THE METHOD OF CHARACTERISTIC FUNCTIONALS

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1. Introduction

The infinite-dimensional analogue of the concept of characteristic function, namely the characteristic functional (ch.f.), was first introduced by A. N. Kolmogorov as far back as 1935 [31], for the case of distributions in Banach space. This remained an isolated piece of work for a long time. Only in the last decade have characteristic functionals again attracted the attention of mathematicians.

Among the works devoted to this subject I shall note here in the first place those of E. Mourier and R. Fortet (see for instance [14], [15], [32], where further references are given) and the fundamental investigations of L. Le Cam [1]. The special case of distributions in Hilbert space is considered in detail, for instance, in the author's work [22].

Let X be a linear space and let $\mathfrak{J}$ be a locally convex topology in this space, let X^* be the space dual to $(X, \mathfrak{J})$, that is, the linear space whose elements x^* are continuous linear functionals over $(X, \mathfrak{J})$. It is quite natural that the following two questions occupy a central position in the general theory. In the first place, when is a nonnegative definite function $\chi(x^*)$ a ch.f. of some σ-additive measure? Secondly, how can the conditions of weak convergence of distributions be expressed in terms of ch.f.? The content of this article is, in fact, connected with these two questions.

Sections 2 and 3 are of auxiliary character. In section 2 are given some facts about measures in completely regular spaces. Here is introduced the notion of tightness of the measure, which is one of the fundamental concepts of the whole theory.

Section 3 contains an enumeration of some needed results from the theory of locally convex spaces. Particular attention is given to spaces that are the dual of Fréchet spaces, since the latter possess many "good" properties from the point of view of this theory.

In section 4 there is introduced the concept of a weak distribution P in a linear topological space $(X, \mathfrak{J})$. Roughly speaking, the problem is as follows. In every "real" distribution in X the linear functionals become random variables and the joint distribution of any finite number of them

$$(1) \qquad x_1^*, x_2^*, \cdots, x_n^*$$

is a σ-additive measure in the n-dimensional Euclidean space. But this can also hold for some finitely additive P, which are called weak distributions in this case. We have borrowed the nomenclature from a paper by I. E. Segal [9] (see also [10]). In a slightly altered form the concept of weak distribution was used by I. M. Gelfand [19] and by K. Itô [20] in developing the theory of generalized random processes. Moreover, in section 4 the one-to-one correspondence between weak distributions and nonnegative definite functionals, which are continuous in all directions, is established.

The two main problems connected with weak distributions, namely conditions for σ-additivity and for the extension to a Baire measure, are discussed in section 5. Here again is seen the fundamental role played by the concept of the tightness of a measure. One of the fundamental results consists in showing that one can obtain an analogue of Bochner's theorem (that is, an assertion of the type: the functional χ is a ch.f. of a σ-additive distribution if and only if it is nonnegative definite, is equal to one at zero, and is continuous in a certain topology) in spaces $X = Y^*$, where the topology in Y is introduced by a system of scalar products. The theorem of uniqueness is also treated in section 5, namely that every tight Baire or Borel measure is uniquely defined by its ch.f.

In section 6 problems connected with weak compactness of families of distributions are considered. It is known that in the finite-dimensional case the following three statements are equivalent:

(a) the set $\{P_\alpha\}$ is relatively compact,

(b) the set $\{P_\alpha\}$ is tight (for definition see section 6),

(c) the set $\{\chi_\alpha\}$ of the corresponding ch.f. is equicontinuous at zero.

In the infinite case these equivalences do not hold in general. As a curious fact we can mention the following. In a separable Hilbert space, where (a) and (b) are equivalent, there exists no locally convex topology with the property that the equicontinuity in it of $\{\chi_\alpha\}$ is equivalent to the relative compactness of $\{P_\alpha\}$. However, the equivalence of (b) and (c) is preserved in spaces $X = Y^*$, where Y is a nuclear space.

I have not considered the possible applications of the method. I doubt that the problem of summing independent random elements with values in linear spaces will play as important a role as the corresponding problem for independent random variables. As a nontrivial example we can note the analysis of empirical distributions (see for example [33]), the addition of "rare sequences of events" (in the sense of [34], [35]) and the study of "random curves" constructed from sums of independent random variables (see [22]). I also note that at the present time there is a strong development of the technique of calculation with moments and semi-invariants of infinite-dimensional distributions (see for example [36]).

The present paper was influenced essentially by the investigations on the theory of measure in topological spaces which were systematized by Varadarajan [2] and by the work of Le Cam [1].

I wish to thank A. N. Kolmogorov for a number of comments and V. V. Sazonov for numerous discussions of the problems under consideration.

2. Measures in topological spaces

For the purposes of this paper it will be sufficient to consider completely regular topological spaces.

Let X be a set of points and $\mathfrak{J}$ a class of *open* subsets of X such that $(X, \mathfrak{J})$ is a completely regular topological space. We introduce the notation

$\mathcal{C}$ is the set of all real functions that are continuous and bounded over all X;

$\mathcal{B}$ is the smallest σ-algebra with respect to which all $f \in C$ are measurable (that is, the σ-algebra of *Baire* sets);

$\hat{\mathcal{B}}$ is a σ-algebra generated by the class $\mathfrak{J}$ of open sets (that is, the σ-algebra of *Borel* sets).

By a *measure* we shall understand in what follows a finite nonnegative σ-additive set function, defined over some algebra (or σ-algebra) of subsets of X.

In studying measures in topological spaces it is expedient from many points of view to restrict ourselves to measures with domain of definition $\mathcal{B}$, that is, to the so-called *Baire* measures, while strengthening the condition of σ-additivity by replacing it by the condition of tightness (see [1], [2], and [3]).

