CHARACTERIZATION OF SAMPLE FUNCTIONS OF STOCHASTIC PROCESSES BY SOME ABSOLUTE PROBABILITIES

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1. Introduction

The investigation of properties of sample functions of different stochastic processes has attracted much attention. It seems, indeed, that the characterization of a stochastic process is not exhaustive unless such basic properties of sample functions as continuity, kind of discontinuity (if any), integrability, and so on, are known. The most advanced investigations in this field are for certain particularly distinguished classes of stochastic processes, namely Markov processes, processes with independent increments, and martingales. There are important results due to Doeblin, Doob, Lévy, Wiener, and others. For arbitrary stochastic processes, without the assumption that they belong to some traditionally distinguished class of stochastic processes, conditions have been given, expressed in terms of the moments of the random variables of the processes considered, under which almost all sample functions are continuous (Kolmogorov, [11]) or have no discontinuities of the second kind [12]. The author [7] has given conditions expressed in terms of some absolute probabilities under which almost all sample functions of the process are jump functions with a finite expected number of discontinuities. The object of this note is to strengthen these results and to obtain other related results.

2. Notation and summary

We consider the real separable (see p. 51, [4]) stochastic process $\{x_t, t \in I_0\}$ where I_0 is a closed interval. We denote by Ω the set of elementary events ω, by $\mathcal{B}$ the smallest Borel field of ω-sets with respect to which all the random variables x_t where $t \in I_0$, are measurable, and by P the probability measure on $\mathcal{B}$. As is known [9], the Borel field $\mathcal{B}$ is generated by the aggregate of ω-sets of the form $\{[x_{t_1}(\omega), \cdots, x_{t_n}(\omega)] \in A\}$, where A is any right semiclosed interval, and $(t_1, \cdots, t_n)$ is any finite set of values of $t \in I_0$ and $n = 1, 2, \cdots$. The measure P on $\mathcal{B}$ is uniquely determined by the finite-dimensional probability measures of ω-sets of the above form. Denote by x_I the increment of x_t in the interval $I \subset I_0$ and by $|I|$ the length of I. We shall consider the following functions.

$$a(I) = P\{x_I \neq 0\}, \tag{1}$$

$$b(I, \epsilon) = P\{|x_I| > \epsilon\}, \tag{2}$$

$$A(I) = \int_I a(J) = \lim_{n\to\infty} \sum_{k=1}^{n} a(I_{nk}) \tag{3}$$

$$B(I, \epsilon) = \int_I b(J, \epsilon) = \lim_{n\to\infty} \sum_{k=1}^{n} b(I_{nk}\epsilon) \tag{4}$$

as $\max_{1\leq k\leq n} |I_{nk}| \to 0$, where $\{I_{nk}\}$ is a partition of I into nonoverlapping intervals I_{nk}, with $k = 1, 2, \cdots, n$. The expressions $A(I)$ and $B(I, \epsilon)$ are the Burkill integrals of $a(I)$ and $b(I, \epsilon)$ respectively. The corresponding upper Burkill integrals $\bar{A}(I)$ and $\bar{B}(I, \epsilon)$ are obtained by replacing in (3) and (4) the symbol lim by $\overline{\lim}$. Finally we define

$$Q(t) = \lim \frac{a(I)}{|I|} \tag{5}$$

as I contracts to a fixed point $t \in I$.

Conditions expressed in terms of the functions (1) to (5) are given under which almost all sample functions of the process

(a) are jump functions where locally monotonic, as defined below,

(b) have no discontinuity of the first kind at a given but arbitrary point $t \in I_0$,

(c) are jump functions without fixed discontinuities and with a finite expected number of discontinuities equal to $\bar{A}(I)$,

(d) are constant on the whole interval I_0,

(e) have no discontinuities of the first kind.

A condition is given in terms of the covariance function of a process stationary in the wide sense, under which almost all sample functions are continuous.

3. General results

We begin with the following definitions.

