STOCHASTIC APPROXIMATION

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1. Introduction

The purpose of this paper is to review the development of the so-called Robbins-Monro (RM) process. Moreover, some of the results presented here seem to be new. A summary of some results in stochastic approximation, including papers up to 1956, has been given by C. Derman [1].

The idea of stochastic approximation had its origin in the framework of sequential design (H. Robbins [2]). There are not only important applications in fields like biology, metallurgy, and so on, but also it is becoming increasingly clear that stochastic approximation is related to interesting questions in other fields of mathematics.

Let us first recall the well-known classical approach to the iterative solution of an equation of the simplest type. Suppose that M is a mapping from Euclidean space R_1 into R_1 and let α be a real number. We are interested in solutions of the equation

$$M(x) = \alpha. \tag{1}$$

It is well known that under weak assumptions on M the following is true. Let x_1 be any real number. Let us define a sequence x_n by induction,

$$x_{n+1} = x_n + a_n[\alpha - M(x_n)], \qquad n \geqq 1, \tag{2}$$

where a_n is a given sequence of real numbers which has to satisfy some conditions not enumerated here. Then x_n converges to a solution of (1); moreover, this solution is the one with the smallest distance from x_1 (R. von Mises and H. Pollaczek-Geiringer [3]).

In many practical applications it happens that the function M is only empirically given; that is to say, for every real number x the value of the function $M(x)$ is subject to an error. We suppose that for every x this error can be represented by a random variable $y(x)$ with distribution function F_x in such a way that $M(x)$ is the mathematical expectation of $y(x)$ for every real number x, so that M can be considered as a regression line. The problem is again to find a solution of equation (1). Whether one knows the error law given by F_x or not, the procedure given by (2) does not work, because we have made the assumption that $M(x)$ cannot be determined exactly. What we really can obtain is a realization for every real x of the random variable $y(x)$ whose mathematical expectation is $M(x)$. Under these circumstances one could try to define, instead of the sequence of real numbers given by (2), a sequence of random variables which converges, for instance, in probability or with probability 1, to a solution of (1). This objective was achieved by Robbins and Monro, who invented the so-called stochastic approximation method which modifies (2) for random variables. This work makes it possible to find under weak assumptions interesting convergence theorems of the kind mentioned above, and gives other interesting results. Moreover, it should be pointed out that stochastic approximation methods are of the nonparametric type.

The following assumptions are made throughout this paper. (R, S, P) is a fixed probability field and a typical element of R is denoted by ω. Let $y(x)$ be a set of random variables, $-\infty < x < \infty$ [that is, a set of real-valued S-measurable mappings $\omega \rightarrow y(x, \omega)$], whose expectations exist for every x and are equal, say, to $M(x)$. M might satisfy different conditions from case to case. $C_1, C_2, \cdots$ denote positive constants, $g_1, g_2, \cdots$ denote arbitrary constants throughout.

2. Convergence theorems

2.1. *The Robbins-Monro method.* In their pioneering paper Robbins and Monro [4] define a nonstationary real-valued Markov chain x_n in the following manner. Let α be a real number and let a_n be a sequence of real positive numbers. Let x_1 be an arbitrary random variable, and define for $n \geqq 1$ and every $\omega \in R$,

$$x_{n+1}(\omega) = x_n(\omega) + a_n[\alpha - y_n(\omega)], \tag{3}$$

where y_n is a random variable whose conditional distribution, given $x_1, \cdots, x_n$, coincides with the distribution of $y(x_n)$. It follows that

$$E(y_n|x_n) = M(x_n), \qquad n \geqq 1. \tag{4}$$

The following results of [4] are cited here as the historically first theorems.

THEOREM 1. *Suppose that there exists a C_1 such that*

$$P\{|y(x)| < C_1\} = 1 \tag{5}$$

for all real x. Suppose further that M satisfies the following conditions. There exists a real number ϑ and an $\eta > 0$ such that

$$M(x) < \alpha - \eta \quad \text{for} \quad x < \vartheta \quad \text{and} \quad M(x) > \alpha + \eta \quad \text{for} \quad x > \vartheta, \tag{6}$$

and there exist C_2, C_3 such that $C_2/n \leqq a_n \leqq C_3/n$ for $n \geqq 1$. Then, if x_1 is of finite variance,

$$E[(x_n - \vartheta)^2] \rightarrow 0 \quad \text{when} \quad n \rightarrow \infty. \tag{7}$$

(This implies of course the convergence of x_n to ϑ in probability.)

Robbins and Monro also proved the following result.

THEOREM 2. *If* (6) *is replaced by the following conditions: M is nondecreasing,*

$$M(\vartheta) = \alpha, \tag{8}$$

and the first derivative of M at ϑ exists and is > 0, then the conclusion of theorem 1 remains valid.

J. Wolfowitz [5] showed, in answer to a question raised in [4], that convergence in probability can be proved under weaker conditions.

From the point of view of a statistician who is mainly interested in applications these two theorems are quite sufficient. From a more mathematical point of view questions about convergence of x_n with probability 1 are of interest. The first theorem concerned with convergence with probability 1 was given by J. R. Blum [6] (see also G. Kallianpur [7]). Using and extending Blum's method we can prove

THEOREM 3. *Suppose that the following conditions are satisfied.*

$$\text{(9)} \qquad M \text{ is Borel-measurable}$$

and locally bounded. There exists a ϑ such that for every $\delta > 0$ and every $N > 0$

$$\text{(10)} \qquad \inf_{\delta \leqq \vartheta - x \leqq N} [\alpha - M(x)] > 0 \quad \text{and} \quad \inf_{\delta \leqq x - \vartheta \leqq N} [M(x) - \alpha] > 0.$$

There exists a C_4 such that

$$\text{(11)} \qquad E\{[y(x) - M(x)]^2\} \leqq C_4 \quad \text{for} \quad -\infty < x < \infty.$$

Further,

$$\text{(12)} \qquad \sum_{n=1}^{\infty} a_n = \infty,$$

and

$$\text{(13)} \qquad \sum_{n=1}^{\infty} a_n^2 < \infty.$$

Then either $x_n(\omega)$ converges to ϑ with probability 1 or for a set of positive probability $x_n(\omega)$ oscillates in such a manner that $\underline{\lim}\, x_n(\omega) = -\infty$, $\overline{\lim}\, x_n(\omega) = \infty$, and no finite limit point of $x_n(\omega)$ exists.

We shall indicate the proof, which uses the following important lemma.

LEMMA 1. *Let v_n be a sequence of random variables and assume that the conditional expectation $E(v_n|v_1, \cdots, v_{n-1}) = 0$, for $n \geqq 1$, with probability 1. If $\sigma^2(v_n)$, the variance of v_n, exists for $n \geqq 1$ and if $\sum_{n=1}^{\infty} \sigma^2(v_n)$ converges, then $\sum_{n=1}^{\infty} v_n$ converges with probability 1 to a random variable* [8].

Using (4), (9), (11), and (13) it can be seen that lemma 1 can be applied to the random variables $v_n = a_n[M(x_n) - y_n]$, so that

$$\text{(14)} \qquad \sum_{j=1}^{\infty} a_j[M(x_j) - y_j]$$

converges to a random variable with probability 1.

Now (3) gives for every $\omega \in R$

$$\text{(15)} \qquad x_n(\omega) - x_m(\omega) + \sum_{j=m}^{n-1} a_j\{M[x_j(\omega)] - \alpha\} = \sum_{j=m}^{n-1} a_j\{M[x_j(\omega)] - y_j(\omega)\}, \qquad n > m \geqq 1.$$

$x_n(\omega)$ cannot diverge in a set of positive probability measure to $+\infty$ or $-\infty$ because if it did then (10) and the convergence of (14) with probability 1 and (15) would together lead to a contradiction. Further, it follows easily from (15), using the convergence with probability 1 of (14) and the local boundedness of M, that each of the statements $\overline{\lim}\, x_n(\omega) = \infty$ and $-\infty < \underline{\lim}\, x_n(\omega) < \infty$ and $-\infty < \overline{\lim}\, x_n(\omega) < \infty$ and $\underline{\lim}\, x_n(\omega) = -\infty$ can only be true in a set of probability 0.

