PROBABILITY METHODS APPLIED TO THE FIRST BOUNDARY VALUE PROBLEM

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1. Introduction

The first boundary value problem (sometimes called the Dirichlet problem) is, in its most restricted form, that of finding a function u, defined on a specified open set D, such that u is a member of some specified linear class, whose members we shall call *regular* functions, and that u has a prescribed continuous boundary function. In the more general form of the problem, the hypothesis of continuity of the boundary function is dropped, and the connection between the function u and the boundary function f is correspondingly loosened. Even the restricted problem, however, cannot usually be solved in its original form without either imposing restrictions on the boundary of D or loosening the relation between u and f.

In many classical cases, $u(z)$, the value of the solution at z, becomes a positive linear functional of the continuous boundary function f, for each point z of D, with value 1 when f is the constant function 1. In view of the Riesz representation theorem, or of the Daniell approach to integration if the Riesz theorem is not available, $u(z)$ can then be expressed as a weighted average of f. In this paper, we shall treat the first boundary value problem for functions defined in the first place by the property that they are, in specified domains, weighted averages of their values over the boundaries. The results will have wide applicability because very few conditions are imposed on the underlying space, the specified domains, or the averaging method. The Perron-Wiener-Brelot (PWB) method is applied to solve a generalized version of the first boundary value problem. Not enough hypotheses are imposed, however, to make the method lead to a solution for all continuous bounded boundary functions. That is, in Brelot's terminology, not all such boundary functions need be resolutive. The PWB method is then slightly generalized, using probability methods, to obtain a slightly larger and more manageable class of resolutive boundary functions, and thus to put in a more general setting the fact that the original PWB solutions are simple weighted averages of the specified boundary functions.

The preceding results, given the domain and class of functions under consideration, are substantially independent of the way in which the domain boundary is defined, even though a PWB solution is shown to have the specified boundary function as a limit along appropriate paths to the boundary. This fact is put in a more general setting by defining and solving a new type of first boundary value problem, an intrinsic problem, in which there is no specified boundary, but in which the general properties of regular functions are used to define a specified limit behavior of a regular function outside compact sub-

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sets of its domain. In this way, it becomes possible to define nontrivial boundary functions without reference to a boundary! It is shown that this is the most general first boundary value problem approach, in the sense that the class of solutions is maximal.

The probability treatment we use makes possible the formulation and proof of the generalization to regular functions of the whole class of theorems centering about and extending Fatou's boundary value theorem for functions harmonic on a plane disc. In particular, therefore, these theorems will be proved valid for such widely differing classes of functions as the solutions of linear second order differential equations in one variable, harmonic functions on a Riemann surface and solutions of the heat equation on an arbitrary open set. The results also make it possible to give intrinsic characterizations of the solutions obtained by the PWB method.

The three classes of functions just described are solutions of differential equations of the form $Wu = 0$, on appropriate spaces. When W is the Laplace operator, the Fatou and related theorems have been extended to the subharmonic functions, that is, to solutions of the inequality $Wu \geqq 0$ and to the usual possibly discontinuous generalizations of such solutions. The corresponding generalizations will be obtained in our development.

An essential problem in such a development is to choose appropriate paths of approach to the boundary of a given set D. If D is a plane disc, and if the class of functions under examination is the class of harmonic functions, the classical approach paths have been radii. Since our approach can only depend on the general properties of regular functions, rather than on the particular properties of a given domain, our paths must have an invariant significance relative to the natural group of transformations under which the class of regular functions is invariant. Since the radii of a disc are not invariant under conformal mapping, the natural mapping in any discussion of harmonic functions, the appropriateness of the radii as the approach curves to the perimeter of a disc when discussing harmonic functions is, in a not too unreasonable sense, accidental rather than intrinsic. In any event, the use of radii in the case of a disc is not very suggestive when D is an arbitrary open plane set, or, even worse, when D is an open Riemann surface. A natural family of paths from a point z is the family of orthogonal trajectories of the niveau curves of the Green's function with pole z, but this choice has not yet led to results of the (Fatou) type desired here. An intrinsic system of approach to the boundary, natural in an investigation of harmonic functions, is a formalization of the idea that a trajectory should approach the boundary in such a way that the instantaneous motion at any point should be directed by harmonic measure. More precisely, the idea is that the trajectories are to be chosen in such a way that the distribution of the first point in which a trajectory meets the boundary of a domain is the harmonic measure on the boundary, relative to the initial point of the trajectory (supposed in the domain). The extension and application of this idea is the motivation for the work in this paper. The ideas, and some of the results, under unnecessarily strong hypotheses, have been discussed without proofs in [8].