DEFINITION 1. *The measure μ defined over the subalgebra $\mathcal{E}$ of the σ-algebra $\hat{\mathcal{B}}$ is called tight if it satisfies the following two conditions.*

(1) *For every $E \in \mathcal{E}$*

$$\mu(E) = \inf_{E \subset G \in \mathcal{E}} \mu(G), \tag{2}$$

where G is an open set of X.

(2) *For every $\epsilon > 0$, there exists a compact $K = K_\epsilon$ such that*

$$\sup_{\mathcal{E} \ni E \subset x \setminus K} \mu(E) < \epsilon. \tag{3}$$

For the sake of brevity we shall sometimes call such compacts ϵ-compacts for μ.

REMARK 1. If $\mathcal{E} = \mathcal{B}$, then the property (1) is automatically satisfied (see for example [6] or [2], theorem 2.7.1).

REMARK 2. Let $\mathcal{E}$ be a subalgebra of $\hat{\mathcal{B}}$ and μ a real nonnegative finite and *finitely additive* set function defined over $\mathcal{E}$. We shall say that μ is tight if it satisfies conditions (1) and (2) of definition 1. It is not hard to show that in this case μ is σ-additive over $\mathcal{E}$ and that its extension over $\mathcal{S}(\mathcal{E})$, the σ-algebra generated by $\mathcal{E}$, is a tight measure in the sense of definition 1.

The consideration of tight measures only avoids a number of pathological cases and in this sense is quite analogous to the limitation to perfect measures proposed in monograph [4]. One should note that the properties of tightness of a measure and its perfection are closely connected (see [5]). On the one hand, every tight measure in the sense of definition 1 is perfect; on the other hand, for instance, in metric spaces of not too high a power, every perfect Borel measure is tight (this will be the case when the space does not contain a system of non-intersecting open sets of cardinality greater than the power of the continuum).

Tight measures possess the property of extension which can be expressed as

THEOREM 1. *Every tight measure* μ *having the property that the open sets of* $\mathcal{E}$ *form an open basis of the space can be uniquely extended to a tight Borel measure* $\hat{\mu}$. *The extension is achieved by the formulas*

(1) $\hat{\mu}(G) = \sup_{O \subset G} \mu(O),$

where $G \in \hat{\mathfrak{B}}$ *and* $O \in \mathcal{E}$ *are open sets.*

(2) $\hat{\mu}(B) = \inf_{G \supset B} \hat{\mu}(G),$

for every $B \in \hat{\mathfrak{B}}$. *Here it is sufficient to assume that* $(X, \mathfrak{J})$ *is regular.*

It follows from this theorem, in particular, that in a completely regular space every tight Baire measure can be uniquely extended into a tight Borel measure. The last assertion can be found, for instance, in [6] and, for compacts, in [7].

The theorem about the extension of Baire measures to Borel measures enables us, among other things, to define the probability of continuity of sample functions of random processes and allied events by a method which differs from the one used by J. L. Doob [27], [21] (see [8] and [12]).

3. Some remarks about linear spaces

Let $(X, \mathfrak{J})$ be a real linear locally convex Hausdorff space (for brevity we shall say simply a locally convex space). The continuous linear functionals over $(X, \mathfrak{J})$ form a linear space $X^* = X^*_{\mathfrak{J}}$. It is known that a locally convex space is completely regular and therefore all the considerations of section 2 are applicable to it.

In what follows we shall always understand by "topology" a *locally convex separated topology*. Together with the topology $\mathfrak{J}$ we shall also consider other topologies $\mathfrak{J}'$, satisfying the condition $X^*_{\mathfrak{J}} = X^*_{\mathfrak{J}'}$. To avoid ambiguity we shall talk in that case of $\mathfrak{J}'$-Baire sets, $\mathfrak{J}'$-tight measures, and so forth.

Every finite subset

$$\lambda = \lambda_n = (x_1^*, x_2^*, \cdots, x_n^*) \tag{4}$$

of elements of X^* defines a mapping

$$\pi_\lambda : x \to \{x_1^*(x), x_2^*(x), \cdots, x_n^*(x)\} \tag{5}$$

of the space X into the n-dimensional Euclidean space R^n. A subset of the set X of the form

$$A = \pi_{\lambda_n}^{-1}(A_n), \tag{6}$$

where n is a natural number and A_n is any n-dimensional Borel set, will be called *cylindrical*. The algebra formed by the cylindrical sets will be denoted by $\mathfrak{A}$. This algebra is the union of all the σ-algebras $\mathfrak{L}^{\lambda_n}$ generated by the "strips"

$$\bigcap_{j=1}^{n} \{x: -\infty < a_j \leqq x_j^*(x) < b_j < \infty\}. \tag{7}$$

In addition to the σ-algebras $\mathcal{B}$ and $\hat{\mathcal{B}}$ of section 2 we also have the σ-algebra $\mathcal{L}$ generated by $\mathcal{A}$. It is obvious that

$$\mathcal{A} \subset \mathcal{L} \subset \mathcal{B} \subset \hat{\mathcal{B}}. \tag{8}$$

We denote as usual by $\mathcal{S} \cap K$ the class of sets of the form $\mathcal{S} \cap K$, $S \in \mathcal{S}$. We shall need a lemma due to V. V. Sazonov

