(\zeta) + \sum_{\zeta \in \Omega, \gamma \in \Gamma_+} r_\zeta b_{\zeta,\gamma} \frac{\partial F}{\partial n}(\gamma) = 0$$

where $F = G_\lambda f$, $\nu = \xi + \sum_{\zeta \in \Omega} r_\zeta \nu_\zeta$. Let $\hat{\mathfrak{D}}$ denote the set of all functions $F \in \mathfrak{D}$, which equal zero on Γ_+. According to (2.6.A), every function F from $\hat{\mathfrak{D}}$ may be written in the form $G_\lambda f$, $(f \in B)$, where $f = \lambda F - \Delta F$. Hence, for any function $F \in \hat{\mathfrak{D}}$ the following corollary of equality (6.22) is satisfied:

$$(6.23) \qquad (F, \nu) + \sum_\zeta r_\zeta \sigma_\zeta [\lambda F(\zeta) - \Delta F(\zeta)] + \sum_{\zeta,\gamma} r_\zeta b_{\zeta,\gamma} \frac{\partial F}{\partial n}(\gamma) = 0.$$

2. Let us prove that $r_\omega = 0$ for all $\omega \in \Omega \backslash \Omega'$. Let $b_{\omega,\gamma} > 0$. Let us consider the function B_ϵ, constructed in lemma 6.2. It is easy to see that for sufficiently small $\epsilon > 0$ the function $F_\epsilon = B_\epsilon(1 - p_\omega)$ belongs to $\hat{\mathfrak{D}}$. For this function the relation (6.23) becomes

$$(6.24) \qquad (F_\epsilon, \nu) - r_\omega b_{\omega,\gamma} \frac{\partial p_\omega}{\partial n}(\gamma) = 0.$$

Since $(\partial p_\omega/\partial n)\gamma \neq 0$, and $(F_\epsilon, \nu) \to 0$ as $\epsilon \to 0$, then $r_\omega = 0$.

Analogously, considering the function $F_\epsilon = B_\epsilon G_\lambda 1$, we arrive at the relation $r_\zeta \sigma_\zeta = 0$. Therefore, $r_\zeta = 0$ if $\sigma_\zeta > 0$.

It is now seen from (6.23) that $(F, \nu) = 0$ for all $F \in \hat{\mathfrak{D}}$. Since $\hat{\mathfrak{D}}$ contains all smooth functions which equal zero near L, then ν is concentrated on $\mathcal{E} \backslash E$. Considering the functions $A(z)$ and $C_\gamma(z)$ constructed in lemma 6.2, we conclude that ν is concentrated on $\Omega \cup \Pi_-$, where $\nu(\gamma^+) + \nu(\gamma^-) = 0$ for all $\gamma \in \Gamma_-$. Hence, the validity of the conditions of lemma 6.3 results from the equality $(Gf, \nu) = 0$ (for $k_\gamma = \nu(\gamma^+)$). From lemma 6.3, it follows that $k_\gamma = 0$, $(\gamma \in \Gamma_-)$. This means the measure ν is concentrated on Ω.

Since the set Ω is finite, the measure ν is also finite. Therefore, $\infty > (p_\omega, \nu) = (p_\omega, \xi) + \sum_\zeta r_\zeta (p_\omega, \nu_\zeta)$. For $\omega \neq \zeta$, $(p_\omega, \nu_\zeta) < \infty$ (see (1.3.D)). Hence, if $r_\omega \neq 0$, then $(p_\omega, \nu_\omega) < \infty$ and therefore, the measure ν_ω is finite and $\omega \in \Omega'$.

3. Since the measure ν is concentrated on Ω, then $(F, \nu) = 0$ for any function F equal to zero on Ω, and therefore

$$(6.25) \qquad (F, \xi) = -\sum_\zeta r_\zeta (F, \nu_\zeta).$$

For any $F \in P_\Omega$ the function $\tilde{F} = F - \sum_\omega F(\omega)p_\omega^\lambda$ vanishes on Ω, and hence

$$(6.26) \qquad (\tilde{F}, \xi) = -\sum_\zeta r_\zeta(\tilde{F}, \nu_\zeta) = -\sum_\zeta r_\zeta(F, \nu_\zeta) + \sum_\omega F(\omega) \sum_\zeta r_\zeta(p_\omega^\lambda, \nu_\zeta).$$