2. Regular functions

The basic space in this paper will be a Hausdorff space, which we write in the form $R \cup R'$. Here R is open and dense in the space, and R' is its complement. Thus R', which may be the null set, is the boundary of R, and, in general, we shall always denote the boundary of a set by priming. In the applications, R is usually given as a metric space, and R' is obtained by completing the metric space, or, more simply, R is given as an open set of some larger space, and R' is its boundary. Whenever R can be metrized, different metrizations may lead to quite different boundaries, and this fact is used in discussing the ramified first boundary value problem. In this paper, we shall always suppose that $R \cup R'$ is separable, and that R is locally compact. It follows that R is metrizable, and can be represented as the union of a sequence of open sets with compact closures. The topology of $R \cup R'$ is critically important in our discussion, and we shall be forced to introduce rather clumsy hypotheses rather than the undesirable stronger one that $R \cup R'$ is compact.

If D is an open subset of R, with a nonnull boundary, we shall call a family $\{\mu_z, z \in D\}$ of complete measures a *transition measure* on D, if the following conditions are satisfied.

TM1. *The domain of μ_z is a Borel field of subsets of D', which may depend on z, and μ_z is the completion of itself considered only on the Borel sets of its domain.*

TM2. $\mu_z(D') = 1$.

TM3. *If f is nonnegative, defined on D', and measurable with respect to the measure μ_z, for every z in D, the function u defined by*

$$u(z) = \int_{D'} f(\zeta)\, \mu_z(d\zeta) \tag{2.1}$$

is continuous, if finite valued on D.

TM4. *In TM3, the integral is finite valued on D, if it is finite valued on a dense subset of D.*

The condition TM4, which we have separated from TM3 for convenience, plays an essential role. In the applications, the verification of TM3 and TM4 is usually trivial.

We shall suppose that there is a distinguished class of open subsets of R, with nonnull boundaries, called *regular* sets. To each regular set D there is to correspond a transition measure $\{\mu(z, D, \cdot), z \in D\}$, where each measure in the family is defined (at least) on all the Borel subsets of D'. The class of functions on D', measurable and integrable (and by this we shall always mean that the integral in question is finite valued), will be denoted by $L(z, D)$, and the integral of a function f in this class will be called the *regular average* of f relative to D, at z. The intersection of the classes $L(z, D)$ for z in D will be denoted by $L(D)$. The following hypotheses are made, and a further one will be added later.

RS1. *The closure of a regular set is a compact subset of R. Every open subset of R is the union of its regular subsets.*

RS2. *If D is regular, if f is a function defined and continuous on D', and if the function u is defined on D by (2.1), on D' by f, then u is continuous on $D \cup D'$.*

RS3. *Suppose that D_1, D_2 are regular, and that $D_1 \cup D_1' \subset D_2$. Suppose that $f \in L(D_2)$, and that u is defined on D_2 as the regular average of f on D_2', relative to D_2. Then u on D_1' defines a function f_1, and it is supposed that the regular average of f_1 relative to D_1 coincides with u on D_1.*

A function u, defined on an open subset D of R will be called *regular* if it is continuous, and if, whenever D_1 is a regular set whose closure is a subset of D, u on D_1 is the regular average of u on D_1', relative to D_1. If D is regular, and if $f \in L(D)$, the function u defined on D by (2.1) is regular, according to RS3. The condition RS3 is equivalent to the condition that, if D is regular, and if A is a Borel subset of D', then $\mu(z, D, A)$ defines a regular function of z on D. The condition is also equivalent to the same condition with the restriction that f is supposed continuous.