Suppose that for a set G of positive probability measure $g_1 = g_1(\omega) = \underline{\lim}\, x_n(\omega) < \overline{\lim}\, x_n(\omega) = g_2(\omega) = g_2$ is true. We treat only the case $\overline{\lim}\, x_n(\omega) \leqq \vartheta$, the case $\overline{\lim}\, x_n(\omega) > \vartheta$ being managed in exactly the same manner. It follows from (15) and the convergence of (14) with probability 1 that

$$x_n(\omega) - x_m(\omega) \geqq \sum_{j=m}^{n-1} a_j\{\alpha - M[x_j(\omega)]\} - \eta \tag{16}$$

with probability 1 for every $\eta > 0$ and all $m, n > N(\eta)$. Fix an $\omega \in G$ such that (16) is valid. There is a finite interval I which depends on g_1 and g_2 such that $x_n(\omega) \in I$ for all $m, n > N(\eta)$. Due to the local boundedness of M there exists a C_5 such that

$$\sup_{x \in I} |M(x)| = C_5. \tag{17}$$

As a consequence of (13) we can choose an $N_1(\epsilon) \geqq N(\eta)$ for every $\epsilon > 0$ such that

$$a_l \leqq \frac{\epsilon}{C_5 + |\alpha|} \quad \text{for all} \quad l \geqq N_1(\epsilon). \tag{18}$$

There always exist g_3 and g_4 with $g_3 < g_4 < \vartheta$ such that for infinitely many pairs of natural numbers $n, m \geqq N_1(\epsilon)$, we have $x_n(\omega) < g_3$, $x_m(\omega) > g_4$, and $x_j(\omega) \leqq g_4$ if $m < j < n$, where it may be that $n = m + 1$ and no such j exists. It follows that

$$x_n(\omega) - x_m(\omega) < g_3 - g_4. \tag{19}$$

Now using (10), from (16) we get

$$x_n(\omega) - x_m(\omega) \geqq -a_m\{|\alpha| + |M[x_m(\omega)]|\} - \eta, \tag{20}$$

and this is $\geqq -(\epsilon + \eta)$ by (17) and (18). This contradicts (19).

Next, suppose that for a set G of positive probability measure the following is true: $\overline{\lim}\, x_n(\omega) = \infty$, $\underline{\lim}\, x_n(\omega) = -\infty$, and there exists at least one finite limit point h. Let C_6, C_7 be such that $C_6 > \max(|\vartheta|, |h|)$ and $C_7 > C_6$ and let

$$C_8 = \sup_{x \in (-C_7, C_7)} |M(x)|. \tag{21}$$

Fix an $\omega \in G$ such that (14) converges. Then either there exist infinitely many pairs (n, m) of natural numbers such that $x_n(\omega) > C_7$ while $-C_7 \leqq x_m(\omega) < C_6$, and $x_j(\omega) \geqq C_6$ for $m < j < n$ or there exist infinitely many pairs (n, m) such that $x_n(\omega) < -C_7$, $-C_6 < x_m(\omega) \leqq C_7$, and $x_j(\omega) \leqq -C_6$ for $m < j < n$. It suffices to treat the first case. Exactly in the same manner as above we get on the one hand,

$$x_n(\omega) - x_m(\omega) > C_7 - C_6, \tag{22}$$

and on the other hand, $x_n(\omega) - x_m(\omega) \leqq a_m(|\alpha| + C_8) + \eta$, which contradicts (22) if a_m is small enough. Hence x_n converges with probability 1 to a random variable if it is not the case that $\overline{\lim}\, x_n(\omega) = \infty$ and $\underline{\lim}\, x_n(\omega) = -\infty$ in a set of positive probability measure. But this random variable must be equal to ϑ with probability 1; otherwise the convergence of $\sum_{j=1}^{\infty} a_j[M(x_j) - \alpha]$ to a random variable with probability 1, which now follows from (15), would lead to a contradiction with (12). Note that only for this last conclusion has condition (12) been used.

A counterexample given by Wolfowitz [5] shows that condition (11) cannot be omitted. Another counterexample also given in [5] shows that (10) cannot be removed. Moreover, it is easy to modify this counterexample in an obvious manner so as to show that the local boundedness of M is essential. However, we can give a further result which contains, so far as I know, all previously published theorems in this direction.

COROLLARY 1. *If the conditions of theorem 3 are satisfied and if further for* $-\infty < x < \infty$

$$M(x) \geqq g_5|x| + g_6 \quad (or \quad M(x) \leqq g_7|x| + g_8), \tag{23}$$

then the convergence of x_n to ϑ with probability 1 follows.

We observe first that (10) implies $g_5 \leqq 0$ or $g_7 \geqq 0$. Suppose for instance that $M(x) \geqq g_5|x| + g_6$ for $-\infty < x < \infty$ and that in a set of positive probability $x_n(\omega)$ has only the limit points ∞ and $-\infty$. Fix an ω such that (14) converges. Let n be so large that

$$a_n(|\alpha| + |g_6|) < \epsilon, \qquad a_n|g_5| < 1, \qquad |a_n\{M[x_n(\omega)] - y_n(\omega)\}| < \eta, \tag{24}$$

$$x_{n+1}(\omega) > C_9, \qquad x_n(\omega) < -C_9, \qquad C_9 > \epsilon + \eta. \tag{25}$$

Then it follows easily from (3) that $x_{n+1}(\omega) < \epsilon + \eta$, and this contradicts (25). It is easy to show that corollary 1 can be falsified if (23) is omitted (Dvoretzky [9]).

Condition (23) is in a certain sense a Tauberian condition, as can be seen by the following considerations. Suppose that $a_n = 1/n$, for $n \geqq 1$, that (14) converges with probability 1, and that nothing is known about M except (9) and (23). Suppose further that it is known that x_n is bounded with probability 1 and that $(x_1 + \cdots + x_n)/n$ converges to a random variable with probability 1. It follows from (15), from the convergence of (14) with probability 1, and from the regularity of the C_1-summability that $\sum_{j=1}^{\infty} [\alpha - \mathrm{M}(x_j)]/j$ is also C_1-summable with probability 1. From the boundedness of x_n with probability 1 and from (23) follows

$$\frac{1}{n}[\alpha - M(x_n)] < \frac{1}{n}(\alpha - g_5|x_n| - g_6) < \frac{C_{10}}{n}, \qquad n \geqq 1, \tag{26}$$

with probability 1. But this is a well-known one-sided Tauberian condition for C_1-summability which implies the convergence of $\sum_{j=1}^{\infty} [\alpha - M(x_j)]/j$ and so that of x_n with probability 1. The same conclusion holds if (23) and the boundedness of x_n with probability 1 are replaced by the one-sided boundedness of M.

2.2. *The Kiefer-Wolfowitz (KW) process.* Stimulated by the Robbins-Monro paper [4], J. Kiefer and J. Wolfowitz [10] constructed a process which enables the determination of an unknown maximum (or minimum) of a function. This process is exactly the stochastic version of a classical iterative procedure first defined by B. Germansky [11]. More precisely, they have proved the following theorem, which was the first result on stochastic approximation to the maximum of an unknown function.

THEOREM 4. *Let x_1 be an arbitrary random variable and define x_n for $n \geqq 1$ and every ω by*

$$x_{n+1}(\omega) = x_n(\omega) + a_n \frac{y_{2n}(\omega) - y_{2n-1}(\omega)}{c_n}, \tag{27}$$

where y_{2n-1} and y_{2n} are random variables whose respective conditional distribution, given $x_1, \cdots, x_n, y_1, \cdots, y_{2n-2}$, are independent and coincide with the distribution of $y(x_n - c_n)$ and $y(x_n + c_n)$ respectively. The sequences of positive numbers a_n and c_n satisfy the following conditions

$$c_n \to 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} a_n c_n < \infty, \qquad \sum_{n=1}^{\infty} a_n^2 c_n^{-2} < \infty. \tag{28}$$

Suppose that M satisfies condition (11) and that there exists a real number ϑ such that M is strictly increasing for $x < \vartheta$ and strictly decreasing for $x > \vartheta$. Suppose further that M satisfies the following conditions. There exist positive numbers β and C_{11} such that $|x_1 - \vartheta| + |x_2 - \vartheta| < \beta$ implies $|M(x_1) - M(x_2)| < C_{11}|x_1 - x_2|$. There exist positive numbers ρ and C_{12} such that $|x_1 - x_2| < \rho$ implies that $|M(x_1) - M(x_2)| < C_{12}$. For every $\delta > 0$ there exists a positive $\eta(\delta)$ such that $|x - \vartheta| > \delta$ implies

$$\inf_{\delta/2 > \epsilon > 0} \frac{|M(x + \epsilon) - M(x - \epsilon)|}{\epsilon} > \eta(\delta). \tag{29}$$

Then x_n converges in probability to ϑ.