From (6.21) it follows that $\sum_\omega a_{\omega,\zeta} r_\omega = q_\zeta$. Hence,

$$(6.27) \qquad (F, \xi) = (\tilde{F}, \xi) + \sum_\omega F(\omega) q_\omega = (\tilde{F}, \xi) + \sum_\omega F(\omega) \sum_\zeta a_{\zeta,\omega} r_\zeta.$$

From (6.26) and (6.27),

$$(6.28) \qquad (F, \xi) = \sum_\zeta r_\zeta\{-(F, \nu_\zeta) + \sum_\omega F(\omega)[(p_\omega^\lambda, \nu_\zeta) + a_{\zeta,\omega})]\}.$$

From (6.1) we remark that for $\zeta \in \Omega'$,

$$(6.29) \qquad a_{\zeta,\omega} = \begin{cases} -(p_\omega^\lambda, \zeta_\zeta) & \text{for } \zeta \neq \omega, \\ (1 - p_\omega^\lambda, \nu_\omega) + c_\omega & \text{for } \zeta = \omega. \end{cases}$$

Substituting (6.29) into (6.28) and taking into account that $r_\zeta = 0$ for $\zeta \in \Omega \backslash \Omega'$, we have

$$(6.30) \qquad (F, \xi) = -\sum_{\zeta \in \Omega'} r_\zeta\{(F - F(\zeta), \nu_\zeta) - c_\zeta F(\zeta)\}.$$

It is clear that this expression is zero for all $F \in P(\mathfrak{U})$.

7. Proof of the fundamental theorems

Theorems 7.1–7.2 refining theorems 1.1–1.3 formulated in section 1 will be proved in this section.

THEOREM 7.1. *Every* $\mathfrak{A}$*-semigroup satisfies some special lateral condition* $\mathfrak{U}$. *Its resolvent is determined by* (6.6)–(6.8). *The domain of the infinitesimal operator* A *is* $\mathfrak{D}(\mathfrak{U})$ *and* $B_0 = P(\mathfrak{U})$.

PROOF 1. Let some $\mathfrak{A}$-semigroup be given, and let R_λ' be its resolvent. Let us consider the set $\mathfrak{U} = \{c_\omega, \sigma_\omega, b_{\omega,\gamma}, \nu_\omega\}$ defined in theorem 5.1, and let R_λ denote the operator given by (6.6)–(6.8). The operator R_λ is defined in the space $\tilde{B} = B(\tilde{\mathcal{E}})$. Define the operator R_λ' also in $\tilde{B}$ by putting $R_\lambda' f = R_\lambda' f_0$, where f_0 is the restriction of f to E^*.

Let $f \in B_0$. Then $F = R_\lambda f \in \mathfrak{D}_A$ and $f = \lambda F - AF$. According to lemma 4.4, there exists a sequence $F_n \in \mathfrak{D}_A$ such that $\|F - F_n\| \to 0$ and $f_n = \lambda F_n - AF_n$ converges uniformly to f. According to theorem 6.1, $F_n = R_\lambda f_n$. Passing to the limit, we obtain that $F = R_\lambda f$. On the other hand, according to (4.2.C), $F = R_\lambda' f$. Therefore

$$(7.1) \qquad R_\lambda' f = R_\lambda f$$

for all $f \in B_0$. It follows that $\mathfrak{D}_A \subseteq \mathfrak{Z}(\mathfrak{U})$.

2. According to (4.10), $\mathfrak{D}_A \subseteq P_\Omega$. Therefore, $\mathfrak{D}_A \subseteq P_\Omega \cap \mathfrak{Z}(\mathfrak{U}) \subseteq P(\mathfrak{U})$. Since $\mathfrak{D}_\mathfrak{A}$ is everywhere dense in B_0 in the sense of uniform convergence, then $B_0 \subseteq P(\mathfrak{U})$. Relying on (4.2.C), (4.2.E) and 1, we have

$$(7.2) \qquad \mathfrak{D}_A = R_\lambda'(B_0) = R_\lambda(B_0) \subseteq R_\lambda\{P(\mathfrak{U})\} = \mathfrak{D}_\mathfrak{U} \subseteq B_0.$$

According to theorem 6.2, the strong closure of $\mathfrak{D}(\mathfrak{U})$ equals $P(\mathfrak{U})$, and therefore formula (7.2) implies that $B_0 = P(\mathfrak{U})$. Comparing (6.13) and (4.2.C) we have $\mathfrak{D}_A = \mathfrak{D}(\mathfrak{U})$.