Combining conditions, we find that the regular average of a nonnegative function on the boundary D' of a regular set D, relative to D, is finite valued and regular on an open subset D_1 of D if it is finite valued on a dense subset of D_1. The average is therefore finite valued on the union of an open and a nowhere dense subset of D and is $+\infty$ elsewhere on D.

A function u defined on an open subset D of R will be called *subregular* if $-\infty \leqq u < \infty$, if u is finite on a dense subset of D, if u is upper semicontinuous, and if, whenever D_1 is a regular set whose closure is a subset of D, u on D_1 is less than or equal to the regular average of u on D_1' relative to D_1,

$$u(z) \leqq \int_{D_1'} u(\zeta)\, \mu(z, D_1, d\zeta)\,. \tag{2.2}$$

A function will be called *superregular* if its negative is subregular.

The subregular function u in (2.2) is bounded from above on D_1', so that the integral in (2.2) either is finite or has the value $-\infty$. Since a subregular function is, by definition, finite on a dense set of its domain, the integral in (2.2) is finite on a dense subset of D_1, and therefore, according to TM4, defines a regular function on D_1. That is, any subregular function u, defined on a set including the closure of a regular set D_1, defines on D_1' a boundary function in the class $L(D_1)$, and u in D_1 is less than or equal to its regular average on D_1', relative to D_1. We now conclude that u on D_1 is less than or equal to any regular function v, defined on a set including the closure of D_1, if $u \leqq v$ on D_1'.

It is clear from this definition that the limit of a monotone sequence of subregular [regular] functions is itself subregular [regular] if it satisfies the finiteness and continuity conditions of the relevant definition. For example, if the functions are subregular [regular] and the sequence is nonincreasing, the limit is subregular [regular] if it is finite on a dense set of its domain.

The choice of further hypotheses is dictated by the demand that certain theorems be true. One of the most important of these is the following.

THEOREM 2.1. *Let u be a function defined and subregular on an open subset D of R, and let D_1 be a regular set whose closure is a subset of D. Then, replacing u on D_1 by the regular average of u on D_1' relative to D_1 yields a function $u_1 \geqq u$ on D, regular on D_1, subregular on D.*

There is a standard technique, that used in subharmonic function theory, for proving such theorems. We shall not review this technique here, except to remark that it is based in the last analysis on the maximum principle, and in fact on the maximum principle as applied to functions which do not satisfy the full defining conditions of these functions, but only local conditions. To analyze this situation in the present context, we shall say that a function u satisfies *local hypotheses* of subregularity if u is defined on an open subset D of R, if $-\infty \leqq u < \infty$, if u is finite on a dense subset of D, if u is upper semicontinuous, and if the following condition is satisfied. It is supposed that to each point z of D there is assigned a neighborhood of z, whose closure is a compact subset of D. The hypothesis states that, if $z \in D$, and if D_1 is a regular set contained in the assigned neighborhood of z, then $u(z)$ is less than or equal to the regular average of u on D_1', relative to D_1, at z. With this definition, a second theorem we wish to be true is the following.

THEOREM 2.2. *If u satisfies local hypotheses of subregularity, for some assigned neighborhood system, u is subregular.*

We have already remarked that an extended version of the maximum principle is used to prove such theorems when the regular functions are the harmonic functions. In the present context, the maximum principle will be obtained in terms of a new concept. As usual, we shall describe as the *support* of the measure function $\mu(z, D, \cdot)$ the closed subset of D', of measure 1, which is the intersection of all the closed subsets of D which have measure 1. Let R_1 be an open subset of R, and suppose that to each point z of R_1 there is assigned a neighborhood $N(z)$ of itself, whose closure is a subset of R_1. Then we shall say that the point z' of R_1 covers the point z'' of R_1, relative to R_1 and the assigned neighborhood system $\{N(z),\ z \in R_1\}$ if there is a finite succession $z_1 = z', \cdots, z_{n+1} = z''$ of points of R_1, $n \geqq 1$, and of regular sets $D_1, \cdots, D_n$, such that $D_j \subset N(z_j)$ and that z_{j+1} is a point of the support of the measure function $\mu(z_j, D_j, \cdot)$ for $j \leqq n$. The covering relationship is obviously transitive.