DEFINITION 1. *The sample function $x_t(\omega)$ will be called a jump function where locally monotonic (briefly, a jump function wlm) if at any continuity point $t \in I_0 - S(\omega)$ of $x_t(\omega)$, where $S(\omega)$ is some possibly empty set nowhere dense in I_0, at which $x_t(\omega)$ is locally monotonic, the equality $x_{t'}(\omega) = x_t(\omega)$ holds for all t' of some interval $(t - \theta(t, \omega), t + \theta(t, \omega))$.*

DEFINITION 2. *The sample function $x_t(\omega)$ will be called a jump function if at any continuity point $t \in I_0$ of $x_t(\omega)$ the equality $x_{t'}(\omega) = x_t(\omega)$ holds for all t' from some interval $(t - \theta(t, \omega), t + \theta(t, \omega))$.*

It is evident how definitions 1 and 2 should be modified for the endpoints of the interval I_0.

THEOREM 1. *Let the process* $\{x_t, t \in I_0\}$ *be separable and let the relation*

$$\lim_{|I| \to 0} a(I) = 0 \tag{6}$$

hold where $I \subset I_0$. *Then almost all* (P) *sample functions of the process are jump functions wlm.*

PROOF. Let relation (6) hold. We can then find for any fixed point $t \in I_0$ a sequence of parameter points $t'_r \neq t$ for $r = 1, 2, \cdots$, converging to t and such that the relations

$$P\{x_t - x_{t'_r} \neq 0\} < \frac{1}{r^2} \tag{7}$$

hold for $r = 1, 2, \cdots$. In virtue of the Borel-Cantelli lemma, the ω-set Λ_t which is such that for any $\omega \in \Lambda_t$ there exists an $R(\omega)$ such that for all $r > R(\omega)$ the equality $x_t(\omega) = x_{t'_r}(\omega)$ holds, satisfies the equality

$$P(\Lambda_t) = 1. \tag{8}$$

Denote by Ξ_t the ω-set such that for any $\omega \in \Xi_t$ there exists for every $\theta > 0$ a point $t'(\omega)$ of the interval $(t - \theta, t + \theta)$ satisfying the equality $x_t(\omega) = x_{t'}(\omega)$. Since $\Lambda_t \subset \Xi_t$ we have by virtue of (8) the equality

$$P(\Xi_t) = 1. \tag{9}$$

Let now $T = \{t_j\}$ for $j = 1, 2, \cdots$, be some denumerable dense set in I_0. Relation (9) implies

$$P\left(\bigcap_{j=1}^{\infty} \Xi_{t_j}\right) = 1. \tag{10}$$

Since the process is separable and, in virtue of (6), T satisfies the separability conditions it follows from (10) that almost all sample functions $x_t(\omega)$ are jump functions *wlm.*

Before formulating theorem 2 let us remember that the function $x_t(\omega)$ is said to have a discontinuity of the first kind at some point t if $x_t(\omega)$ has at t both onesided limits and they are not equal. If at least one onesided limit at t does not exist the discontinuity is said to be of the second kind.

THEOREM 2. *Let* $\{x_t, t \in I_0\}$ *be a real, separable stochastic process and let the relation*

$$\lim_{|I| \to 0} b(I, \epsilon) = 0 \tag{11}$$

hold for any $\epsilon > 0$, *where* $I \subset I_0$. *Then almost all sample functions have no discontinuity of the first kind at any fixed but arbitrary point* $t \in I_0$.

PROOF. Let relation (11) hold and let the assertion of the theorem not be true. We have then for some point $t \in I_0$ (different from both endpoints of I_0) the equality

$$P\{\lim_{t\uparrow t_0} x_t = x_{t_0-0} \neq x_{t_0+0} = \lim_{t\downarrow t_0} x_t\} = \alpha, \tag{12}$$

where $\alpha > 0$.