3. Let $f_n \xrightarrow{w} f$ (w-convergence in the space $\tilde{\mathcal{E}}$). By virtue of (6.3.B), $\|R_\lambda f_n - R_\lambda f\| \to 0$. According to (4.2.D), $R'_\lambda f_n \xrightarrow{w} R'_\lambda F$. Relying on (6.3.A) we conclude that the equality (7.1) holds for all $f \in \tilde{B}$.

In particular, let $f(\omega) = 1$ for $\omega \in \tilde{\Omega}$ and $f(z) = 1$ for all $z \in E^*$. Then $R_\lambda f = \sum_{\omega,\zeta} r^\lambda_{\omega,\zeta}\sigma_\zeta p^\lambda_\omega$. But evidently $R'_\lambda f = 0$. Hence, $\sum_\omega r^\lambda_{\omega,\zeta}\sigma_\zeta = 0$ and $\sigma_\zeta = 0$. The condition $\mathfrak{U}$ is special.

THEOREM 7.2. *For every set $\mathfrak{U}$ satisfying conditions* (1.3.A)–(1.3.H), *there exists one and only one Markov semigroup T_t in the space $\tilde{\mathcal{E}}$ for which the infinitesimal operator A is a contraction of the operator $\tilde{\mathfrak{A}}$ and $\mathfrak{D}_{\tilde{A}} \subseteq \mathfrak{J}(\mathfrak{U})$. The resolvent for this semigroup is determined by the formula*

$$R_\lambda f(z) = \int_{E^*} r_\lambda(z, w)f(w)\, dw + \sum_{\zeta\in\Omega} r_\lambda(z, \zeta)f(\zeta) \tag{7.3}$$

where

$$r_\lambda(z, w) = g_\lambda(z, w) + \sum_{\omega,\zeta} r^\lambda_{\omega,\zeta}p^\lambda_\omega(z)[\sum_\gamma b_{\zeta,\gamma}B^\lambda_\gamma(w) + \int_{E^*} g_\lambda(z, w)\nu_\zeta(dz)] \tag{7.4}$$

$$r_\lambda(z, \zeta) = \sigma_\zeta \sum_\omega p^\lambda_\omega(z)r^\lambda_{\omega,\zeta}.$$

We have $B_0(\tilde{\mathcal{E}}) = P(\mathfrak{U}, \mathfrak{D}_A = \mathfrak{D}(\mathfrak{U}))$. If the set $\mathfrak{U}$ is special, then the semigroup T_t may be considered in the space E^*, and it is an $\mathfrak{A}$-semigroup.

PROOF 1. It is easy to see that (7.3)–(7.4) define the same operator as do (6.6)–(6.8). Let us show that this operator satisfies conditions (4.2.A)–(4.2.B). According to lemma 4.3, it is sufficient to verify conditions (4.5.A)–(4.5.B). The first of these conditions is obvious. Let us verify the second.

From (6.1)

$$\sum_{\zeta\in\Omega} a^\lambda_{\omega,\zeta} = \sum_{\gamma\in\Gamma_+} b_{\omega,\gamma}\lambda\{B^\lambda_\gamma, 1\} + (\lambda G_\lambda 1, \nu_\omega) \tag{7.5}$$

from which

$$1 = \sum_{\zeta\in\Omega} r^\lambda_{\omega,\zeta}\{\sum_{\gamma\in\Gamma_+} b_{\zeta,\gamma}\lambda\{B^\lambda_\gamma, 1\} + c_\zeta + (\lambda G_\lambda 1, \nu_\zeta)\}. \tag{7.6}$$

Now, let us put $f = 1$ into (6.7), multiply the equality obtained by λ, and subtract from (7.6). We obtain

$$1 - \lambda Q^\lambda_\omega(1) = \sum_{\zeta\in\Omega} r^\lambda_{\omega,\zeta}c_\zeta \geq 0. \tag{7.7}$$

2. According to the Hille-Yosida theorem (see [7], section 21, say), the operator A given in the set $\mathfrak{D}_A$ of a functional Banach space L is an infinitesimal operator of some semigroup T_t satisfying conditions (1.1.A)–(1.1.B), and such that as $t \to 0$, $\|T_tf - f\| \to 0$ for all $f \in L$ if

(a) $\mathfrak{D}_A$ is everywhere dense in L (in the strong sense);

(b) the operator $(\lambda\mathfrak{J} - A)^{-1}$ is defined for $\lambda > 0$ in the whole space L and satisfies requirements (4.2.A)–(4.2.B).