Suppose that u is subregular on the open set D containing the closure of the regular set D_1. Since u is upper semicontinuous, it has a finite maximum value c_1 on $D_1 \cup D_1'$, and it is obvious from (2.2) that c_1 is assumed by u at some point of D_1'. (It is not necessarily true, however, under our present hypotheses, that $u < c_1$ on D_1 unless $u = c_1$ identically, even if D_1 is connected. We shall not impose such a condition in this paper, because it would exclude important applications, for example, applications to the solutions of parabolic partial differential equations.) Suppose now that u is subregular on D, or if not subregular at least satisfies local conditions of subregularity relative to some specified neighborhood system. Then the preceding version of a maximum principle is no longer necessarily valid. Let c be the supremum of u on D. Since u is upper semicontinuous, if u never takes on the value c, u has the limit value c at the boundary. On the other hand, if u takes on the value c, say at the point z_0, then the subset A of D on which $u(z) = c$ is closed relative to D, and obviously contains every point covered by z_0, or by any other point of A, relative to the assigned neighborhood system. In order to obtain a useful theory of regular functions, we shall need the validity of a maximum principle for functions satisfying local hypotheses of subregularity. It will be sufficient for our purposes to impose the condition that a set like A stretches to the boundary of D (at least if D is regular) in the sense of the following condition which is the last we shall always impose on the system of transition measures and regular sets.

RS4. *Let D be a regular set, let z be a point of D, and let A be the set of points covered by z, relative to D and any specified neighborhood system. Then some point of D' is a limit point of A.*

We give two illustrations of this condition. Let R be a Euclidean space, and let the regular functions be the harmonic functions. The regular sets can be taken as the open spheres, if a small class is desired, or as the bounded open sets whose boundary points are all regular in the usual potential-theoretic sense, if a large class is desired. In either case, in the notation of RS4, $A = D$, so that RS4 is obviously satisfied. Again let R be a Euclidean space, and let the regular functions be the solutions of the heat equation, so that there is one distinguished (time) coordinate axis. The regular sets can be taken, for example, as the bounded open sets whose boundaries are pieces of finitely many hyperplanes, none of which is perpendicular to the distinguished axis. In this case, in the notation of RS4, A is the class of those points which are endpoints of continuous curves in D, with initial point z, along which the distinguished coordinate is monotone decreasing when a point moves from initial to endpoint [7]. Thus A and D are never the same, but RS4 is satisfied.

Under RS4, it is clear that a subregular function, or even a function satisfying only local subregularity conditions relative to some system of neighborhoods, defined on a regular set D, has its supremum as a limit at some point of D'. With the help of this version of the maximum principle, the desired theorems 2.1 and 2.2 above are easily proved by the standard methods of subharmonic function theory. If D is no longer supposed regular, but if D has a compact closure, and if it is supposed that D is the union of a monotone sequence of regular sets, the above version of the maximum principle remains true. If D is no longer supposed to have a compact closure, but is still supposed to be the union of a monotone sequence of regular sets, a function u with the above property has its supremum as a limit on D' in the sense that this supremum is a limit of u on a sequence of points of D, only finitely many of the points lying in any compact subset of D. Finally, if D is only an open subset of R, we have no version of the maximum principle for a function subregular on D. In the classical cases, every open subset of R is the union of a monotone sequence of regular sets, if the class of regular sets is suitably defined. The latter qualification may require a few remarks. It is easily seen that the classes of regular and subregular functions are not changed if the class of regular sets is reduced, as long as the class is large enough to make RS1 remain valid. At the other extreme, once the theory has been set up, the class of regular sets can be augmented by any open set whose closure is a compact subset of R, as long as RS4 is valid for the set, and as the restricted first boundary value problem always has a solution. That is, to each specified continuous boundary function must correspond a regular function in the set, with the specified function as boundary function in the usual limit sense. No change in the class of regular sets in either direction, as just described, will affect any of the later work. For example, the classes of PWB resolutive boundary functions of an open set, of PWB resolutive open sets, and so on, will not be affected.