Denote by $J \subset I_0$ an interval having t_0 as its midpoint, by $\Lambda_n(J)$ an ω-set for which $|x_J(\omega)| > 1/n$, and by $\Lambda(J)$ an ω-set for which $x_J(\omega) \neq 0$. The sequence $\{\Lambda_n(J)\}$ is increasing, hence

$$\begin{aligned} \Lambda(J) &= \bigcup_{n=1}^{\infty} \Lambda_n(J) = \lim_{n\to\infty} \Lambda_n(J), \\ P\{x_J \neq 0\} &= P\{\Lambda(J)\} = \lim_{n\to\infty} P\{\Lambda_n(J)\}. \end{aligned} \tag{13}$$

Taking into account (12) and (13) we obtain $\alpha \leqq \lim_{|J|\to 0} P\{x_J \neq 0\} = \lim_{|J|\to 0} \lim_{n\to\infty} P\{\Lambda_n(J)\}$. Consequently for $|J|$ sufficiently small and n sufficiently large, that is, for ϵ sufficiently small, $P\{\Lambda_n(J)\}$ will be at least near α, which contradicts relation (11). Hence $\alpha = 0$, and theorem 2 is thus proved.

Let us remark that if (11) holds, any set of parameter points dense in I_0 satisfies the separability conditions, and consequently for any fixed but arbitrary $t \in I_0$ the equation

$$P\{x_{t-0} = x_{t+0} \neq x_t\} = 0 \tag{14}$$

holds. Now let (11) be satisfied and let the process $\{x_t, t \in I_0\}$ have a fixed discontinuity point t', that is, for some $\beta > 0$, the equation

$$P(\lim_{t\to t'} x_t = x_{t'}) = 1 - \beta \tag{15}$$

holds. We then obtain from theorem 2 and (14) the following.

COROLLARY. *Let the real, separable process $\{x_t, t \in I_0\}$ satisfy* (11) *and let it have a fixed discontinuity point $t' \in I_0$. Then the probability that x_t has at t' a discontinuity of the second kind equals β where β is given by relation* (15).

Theorem 3 which we shall now formulate is a stronger version of some results obtained by the author [7]. It was assumed in that paper that the process under consideration had no fixed discontinuities. Now this assumption is entirely omitted in part (b) and is replaced in part (a) of theorem 3 by the weaker assumption (11). The former assumption is now obtained as a consequence (ii). In obtaining these strengthened results essential use will be made of the corollary to theorem 2.

THEOREM 3. *Let the stochastic process $\{x_t, t \in I_0\}$ be separable.*

(a) *If relation* (11) *holds for any $\epsilon > 0$ and if moreover*

$$\bar{A}(I_0) < \infty, \tag{16}$$

then for every open interval $I \subset I_0$,

(i) *almost all* (*P*) *sample functions of the process are jump functions,*

(ii) *the process has no fixed points of discontinuity,*

(iii) *the relation*

$$E\xi(I) = \bar{A}(I) = A(I) \tag{17}$$

holds, where $\xi(I)$ is the number of discontinuities of $x_t(\omega)$ on the interval I,

(iv) *the ω-set such that at the discontinuities, if any, of $x_t(\omega)$ both onesided limits exist, has probability* 1,

(v) *the function $Q(t)$ exists almost everywhere in I and satisfies the relation*

$$\int_I Q(t)\, dt \leqq A(I). \tag{18}$$

(b) *If $a(I)$ is an absolutely continuous function of an interval or $a(I)$ satisfies the Lipschitz condition, then the assertions* (i) *to* (v) *are satisfied, and moreover*

$$\int_I Q(t)\, dt = A(I). \tag{19}$$

Proof. In the proof of assertion (i) given formerly in [7] the assumption that the process has no fixed points of discontinuity was used only for stating that any denumerable dense set of points $t \in I$ satisfies the separability conditions. Now relation (11) is sufficient for this purpose and therefore assertion (i) is proved. Moreover, in proving assertion (i), in [7] it was also proved that almost every sample function $x_t(\omega)$ has a finite number of discontinuities. Hence by virtue of relation (11) and the corollary to theorem 2, the process has no fixed points of discontinuity, which proves assertion (ii). Once this fact has been established the proof of the remaining assertions of part (a) given in [7] remains valid.