A very simple and typical example of a function M which satisfies these conditions is $M(x) = C_{13}|x - \vartheta|$ for $-\infty < x < \infty$. Note that $M(x) = (x - \vartheta)^2$ for $-\infty < x < \infty$ does not satisfy these conditions.

Later these conditions were considerably weakened by J. R. Blum [6], D. L. Burkholder [12], and by A. Dvoretzky [9] (see below), but Dvoretzky's approach goes much further. He introduced a general stochastic approximation process which embraces as special cases the RM and the KW processes.

2.3. *Process of Dvoretzky.* Let $(x_1, \cdots, x_n) \to T_n(x_1, \cdots, x_n)$, for $n \geqq 1$, be a sequence of Borel-measurable mappings from R_n to R_1. Let x_1 and u_n, for $n \geqq 1$, be random variables and define for $n \geqq 1$ and every ω

$$x_{n+1}(\omega) = T_n[x_1(\omega), \cdots, x_n(\omega)] + u_n(\omega). \tag{30}$$

Dvoretzky [9] proved the following convergence theorem for this process.

THEOREM 5. *Let* $\alpha_n, \beta_n, \gamma_n$ *be nonnegative functions from* R_n *to* R_1 *which satisfy the conditions* $\lim_{n\to\infty} \alpha_n(x_1, \cdots, x_n) = 0$ *uniformly for all sequences* $x_1, x_2, \cdots$; *the functions* β_n *are Borel-measurable and* $\sum_{n=1}^{\infty} \beta_n(x_1, \cdots, x_n)$ *is uniformly convergent and uniformly bounded for all sequences* $x_1, x_2, \cdots$; $\sum_{n=1}^{\infty} \gamma_n(x_1, \cdots, x_n)$ *diverges to* ∞ *uniformly for all sequences* $x_1, x_2, \cdots$ *with* $\sup_n |x_n| \leqq C_{14}$ *for every* C_{14}. *Let* ϑ *be a real number such that*

$$\begin{aligned}(31)\quad &|T_n(x_1, \cdots, x_n) - \vartheta| \\ &\leqq \max \{\alpha_n(x_1, \cdots, x_n)[1 + \beta_n(x_1, \cdots, x_n)]|x_n - \vartheta| - \gamma_n(x_1, \cdots, x_n)\}\end{aligned}$$

for all $(x_1, \cdots, x_n) \in R_n$ *where* $n \geqq 1$. *Suppose further that* $E(u_n|x_1, \cdots, x_n) = 0$ *with* $n \geqq 1$, *with probability* 1, *and that* $\sum_{n=1}^{\infty} E(u_n^2) < \infty$. *Then* x_n *converges to* ϑ *with probability* 1.

Later Wolfowitz [13] gave another proof of this theorem which is simpler than Dvoretzky's proof and similar to the proof given for theorem 3.

A specialization of theorem 5 to the case of the *RM* process, with the relation $T_n(x_1, \cdots, x_n) = x_n + a_n[\alpha - M(x_n)]$ for $(x_1, \cdots, x_n) \in R_n$ and with $u_n(\omega) = a_n\{M[x_n(\omega)]\} - y_n(\omega)$, for $\omega \in R$, $n \geqq 1$ gives a theorem like corollary 1 but with the one-sided condition (23) replaced by the less general two-sided condition. A similar specialization to the *KW* process gives

THEOREM 6. *Consider the process defined by* (27) *and suppose that assumptions* (11) *and the first, second, and fourth parts of* (28) *are satisfied; there exist* C_{15} *and* C_{16} *such that* $|M(x + 1) - M(x)| < C_{15}|x| + C_{16}$, *for* $-\infty < x < \infty$; *there exists a real* ϑ *such that for every* $\delta > 0$ *and* $N > 0$ *we have, for the upper and lower derivative of* M,

$$\sup_{\delta < x - \vartheta < N} \bar{D}M(x) < 0, \qquad \inf_{\delta < \vartheta - x < N} \underline{D}M(x) > 0. \tag{32}$$

Then x_n *converges to* ϑ *with probability* 1.

$M(x) = -(x - \vartheta)^2$ obviously satisfies the conditions of theorem 6. Note that the third part of (28) is superfluous. Roughly speaking, a true convergence theorem for the *RM* process "implies" a true convergence theorem for the *KW* process if the conditions imposed on M in the *RM* process are replaced by similar conditions for the derivative of M.

Dvoretzky [9] and Wolfowitz [13] are interested not only in convergence with probability 1 but also in mean convergence. More exactly the following result has been proved.

THEOREM 7. *Consider the process defined by* (30). *Define*

$$b_n = E[(x_n - \vartheta)^2]. \tag{33}$$

$E(x_1^2) < \infty$ *together with the assumptions of theorem* 5 *imply*

$$b_n \to 0. \tag{34}$$

T. Kitagawa [14] gave a slight modification of this result. Instead of a single real number ϑ, he introduced a sequence of real numbers ϑ_n and gets (under weaker assumptions) the conclusion $E[(x_n - \vartheta_n)^2] \rightarrow 0$.

In previous papers the author ([15], [16], and [17]) showed that in the special case of the *RM* process, under stronger conditions, not only can (34) be proved but also the order of magnitude of b_n can be determined. We have

THEOREM 8. *Consider the process defined by* (3) *and suppose that assumptions* (5), (9), (12), *and* (13) *are satisfied, and that* x_1 *is a random variable which is bounded with probability* 1,

$$[M(x) - \alpha](x - \vartheta) > 0 \tag{35}$$

for all $x \neq \vartheta$. *There exists an* $\epsilon > 0$ *such that*

$$|M(x) - \alpha| \geqq C_{17}|x - \vartheta|, \qquad |x - \vartheta| \leqq \epsilon, \tag{36}$$

and

$$|M(x) - \alpha| \geqq C_{18}, \qquad |x - \vartheta| > \epsilon. \tag{37}$$

Then

$$b_n \leqq b_1 \prod_{i=1}^{n-1} \left(1 - \frac{C_{19}a_i}{A_{i-1}}\right) + \prod_{i=1}^{n-1} \left(1 - \frac{C_{19}a_i}{A_{i-1}}\right) \sum_{i=1}^{n-1} a_i^2 e_i \left[\prod_{r=1}^{i} \left(1 - \frac{C_{19}a_r}{A_{r-1}}\right)\right]^{-1}, \tag{38}$$

where

$$e_i = E[(y_i - \alpha)^2], \qquad i \geqq 1;\ A_0 = 1,\ A_n = \sum_{i=1}^{n} a_i. \tag{39}$$

C_{19} has the following significance. From the assumptions made there follows the existence of a C_{19} such that

$$|M(x_n) - \alpha| \geqq \frac{C_{19}}{2A_{n-1}} |x_n - \vartheta|, \qquad n \geqq 1, \tag{40}$$

with probability 1.

A similar theorem is true for the so-called quasi-linear case in which assumptions (5), (36), and (37) are replaced by (11) and the following condition. There exist C_{20} and C_{21} with $C_{20} > C_{21}$ such that

$$C_{20}|x - \vartheta| \geqq |M(x) - \alpha| \geqq C_{21}|x - \vartheta|, \qquad -\infty < x < \infty. \tag{41}$$

Then (38) remains true if one replaces a_i/A_{i-1} by a_i for $i \geqq 1$, and C_{19} by $2C_{21}$. It follows immediately on choosing

$$a_n = \frac{c}{n} \quad \text{for} \quad n \geqq 1, \qquad \text{with} \quad c > \frac{1}{2C_{21}}, \tag{42}$$

that $b_n = O(1/n)$.

For the linear case, in which M is supposed to be of the form $M(x) = g_9x + g_{10}$ for every real x, a result like theorem 8 was also obtained by J. L. Hodges, Jr., and E. L. Lehmann [18].

Condition (42) deserves still some attention. It is obvious from geometrical considerations that, for the classical iteration process given by (2), the speed of the convergence of x_n to ϑ is closely related to the slope of M near ϑ. The greater the slope near ϑ the faster the convergence of x_n provided the sequence a_n in (2) is chosen in a proper way. A very similar situation occurs for the *RM* process. Roughly speaking, the smallest slope of M near ϑ is decisive for the magnitude of the asymptotic variance of the random variable x_n given by (3). The smallest possible order of b_n is only attainable if a proper C_{21} is known. For the *KW* process similar studies of the magnitude of the asymptotic variance of the random variable x_n were made by V. Dupač [19].