In connection with the preceding remarks, it must be added that our theory is a local theory, because the regular sets may be small, and for this reason the theory may be difficult to apply, or even impossible, without further hypotheses, to functions defined on a set which is neither regular nor the union of a monotone sequence of regular sets. Our first token of this fact is that we have obtained a form of the maximum principle only for functions defined on sets of special type. Whether a similar form is valid for functions defined on an arbitrary open subset of R depends on how that set can be expressed as a union of regular sets, and on the nature of the set of points of a regular set covered by a point of that set, relative to specified neighborhood systems. In order to keep the local flavor of our hypotheses, we have restricted D to be regular in RS4. A weakening of this restriction would simplify later work.

We shall use below the following theorem, proved under our hypotheses by the standard methodology of subharmonic function theory.

THEOREM 2.3. *Let u be defined and subregular on an open set D, and let D_1, D_2 be regular sets, with*

$$D_1 \subset D_2, \qquad D_2 \cup D_2' \subset D. \tag{2.3}$$

Then, if $z \in D_1$, the regular average of u on D_1' with respect to D_1 at z is at most that of u on D_2' with respect to D_2 at z.

If u is a function defined and subregular on the set D, and if there is a function v, regular on D, such that $u \leqq v$ and such that any function v_1 with the same properties as v satisfies the inequality $v \leqq v_1$, then v will be called the best regular majorant of u. We can write D as the union of a sequence of regular sets, $D = \bigcup_1^\infty D_n$, with closures in D. If this sequence is monotone, v can be obtained as follows. Define u_n as the regular average of u on D_n', relative to D_n. Then $u_n \leqq u_{n+1}$ on D_n, by the theorem just stated above. Define $v = \lim_{n\to\infty} u_n$. Then, if v is finite on a dense subset of D, v is regular and is the best regular majorant of u. Conversely, if there is a best regular majorant of u, it can be obtained in this way. If the sequence $\{D_n, n \geqq 1\}$ is not monotone, a more complicated procedure can be used to obtain the best regular majorant, as follows. We can and shall assume that each set D_n is repeated infinitely often in the sequence of these sets. Define u_1 on D_1 as the regular average of u on D_1', relative to D_1, and define u_1 as u on $D - D_1$. In general, if $u_1, \cdots, u_{n-1}$ have been defined, define u_n on D_n as the regular average of u_{n-1} on D_n', relative to D_n, and define u_n as u_{n-1} on $D - D_n$. Then u_n is subregular and $u_{n-1} \leqq u_n$. The limit v of the sequence of subregular functions obtained in this way is the best regular majorant, if there is one, and there is one if v is finite valued on a dense subset of D.

3. The Perron-Wiener-Brelot method

The PWB method will now be used to solve the first boundary value problem on an arbitrary open subset D of R. Before solving the problem in anything like its usual formulation, it is obvious that at least one hypothesis is essential, namely that D' have enough points. For example, D may be R itself, and R' may be empty, and, even if R' is not empty, it can be replaced by any subset of itself without invalidating any of our hypotheses. Since the usual formulation of the first boundary value problem (but not that in section 7) involves a preassigned boundary function defined on D', the absence of further hypotheses implies the absence of a problem, or its insolubility, or its absurdity, depending on the formulation. In order to obtain a unique solution, some hypothesis must be added that ensures that the behavior of a regular function near individual boundary points of D determines the function on D, and such a hypothesis will simultaneously ensure that D' is not too small. The basic property of this sort is the maximum principle. We have seen in section 2 that we have no version of the maximum principle unless D is the union of a monotone sequence of regular sets. Yet to restrict our analysis to such sets D would be ill advised, because, in some applications, the regular sets are small sets of special type, for example spheres in a metric space, in which case the unions of monotone sequences of regular sets will be too special. We therefore impose a further restriction on D in discussing the PWB method. It is not necessarily satisfied if D is the union of a monotone sequence of regular sets, because the maximum principle obtained above for such a set does not link the character of a subregular function on D with its limiting character at individual points of D', and it is this linking that we need. Rather than attempting to find and impose more basic conditions sufficient for the validity of the maximum principle in the desired form, conditions which might be undesirable later, we shall simply state the principle, for a specified set D, as an added hypothesis, as follows. This hypothesis will always be mentioned explicitly, when it is needed.