Let us apply this theorem to the space $P(\mathfrak{U})$ and the contraction A of the operator $\tilde{\mathfrak{A}}$ in the domain $\mathfrak{D}(\mathfrak{U})$. By virtue of theorem 6.1 and (6.13), the operator R_λ, defined by (7.3)–(7.4) agrees with $(\lambda\mathfrak{J} - A)^{-1}$. According to 1, conditions (4.2.A)–(4.2.B) are satisfied, and according to theorem 6.2, requirement (a) is satisfied. Therefore, A is an infinitesimal operator of the semigroup T_t satisfying conditions (1.1.A)–(1.1.B) and continuous in $P(\mathfrak{U})$. There it follows from lemma 3.3 that

$$T_t f(x) = \int_{\bar{\mathcal{E}}} P(t, x, dy) f(y). \tag{7.8}$$

Hence, the operator T_t may be extended to the whole space $\tilde{B}$ so that it will be continuous relative to w-convergence. It easily follows that (7.8) defines a semigroup T_t in the space $\tilde{B}$, which satisfies conditions (1.1.A)–(1.1.B).

3. The resolvent of the semigroup T_t is determined in the space $P(\mathfrak{U})$ by (7.3)–(7.4). The arguments of paragraph 3 of the proof of theorem 7.1 show that these formulas remain valid for all $f \in \tilde{B}$. In particular,

$$\int_0^\infty P(t, x, \zeta) e^{-\lambda t}\, dt = r_\lambda(z, \zeta). \tag{7.9}$$

Hence, it is seen that if $\sigma_\zeta = 0$, then $P(t, x, \zeta) = 0$ for all t, and the semigroup T_t may be considered in the space E^*. Evidently, condition (1.1.D) is satisfied here. From (6.3.A) the validity of (1.1.E) follows so that we have an $\mathfrak{A}$-semigroup.

4. The considerations of paragraphs 1–2 of the proof of theorem 7.1 show that if $\mathfrak{D}_A \subseteq \mathfrak{F}(\mathfrak{U})$ and $A \subseteq \tilde{\mathfrak{A}}$ for the Markov semigroup T_t, then $\mathfrak{D}_A = \mathfrak{D}(\mathfrak{U})$. By virtue of theorem 6.2 and proposition (6.3.A), the w-closure of $\mathfrak{D}_A$ coincides with $\tilde{B}$. According to the uniqueness theorem ([1], theorem 1.8), the semigroup T_t is defined uniquely by its infinitesimal operator.

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APPENDIX A. The Minimum Principle

When $\lambda = 0$ and condition (2.1.A) is satisfied with the equality sign, lemma 2.1 is proved in Petrovskii's book, say ([8], lemma 1, section 28). In the general case the proof has also been carried out, only it is necessary to determine the auxiliary function w by the formula

$$w(x, y) = u(x, y) - u(x_0, y_0) + \frac{\alpha}{v(e^{-1/\lambda}, 0)} v(x, y). \tag{1}$$

In order to prove theorem 2.1, it is necessary to replace the assumption $\Delta h = 0$ by the assumption $\lambda h - \Delta h \geq 0$ in the formulations and proofs of lemmas 5.1–5.4 of [2]. Hence, the proof of lemma 5.1 does not change. The coincidence of the exact lower bounds of the function h on the two sets is stated in each of the lemmas 5.2–5.4. In our case, these statements remain valid under the additional assumption that, each time, at least one of the two lower bounds under consideration is negative. Hence, in the proofs of lemmas 5.2 and 5.3 it is necessary to consider the auxiliary function

$$\mathcal{H}_\varepsilon(z) = h(z) - \tilde{k} - q\frac{b(z)}{b(\epsilon)}, \tag{2}$$

and the function

$$H(z) = h(z) - u - \tilde{k} + u\frac{g(z)+1}{q+1} \tag{3}$$

in the proof of lemma 5.4, where $\tilde{k} = \min(k, 0)$. After these modifications, theorem 2.1 is derived from lemmas 5.1–5.2 exactly as theorem 5.1 has been derived in [2].