LEMMA 1. *Let K_n be an increasing sequence of compacts in X and let $C = \bigcup_{n=1}^{\infty} K_n$. Then*

$$\mathcal{B} \cap C = \mathcal{L} \cap C. \tag{9}$$

PROOF. We denote by $\mathcal{C}'$ the subset of $\mathcal{C}$ composed of all $\mathcal{L}$-measurable functions. It is obvious that $\mathcal{C}'$ is an algebra which separates points and contains constants. Let K be a compact in X, let $\mathcal{C}'_K$ be the set of restrictions to K of the functions in $\mathcal{C}'$. Then by the Stone-Weierstrass theorem ([29], chapter 1), the uniform closure of $\mathcal{C}'_K$ coincides with the algebra $\mathcal{C}_K$ of all continuous functions on K. All the functions of $\mathcal{C}'_K$ and therefore all those of $\mathcal{C}_K$ are measurable with respect to $\mathcal{L} \cap K$. Since $\mathcal{B}$ is a σ-algebra generated by the sets

$$Z = \{x : f(x) = 0, f \in \mathcal{C}\} \tag{10}$$

and $Z \cap K \in \mathcal{L} \cap K$, then $\mathcal{B} \cap K \subset \mathcal{L} \cap K$. Taking into consideration that the inverse inclusion always holds we have $\mathcal{B} \cap K = \mathcal{L} \cap K$.

Let now K_n be the compacts satisfying the conditions of the lemma. Let

$$A = B \cap C, \qquad B \in \mathcal{B}. \tag{11}$$

For every natural number n we can find an $L_n \in \mathcal{L}$ such that $B \cap K_n = L_n \cap K_n$. Then

$$A = \left(\bigcup_{n=1}^{\infty} \bigcap_{k \geq n} L_k \right) \cap \left(\bigcup_{n=1}^{\infty} K_n \right) \in \mathcal{L} \cap C, \tag{12}$$

which remained to be proved.

In many examples encountered in this paper, X will be the *dual of a Fréchet space*. Therefore we give below the definition and the needed properties of these spaces. The proofs can be found in [28], [30], and [23].

A Fréchet space $(Y, \mathcal{H})$ is a complete metrizable locally convex space. In the dual space X we consider three topologies, the weak topology $\mathfrak{T}_s$, the compact topology $\mathfrak{T}_c$, and the bounded topology $\mathfrak{T}_b$. In the topology $\mathfrak{T}_s$ a basis for the neighborhoods of zero is given by the sets

$$U_{\epsilon, y_1, \cdots, y_n}(0) = \bigcap_{j=1}^{n} \{x : |x(y_j)| < \epsilon\}, \tag{13}$$

in the topology $\mathfrak{T}_c$ by the sets

$$V_{\epsilon, K}(0) = \{x : \sup_{y \in K} |x(y)| < \epsilon\}, \tag{14}$$

where K is an arbitrary *compact* in Y, and in the topology $\mathfrak{T}_b$ by the sets

$$W_{\epsilon,A}(0) = \{x: \sup_{y \in A} |x(y)| < \epsilon\}, \tag{15}$$

where A is an arbitrary *bounded* set in Y.

(1) It is always true that $\mathfrak{J}_s \subseteq \mathfrak{J}_c$. The equality is possible only in the case of a finite dimensional Y.

It is always true that $\mathfrak{J}_c \subseteq \mathfrak{J}_b$. The equality holds only in the case when Y is a Montel space (a Fréchet space is called a Montel space if its closed bounded sets are compact).

(2) The topologies induced by $\mathfrak{J}_s$ and $\mathfrak{J}_c$ on $\mathfrak{J}_s$-compacts coincide. In particular the compacts in these topologies are the same.

(3) The spaces with the topologies $\mathfrak{J}_s$ and $\mathfrak{J}_c$ are hemicompact, that is,

(α) there exists a sequence of compacts C_n such that every compact $C \subset X$ lies in one of the C_n; in particular,

$$X = \bigcup_{n=1}^{\infty} C_n. \tag{16}$$

The space X with topology $\mathfrak{J}_c$ is a k-space, that is,

(β) the set is closed if and only if its intersection with the compacts is closed. This terminology is taken from [1].

(4) The space dual to $(X, \mathfrak{J}_s)$ and to $(X, \mathfrak{J}_c)$ is the space Y itself.

From lemma 1 and property (3α) it follows that

(5) The classes of Baire sets in the topologies $\mathfrak{J}_s$ and $\mathfrak{J}_c$ coincide with $\mathfrak{L}$.

(6) In the case of *separable* $(Y, \mathfrak{K})$ the classes of Borel sets in the topologies $\mathfrak{J}_s$ and $\mathfrak{J}_c$ also coincide with $\mathfrak{L}$.

We now consider some special cases.

EXAMPLE 1. Let $(Y, \mathfrak{K})$ be a Banach space. Then for every $x \in X$ we can introduce in the usual way the norm $||x||$. The topology induced by this norm coincides with $\mathfrak{J}_b$. For the compacts C_n in condition (3α) we can take spheres of radius n

$$C_n = \{x: ||x|| \leqq n\}. \tag{17}$$

EXAMPLE 2. Let Y be a linear space in which the particular countable set of scalar products

$$(y, y)_n, \qquad n = 1, 2, \cdots, \tag{18}$$

satisfies the condition

$$(y, y)_n \leqq (y, y)_{n+1} \tag{19}$$

for every $y \in Y$ and for every n. We denote by $\mathfrak{K}$ the topology in which the basis for the neighborhoods of zero is formed by the sets

$$O_{n,\epsilon}(0) = \{y: (y, y)_n < \epsilon\}. \tag{20}$$

The space $(Y, \mathfrak{K})$ is metrizable. For the distance we can take the function

$$\rho(y', y'') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||y' - y''||_n}{1 + ||y' - y''||_n}, \tag{21}$$

where

(22) $$\|y\|_n = [(y, y)_n]^{1/2}.$$

We shall call $(Y, \mathfrak{H})$ a *countably Hilbertian space* if

(a) The scalar products are concordant, that is, if for each sequence $\{y_j\}$ with $y_j \in Y$, which is a Cauchy sequence simultaneously in the nth and in the $(n + 1)$st norm, the convergence to zero in the nth norm implies the convergence to zero in the $(n + 1)$st norm.