2.4. *Other convergence theorems.* A generalization of the *RM* process has been given recently by Blum [20]. Let $\{T_n: n \geqq 1\}$ be a sequence of Borel-measurable mappings from R_n to R_1 which satisfy the condition

$$\sum_{n=1}^{\infty} |T_n(x_1, \cdots, x_n)| < \infty \tag{43}$$

for every sequence $x_1, x_2, \cdots$ of real numbers. Let $\{h_n: n \geqq 1\}$ be a sequence of Borel-measurable mappings from R_n to R_1 with the following property. There exists an integer $k > 0$ such that for every real number r and every $\epsilon > 0$ and all $n \geqq N(r, \epsilon)$, we have $x_i \geqq r + \epsilon$ [respectively $x_i \leqq r - \epsilon$] for $i = n - k + 1, \cdots, n$ implies $h_n(x_1, \cdots, x_n) > r$ [respectively $h_n(x_1, \cdots, x_n) < r$]. Then the following result is true.

THEOREM 9. *Suppose that M is bounded and satisfies conditions* (9) *and* (10) *with $\alpha = 0$ and* (11). *Suppose further that T_n satisfies* (43). *Let x_1 be any random variable and define by induction $x_{n+1}(\omega) = x_n(\omega) + T_n[x_1(\omega), \cdots, x_n(\omega)] - a_n w_n(\omega)$ for every $\omega \in R$, for $n \geqq 1$, where a_n is a sequence of positive numbers satisfying* (12) *and* (13) *and where w_n is a random variable whose conditional distribution, given $x_1, \cdots, x_n$, coincides with the distribution of $y[h_n(x_1, \cdots, x_n)]$ for all $n \geqq 1$, and where h_n may satisfy the condition listed above. Then x_n converges to ϑ with probability* 1.

The proof uses lemma 1 and the following result.

LEMMA 2. *Let u_n, v_n be two sequences of random variables. Denote for every $\epsilon > 0$ the characteristic function of the set $\{\omega: u_n(\omega) \geqq \epsilon, \cdots, u_{n-k+1}(\omega) \geqq \epsilon\}$ by $c_{\epsilon,n,k}$ and by $c_{-\epsilon,n,k}$ the characteristic function of the set $\{\omega: u_n(\omega) \leqq -\epsilon, \cdots, u_{n-k+1}(\omega) \leqq -\epsilon\}$. Suppose further that for some positive integer k and every $\epsilon > 0$ we have $\sum_{n=k}^{\infty} c_{\epsilon,n,k}(v_n + |v_n|)/2$ and $\sum_{n=k}^{\infty} c_{-\epsilon,n,k}(v_n - |v_n|)/2$ converge with probability* 1. *Then u_n converges with probability* 1 *if and only if $\lim_{n\to\infty} v_n = 0$ with probability* 1 [20].

Let us go back once more to the *KW* process. Theorem 6 shows how to determine under weak conditions the location ϑ of an unknown maximum or minimum of a function M. But it may also be of interest to find out the value $M(\vartheta)$ of the extremum. In this direction we note a convergence theorem given by Burkholder [12].

THEOREM 10. *Suppose that the conditions of theorem* 6 *are satisfied. Suppose furthermore that M is continuous at ϑ. Then, if the random variables y_j for $j \geqq 1$ have the same significance as in theorem* 4, $\sum_{j=1}^{n} y_j/n$ *converges to $M(\vartheta)$ with probability* 1.

The proof depends on the following lemma, which is only a reformulation of lemma 1.

LEMMA 3. *Let* $\{v_n : n \geqq 1\}$ *be a sequence of random variables and assume* $E(v_n | v_1, \cdots, v_{n-1}) = 0$ *for* $n \geqq 1$, *with probability* 1. *If* $\sigma^2(v_n)$ *exists for* $n \geqq 1$ *and if* $\sum_{n=1}^{\infty} \sigma^2(v_n)/n^2$ *converges, then* $\sum_{n=1}^{\infty} v_n/n$ *converges with probability* 1 *to a random variable.*

Apply now lemma 3 to the random variables $v_{2n-1} = y_{2n-1} - M(x_n - c_n)$ and $v_{2n} = y_{2n} - M(x_n + c_n)$.

3. Higher moments and the asymptotic distribution

The first results in this direction were given by K. L. Chung [21] for the case of the *RM* process. He developed an interesting method which was used in a very rudimentary form earlier in [15], [16], and [17]. The method depends on the study of some linear difference inequalities and can be roughly described in the following way. Using the notations (33), (39), and $d_n = E\{(x_n - \vartheta)[M(x_n) - \alpha]\}$ it is easy to derive from (3) the following equation

$$b_{n+1} = b_n - 2a_n d_n + e_n a_n^2. \tag{44}$$

Suppose that it is known that $d_n \geqq 0$. This is the case, for instance, if condition (35) is satisfied. Clearly, if a lower [upper] bound for $d_n (n \geqq 1)$ and an upper [lower] bound for $e_n (n \geqq 1)$ can be found, one obtains from (44) linear difference inequalities for b_n and derives from them upper [lower] bounds for b_n. In the same manner, using induction too, one can get bounds also for the higher moments of $x_n - \vartheta$. Besides this, Chung [21] obtained with the help of this method precise results about the asymptotic behavior of the higher moments of $x_n - \vartheta$ as $n \to \infty$. In [21] the following theorem is actually given.

THEOREM 11. *Suppose that* (5), (8), (9), *and* (35) *are satisfied and also assume that*

$$M(x) = \alpha + \alpha_1(x - \vartheta) + o(|x - \vartheta|) \quad as \quad x - \vartheta \to 0, \ 0 < \alpha_1 < \infty. \tag{45}$$

For every $\delta > 0$ *we have*

$$\inf_{|x-\vartheta| > \delta} |M(x) - \alpha| = K_0(\delta) > 0. \tag{46}$$

For all x *we have*

$$E\{[y(x) - M(x)]^2\} = \sigma^2. \tag{47}$$

Choose $a_n = 1/n^{1-\epsilon}$ *for* $n \geqq 1$, *with* $1/2(1 + C_{22}) < \epsilon < 1/2$, *where* C_{22} *has a significance similar to that of* C_{19} *in theorem* 8. *Then we have for every integer* $r \geqq 1$

$$\lim_{n\to\infty} n^{(1-\epsilon)r/2} b_n^{(r)} = \begin{cases} 0 & if \quad r = 2s - 1, \\ (\sigma^2/2\alpha_1)^s (2s-1)(2s-3) \cdots 3\cdot 1 & if \quad r = 2s, \end{cases} \tag{48}$$

where $b_n^{(r)}$ *denotes the* rth *moment of* $x_n - \vartheta$.

Using the well-known theorem of Fréchet and Shohat one gets the following result.

COROLLARY 2. *The random variable* $n^{(1-\epsilon)/2}(x_n - \vartheta)$ *has an asymptotically normal distribution with mean* 0 *and variance* $\sigma^2/2\alpha_1$.

A similar theorem is true for the quasi-linear case in which (5) and (46) are replaced by (41) and by the condition that

$$E[|y(x) - M(x)|^p] \leqq C_{23}(p) \tag{49}$$

for all x and all $p \geqq 1$. Then, if (42) is satisfied, we have for every integer $r \geqq 1$

(50)

$$\lim_{n\to\infty} n^{r/2}b_n^{(r)} = \begin{cases} 0 & \text{if } r = 2s - 1, \\ [\sigma^2c^2/(2\alpha_1 c - 1)]^s(2s - 1)(2s - 3) \cdots 3\cdot 1 & \text{if } r = 2s. \end{cases}$$

Obviously we have also a result on the asymptotic normality of $n^{1/2}(x_n - \vartheta)$ which corresponds to corollary 2. But condition (41) is of course very restrictive; it is not satisfied, for instance, if M is bounded. Hodges and Lehmann [18] introduced a nice idea which shows that (41) can be replaced by

$$|M(x) - \alpha| \leqq C_{24}|x - \vartheta|, \qquad -\infty < x < \infty, \tag{51}$$

if one is only interested in the asymptotic distribution of $x_n - \vartheta$ at the sacrifice of knowledge about the behavior of the moments of $x_n - \vartheta$. Let us describe this without going into details. It follows from (45) that for small $\eta > 0$ there exists a C_{25} with $C_{25} = \inf_{|x-\vartheta|<\eta} |[M(x) - \alpha]/(x - \vartheta)|$. Choose $a_n = c/n$ for $n \geqq 1$, with $c > 1/2C_{25}$. Then (8), (9), (35), (49), and (51) imply the convergence of x_n to ϑ with probability 1 as a consequence of corollary 1. This implies that for every $\eta > 0$ and $\epsilon > 0$ we have $|x_n(\omega) - \vartheta| < \eta$ with probability $\geqq 1 - \epsilon$ for $n \geqq N(\epsilon, \eta)$. It follows from (45) that (41), with $x = x_n$ when $n \geqq N(\epsilon, \eta)$, is satisfied with probability $1 - \epsilon$. Now a truncation argument is used. If we remove a set of probability $\leqq \epsilon$ the conditions cited above for the quasi-linear case are satisfied. Hence the asymptotic normality of $n^{1/2}(x_n - \vartheta)$ is proved. This is, especially from the point of view of a statistician, a satisfactory result. Even in the case of a bounded M we can get an "asymptotic variance" for x_n of order $1/n$ whereas corollary 2 gives only order $1/n^{1-\epsilon}$.