M(D, D'). *The set D is an open subset of R, and, if u is subregular and bounded from above on D, the supremum of u is a limiting value of u at some point of D'.*

Note that D' cannot be the null set, if M(D, D') is satisfied. If D has a compact closure, either of the following conditions is sufficient for the validity of M(D, D'): (a) D is the union of a monotone sequence of regular sets; (b) RS4 is satisfied for D (even if D is not regular).

In the following, we shall use the PWB method to solve, in its usual generalization, the first boundary value problem on R. The substitution of R for a general open subset D

of R is no real restriction, because such a set D itself can be taken as the space of the discussion, if we define the regular sets of this space as the regular sets of R whose closures are subsets of D.

In the following discussion, we suppose throughout that $\mathrm{M}(R, R')$ is satisfied. Let f be any function, not necessarily finite valued, defined on R'. The lower [upper] PWB class of functions for f is defined, following Brelot [1], as the class of functions on R, containing, together with the function $-\infty$ $[+\infty]$, every subregular [superregular] function on R which is bounded from above [below] and whose upper [lower] limit at each point of R' is at most [least] the value of f at that point. Then, applying $\mathrm{M}(R, R')$, we find that every function in the upper class is greater than or equal to every function in the lower class. The upper [lower] limit u_f $[u^f]$ of the lower [upper] class of functions is called the lower [upper] PWB solution for f. The following cases can arise.

(a) The function $-\infty$ is the only function in the lower class, and the function $+\infty$ is the only function in the upper class. Then $u_f = -\infty$, $u^f = +\infty$.

(b) There are at least two functions in the lower class, but $+\infty$ is the only function in the upper class. Then $u^f = +\infty$, and there is a possibly empty open set $R_0 \subset R$ such that u_f is regular on R_0, if R_0 is nonnull, is finite on a possibly empty additional set, which is nowhere dense, and is $+\infty$ elsewhere on R.

(c) Same as (b), with interchange of the roles of lower and upper classes, of $-\infty$ and $+\infty$.

(d) There are at least two functions in each class. Then u_f and u^f are regular on R, with $u_f \leqq u^f$.

No comment is necessary in case (a). The characterizations of the lower and upper solutions in the other cases follow easily from the following assertion, whose truth we shall now prove. *If the lower class contains at least two functions, there is a possibly empty open set $R_0 \subset R$, such that u_f is regular on R_0, if R_0 is nonnull, is finite on a possibly empty additional set, which is nowhere dense, and is $+\infty$ elsewhere on R.*

According to our definitions, the maximum of any finite number of subregular functions defined on the same domain is subregular. There is therefore, under the italicized hypotheses, a monotone nondecreasing sequence $\{u_n, n \geqq 1\}$, of subregular functions in the lower class, converging to u_f on a denumerable set A, dense on R. Let D be any regular set. Replacing u_n on D by its regular average on D', relative to D, we can suppose that u_n is regular on D. Then $u_f \geqq u_n > -\infty$ on D, so that, since R can be covered by regular sets, $u_f > -\infty$ on R. Let $u_\infty = \lim_{n\to\infty} u_n$. Since the u_n sequence is monotone, and since u_n is integrable on D'_n with respect to $\mu(z, D, \cdot)$ measure, for z in D,