(b) Y is complete with respect to the metric ρ.

Every countably Hilbertian space Y can be represented as an intersection

(23) $$Y = \bigcap_{n=1}^{\infty} Y_n, \qquad Y_n \downarrow,$$

where Y_n is a Hilbert space, which is the completion of Y with respect to the nth scalar product. In the dual space $Y^* = X$ the sets

(24) $$C_{n,m} = \left\{x\colon \sup_{\|y\|_n \leq 1} |x(y)| \leq m\right\},$$

$n, m = 1, 2, \cdots$, are compacts in the topologies $\mathfrak{I}_s$ and $\mathfrak{I}_c$ and their union is equal to X.

We now introduce the concept of nuclear space [25], [24], [11]. We shall call *S-operator* in Hilbert space every linear symmetric nonnegative completely continuous operator with finite trace. A countably Hilbertian space in $(Y, \mathfrak{H})$ is called *nuclear* if for every n there can be found an $m > n$ and an S-operator $S_{n,m}$ in X_m such that

(25) $$(y', y'')_n = (S_{n,m}y', y'')_m.$$

It is not hard to see that every nuclear space is separable. We note, although we will not make use of it, that every nuclear space is a Montel space. Therefore, in the conjugate space the classes of Baire and Borel sets in the topologies $\mathfrak{I}_s$ and $\mathfrak{I}_c = \mathfrak{I}_b$ coincide with $\mathfrak{L}$.

4. Weak distributions and their characteristic functionals

Let $(X, \mathfrak{I})$ be a locally convex space and let P be a real nonnegative finite and finitely additive function given over an algebra $\mathfrak{A}$ of cylindrical sets, and let $P(X) = 1$.

DEFINITION 2. *P is called a weak distribution in X if it is σ-additive on each of the classes $\mathfrak{L}^{\lambda_n}$.*

To every weak distribution P corresponds its *characteristic functional* (ch.f.) defined by

(26) $$\chi(x^*, P) = \int_X e^{ix^*(x)}\, dP.$$

The integral $\mathfrak{A}$ of a measurable bounded function with respect to a finitely additive P is defined, as usual, as the limit of Lebesgue sums. It is clear that $\chi(x^*, P)$ can be calculated as $\int \exp(iu)\, dF^{x^*}(u)$, where $F^{x^*}(u) = P\{x: x^*(x) < u\}$.

If $\hat{P}$ is a probability measure, whose domain of definition contains $\mathcal{A}$, then its ch.f. is defined analogously. It is clear that to each such measure $\hat{P}$ corresponds a weak distribution P (simply by restricting $\hat{P}$ to $\mathcal{A}$). It is well known that a weak distribution need not necessarily be σ-additive. The simplest example is $\chi(x^*, P) = \exp\{-(1/2)\|x^*\|^2\}$ in a separable Hilbert space X.

The ch.f. possesses the following fundamental properties.

(1) It is nonnegative definite, that is, for any $x_1^*, x_2^*, \cdots, x_n^* \in X^*$ and any complex numbers $c_1, c_2, \cdots, c_n$

$$\sum_{k,l=1}^{n} \chi(x_k^* - x_l^*, P) c_k \bar{c}_l \geqq 0. \tag{27}$$

(2) Continuity, that is, for every fixed $x^* \in X^*$, the function $\chi(tx^*, P)$ of a real argument t is continuous.

(3) $\chi(\theta^*) = 1$, where θ^* is the zero of the space X^*.

It is not hard to show that the converse is true.

THEOREM 2. *Every functional $\chi(x^*)$ satisfying conditions* (1) *to* (3) *is a ch.f. of some weak distribution. Moreover, P is uniquely defined by χ.*

PROOF. The proof follows, on the whole, the same plan as the proof of Kolmogorov's theorem about probabilities in infinite-dimensional spaces ([13], chapter 3; compare also closely related statements in [10], section 3). In addition we make use of the following elementary

LEMMA 2. *Let $\psi(\vec{\alpha}) = \psi(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a nonnegative definite function in R^n satisfying the conditions*

(1) $\psi(\vec{0}) = 1$,

(2) *$\psi(\vec{\alpha})$ is continuous at zero in n linearly independent directions $\{t_j\vec{e}_j\}$, that is, $\psi(t_j\vec{e}_j) \to 1$ for $t_j \to 0$, $1 \leqq j \leqq n$.*

Then ψ is an n-dimensional characteristic function.

After introducing the concept of weak distributions two questions arise naturally.

(1) When is a function P σ-additive over $\mathcal{A}$, and therefore when can it be extended uniquely into a probability measure $\hat{P}$ on $\mathcal{L}$?

(2) When can the probability distribution $\hat{P}$ thus obtained be extended uniquely into a measure over $\mathcal{B}$ (this gives, for example, a way of defining from χ the integrals of all continuous functions, and so forth).

The answers to these questions will be found in section 5.

5. The extension of a weak distribution to probability measure

The question of σ-additivity of P on $\mathcal{A}$ is not directly connected with topological considerations. However, it is possible to give sufficient conditions for the

σ-additivity of P in topological terms. Moreover in some important special cases these conditions become necessary and sufficient.