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APPENDIX B. The Functions $p_\gamma(z)$ and $g_w(z)$

1. Functions $p_\gamma(z)$, ($\gamma \in \Gamma_+$) satisfying conditions (2.3.A)–(2.3.D), have been constructed in [2] (see theorem 5.2 and section 5.7). The property (2.3.E) results from formulas of section 4.8 of [2].

The function $g_w(z)$ satisfying conditions (2.2.A)–(2.2.D) has been constructed in section 6 of [2]. This function is nonnegative. It is determined by the formulas

$$2\pi g_w(z) = q_w(z) - \sum_{\gamma\in\Gamma_+} q_w(\gamma)p_\gamma(z), \tag{1}$$

$$q_w(z) = \operatorname{Re}\int_0^z G_w(z)\,dz. \tag{2}$$

The form of the function $G_w(z)$ depends on the sign of the index ℓ of the vector field $v(z)$. Let us first assume that $\ell \geq 0$. Then for $w \neq 0$,

$$G_w(z) = e^{-i\sigma(z)}z^\ell\left\{e^{i\sigma(w)}w^{-\ell-1}\tfrac{1}{2}\frac{w+z}{w-z} - e^{-i\overline{\sigma(w)}}w^{*\ell+1}\tfrac{1}{2}\frac{w^*+z}{w^*-z}\right\} \tag{3}$$

where $w^* = \overline{w}^{-1}$ and $\sigma(z)$ is an analytic function in the circle E, which has a Hölder-continuous derivative $\sigma'(z)$ in the closed circle $E \cup L$ (see section 4.4 of [2]). This formula is not suitable for studying $g_w(z)$ for values of w near zero. Let us put

$$\tilde{G}_w(z) = e^{-i\sigma(z)}\left\{\frac{e^{i\sigma(w)}}{w-z} - z^{2\ell}w^*\frac{e^{-i\sigma(w)}}{w^*-z}\right\}. \tag{4}$$

It is easy to verify that the difference $f(z) = G_w(z) - \tilde{G}_w(z)$ is regular in the circle E for any $w \neq 0$, continuous on E^*, bounded in $E \cup L$, and satisfies the relation $\operatorname{Re}\{f(z)e^{i\sigma(z)}z^{-\ell}\} = 0$ for $z \in L\backslash\Gamma$. Hence it follows that the function

$$\tilde{q}_w(z) = \operatorname{Re}\int_{z_0}^z \tilde{G}_w(z)\,dz, \qquad (z_0\in E) \tag{5}$$

differs from $q_w(z)$ by a bounded harmonic function satisfying the boundary condition A, and by virtue of the minimum principle it follows from (1) that

$$2\pi g_w(z) = \tilde{q}_w(z) - \sum_{\gamma\in\Gamma_+} \tilde{q}_w(\gamma)p_\gamma(z). \tag{6}$$

2. Let us select some $\rho \in (0, 1)$ and let $z_0 = \frac{3}{2}\rho$. Let us put $E_\rho = \{z: |z| \leq \rho\}$; $E^\rho = \{z: \rho < |z| < 1\}$. Let us consider the functions

$$R_w(z) = Q_w(z) + \frac{1}{z - w}; \qquad \tilde{R}_w(z) = \tilde{Q}_w(z) + \frac{1}{z - w}; \tag{7}$$

$$r_w(z) = \text{Re} \int_0^z R_w(z)\, dz = q_w(z) + \ln |z - w| - \ln |w|, \tag{8}$$

$$\tilde{r}_w(z) = \text{Re} \int_{z_0}^z \tilde{R}_w(z)\, dz = \tilde{q}_w(z) + \ln |z - w| - \ln |z_0 - w|. \tag{9}$$