C. Derman [22] and V. Dupač [19] obtained similar results with the help of Chung's method for the KW process. Using and refining this method, Burkholder [12] obtained more general and more sophisticated results for a more general stochastic approximation process which contains the processes of RM and of KW as special cases. This process has the following form. For every natural number n let M_n be a mapping from R_1 to R_1. For every n and every real x let $z_n(x)$ be a random variable, $[\omega \to z_n(x, \omega)]$, whose expectation exists and is equal to $M_n(x)$. Let x_1 be a random variable and k_n a sequence of positive real numbers and define x_n for $n \geqq 1$ and every ω by

$$x_{n+1}(\omega) = x_n(\omega) - k_n z_n(\omega), \tag{52}$$

where the conditional distribution of z_n given $x_1, \cdots, x_n$ is the distribution of $z_n(x_n)$. There exists an analogue to theorem 11 for this process. Moreover, the quasi-linear case can also be treated and the truncation device of Hodges and Lehmann [18] is useful also if one is only interested in the asymptotic distribution of x_n.

The most general and complete results concerning asymptotic distribution of x_n for the RM and KW processes have been obtained by J. Sacks [23]. He does not use the method of moments and gives theorems which are modeled on the Lindeberg-Feller central limit theorem. He observes that his method works under suitable conditions also for the process defined by (52). We will now formulate such a theorem and give the main lines of the proof.

THEOREM 12. *Let there be given a process of type* (52). *Suppose that conditions* (53) *to* (62) *are satisfied.* M_n *is Borel-measurable for* $n \geqq 1$; *there exists a sequence of real numbers* μ_n *with* $M_n(\mu_n) = 0$ *and there exists a* ϑ *with*

$$\mu_n - \vartheta = O(n^{-\gamma}), \qquad \gamma > 0. \tag{53}$$

$$(x - \mu_n)M_n(x) \geqq 0 \tag{54}$$

for all real x. *There exist* C_{26} *and* C_{27} *and a bounded sequence of positive numbers* l_n *with*

$$C_{26} \leqq \frac{M_n(x)}{l_n(x - \mu_n)} \leqq C_{27} \tag{55}$$

for all $x \neq \mu_n$;

$$M_n(x) = \beta_1 l_n(x - \mu_n) + l_n\tau(x - \mu_n) \tag{56}$$

with $\lim_{x\to 0} \tau(x)/x = 0$. *For a* ξ *with*

$$0 < \xi < \frac{1}{2}, \qquad \xi < \gamma, \tag{57}$$

we have

$$n^{\xi+1/2}k_n \to c > 0 \tag{58}$$

and

$$nk_nl_n \to d \tag{59}$$

with

$$d > \xi/C_{26}, \tag{60}$$

$$\sup_{x,n} E\{[z_n(x) - M_n(x)]^2\} < \infty. \tag{61}$$

The sequence of mappings $x \to E\{[z_n(x) - M_n(x)]^2\}$ *converges continuously to a* $\sigma^2 > 0$ *at* $x = \vartheta$ *for* $n \to \infty$. *Denote the set* $\{\omega: |z_n(x, \omega) - M_n(x)| > R\}$ *by* $\Lambda_{n,x,R}$ *and take* $\epsilon > 0$. *Suppose that*

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \sup_{|x-\vartheta|<\epsilon} \int_{\Lambda_{n,x,R}} \{[z_n(x, \omega) - M_n(x)]^2\}\, dP = 0 \tag{62}$$

uniformly in n. *Then* $n^{\xi}(x_n - \vartheta)$ *is asymptotically normally distributed with mean* 0 *and variance* $\sigma^2c^2/(2\beta_1 d - 2\xi)$.

An important tool for the proof is

LEMMA 4. [8]. *Let* u_{nk} *for* $1 \leqq k \leqq n$ *and* $n \geqq 1$ *be a set of random variables whose distribution function will be denoted by* F_{nk} *and whose variance* σ_{nk}^2 *exists. Suppose that* $E(u_{nk}|u_{n1}, \cdots, u_{nk-1}) = 0$ *with probability* 1. *Define* $E(u_{nk}^2|u_{n1}, \cdots, u_{nk-1}) = s_{nk}^2$. *Under the assumptions*

$$\sum_{0 \leqq k \leqq n} \int_{|x| \geqq \epsilon} x^2\, dF_{nk} \to 0, \qquad \epsilon > 0. \tag{63}$$

$$\sigma_n^2 = \sum_{k \leqq n} \sigma_{nk}^2 \leqq C_{28} \tag{64}$$

for $n \geqq 1$, *and*

$$\sum_{k=1}^{n} E|s_{nk}^2 - \sigma_{nk}^2| \to 0 \tag{65}$$

for $n \to \infty$, *we have* $u_n/\sigma_n = \sum_{k \leqq n} u_{nk}/\sigma_n$ *is asymptotically distributed according to a normal distribution with mean* 0 *and variance* 1.

Now for simplicity let x_1 be a constant with probability 1. We observe that

$$\beta_1 d - \xi > 0 \tag{66}$$

as a consequence of (55), (56), and (60). From (52) and (56) we immediately obtain

$$\begin{aligned} x_{n+1} - \vartheta = (x_1 - \vartheta)\beta_{0n} &+ \sum_{m=1}^{n} \beta_1 k_m l_m \beta_{mn}(\mu_m - \vartheta) \\ &- \sum_{m=1}^{n} k_m[z_m - M_m(x_m)]\beta_{mn} - \sum_{m=1}^{n} k_m l_m \beta_{mn} \tau(x_m - \mu_m), \end{aligned} \tag{67}$$

where $\beta_{mn} = \prod_{i=m+1}^{n}(1 - \beta_1 k_i l_i)$. It is easy to see from (59) that

$$\beta_{mn} = m^{\beta_1 d} n^{-\beta_1 d} + o(m^{\beta_1 d} n^{-\beta_1 d}), \qquad n \geqq m; \qquad m \to \infty. \tag{68}$$

Secondly, using (68) and (58) we get

$$h_n = \left(\sum_{m=1}^{n} k_m^2 \beta_{mn}^2 \right)^{-1/2} = \left[\frac{2(\beta_1 d - \xi)}{c^2} \right]^{1/2} n^{\xi} + o(n^{\xi}) \tag{69}$$

and it is enough to prove that $h_n(x_n - \vartheta)$ is asymptotically normally distributed with mean 0 and variance σ^2. For this it suffices to show that

(a) $h_n \beta_{0n} \to 0$;

(b) $h_n \sum_{m=1}^{n} k_m l_m (\mu_m - \vartheta)\beta_{mn} \to 0$;

(c) $h_n \sum_{m=1}^{n} k_m[z_m - M_m(x_m)]\beta_{mn}$ is asymptotically normally distributed with mean 0 and variance σ^2;

(d) $\sum_{m=1}^{n} k_m l_m \beta_{mn} \tau(x_m - \mu_m)$ converges to 0 in probability.