Let $\mathfrak{J}'$ be some topology in X with $X^*_{\mathfrak{J}'} = X^*_{\mathfrak{J}}$. If the weak distribution P over $\mathfrak{A}$ is $\mathfrak{J}'$-tight, then in accordance with remark 2 after definition 1, it is possible to extend P into a σ-additive, and what is more into a $\mathfrak{J}'$-tight, measure over $\mathfrak{L}$. A weak distribution is automatically regular with respect to $\mathfrak{J}'$, that is, condition (1) of definition 1 is satisfied for P. Therefore condition (2) of definition 1 is necessary and sufficient for P to be $\mathfrak{J}'$-tight. Suppose now that the weak distribution P is fixed, $\lambda = (x_1^*, x_2^*, \cdots, x_n^*)$, let P^λ be the joint distribution of $x_1^*, x_2^*, \cdots, x_n^*$,

$$P^\lambda(A_n) = P[\pi_\lambda^{-1}(A_n)], \tag{28}$$

where A_n is an arbitrary Borel set in R^n. The image of the set $C \subset X$ under the mapping π_λ will be called the λ-projection of C. Obviously

$$C^\lambda = \pi_\lambda^{-1}[\pi_\lambda(C)] \tag{29}$$

is the least cylindrical set containing C and definable by $x_1^*, x_2^*, \cdots, x_n^*$. From the previous remarks we have

LEMMA 3. *In order that a weak distribution P be extended into a $\mathfrak{J}'$-tight measure $\hat{P}$ over $\mathfrak{L}$ it is necessary and sufficient that for any $\epsilon > 0$ a $\mathfrak{J}'$-compact $C_\epsilon \subset X$ can be found such that for all λ*

$$1 - P^\lambda[\pi_\lambda(C_\epsilon)] < \epsilon. \tag{30}$$

Naturally, one can verify (30) in principle with finite-dimensional characteristic functions

$$\psi^\lambda(t, P) = \psi^{x_1^*, \cdots, x_n^*}(t_1, \cdots, t_n, P) = \chi(t_1 x_1^* + \cdots + t_n x_n^*, P). \tag{31}$$

We now consider some examples of the application of lemma 3. We shall take the case when X is the dual of some Fréchet space $(Y, \mathfrak{H})$. In this case it can be seen from property (3α) after equation (15) that the $\mathfrak{J}_s$-tightness, the $\mathfrak{J}_c$-tightness, and the countable additivity of P are equivalent ([1], p. 232).

EXAMPLE 1. If $(Y, \mathfrak{H})$ is a Banach space we can take for the compacts C_ϵ the spheres

$$C_\epsilon = \{x: ||x|| \leqq r_\epsilon\}. \tag{32}$$

Lemma 3 then gives us the result of papers [15] and [14] (chapter 3). Unfortunately the λ-projection of spheres cannot be simply described and therefore the verification of relation (30) in terms of ch.f. is difficult (see theorem 5 below). We note in passing that if $(Y, \mathfrak{H})$ is separable then in X the classes of Baire and Borel sets with respect to all three topologies $\mathfrak{J}_s$, $\mathfrak{J}_c$, and $\mathfrak{J}_b$ coincide with $\mathfrak{L}$. In this case lemma 3 answers both questions proposed at the end of section 4.

If the topology $\mathfrak{H}$ in Y is defined by a *scalar product* or by a sequence of scalar products, then the conditions of lemma 3 can be expressed in terms of the continuity of the ch.f. $\chi(y, P)$, with $y \in Y$ in some topology, which gives an analogue of the classical theorem of Bochner. This topology is defined by a suitably chosen system of completely continuous quadratic forms with a finite trace. To this field of ideas belong the results of papers [17], [16], [18], and [11], where the cases in which $(Y, \mathcal{H})$ is a Hilbert space, a countably Hilbertian space, and a nuclear space are considered. In the proofs one should mention two main points. The first point is that the existence of scalar products enables us to consider only those λ-projections of the compacts which are finite-dimensional spheres. The second point is the application of a certain lemma (lemma 4 below) about finite-dimensional distributions which was not explicitly used in [22]. The importance of the lemma was underscored in [18].

EXAMPLE 2. To illustrate the above, consider the case of a separable Hilbert space $(Y, \mathcal{H})$. For the sake of what follows we shall not identify $X = Y^*$ and Y. We shall call the topology $\mathcal{J}'$ in Y *admissible* if the following statement holds.

In order that χ be the ch.f. of some σ-additive distribution over $\mathcal{A}$ (or $\mathcal{L}$) it is necessary and sufficient that χ be continuous in the topology $\mathcal{J}'$. We shall suppose that the letter χ denotes only nonnegative definite functionals, equal to unity at zero. For every such functional continuity is equivalent to continuity at zero.

We shall denote by $\mathcal{J}$ the topology in which the basis for the neighborhoods of zero is given by the sets

$$\mathcal{J}_S(0) = \{y: (Sy, y) < 1\}, \tag{33}$$

where S is an arbitrary S-operator in Y, with the property (25). The result of Sazonov [17] can be formulated in these terms as

THEOREM 3. *The topology $\mathcal{J}$ is admissible, and is the weakest of all admissible topologies.*

REMARK. The author does not know of any other admissible topologies.