In order to prove part (b) of the theorem we remark that if $a(I)$ satisfies the Lipschitz condition it is (see p. 287, [2]) an absolutely continuous function of an interval and this implies (see p. 287, [2]) that relation (16) holds. Since (11) evidently holds then also, all the assumptions of part (a) are fulfilled and consequently so are all its assertions. Moreover the absolute continuity of $a(I)$ implies (see p. 289, [2]) the absolute continuity of $A(I)$, hence relation (19) is true. Theorem 3 is thus proved.

Theorem 4. *Let $\{x_t, t \in I_0\}$ be a real, separable stochastic process. The relation*

$$P\{x_t = \text{const.}, t \in I_0\} = 1 \tag{20}$$

holds if and only if the relation

$$Q(t) \equiv 0 \tag{21}$$

holds uniformly with respect to $t \in I_0$.

Proof. Let relation (21) hold uniformly with regard to $t \in I_0$. We shall show that $a(I)$ is then an absolutely continuous function of an interval. Indeed, relation (21) implies that for any $\alpha > 0$ there exists such a $\beta > 0$ that if $|I| < \beta$ the inequality $a(I) < \alpha|I|$ is true. Let us now divide the interval I into a finite number, say n, of nonoverlapping intervals I_k with $|I_k| < \beta$ for $k = 1, 2, \cdots, n$. The function $a(I)$ then satisfies the Lipschitz condition in every interval I_k and consequently $a(I)$ is an absolutely continuous function of an interval in I_k. It then follows that $a(I)$ has the same property in the whole interval I_0. By virtue of part (b) of theorem 3 the relation (19) holds. Taking into account (21) we obtain $A(I) = 0$, which in virtue of (17) implies relation (20). The "if" assertion is thus proved, the "only if" assertion is evident.

Let us remark that if we assume that for any $\epsilon > 0$ relation (11) holds and moreover $\overline{B}(I, \epsilon) < \infty$, we cannot obtain any new properties of the sample functions of the process in addition to the property established in theorem 2. However, the following theorem is true.

THEOREM 5. *Let the stochastic process* $\{x_t, t \in I_0\}$ *be separable and let the relation*

$$\overline{B}(I, \epsilon) = 0 \tag{22}$$

hold for any $\epsilon > 0$. *Then almost all* (P) *sample functions* $x_t(\omega)$ *have no discontinuities of the first kind.*

PROOF. Let us at first remark that relation (11) follows from relation (22). Let $T = \{t_j | j = 1, 2, \cdots\}$ be a denumerable dense set in I_0 and let $\tau_{n1}, \cdots, \tau_{nn}$ for $n = 1, 2, \cdots$ denote the points $t_1, \cdots, t_n$ arranged in increasing order, $\tau_{n1} < \cdots < \tau_{nn}$. Denote by I_{nk} the interval $[\tau_{nk}, \tau_{n,k+1})$ and by $\rho_n(T, \epsilon)$ the number of intervals I_{nk} for which $|x_{I_{nk}}| > \epsilon$. We have

$$E\rho_n(T, \epsilon) = \sum_{k=1}^{n} b(I_{nk}, \epsilon). \tag{23}$$

It follows from relation (22) that as $\max_{1 \leqq k \leqq n} |I_{nk}| \to 0$ we have $\lim_{n\to\infty} E\rho_n(T, \epsilon) = 0$. The last relation implies the existence of a subsequence, $n_j \uparrow \infty$, for which the relation

$$P\{\lim_{j\to\infty} \rho_{n_j}(T, \epsilon) = 0\} = 1 \tag{24}$$

is true. Since ρ_n can take only noninteger values, relation (24) implies that the ω-set $\Lambda(\epsilon)$, which is such that for any $\omega \in \Lambda(\epsilon)$ there exists a j_0 which may depend on ϵ and ω such that for $j > j_0$ the equality $\rho_{n_j}(T, \epsilon) = 0$ holds, satisfies the relation $P\{\Lambda(\epsilon)\} = 1$. Hence, taking into account the method of constructing the intervals I_{nk}, we see that almost all $(P) x_t(\omega)$ have no discontinuities of the first kind with a jump greater than ϵ. Consider now a sequence of positive constants $\{\epsilon_n\}$ where $\epsilon_n \downarrow 0$. The set Λ, such that the $x_t(\omega)$ corresponding to $\omega \in \Lambda$ have no discontinuities of the first kind, is given by the formula

$$\Lambda = \bigcap_{n=1}^{\infty} \Lambda(\epsilon_n). \tag{25}$$

Consequently, since $P\{\Lambda(\epsilon_n\} = 1$, we obtain $P\{\Lambda) = 1$. Theorem 5 is thus proved.