(a) is a consequence of (68), (69), and (66). (b) follows from (53), (68), (69), (59), and (57). Denote the random variable $h_n k_j \beta_{jn}[z_j - M_j(x_j)]$ by u_{nj} for $1 \leqq j \leqq n$, $n \geqq 1$. To prove (c) it is enough to show that u_{nj} satisfies the conditions of lemma 4. First of all it is easy to see that

$$E(u_{nj}|u_{n1}, \cdots, u_{nj-1}) = 0 \tag{70}$$

with probability 1. Then we will show that

(71)
$$\sum_{j=1}^{n} \int_{|u_{nj}| \geqq \epsilon} u_{nj}^2 \, dP \to 0$$

for every $\epsilon > 0$. But $|u_{nj}| \geqq \epsilon$ together with (58), (68), (69), and (66) implies that for a proper $\delta(\epsilon) > 0$

(72)
$$|z_j - M_j(x_j)| \geqq \delta(\epsilon) j^{1/2}.$$

From the assumptions made it follows by theorem 5 that x_n converges to ϑ with probability 1, and so as a consequence of (61) and (62)

(73)
$$\lim_{R \to \infty} \sup_j \int_{\Lambda_{j,x_j,R}} \{[z_j - M_j(x_j)]^2\} \, dP \to 0,$$

where $\Lambda_{j,x_j,R} = \{\omega: |z_j(\omega) - M_j[x_j(\omega)]| > R\}$. Hence using (61) and (72), together with (58), (66), (68), and (69) we have, for a suitable natural number N and $n > N$ large enough,

(74)
$$\sum_{j=1}^{n} \int_{|u_{nj}| \geqq \epsilon} u_{nj}^2 \, dP = \sum_{j=1}^{N} \int_{|u_{nj}| \geqq \epsilon} u_{nj}^2 \, dP + \sum_{j=N+1}^{n} \int_{|u_{nj}| \geqq \epsilon} u_{nj}^2 \, dP$$
$$= O\left[n^{-2(\beta_1 d - \xi)} \sum_{m=1}^{N} m^{2\beta_1 d - 2\xi - 1} \right] + o\left[n^{-2(\beta_1 d - \xi)} \sum_{m=N+1}^{n} m^{2\beta_1 d - 2\xi - 1} \right] = o(1)$$

for every $\epsilon > 0$. Next we have to prove (65). As the last part of the above proof of (71) shows, it is enough to prove that $E[|E\{[z_k - M_k(x_k)]^2 | [z_1 - M_1(x_1)], \cdots, [z_{k-1} - M_{k-1}(x_{k-1})]\} - E\{[z_k - M_k(x_k)]^2\}|]$ converges to 0 for $k \to \infty$. But this follows from

(75)
$$E\{[z_k - M_k(x_k)]^2 | [z_1 - M_1(x_1)], \cdots, [z_{k-1} - M_{k-1}(x_{k-1})]\} - E\{[z_k - M_k(x_k)]^2\} \to 0$$

with probability 1 as a consequence of (61) and of the well-known theorem on the term by term integration of boundedly convergent sequences. (75) follows easily from the convergence of x_n to ϑ with probability 1 and the continuous convergence of $E\{[z_n(x) - M_n(x)]^2\}$ at $x = \vartheta$.

Now we have to show that (64) is satisfied but it can easily be proved, in the same way that (75) was proved, that σ_n^2 converges to σ^2. Therefore, some positive bound C_{28} for σ_n^2, with $n \geqq 1$, exists.

(d) is proved in the following way. For every $\eta > 0$ there exists an $\epsilon > 0$ such that

(76)
$$|\tau(x - \vartheta)| < \eta^2 |x - \vartheta|$$

for $|x - \vartheta| < \epsilon$. On the other hand, for every $\delta > 0$ and $\epsilon > 0$, we have

(77)
$$|x_n - \vartheta| < \epsilon$$

for all $n \geqq N(\epsilon, \delta)$ with probability $1 - \delta$. According to the assumptions made it is possible to choose a natural number $N_1 = N_1(\eta) \geqq N(\epsilon, \delta)$ such that for $n > N_1(\eta)$ we have $|h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} m^{-\gamma}| \leqq \eta/2$. Then we have for $n > N_1(\eta)$, using (53),

$$(78)\quad P\left\{|h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} \tau(x_m - \mu_m)| > \eta\right\}$$
$$\leqq P\left\{|h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} \tau(x_m - \vartheta)| > \eta - o\left(h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} m^{-\gamma}\right)\right\}$$
$$\leqq P\left\{|h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} \tau(x_m - \vartheta)| > \eta/2\right\}$$
$$\leqq \delta + P\left\{h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} |x_m - \vartheta| > 1/2\eta\right\}$$
$$\leqq \delta + 2\eta E\left\{h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} |x_m - \vartheta|\right\},$$

using (77) and (76). Using the modified form of (38) for the quasi-linear case we obtain, under assumption (58), $E[(x_n - \vartheta)^2] = O(n^{-\xi-1/2})$. With the help of this and of Schwarz's inequality we have from (78)

$$(79)\quad P\{|h_n \sum_{m=N_1}^{n} k_m l_m \beta_{mn} \tau(x_m - \mu_m)| > \eta\} \leqq \delta + O(2\eta).$$

Moreover, for a fixed N_1, $P\{h_n|\sum_{m=1}^{N} k_m l_m \beta_{mn}(x_m - \mu_m)| > \eta\}$ becomes arbitrarily small for every $\eta > 0$ if n is large enough. This proves (d). It should still be mentioned that the truncation device of Hodges and Lehmann [18] can again be used and (55) can be relaxed.

We shall discuss the application of theorem 12 to the KW process defined by (27). This may be done by introducing the following substitutions

$$(80)\quad k_n = a_n/c_n, \qquad n \geqq 1,$$
$$l_n = c_n, \qquad n \geqq 1,$$

$$(81)\quad z_n(x) = y(x - c_n) - y(x + c_n), \qquad -\infty < x < \infty; \qquad n \geqq 1,$$

and therefore

$$(82)\quad M_n(x) = M(x - c_n) - M(x + c_n), \qquad -\infty < x < \infty; \qquad n \geqq 1.$$

One must first check whether all conditions of theorem 12 are satisfied. Here we are concerned especially with condition (54). Suppose there exists a real number ϑ such that

$$(83)\quad M(x_1) < M(x_2) \quad \text{for} \quad \begin{cases} x_1 < x_2 \leqq \vartheta \\ x_1 > x_2 > \vartheta. \end{cases}$$

Then it is easy to see that for every $\epsilon > 0$ there exists a $\mu(\epsilon)$ such that

$$(84)\quad [x - \mu(\epsilon)][M(x - \epsilon) - M(x + \epsilon)] > 0 \qquad \text{for all} \quad x \neq \mu(\epsilon).$$

Burkholder [12] has introduced the following concept. Suppose that M satisfies (83). M is called η-locally even at ϑ if

$$(85)\quad \vartheta - \mu(\epsilon) = O(\epsilon^{1+\eta}), \qquad \eta \geqq 0,\ \epsilon \to 0.$$

It is not difficult to see that any M satisfying (83) is 0-locally even at ϑ.

Intuitively it seems very likely that the "asymptotic variance" of $x_n - \vartheta$ for the KW process is smaller the greater η is. Let us make this more precise. Suppose that M is η-locally even. After making the substitutions (80) to (82) we immediately recognize from (84) that condition (54) is satisfied with

$$\mu_n = \mu(c_n), \qquad n \geqq 1. \tag{86}$$

Now, from conditions (58) and (59) it follows easily that $c_n = O(n^{\xi-1/2})$. From this result, (85), and (86), it follows that condition (53) is satisfied with $\gamma = (\xi - 1/2)(1 + \eta)$. According to (57) we must have $0 < \xi < (1/2 - \xi)(1 + \eta)$, which is equivalent to

$$0 < \xi < \frac{1}{2} - \frac{1}{4 + 2\eta}. \tag{87}$$

Summing up, we have the following result.

COROLLARY 3. *Consider the KW process defined by* (27). *Suppose that M is η-locally even at ϑ. If* (87) *is satisfied, then, under some more conditions which can easily be found by comparison with the condition of theorem* 12, $n^{\xi}(x_n - \vartheta)$ *is asymptotically normally distributed.*

Sacks [23] has shown that this corollary is still true if in the conclusion n^{ξ} is replaced by $n^{1/2-1/(4+2\eta)}(\log n)^{-1}$. The question arises whether the factor $(\log n)^{-1}$ can be omitted. The answer is, in general, no, as has been shown by Sacks [23].

Let us go back for a moment to theorem 12. On the assumptions of this theorem two important and, in most practical applications, unknown parameters occur: β_1 and σ^2. This means that the result of theorem 12 can in general not be used for the construction of asymptotic confidence intervals for ϑ. But it is easy to see, using lemma 3, that under the assumptions of theorem 12

$$s_n^2 = \sum_{j=1}^{n} z_j^2/n \tag{88}$$

converges to σ^2 with probability 1. Let us introduce the following notations: f_n is a sequence of positive numbers; z_n^* is a random variable whose conditional distribution given $x_1, \cdots, x_n, z_1, \cdots, z_{n-1}$ is independent of the corresponding conditional distribution of z_n and is given by the distribution of $z(x_n + f_n)$ for all $n \geqq 1$. Now the following theorem is given in [12], which is of some practical interest.