5.1. *Sufficiency of the continuity of χ in the $\mathcal{J}$-topology.*

(1) From the $\mathcal{J}$-continuity of $\chi(y)$ it follows that χ is a ch.f. of some weak distribution P

$$\chi(y) = \chi(y; P). \tag{34}$$

(2) Let Q be the probability measure in R^n

$$\psi(t_1, t_2, \cdots, t_n) = \int \exp [i(t_1\alpha_1 + \cdots + t_n\alpha_n)]\, dQ \tag{35}$$

and

$$\sum a_{k,l}t_kt_l, \qquad 1 \leqq k, \quad l \leqq n, \tag{36}$$

be a nonnegative quadratic form. Then lemma 4 holds.

LEMMA 4. *If the inequality $\sum a_{k,l}t_kt_l \leqq 1$ implies the inequality*

$$1 - \operatorname{Re} \psi(t_1, t_2, \cdots, t_n) \leqq \epsilon, \tag{37}$$

then

$$Q\{\alpha: \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 \geqq c^2\} \leqq \frac{\sqrt{e}}{\sqrt{e} - 1}\left(\epsilon + \frac{a^2}{c^2}\right), \tag{38}$$

where

$$a^2 = \sum_k a_{k,k}. \tag{39}$$

(3) Let $\chi(y, P)$ be continuous in the $\mathcal{J}$-topology and let $\epsilon > 0$ be a given positive number. Let $\eta = (\sqrt{e} - 1)\epsilon/4\sqrt{e}$ and let the S-operator be such that for $(Sy, y) < 1$,

$$1 - \operatorname{Re} \chi(y, P) < \eta. \tag{40}$$

Denote by A^2 the trace of S and define c by the relation

$$\frac{2\sqrt{e}}{\sqrt{e} - 1} \frac{A^2}{c^2} = \frac{\epsilon}{2} \tag{41}$$

Let us verify that for the sphere

$$C_\epsilon = \{x: \|x\| \leqq c\} \tag{42}$$

the inequality (30) of lemma 3 is satisfied.

Let $\lambda = \lambda_m = (y_1, y_2, \cdots, y_m)$ be an arbitrary finite subset of elements of Y and let $\mu = \mu_n = (y_1', y_2', \cdots, y_n')$ be an orthonormal system equivalent to λ, in the sense that the linear envelopes of λ and μ coincide. Then

$$P^\lambda\{\pi_\lambda(C_\epsilon)\} = P^\mu\{\pi_\mu(C_\epsilon)\}, \tag{43}$$

and the μ-projection of C_ϵ is the n-dimensional sphere

$$\{\alpha: \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 \leqq c^2\}. \tag{44}$$

The ch.f. ψ^μ of the distribution P^μ over the ellipsoid

$$[S(t_1y_1' + \cdots + t_ny_n'), (t_1y_1' + \cdots + t_ny_n')] \leqq 1 \tag{45}$$

satisfies the inequality

$$1 - \operatorname{Re} \psi^\mu(t_1, \cdots, t_n) \leqq \eta. \tag{46}$$

Therefore by lemma 4

$$1 - P^\mu\{\pi_\mu(C_\epsilon)\} = P^\mu\{\alpha: \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 > c^2\} < \epsilon, \tag{47}$$

that is, condition (30) of lemma 3 is satisfied, which remained to be proved.

5.2. *Necessity of continuity in $\mathcal{J}$-topology.* Suppose that P is countably additive over $\mathfrak{L}$. Select a sphere $C \subset X$ with center at the origin so that $1 - P(C) < \epsilon/2$. The equation

$$\epsilon^{-1} \int_C x^2(y)\, dP = (Sy, y) \tag{48}$$

defines, as can be easily seen, an S-operator in Y. In the $\mathcal{J}$-neighborhood of zero $\mathcal{J}_S(0)$

$$1 - \operatorname{Re} \chi(y) = \int_X [1 - \cos x(y)]\, dP < \frac{\epsilon}{2}(Sy, y) + \frac{\epsilon}{2} < \epsilon. \tag{49}$$

EXAMPLE 3. Let $(Y, \mathcal{H})$ be a separable, countably Hilbertian space. Denote by $\mathcal{G}$ the topology in which the basis for the neighborhoods of zero is given by the sets

$$\mathcal{G}_{nS} = \{y: (Sy, y)_n < 1\}, \tag{50}$$

where n is any natural number and S is some S-operator in Y_n. Here an assertion similar to theorem 3 holds (see [16]). It is not hard to see that $\mathcal{H} \subseteq \mathcal{G}$ always. Moreover the equality holds if and only if $(Y, \mathcal{H})$ is nuclear. Therefore in the space X, dual to the nuclear space the necessary and sufficient conditions for the function $\chi(y)$ with $\chi(0) = 1$ to be a ch.f. of a countably additive distribution over $\mathcal{L}$ reduces, as in Bochner's theorem, to the conditions (a) nonnegative definiteness, (b) continuity.

We now consider the second of the questions proposed at the end of section 4. Little is known that is applicable to the case of general locally convex spaces. We give below a lemma of Sazonov and a theorem of Le Cam [1] (theorem 8), which exhaust practically everything that is known in this direction.

LEMMA 5. *Let $(X, \mathfrak{J})$ be a locally convex space and let P be a $\mathfrak{J}$-tight weak distribution. Then $\hat{P}$ can be uniquely extended into a Baire measure (which is automatically tight).*

PROOF. (1) We can assume at once that P is a σ-additive $\mathfrak{J}$-tight measure on $\mathcal{L}$. Let $K_1 \subset K_2 \subset \cdots$ be a sequence of compacts in X such that

$$P^*(K_n) > 1 - \frac{1}{n}, \tag{51}$$

where P^* is the outer measure induced by P. Then by lemma 1 $\mathcal{B} \cap C = \mathcal{L} \cap C$ where $C = \cup_{n=1}^{\infty} K_n$. For every $B \in \mathcal{B}$ we define a set $L \in \mathcal{L}$ such that

$$B \cap C = L \cap C. \tag{52}$$

Since $P^*(C) = 1$, L is defined uniquely by this equation up to a set of measure zero. Let $\hat{P}(B) = P(L)$. From this definition it follows immediately that $\hat{P}$ is a tight measure on $\mathcal{B}$.