It is easy to verify that the functions $\tilde{R}_w(z)$ and $d\tilde{R}_w(z)/dz$ are continuous and, therefore, bounded in the domain $z \in E \cup L$, $w \in E^\rho$. The functions $R_w(z)$ and $dR_w(z)/dz$ are continuous in the domain $z \in E \cup L$, $w \in E^\rho$ and the estimates

$$\begin{aligned} |R_w(z)| &\leq \mathcal{K}_1 + \frac{\mathcal{K}_2}{|z - w^*|}, \\ \left|\frac{dR_w(z)}{dz}\right| &\leq \frac{\mathcal{K}_3}{|z - w|} + \frac{\mathcal{K}_4}{|z - w^*|^2}, \end{aligned} \tag{10}$$

are satisfied for them ($\mathcal{K}_i$ are constants dependent on ρ). From the relations

$$\frac{\partial r_w}{\partial x} - i\frac{\partial r_w}{\partial y} = R_w; \qquad \frac{\partial \tilde{r}_w}{\partial x} - i\frac{\partial \tilde{r}_w}{\partial y} = \tilde{R}_w \tag{11}$$

there results that the functions $(\partial\tilde{r}_w/\partial x)$, $(\partial\tilde{r}_w/\partial y)$, $(\partial^2\tilde{r}_w/\partial x^2)$, $(\partial^2\tilde{r}_w/\partial x\partial y)$, $(\partial^2\tilde{r}_w/\partial y^2)$ are continuous in the domain $z \in E \cup L$, $w \in E^\rho$; the functions $(\partial r_w/\partial x)$, $(\partial r_w/\partial y)$, $(\partial^2 r_w/\partial x^2)$, $(\partial^2 r_w/\partial x\partial y)$, $(\partial^2 r_w/\partial y^2)$ are continuous in the domain $z \in E \cup L$, $w \in E^\rho$, and the first two are majorized in absolute value by the function $\mathcal{K}_1 + (\mathcal{K}_2/|z - w^*|)$, and the last three functions by $(\mathcal{K}_3/|z - w|) + (\mathcal{K}_4/|z - w^*|^2)$. Let us note that the functions $r_w(z)$ and $\tilde{r}_w(z)$ are harmonic in the domain E.

3. It has been shown in ([2], §7) that $q_w(\gamma) \to 0$ as $w \to \gamma$, $(\gamma \in \Gamma_+)$ and therefore, $q_w(\gamma)$ is bounded in E. Hence, there results from (1), (8), and (10) that the function $|z - w|g_w(z)$ is bounded in the domain $z \in E \cup L$, $w \in E^\rho$. From (6) and (9) and the boundedness of $\tilde{R}_w(z)$ it follows that $|z - w|g_w(z)$ is bounded in the domain $z \in E \cup L$, $w \in E_\rho$. Hence, the statement (2.2.E) is valid.

4. Let f be a bounded measurable function in E. Let us put $F(z) = Gf(z) = \int_E g_w(z)f(w)\, dw$. From (1) and (6) there results that

$$F(z) = \varphi(z) - \sum_{\gamma\in\Gamma_+} \varphi(\gamma)p_\gamma(z) \tag{12}$$

where

$$2\pi\varphi(z) = \int_{E^\rho} q_w(z)f(w)\, dw + \int_{E_\rho} \tilde{q}_w(z)f(w)\, dw. \tag{13}$$

By virtue of (8) and (9), $2\pi\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where

$$\varphi_1(z) = -\int_E \ln |z - w| f(w)\, dw,$$

$$\varphi_2(z) = \int_{E_\rho} [\tilde{r}_w(z) + \ln |z_0 - w|] f(w)\, dw, \tag{14}$$

$$\varphi_3(z) = \int_{E^\rho} [r_w(z) + \ln |w|] f(w)\, dw.$$

It is known from potential theory (see [8], §35, say) that (1) φ_1 has Hölder-continuous first derivatives in $E \cup L$; they may be found by differentiation under the integral sign; (2) if f is Hölder-continuous in E, then the second derivatives of φ_1 exist and are Hölder-continuous in E, and $\Delta\varphi_1 = -f$.

There results from the properties of r_w and $\tilde{r}_w$ derived in section 2, that (a) the first and second derivatives of the function φ_2 exist and are Hölder-continuous in $E \cup L$, and they may be obtained by differentiating under the integral sign; (b) the first derivatives of φ_3 exist and are Hölder-continuous in $E \cup L$ and may be obtained by differentiation under the integral sign; and (c) the functions φ_2 and φ_3 are harmonic in E.