THEOREM 13. *Suppose that the conditions of theorem* 12 *are satisfied. If* $f_n \to 0$ *and* $f_n^{-1} = O(n^{\xi_0})$ *for some* ξ_0 *with* $0 < \xi_0 < \xi$ *then* $t_n = \sum_{j=1}^{n} [(z_j^* - z_j)/l_j f_j n]$ *converges to* β_1 *with probability* 1. *It follows from theorem* 12 *and* (88) *that* $n^{\xi}|2t_n d - 2\xi|^{1/2}(cs_n)^{-1}(x_n - \vartheta)$ *is asymptotically normally distributed with mean* 0 *and variance* 1.

The problem of what happens to the asymptotic distribution of x_n, if there is one, without conditions like (61) and (62), is not yet attacked. But an interesting investigation of Chung [21] is the linear case for the RM process should nevertheless be mentioned. Suppose that $M(x) = x - \vartheta$ for all x and some real ϑ such that ϑ is the solution of the equation $M(x) = 0$. Suppose further that F is any distribution function and that the distribution function F_x of $y(x)$ is given by $F_x = P[y(x) \leqq y] = F[y - M(x)]$ for all real y and every fixed real x. Assume further that in the RM process given by (3) $\alpha = 0$ and $a_n = 1/n$, that is, $x_{n+1}(\omega) = x_n(\omega) - y_n(\omega)/n$ for all $n \geqq 1$. Under these assumptions we have

THEOREM 14. *All possible limit distributions of $x_n - \vartheta$ are stable laws.*

The proof can be based on considerations of characteristic functions. If φ is the characteristic function of F we find that

$$E[e^{i(x_{n+1}-\vartheta)t}] = \left[\varphi\left(-\frac{t}{n}\right)\right]^n, \qquad -\infty < t < \infty. \tag{89}$$

It follows in particular that every stable law of exponent a, where $1 < a \leqq 2$, is the limit distribution of $x_n - \vartheta$ for a suitable choice of F.

4. Optimal property of the RM process and stopping rules

Chung [21] has proved with the help of an argument due to Wolfowitz [24] that the RM process is asymptotically minimax in the quasi-linear case under certain conditions. We shall clarify this below. Denote by F the mapping $x \to F_x$, with $-\infty < x < \infty$, where F_x is the distribution function of the random variable $y(x)$. It is assumed that for every F there exists a corresponding M_F. Clearly, M_F is defined for every x by $M_F(x) = \int_{-\infty}^{+\infty} y\, dF_x(y)$ according to our basic assumptions. Let H be a class of mappings F which have the property that for the corresponding M_F and ϑ_F the conditions (8), (9), (41), (45) with α_{1F}, (47) with σ_F^2, and (49) are satisfied. Suppose furthermore that there exist C_{29} and C_{30} such that $\inf_H \alpha_{1F} \geqq C_{29}$ and $\sup_H \sigma_F^2 \leqq C_{30}$. Suppose further that H contains the set of all mappings $x \to N[C_{29}(x - a), C_{30}]$ where $N(m, \sigma^2)$ denotes a normal distribution with mean m and variance σ^2 and where $g_9 < a < g_{10}$ with $g_9 < g_{10}$. Choose in the RM process given by (3)

$$a_n = \frac{1}{C_{29}n}, \qquad n \geqq 1. \tag{90}$$

Now let W be a weight function defined over $H \times R_1$ as follows. $W(F, t) = C_{31}|\vartheta_F - t|^r$ for all $(F, t) \in H \times R_1$ where $r \geqq 2$ is an integer.

THEOREM 15. *If D_n is the set of all sequential estimates T_n for ϑ_F based on a sample of size n and if x_{n+1} is the estimate given by the RM process with* (90), *then under the assumptions made*

$$\lim_{n\to\infty} \frac{\sup_{F\in H} E[W(F, x_{n+1})]}{\inf_{T_n \in D_n} \sup_{F\in H} E[W(F, T_n)]} = 1. \tag{91}$$

From the point of view of application this means that in the large sample case the problem of an optimal choice of the sequence a_n is solved to a certain extent. But very little is known about "good" stopping rules for stochastic approximation processes in the small sample case. For the case of a fixed sample size a result of Dvoretzky should be mentioned.

THEOREM 16. *Consider a RM process given by* (3) *and suppose that* (10), (9), (11), *and* (41) *are satisfied. Suppose further that* $E[(x_1 - \vartheta)^2] \leqq C_{32}$ *and that*

$$C_{32} \leqq \frac{2C_4}{C_{21}(C_{20} - C_{21})}. \tag{92}$$

Then, if

$$a_n = \frac{C_{21}C_{32}}{C_4 + C_{21}^2 C_{32} n}, \qquad n = 1, \cdots, p, \tag{93}$$

it follows that

$$E[(x_{p+1} - \vartheta)^2] \leqq \frac{C_{32}C_4}{C_4 + C_{21}^2 C_{32} p} \tag{94}$$

and the choice (93) *of the* a_n *is an optimal one in the sense that if* (93) *does not hold then there exists a process given by* (3) *satisfying* (92) *for which* (94) *is false.*

Later Block [25] considered some generalizations of theorem 16 (see also Schmetterer [26]). Obviously, theorems of this kind are closely related to theorem 8.

5. Accelerated stochastic approximation

Consider the process defined by (30). One might have the following idea. As long as $x_n - x_{n-1}$ always has the same sign, the approximation to ϑ cannot be very good. If x_n is near ϑ then the sign of $x_n - x_{n-1}$ will in general fluctuate frequently. Kesten [27] recently used this idea to obtain a process which may accelerate the convergence of x_n to ϑ. Let T_n have the same significance as in the process of Dvoretzky. Let x_1 and u_n for all $n \geqq 1$ be random variables and let a_n be a sequence of positive numbers. Define x_n for $n \geqq 1$ by

$$x_{n+1}(\omega) = T_n[x_1(\omega), \cdots, x_n(\omega)] + d_n(\omega)u_n(\omega), \tag{95}$$

where $d_1 = a_1$, $d_2 = a_2$, $d_n = a_{s(n)}$ with

$$s(n) = 2 + \sum_{i=1}^{n} j[(x_i - x_{i-1})(x_{i-1} - x_{i-2})], \qquad j(x) = \begin{cases} 1, & x \leqq 0, \\ 0, & x > 0. \end{cases} \tag{96}$$

This means that d_n is constant so long as $x_n - x_{n-1}$ and $x_{n-1} - x_{n-2}$ have the same sign. Let $\alpha_n, \beta_n, \gamma_n$ be functions defined as in theorem 5. Let

$$\rho(\delta) = \inf_n \inf_{\substack{x_1, \cdots, x_{n-1} \\ |x_n - \vartheta| < \delta}} \frac{\gamma_n(x_1, \cdots, x_n)}{d_n}. \tag{97}$$

Kesten [27] proved the following result.

THEOREM 17. *Suppose that* (12) *and* (13) *are satisfied and that* $a_{n+1} \leqq a_n$ *for all* $n \geqq 1$. *If* $|T_n(x_1, \cdots, x_n) - \vartheta| \leqq [1 + \beta_n(x_1, \cdots, x_n)|x_n - \vartheta| - \gamma_n(x_1, \cdots, x_n)]$ *when* $[T_n(x_1, \cdots, x_n)](x_n - \vartheta) > 0$; $|T_n(x_1, \cdots, x_n) - \vartheta| \leqq \alpha_n(x_1, \cdots, x_n)$

when $[T_n(x_1, \cdots, x_n) - \vartheta](x_n - \vartheta) \leqq 0$; $\lim_{s(n)\to\infty} \alpha_n(x_1, \cdots, x_n) = 0$ *uniformly for all sequences* $x_1, x_2, \cdots$, *with* $s(n) \to \infty$; $\lim_{n\to\infty} (x_n - \vartheta)\beta_n(x_1, \cdots, x_n)/d_n = 0$ *uniformly for all sequences* $x_1, x_2, \cdots$; $\sum_{n=1}^{\infty} \beta_n(x_1, \cdots, x_n)$ *converges uniformly for all sequences* $x_1, x_2, \cdots$; $\rho(\delta) > 0$ *for every positive* δ; $E(u_n|x_1, \cdots, x_n) = 0$, $E(u_n^2|x_1, \cdots, x_n) \leqq C_{33}$ *with probability* 1;

$$\lim_{n\to\infty} \lim_{\tau\to 0} \inf_{\substack{0<|x_n-\vartheta|\leqq\tau \\ x_1,\cdots,x_{n-1}}} P\{T_n(x_1, \cdots, x_n) + d_n u_n \geqq x_n|x_1, \cdots, x_n\} > 0 \tag{98}$$

and

$$\lim_{n\to\infty} \lim_{\tau\to 0} \inf_{\substack{0\leqq|x_n-\vartheta|\leqq\tau \\ x_1,\cdots,x_{n-1}}} P\{T_n(x_1, \cdots, x_n) + d_n u_n < x_n|x_1, \cdots, x_n\} > 0, \tag{99}$$

then x_n, *given by* (95), *converges to* ϑ *with probability* 1.