(2) Let $\hat{P}_1$ and $\hat{P}_2$ be two tight extensions of P on $\mathcal{B}$ and let $B \in \mathcal{B}$. For an arbitrary $\epsilon > 0$ select a compact K such

$$\hat{P}_i(K) > 1 - \frac{\epsilon}{2}, \qquad i = 1, 2. \tag{53}$$

Let $L \in \mathcal{L}$ be such that $L \cap K = B \cap K$. Then

$$|\hat{P}_1(B) - \hat{P}_2(B)| \leqq |\hat{P}_1(B) - \hat{P}_1(L)| + |\hat{P}_2(B) - \hat{P}_2(L)| < \epsilon. \tag{54}$$

Hence $\hat{P}_1 \equiv \hat{P}_2$.

As a corollary we obtain

THEOREM 4. *Every tight Baire or Borel measure P in $(X, \mathfrak{J})$ is uniquely defined by its ch.f.*

We next give Le Cam's theorem together with a proof based on the use of lemma 3 and different from Le Cam's proof.

We shall call every finite sum of the form

$$q(x) = \sum c_k e^{ix_k^*(x)} \tag{55}$$

a trigonometric polynomial.

THEOREM 5. *In order that the functional $\chi(x^*, P)$ be a ch.f. of a tight measure on $\mathfrak{L}$ it is necessary and sufficient that for every $\epsilon > 0$ there exist a compact K in X and a number $\delta > 0$ such that the inequalities*

$$\sup_{x \in X} |q(x)| \leqq 1, \tag{56}$$

$$\sup_{x \in K} |q(x)| < \delta, \tag{57}$$

imply

$$|\sum c_k \chi(x_k^*, P)| < \epsilon. \tag{58}$$

PROOF OF NECESSITY. Let P be a tight measure on $\mathfrak{L}$, and $\epsilon > 0$. Let K be an $\epsilon/2$ compact, $\tilde{K}$ a measurable envelope of K, and $\delta = \epsilon/2$. Then for every trigonometric polynomial q satisfying the conditions of the theorem with K and δ selected as above

$$|\sum c_k \chi(x_k^*)| \leqq \int_{\tilde{K}} |q(x)|\, dP + \int_{X \setminus \tilde{K}} |q(x)|\, dP < \epsilon. \tag{59}$$

PROOF OF SUFFICIENCY.

LEMMA 6. *Let Q be the probability distribution in $R^n = \{\alpha\}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$. If the compact $\hat{K}$ is such that for every trigonometric polynomial*

$$\hat{q}(\alpha) = \sum c_k e^{i(t^k, \alpha^k)}, \qquad t^k, \alpha^k \in R^n, \tag{60}$$

the inequalities

$$\sup_{\alpha} |\hat{q}(\alpha)| \leqq 1, \qquad \sup_{\alpha \in K} |\hat{q}(\alpha)| < \delta, \tag{61}$$

imply the inequality

$$\left| \int \hat{q}(\alpha)\, dQ \right| < \epsilon, \tag{62}$$

then

$$1 - Q(K) \leqq \epsilon. \tag{63}$$

PROOF. Lemma 6 is based on the fact that by means of the classical Weierstrass theorem (see, for example [3], chapter 1), we can construct a sequence $\tilde{q}_m(\alpha)$ of trigonometric polynomials with the properties (a) $0 \leqq \tilde{q}_m(\alpha) \leqq 1$, (b) as $m \to \infty$ we have for every α

$$\tilde{q}_m(\alpha) \to \varphi_k(\alpha) = \begin{cases} 1, & \alpha \in K, \\ 0, & \alpha \notin K, \end{cases} \tag{64}$$

where the convergence on K is uniform.

Suppose now that the ch.f. of the weak distribution P satisfies the conditions of theorem 5. Let $\lambda = (x_1^*, x_2^*, \cdots, x_n^*)$. For an arbitrary $\epsilon > 0$ and a corresponding $K \subset X$ and $\delta > 0$ let

$$\hat{q}(\alpha) = \sum c_k e^{i(t^k,\alpha^k)} \tag{65}$$

be a trigonometric polynomial satisfying the conditions

$$\sup_{\alpha} |\hat{q}(\alpha)| \leqq 1, \qquad \sup_{\alpha \in \pi_\lambda(K)} |\hat{q}(\alpha)| < \delta. \tag{66}$$

Letting

$$q(x) = \hat{q}[\pi_\lambda(x)] = \sum c_k e^{iy_k^*(x)}, \tag{67}$$

where $y_k^* = t_1^k x_1^* + \cdots + t_n^k x_n^*$, we obtain a trigonometric polynomial $q(x)$ in X satisfying the conditions of the theorem. Therefore

$$\left| \int_{R^n} \hat{q}(\alpha)\, dP^\lambda \right| = \left| \sum c_k \psi^\lambda(t^k) \right| = \left| \sum c_k \chi(y_k^*) \right| < \epsilon, \tag{68}$$

and by lemma 6

$$1 - P^\lambda[\pi_\lambda(K)] \leqq \epsilon. \tag{69}$$

It remains to apply lemma 3.