The propositions (2.5.A)–(2.5.G) are derived without difficulty from these results and the known properties of the functions $p_\gamma(z)$.

5. When $\ell < 0$, the functions $G_w(z)$ and $\tilde{G}_w(z)$ are no good in that they have a pole at zero. Therefore, an expression of the form

$$\sum_{\gamma \in \Gamma_1} a_\gamma(w) \frac{z^\ell e^{-i\sigma(z)}}{2i} \frac{z + \gamma}{z - \gamma} \tag{15}$$

is added to G_w, and the expression

$$\sum_{\gamma \in \Gamma_1} \tilde{a}_\gamma(w) \frac{z^\ell e^{-i\sigma(z)}}{2i} \frac{z + \gamma}{z - \gamma} \sum_{k=0}^{m-1} b_k(w)(z^{-k} + z^k) z^\ell e^{-i\sigma(z)} + \sum_{k=1}^{m-1} b_{-k}(w) i (z^k - z^{-k}) z^\ell e^{-i\sigma(z)} \tag{16}$$

to $\tilde{G}_w$, where Γ_1 is an arbitrary subsystem of the system Γ_- consisting of $2\ell - 1$ points, and a_γ, $\tilde{a}_\gamma$, b_k are bounded harmonic functions. For example,

$$a_\gamma(w) = -\mathrm{Re}\, \{ie^{i\sigma(w)} w^{-\ell-1} P_\gamma(w)\} \tag{17}$$

where

$$P_\gamma(w) = \gamma^{\ell-1} w^{1+\ell} \prod_{\beta \in \Gamma_1, \beta \neq \gamma} \frac{w - \beta}{\gamma - \beta}. \tag{18}$$

This modification to G_w and $\tilde{G}_w$ require no essential changes in the derivations of propositions (2.2.E) and (2.5.A)–(2.5.G) made in sections 2–4.

THEOREM 1. *The functions* $g_w(\gamma^+) - g_w(\gamma^-)$, $(\gamma \in \Gamma_-)$ *are linearly independent.*

PROOF. Let us assume that for some constants m_γ

$$\sum_{\gamma \in \Gamma_-} m_\gamma [g_w(\gamma^+) - g_w(\gamma^-)] = 0. \tag{19}$$

It has been proved in ([2], §7) that if w_n is a sequence approaching $\beta \in \Gamma_+$ along the normal, then $c_n g_{w_n}(z) \to p_\beta(z)$ for a suitable choice of the constants c_n, where the convergence is uniform outside an arbitrary neighborhood of the points β. Hence, it follows from (19) that

$$\sum_{\gamma \in \Gamma_-} m_\gamma [p_\beta(\gamma^+) - p_\beta(\gamma^-)] = 0. \tag{20}$$

First let $\ell \geq 0$. Harmonic functions h_α, $(\alpha \in \Gamma_-)$ have been constructed in section 4.8 of [2] such that $h_\alpha(\gamma^+) - h_\alpha(\gamma^-) = 0$, if $\alpha \neq \gamma$ and $h_\gamma(\gamma^+) - h_\gamma(\gamma^-) \neq 0$. From section 5.6 of [2] it easily follows that the functions h_α are linear combinations of the functions p_β. Hence, it follows from (20) that $\sum_\gamma m_\gamma [h_\alpha(\gamma^+) - h_\alpha(\gamma^-)] = 0$, which means $m_\alpha = 0$.

Now, let $\ell < 0$. Let us fix some $\alpha \in \Gamma_-$, and let us select the subsystem Γ_1 of the system Γ_- such that $\alpha \in \Gamma_-$. From (1), (19), and (20), one obtains

$$\sum_{\alpha \in \Gamma_1} m_\gamma [q_w(\gamma^+) - q_w(\gamma^-)] = 0. \tag{21}$$