The proof of theorem 17 rests on a refinement of Wolfowitz's proof of theorem 5. It would be interesting to find less complicated conditions which guarantee the conclusion of this theorem.

6. Generalizations to more general spaces

It is quite natural to ask whether the definitions and the results so far obtained may be generalized to the case that M is a mapping from R_n to R_n or more generally from some abstract space B into B. It is almost obvious how to generalize, for example, the definition of the RM process given by (3) to the definition of an n-dimensional RM process. Suppose that for every $x = (x_1, \cdots, x_n) \in R_n$ an n-dimensional random variable $y(x) = y(x_1, \cdots, x_n) = [y^{(1)}(x_1, \cdots, x_n), \cdots, y^{(n)}(x_1, \cdots, x_n)]$ is given whose mathematical expectation $M(x) = M(x_1, \cdots, x_n) = [M^{(1)}(x_1, \cdots, x_n), \cdots, M^{(n)}(x_1, \cdots, x_n)]$ say, exists for every $(x_1, \cdots, x_n) \in R_n$. Let $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ be an element of R_n. The problem is, of course, to find a solution of the n equations $M(x) = \alpha$. Let us start with some n-dimensional random variable $x_1^{(n)}$. Let a_j be a sequence of positive numbers and define for $j \geqq 1$ and every $\omega \in R$

$$x_{j+1}^{(n)}(\omega) = x_j^{(n)}(\omega) + a_j[\alpha - y_j^{(n)}(\omega)], \tag{100}$$

where $y_j^{(n)}$ is a random variable whose conditional distribution, given $x_1^{(n)}, \cdots, x_j^{(n)}$, is the distribution of $y(x_j^{(n)})$. Relation (100) is the n-dimensional analogue to the RM process given by (3). The first results in the direction of a generalization of theorem 5 for the process given by (100) and the analogue of the KW process were obtained by Blum [28]. Results concerning asymptotic distributions in the case of the n-dimensional analogue of the RM process and the KW process were given by Sacks [23], who generalized his method to this case.

Some hints for generalizations of the process of Dvoretzky to the case of Banach spaces were given by Dvoretzky [9] himself. A convergence theorem in Banach spaces which generalizes theorem 5 is due to Block [29]. But it should be pointed out that these theorems are not essentially probabilistic. Let us introduce for this the following notations. Let B be a vector space over the real numbers. Let N be an ordered (not necessarily completely ordered) vector space over the real numbers. Suppose further that a Hausdorff topology is introduced in N which is compatible with the structure of N. Suppose that a mapping ρ from B into N is given with the following properties: $\rho(x) \geqq 0$ for all $x \in B$ and $\rho(x) = 0$ implies $x = 0$. Also $\rho(x + y) \leqq \rho(x) + \rho(y)$ for all $x, y \in B$, and $\rho(\alpha x) = |\alpha|\rho(x)$ for all $x \in R$ and all $\alpha \in R_1$. A sequence x_n of elements from B is called convergent to $x \in B$ if $\rho(x_n - x)$ converges to 0 (in the sense of the given topology). ρ is called the generalized norm of B (see Kantorovič [30] and Schröder [31]). Now we can state the following result.

THEOREM 18. *Suppose that the conditions below are satisfied. Q is a positive and linear operator from N into N with the property that $Q^n\nu$ converges to 0 for every $\nu \in N$. If ν_n is a sequence of elements from N which converges to 0 then $\sum_{i=1}^{n} Q^{n-i}\nu_i$ converges also to 0. T_n is a mapping from $B^n = B \times B \times \cdots \times B$ into B for $n \geqq 1$.*

$$\rho[T_n(u_1, \cdots, u_n)] \leqq Q\rho(u_n) \tag{101}$$

for $n \geqq 1$ and for all $(u_1, \cdots, u_n) \in B^n$. Also let z_n be a sequence of elements from B which converges to 0. Let $u_1 \in B$ be any element and define by induction

$$u_{n+1} = T_n(u_1, \cdots, u_n) + z_n. \tag{102}$$

Then u_n converges to 0.

The proof is very simple. It follows from (101) and (102) and the other assumptions made that $\rho(u_{n+1}) \leqq Q^n\rho(u_1) + \sum_{i=0}^{n-1} Q^i\rho(z_{n-i})$ for $n \geqq 1$ where $Q^0 = E$ is the identical mapping. It follows immediately that $\rho(u_n)$ converges to 0. If N is lattice ordered then condition (101) can be replaced by $\rho[T_n(u_1, \cdots, u_n)] \leqq \sup [\alpha_n, Q\rho(u_n)]$, where α_n converges to 0. This is a natural generalization of condition (31). The assumptions made for Q are satisfied if, for example, the following conditions hold: Q is a positive and linear operator from N into N and $\sum_{n=0}^{\infty} Q^n\nu$ converges for every $\nu \in N$. (This means that the operator $(E - Q)^{-1}$ exists for all elements of N.) For all elements of N a norm in the usual sense is defined which is compatible with the given topology. These ideas become important for questions in numerical analysis (see Schröder [31]).

Suppose now that u_1 and z_n, for $n \geqq 1$, are random elements which take values in B. Consider instead of (102) $u_{n+1}(\omega) = T_n[u_1(\omega), \cdots, u_n(\omega)] + z_n(\omega)$. If it is known that $z_n(\omega)$ converges to 0 with probability 1 and that besides this all conditions of theorem 18 are satisfied, then it follows that $u_n(\omega)$ converges to 0, that is, $\rho u_n(\omega)$ converges to 0, with probability 1. This is, however, a theorem on stochastic approximation.

It is well known that classical iteration procedures are closely connected with fixed point theorems, so it is not surprising that stochastic approximation is related to the subject of random fixed point theorems. These theorems, however, are beyond the scope of this paper. General investigations in this direction have been made by Hanš [32], [33] and by Špaček [34].

7. Some applications of stochastic approximation

A well-known application of the RM process first mentioned by Robbins and Monro themselves is of the following kind. Let M be a response curve for a (not necessarily biological) population. This means that to every input x a part $M(x)$ of the population responds and the rest does not. It is natural to suppose that M is nondecreasing, that $M(x) = 0$ for $x \leqq 0$ and $M(x) = 1$ for $x \to \infty$, that is, that M is a distribution function. Very frequently the problem is to determine a quantile of M. In bioassay, for example, one is often interested in the determination of the median ϑ (the so-called LD 50) which is a solution of the equation $M(x) = 1/2$. Suppose now that M is unknown but that for every input x and every element of the population an experiment can be performed whose outcome is a random variable $y(x)$ which takes only two values: 1 (response), 0 (no response). We have $P\{y(x) = 1\} = M(x)$, $P\{y(x) = 0\} = 1 - M(x)$ and therefore $E[y(x)] = M(x)$. Hence a quantile of an unknown response curve M can be obtained by the RM process defined by (3). Another (biological) application has been considered recently by Guttman and Guttman [35].

Finally we consider an application of the RM process to the problem of rounding-off errors. This problem occurs, for example, if one solves equations by classical iteration processes with the help of electronic computers. We shall only discuss the simplest problem. Define for every real number x a random variable $y(x)$ in the following way,

$$P\{y(x) = [x]\} = 1 - (x - [x]), \qquad P\{y(x) = 1 + [x]\} = x - [x]. \tag{103}$$

This can be interpreted as a random rounding-off rule in the following manner. A real number x is replaced by $[x]$ with probability $1 - (x - [x])$ and by $1 + [x]$ with probability $x - [x]$. Note that $E\{y(x)\} = x$. From here we can deduce as a pattern for more general theorems the following result. If one solves a linear equation by an iterative procedure, and modifies it by using for every step of the iteration the random rounding-off rule given by (103), then the modified procedure converges with probability 1 to a solution of the given equation. A very similar result was given by Fabian [36].

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