6. Compactness and tightness of families of distributions

We denote by $\mathcal{P}$ the set of all Baire probability distributions in a locally convex space $(X, \mathfrak{T})$. A weak topology can be defined in $\mathcal{P}$, for example, by means of a system of neighborhoods. An arbitrary neighborhood of the point $P_0 \in \mathcal{P}$ is determined by giving an $\epsilon > 0$ and functions $g_1, g_2, \cdots, g_n \in C$,

$$N(P_0, g_1, \cdots, g_n, \epsilon) = \bigcap_{j=1}^{n} \left\{ P: \left| \int g_j\, dP - \int g_j\, dP_0 \right| < \epsilon \right\}. \tag{70}$$

Correspondingly, a weak convergence $P_n \Rightarrow P$ is introduced as the convergence

$$\int g\, dP_n \to \int g\, dP_0 \tag{71}$$

for every function $g \in C$ (see [1] and [2]). We shall be interested in the relative compactness of subsets of $\mathcal{P}$ and in particular of sequences of elements of $\mathcal{P}$, more specifically not of $\mathcal{P}$ itself but of its subset $\mathcal{P}_t$ composed of all tight distributions.

The connection between the relative compactness of the subsets of $\mathcal{P}$ and the properties of the ch.f. is established by means of the concept of *tightness* of the sets of measures.

DEFINITION 3. *The set* $\{P_\alpha\}$ *of distributions in* X *is called tight if* (a) *every* P_α *is tight,* (b) *for every* $\epsilon > 0$, *the same* ϵ*-compact* K_ϵ *can be selected for all* α.

In the simplest case, for example, when $(X, \mathfrak{T})$ is a separable Banach space, the tightness of the family $\{P_\alpha\}$ is equivalent to its relative weak compactness. But, in general, this is not the case. The relation between these two concepts is studied in detail by Le Cam [1] and Varadarajan [2]. We give below three theorems taken from these papers.

THEOREM 6. *Let $(X, \mathfrak{J})$ be a topological space and let $\mathfrak{M}$ be a tight set of Baire measures. Then $\mathfrak{M}$ is relatively weakly compact and its closure is again tight* (see [2], theorems 6.4.1 and 6.3.4).

THEOREM 7. *Let $(X, \mathfrak{J})$ be a hemicompact k-space* [see properties (1) to (6) after equation (15)] *and let $\{P_n; n = 0, 1, 2, \cdots\}$ be a sequence of Baire measures. If*

$$P_n \Rightarrow P_0 \tag{72}$$

then the set $\{P_n; n = 0, 1, 2, \cdots\}$ is tight (see [1], theorem 9.2).

REMARK. It is easily seen that in $(X, \mathfrak{J})$ every Baire measure is automatically tight.

THEOREM 8. *Let $(X, \mathfrak{J})$ be a metrizable space. The closed set $\mathfrak{M} \subset \mathcal{P}_t$ is compact if and only if it is tight* (see [2], theorem 6.4.4).

From theorem 6 and lemma 3 we deduce

COROLLARY 1. *If condition* (30) *of lemma* 3 *is satisfied for the set of distributions $\{P_\alpha\}$ in the locally convex space $(X, \mathfrak{J})$ uniformly in α,*

$$1 - P_\alpha^\lambda[\pi_\lambda(C_\epsilon)] < \epsilon, \tag{30'}$$

where the $\mathfrak{J}$-compact C_ϵ does not depend on α, then $\{P_\alpha\}$ is relatively weakly compact.

From this in turn follows a sufficient condition for $\chi(x^*)$ to be a ch.f. of some $\mathfrak{J}$-tight distribution.

THEOREM 9. *If $\{P_n; n = 1, 2, \cdots\}$, is a sequence of weak distributions which satisfy the conditions of lemma* 3 *uniformly and if for every $x^* \in X^*$, we have $\chi(x^*, P_n) \to \chi(x^*)$ then there exists a $\mathfrak{J}$-tight measure P_0 such that*

$$\chi(x^*) = \chi(x^*, P_0) \tag{73}$$

and $P_n \Rightarrow P_0$.

As a consequence of this we can obtain theorems of Mourier ([14], chapter 3, theorem 8), Le Cam ([1], theorem 10), and Getoor ([10], theorems 5 and 6). In all cases when the estimate of the probabilities

$$1 - P^\lambda[\pi_\lambda(C_\epsilon)], \tag{74}$$

appearing in lemma 3, can be achieved by means of the ch.f. we can obtain from theorem 9 theorems of the following type.

THEOREM 10. [17]. *Let $(Y, \mathcal{H})$ be a separable Hilbert space $X = Y^*$* [see remark following equation (33) concerning example 2] *and let $\{P_\alpha\}$ be a family of distributions in X. If the corresponding ch.f. are equicontinuous at zero in the $\mathcal{G}$-topology, then $\{P_\alpha\}$ is tight and therefore is weakly relatively compact.*

REMARK. It can be shown that in a separable Hilbert space $(Y, \mathcal{H})$ there does not exist any locally convex topology $\mathcal{G}'$ with the property that the equicontinuity of χ_α for $\mathcal{G}'$ is equivalent to the tightness of the family $\{P_\alpha\}$ or, what is the same thing, to the relative compactness of $\{P_\alpha\}$; here we have in mind a strong topology in X.

This sharply distinguishes the case of the Hilbert space from the finite-dimensional case. Of all the countably Hilbertian spaces only the nuclear space $(Y, \mathcal{H})$

has the property that there exists a topology, namely $\mathfrak{K}$ itself, such that the equicontinuity of $\chi(y, P_\alpha)$ in it is equivalent to a weak sequential compactness of the set of distributions P_α in $(X, \mathfrak{I}_c)$.

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