For $\ell < 0$ the function $G_w(z)$ differs by (15) from the function defined by (3). Hence, it is easy to see that

$$\begin{aligned} q_w(\gamma^+) - q_w(\gamma^-) &= a_\gamma(w) A_\gamma \qquad \text{for } \gamma \in \Gamma_1, \\ q_w(\gamma^+) - q_w(\gamma^-) &= 0 \qquad \text{for } \gamma \in \Gamma_- \backslash \Gamma_1, \end{aligned} \tag{22}$$

where A_γ are real constants different from zero. By virtue of (17) equality (21) becomes

$$\text{Re}\, \{\sum_\gamma i e^{i\sigma(w)} w^{-\ell-1} m_\gamma A_\gamma P_\gamma(w)\} = 0. \tag{23}$$

The function under the sign Re is regular in E and continuous in $E \cup L$. Hence, it follows from (23) that this function equals the pure imaginary constant iA_0. Therefore,

$$\sum_\gamma m_\gamma A_\gamma P_\gamma(w) = A_0 e^{-i\sigma(w)} w^{\ell+1}. \tag{24}$$

From the definition of the function $\sigma(w)$ (see [2], section 4.4) it follows that for $w = e^{it}$ the right side equals $A e^{-i\theta(t)}$ (see section 3.4 for the definition of $\theta(t)$). The left side of (24) is real for $w = e^{it}$; hence $A = 0$. Now, putting $w = \alpha$ in (24), we obtain $A_\alpha m_\alpha = 0$, and therefore, $m_\alpha = 0$.

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APPENDIX C. Lemmas on Inversion of the Matrices

LEMMA 1. *Let $P = (p_{\omega,\zeta})$ be a matrix with nonnegative elements such that for all ω, $s_\omega(P) = \sum_\zeta p_{\omega,\zeta} \leq 1$. Then $s_\omega(P^n) \leq 1$ for all ω and n. Let us put $\omega \in K$ if $s_\omega(P^n) = 1$ for all n. Then $p_{\omega,\zeta} = 0$ for all $\omega \in K$, $\zeta \notin K$. If the set K is empty, then the series*

$$\sum_{n=0}^{\infty} P^n \tag{1}$$

converges and the matrix $I - P$ has an inverse with nonnegative elements.

PROOF. Let us note that for all m and n

$$s_\omega(P^{m+n}) = \sum_\zeta p_{\omega,\zeta}^{(m)} s_\zeta(P^n). \tag{2}$$

Putting $m = 1$, we deduce by induction the first statement of the lemma. Furthermore, if $\omega \in K$, then for any n,

$$\sum_\zeta p_{\omega,\zeta}[1 - s_\zeta(P^n)] = 0. \tag{3}$$

If $\zeta \notin K$, an n may be selected such that $s_\zeta(P^n) < 1$, and then it follows from (3) that $p_{\omega,\zeta} = 0$. Finally, if K is empty, then for any ω there exists an n such that $s_\omega(P^n) < 1$. Relation (2) implies that $s_\omega(P^{m+n}) \leq s_\omega(P^m)$. Hence, for some n_0 the inequality $s_\omega(P^{n_0}) < 1$ is satisfied for all ω and $\max_n s_\omega(P^{n_0}) = c < 1$. From (2) it follows that

$$s_\omega(P^{kn_0}) \leq c s_\omega(P^{(k-1)n_0}). \tag{4}$$

Therefore, $s_\omega(P^n) \leq c^k$ for $kn_0 \leq n < (k + 1)n_0$ and the series (1) converges. Its sum is the inverse matrix for $I - P$.

LEMMA 2. *Let $A = (a_{\omega,\zeta})$ be a matrix satisfying the conditions*

$$a_{\omega,\omega} > 0, \qquad a_{\omega,\zeta} \leq 0 \quad \text{for} \quad \omega \neq \zeta, \quad \sum_\zeta a_{\omega,\zeta} \geq 0. \tag{5}$$

Let us consider the matrix P with elements $p_{\omega,\omega} = 0$, $p_{\omega,\zeta} = -(a_{\omega,\zeta}/a_{\omega,\omega})$ for $\omega \neq \zeta$. If the set K, defined for this matrix in lemma 1 is empty, the matrix A has an inverse with nonnegative elements.

PROOF. According to lemma 1, the matrix $I - P$ has an inverse with nonnegative elements. But $A = \Lambda(I - P)$, where Λ is a diagonal matrix with diagonal elements $a_{\omega,\omega}$. Hence, the statement of lemma 2 follows from the formula $A^{-1} = (I - P)^{-1}\Lambda^{-1}$.

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