A NOTE ON RANDOM TRIGONOMETRIC POLYNOMIALS

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1. General remarks

This note is a postscript to our paper [1]. It deals with a problem having close connection with the topics discussed there, and uses similar methods. However, to make the note more readable, we make it self-contained at the expense of a repetition of some of the arguments in [1]. For the sake of proper perspective we begin by restating some of the results of that paper.

Consider a general trigonometric polynomial of order n,

$$\frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) , \tag{1.1}$$

with, say, real coefficients. Let $\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t), \cdots$ be the Rademacher functions

$$\varphi_n(t) = \text{sign} \sin 2^n \pi t , \qquad 0 \leqq t \leqq 1 , \tag{1.2}$$

which represent independent random variables taking values ± 1, each with probability 1/2. We write

$$P_n(x, t) = \frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \varphi_k(t) , \tag{1.3}$$

$$M_n(t) = \max_x | P_n(x, t) | . \tag{1.4}$$

One of the problems discussed in [1] was that of the order of magnitude of $M_n(t)$ for $n \to \infty$ and almost all t (this presupposes, of course, that the a_k and b_k are defined for all k). It turns out (see pp. 270–271 in [1]) that, *if the series* $\sum (a_k^2 + b_k^2)$ *diverges, and*

$$R_n = \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + b_k^2) , \tag{1.5}$$

then

$$\limsup_{n \to \infty} \frac{M_n(t)}{\sqrt{R_n \log n}} \leqq 2 \tag{1.6}$$

for almost all t.

This result was obtained under the sole assumption that $\sum (a_k^2 + b_k^2)$ diverges. If we want to obtain an estimate for $M_n(t)$ from below we must introduce further restrictions on a_n, b_n. Write

$$T_n = \sum_{k=1}^{n} (a_k^4 + b_k^4) . \tag{1.7}$$

It was shown in [1] that, *if*

$$\frac{T_n}{R_n^2} = o\left(\frac{1}{n}\right), \tag{1.8}$$

then

$$\liminf_{n\to\infty} \frac{M_n(t)}{\sqrt{R_n \log n}} \geq \frac{1}{2\sqrt{6}} \tag{1.9}$$

for almost all t [incidentally, we can also then in the right-hand side of (1.6) replace 2 by 1]. Thus, under the hypothesis (1.8), $M_n(t)$ is for almost all t strictly of order $(R_n \log n)^{1/2}$.

Clearly (1.8) implies the divergence of $\sum (a_k^2 + b_k^2)$, and is, in turn, a consequence of this divergence if $(a_n^2 + b_n^2)^{1/2}$ is bounded above and away from 0. In particular, the $M_n(t)$ for the series

$$\tfrac{1}{2} + \sum_{\nu=1}^{\infty} \varphi_\nu(t) \cos \nu t \tag{1.10}$$

for almost all t are strictly of order $(n \log n)^{1/2}$.

We add, parenthetically, that the problem of whether there exists at least one $t = t_0$ (t_0 not diadically rational), such that (1.10) satisfies $M_n(t_0) = O(\sqrt{n})$, seems to be open.

We now pass to the proper topic of this note. Subdivide the interval $(0, 2\pi)$ into $2n + 1$ equal parts and write

$$\alpha_\nu = \alpha_\nu^{(n)} = \frac{2\pi\nu}{2n+1}, \qquad \nu = 0, 1, \cdots, 2n. \tag{1.11}$$

Consider the trigonometric interpolating polynomial of order n which at the point α_ν takes the value $\varphi_\nu(t)$, $\nu = 0, 1, \cdots, 2n$. Such a polynomial exists and is uniquely determined. We denote it by $I_n(x, t)$ or sometimes, for brevity, by I, and write

$$M_n(t) = \max_x | I_n(x, t) | \tag{1.12}$$

[thus $M_n(t)$ no longer has the meaning (1.4)]. We are going to prove the following result.

THEOREM. *For almost all t we have*

$$\limsup_{n\to\infty} \frac{M_n(t)}{(\log n)^{1/2}} \leq 2. \tag{1.13}$$

2. Proof of the theorem

Denote by $D_n(x)$ the Dirichlet kernel

$$D_n(x) = \frac{\sin (n + \frac{1}{2}) x}{2 \sin \frac{1}{2} x}. \tag{2.1}$$

We have then the classical formula

$$I_n(x, t) = \frac{1}{2n+1} \sum_{\nu=0}^{2n} \varphi_\nu(t) D_n(x - \alpha_\nu). \tag{2.2}$$

For any finite sum $S = \sum a_\nu \varphi_\nu(t)$ of Rademacher functions with real coefficients, and any positive λ, we have

$$\int_0^1 e^{\lambda S} dt = \prod_\nu \int_0^1 e^{\lambda a_\nu \varphi_\nu} dt = \prod_\nu \tfrac{1}{2} (e^{\lambda a_\nu} + e^{-\lambda a_\nu}) \tag{2.3}$$

$$= \prod_\nu \left(1 + \frac{\lambda^2 a_\nu^2}{2!} + \frac{\lambda^4 a_\nu^4}{4!} + \cdots\right),$$

and since $(2p)! \geqq 2^p p!$, the last product does not exceed

$$\prod_\nu \left(\sum_{p=0}^{\infty} \frac{\lambda^{2p} a_\nu^{2p}}{2^p p!}\right) = \prod_\nu e^{\lambda^2 a_\nu^2/2}. \tag{2.4}$$