Let us remark that, as Dobrushin [3] has proved, the assertion of theorem 5 holds under the assumption that as $\Delta t \to 0$ the relation

$$\sup_{\substack{t \in I_0 \\ t+\Delta t \in I_0}} P\{|x_{t+\Delta t} - x_t| > \epsilon\} = o(\Delta t) \tag{26}$$

holds for any $\epsilon > 0$. We shall show that relation (26) implies relation (22). Indeed it follows from (26) that for any $\eta > 0$ there exists such an $\alpha > 0$ that for any $t \in I_0$, $t + \Delta t \in I_0$ with $|\Delta t| \leqq \alpha$ we have $P\{|x_{t+\Delta t} - x_t| > \epsilon\} < \eta|\Delta t|$. In other words for sufficiently large n we shall have for $k = 1, \cdots, n$ the relation $b(I_{nk}, \epsilon) < \eta|I_{nk}|$ and consequently

$$\sum_{k=1}^{n} b(I_{nk}, \epsilon) < \eta |I_0|. \tag{27}$$

Since η is arbitrarily small we obtain relation (22).

It is easy, however, to show that relation (26) is a more restrictive condition than (22). Indeed the following example shows that (22) may be satisfied although (26) is not.

EXAMPLE. Let $\Omega = \{\omega_j | j = 1, 2, \cdots\}$ and let $P(\omega_j) = 1/2^j$. Let $x_t(\omega_j)$ be given by the formula

$$x_t(\omega_j) = \begin{cases} 2^j t, & 0 \leqq t \leqq 1/2^j, \\ 1, & 1/2^j < t \leqq 1. \end{cases} \tag{28}$$

It is evident that for this process relation (26) does not hold. We shall show that relation (22) holds. Let $\eta > 0$ be an arbitrary number. We have to show that for sufficiently small $\max_{1 \leqq k \leqq n} |I_{nk}|$ the inequality

$$\sum_{k=1}^{n} P\{|x_{I_{nk}}| > \epsilon\} < \eta \tag{29}$$

will be true. We have

$$\sum_{k=1}^{n} P\{|x_{I_{nk}}| > \epsilon\} = \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{k=1}^{n} P\{|x_{I_{nk}}| > \epsilon | x_t = x_t(\omega_j)\}. \tag{30}$$

Let j_0 be an integer such that $2^{-j_0} < \epsilon\eta$ and consider the partition $\{I_{nk}\}$ of the interval $[0, 1]$ such that $\max_{1 \leqq k \leqq n} |I_{nk}| < 2^{-j_0}\epsilon$. We have then for $j = 1, 2, \cdots, j_0$

$$P\{|x_{I_{nk}}| > \epsilon | x_t = x_t(\omega_j)\} = 0. \tag{31}$$

Consequently

$$\sum_{k=1}^{n} P\{|x_{I_{nk}}| > \epsilon\} = \sum_{j=j_0+1}^{\infty} \frac{1}{2^j} \sum_{k=1}^{n} P\{|x_{I_{nk}}| > \epsilon | x_t = x_t(\omega_j)\}. \tag{32}$$

Since, as follows from the definition of the $x_t(\omega_j)$, the second sum, where k runs from 1 to n, on the right side of (32) can only be at most equal to $1/\epsilon$, and taking into account the method of choosing j_0, relation (32) implies relation (29). Hence relation (22) holds.

4. Particular cases

We shall deal in this section with stochastic processes of certain special types.

THEOREM 6. *Let $\{x_t, t \in I_0\}$ be a real, separable martingale and let relation* (22) *hold. Then almost all* (P) *sample functions $x_t(\omega)$ of the process are continuous.*

PROOF. The assertion of the theorem follows from theorem 5 and from a theorem of Doob (see chapter 7 [4]) stating that the set of sample functions $x_t(\omega)$ of a separable martingale which have only discontinuities of the first kind, if any, has probability 1.

Dobrushin [3] has proved the assertion of theorem 6 under the stronger assumption (26).

We shall now strengthen slightly a theorem of Dynkin [5] and Kinney [9]. Denote

$$C(h, \epsilon) = \sup_{\substack{t \in I_0 \\ t+\Delta t \in I_0 \\ |\Delta t| \leq h \\ -\infty < x < \infty}} P\{|x_{t+\Delta t} - x_t| > \epsilon | x_t = x\}, \tag{33}$$

$$D(h, \epsilon) = \sup_{\substack{t \in I_0 \\ t+\Delta t \in I_0 \\ |\Delta t| \leq h}} \int_{-\infty}^{\infty} P\{|x_{t+\Delta t} - x_t| > \epsilon | x_t = x\} \, dF_t(x), \tag{34}$$

where $F_t(x) = P(x_t < x)$.

THEOREM 7. *Let* $\{x_t, t \in I_0\}$ *be a separable Markov process. If as* $h \rightarrow 0$ *the relation*

$$D(h, \epsilon) = o(h) \tag{35}$$

holds, then almost all sample functions of the process are continuous.

PROOF. We have

$$P\{|x_{t+\Delta t} - x_t| > \epsilon\} = \int_{-\infty}^{\infty} P\{|x_{t+\Delta t} - x_t| > \epsilon | x_t = x\} \, dF_t(x). \tag{36}$$

Relations (35) and (36) imply relation (26), hence, a fortiori, relation (22). By a theorem of Fuchs [8], the assertion of theorem 7 is obtained.

We remark that Dynkin and Kinney have proved the assertion of theorem 7 under the more restrictive assumption that $C(h, \epsilon) = o(h)$.

Theorem 8 below is known. However, we shall give a simple proof using tools applied in this paper.

THEOREM 8. *Let* $\{x_t, t \in I_0\}$ *be a separable stochastic process with independent increments and let relation* (6) *hold. Then the assertions* (i) *to* (v) *of theorem* 3 *are true.*

PROOF. It has been proved in [6] that the independence of increments and relation (6) imply the relation (16). Consequently if the assumptions of theorem 8 are satisfied, all the assumptions of part (a) of theorem 3 are satisfied and hence all its assertions are true.

THEOREM 9. *Let* $\{x_t, -\infty < t < +\infty\}$ *be a real, separable stochastic process, stationary in the wide sense, and let the covariance function* $R(\tau)$ *be for* $\tau \rightarrow 0$ *of the form*

$$R(\tau) = D^2(x_0) + O(|\tau|^{1+\delta}), \tag{37}$$

where $\delta > 0$. *Then almost all sample functions of the process are continuous.*

PROOF. We have for any real t and τ the relation

$$E(x_{t+\tau} - x_t)^2 = 2[D^2(x_0) - R(\tau)]. \tag{38}$$

Taking into account (37) we obtain for arbitrary t

$$E(x_{t+\tau} - x_t)^2 = O(|\tau|^{1+\delta}). \tag{39}$$

By the result of Kolmogorov (published in [11]), relation (39) implies that almost all sample functions $x_t(\omega)$ of the process are continuous in every finite closed interval. Theorem 9 is thus proved.

We remark that if the covariance function $R(\tau)$ is twice differentiable at $\tau = 0$, then almost all sample functions of the process are absolutely continuous (see p. 536, [41]).

We remark finally that, as Belayev [1] has shown, if the process $\{x_t, -\infty < t < \infty\}$ is real, separable, stationary, and Gaussian, and if the relation

$$R(\tau) = D^2(x_0) + O\left(\frac{1}{\log |\tau|^{1+\delta}}\right) \tag{40}$$

with $\delta > 0$ holds, then almost all sample functions are continuous. Relation (40) is evidently less restrictive than relation (37); however, Belayev considers only Gaussian stationary processes.

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