This leads to the very well known inequality

$$\int_0^1 e^{\lambda \Sigma a_\nu \varphi_\nu} dt \leqq e^{\lambda^2 \Sigma a_\nu^2/2}. \tag{2.5}$$

If we apply it to (2.2) we get

$$\int_0^1 e^{\lambda |I|} dt < \int_0^1 (e^{\lambda I} + e^{-\lambda I})\, dt \leqq 2 \exp\left[\frac{\lambda^2}{2} \frac{4}{(2n+1)^2} \sum_\nu D^2(x - a_\nu)\right]. \tag{2.6}$$

Now, D_n^2 being a trigonometric polynomial of order $2n$, its discrete average over any system of $2n + 1$ equidistant points is the same. Hence

$$\frac{2}{2n+1} \sum_{\nu=0}^{2n} D^2(x - a_\nu) = \frac{2}{2n+1} \sum_{\nu=0}^{2n} D^2(a_\nu) = \frac{2}{2n+1} D^2(0) = \frac{2n+1}{2}, \tag{2.7}$$

by (2.1) and (1.11), so that

$$\frac{4}{(2n+1)^2} \sum_\nu D^2(x - a_\nu) = 1 \tag{2.8}$$

and (2.6) takes the form

$$\int_0^1 e^{\lambda |I|} dt \leqq 2\, e^{\lambda^2/2}. \tag{2.9}$$

Integrating this with respect to x and inverting the order of integration we obtain

$$\int_0^1 dt \int_0^{2\pi} e^{\lambda |I|} dx \leqq 4\pi\, e^{\lambda^2/2}. \tag{2.10}$$

Our next step will be to deduce from this an estimate for the integral

$$\int_0^1 e^{\lambda M_n(t)} dt. \tag{2.11}$$

This deduction is based on the very well known theorem of S. Bernstein which asserts that for any trigonometric polynomial $T(x)$ of order n we have

$$\max_x |T'(x)| \leqq n \max_x |T(x)|. \tag{2.12}$$

[A proof of this theorem may be found, for example, in [2] (see p. 90, Vol. 2).]

Fix t, write $M = M_n(t)$ and denote by $x_0 = x_0(t)$ a point x at which $|I|$ attains its maximum $M_n(t)$. Take any number θ positive and less than 1, and consider the interval $x_0 \leqq x \leqq x_0 + (1 - \theta)/n$. Since the slope of the curve $y = I$ does not exceed M, the value of $|I|$ in the interval just written cannot change more than $(1 - \theta)M$, and so is at least θM in that interval. When in the inner integral (2.10) we replace the interval of integration $(0, 2\pi)$ by $[x_0, x_0 + (1 - \theta)/n]$, it follows that

$$\int_0^1 e^{\theta\lambda M_n(t)} \cdot \frac{1-\theta}{n}\, dt \leqq 4\pi\, e^{\lambda^2/2}, \tag{2.13}$$

or

$$\int_0^1 e^{\theta\lambda M_n(t)} dt \leqq \frac{4\pi n}{1-\theta}\, e^{\lambda^2/2} = \frac{4\pi}{1-\theta}\, e^{\lambda^2/2 + \log n}. \tag{2.14}$$

So far λ has been arbitrary. We now set $\lambda = (2c \log n)^{1/2}$, where c will be determined in a moment. We obtain successively

$$\int_0^1 e^{\lambda\theta M_n(t)}\,dt < \frac{4\pi}{1-\theta}\, e^{(c+1)\log n}, \tag{2.15}$$

and

$$\int_0^1 e^{\lambda\theta M_n(t)-(c+2+\epsilon)}\,dt < \frac{4\pi}{1-\theta}\, e^{-(1+\epsilon)\log n} = \frac{4\pi}{1-\theta}\, n^{-1-\epsilon}, \tag{2.16}$$

where $\epsilon > 0$. Since the series with terms $n^{-1-\epsilon}$ converges, the sum of the integrals on the left of (2.16) is finite. This implies that the series with terms $\exp[\lambda\theta M_n - (c + 2 + \epsilon)]$ converges, for almost all t, and in particular that, for almost all t and n large enough,

$$M_n(t) \leqq \frac{(c+2+\epsilon)\log n}{\theta(2c\log n)^{1/2}} = \frac{1}{\theta\sqrt{2}}\sqrt{\log n}\cdot\frac{c+2+\epsilon}{\sqrt{c}}. \tag{2.17}$$

Selecting now for c the value 2 (which minimizes the sum $c^{1/2} + 2c^{-1/2}$) we deduce that

$$\limsup_{n\to\infty} \frac{M_n(t)}{(\log n)^{1/2}} \leqq \frac{1}{\theta}(2 + \tfrac{1}{2}\epsilon) \tag{2.18}$$

for almost all t. Since we may take ϵ arbitrarily small and θ arbitrarily close to 1, (1.13) follows and the theorem is established.

It is very likely that for almost all t, $M_n(t)$ is exactly of the order $(\log n)^{1/2}$ but, so far, this is an open problem.

REFERENCES

[1] R. Salem and A. Zygmund, "Some properties of trigonometric series whose terms have random signs," *Acta Math.*, Vol. 91 (1954), pp. 245–301.

[2] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vols. 1 and 2, New York, Dover Publications, 1945.