$$u_\infty(z) = \int_{D'} u_\infty(\zeta)\, \mu(z, D, d\zeta)\,. \tag{3.1}$$

In view of our hypotheses on such integrals, u_∞ must be regular on an open possibly empty subset D_0 of D, finite on a possibly empty additional set, and $+\infty$ elsewhere on D. Now u_∞ is uniquely determined on D_0 by its values on $A \cap D_0$, that is, by the values of u_f on $A \cap D_0$, so that adjoining another point of D_0 to A does not change u_∞ on D_0. This means, since any point of D_0 can be so adjoined, that $u_\infty = u_f$ on D_0, so that u_f is regular on D_0 if this set is not empty. Moreover $u_f \geqq u_\infty$, so that $u_f = u_\infty = +\infty$ where the right-hand half of this equality is true. Since some sequence of regular sets D covers R, we have now completed the proof of the above italicized statement.

An easy extension of the reasoning just used shows that, if the lower class contains at least two functions, there is a monotone sequence of functions in the lower class converging uniformly to u_f on every compact subset of the set on which u_f is regular.

If u_f and u^f are finite valued and equal, the regular function u_f will be called the PWB solution of the first boundary value problem on R, and f will be called PWB *resolutive*. The PWB resolutive boundary functions and corresponding solutions satisfy the following relations.

RF1. *If f_1 and f_2 are PWB resolutive, determining solutions u_1 and u_2 respectively, then $c_1f_1 + c_2f_2$ is PWB resolutive, and determines the solution $c_1u_1 + c_2u_2$, for all constants c_1, c_2.*

RF2. *The (finite) constant functions are PWB resolutive, and the solution corresponding to the boundary function* 1 *is the function* 1 *on R.*

RF3. *If f is PWB resolutive, and if $f \geqq 0$, then the corresponding solution is nonnegative.*

RF4. *If f is PWB resolutive, there are Borel measurable PWB resolutive boundary functions f_1, f_2, with the same PWB solution as f, and satisfying the inequality $f_1 \leqq f \leqq f_2$.*

RF5. *If $\{f_n, n \geqq 1\}$ is a monotone nonincreasing [nondecreasing] sequence of PWB resolutive boundary functions, corresponding to the sequence $\{u_n, n \geqq 1\}$ of solutions, then $\lim_{n\to\infty} f_n = f$ is PWB resolutive, if $u_f > -\infty$ $[u^f < +\infty]$, with solution $\lim_{n\to\infty} u_n$, where the latter limit exists uniformly on every compact subset of R.*

Properties RF1, RF2, RF3 are easily verified. To prove RF4 we use a sequence $\{u_n, n \geqq 1\}$ of subregular functions, in the lower class for f, converging monotonely to the solution u for f. Let g_n be the upper limiting function of u_n on R'. Then g_n is upper semicontinuous, and we can take $f_1 = \lim_{n\to\infty} g_n$. The function f_2 can be defined similarly. Finally, to prove RF5, we note first that, according to our analysis of lower and upper functions, u_f and u^f are both regular, under the stated hypotheses, and go on by paraphrasing the proof of a special case (Brelot and Choquet [3]) in which the result is known. We treat only the monotone nonincreasing case. Suppose that $\{f_n, n \geqq 1\}$ is as described in RF5, and that the (monotone) sequence $\{u_n, n \geqq 1\}$ has the limit u. Then

$$u_n \geqq u \geqq u^f \geqq u_f\,. \tag{3.2}$$

We show that $u_f \geqq u$. To show this, let z_0 be any point of R, let ϵ be a positive number, and choose v_j as a lower PWB function for $f_j - f_{j-1}$ and to satisfy

$$v_j(z_0) \geqq u_j(z_0) - u_{j-1}(z_0) - \frac{\epsilon}{2^j}, \tag{3.3}$$

where we have defined $f_0 = 0$, $u_0 = 0$. Then $\sum_1^n v_j$ is a lower PWB function for f_n, and

$$\sum_1^n v_j(z_0) > u_n(z_0) - \epsilon\,. \tag{3.4}$$

By hypothesis, $u_f > -\infty$, so there is a lower PWB function v, not the function $-\infty$, for f. Then v is also a lower PWB function for f_n, so that

$$\max\left[v, \sum_1^n v_j\right] \tag{3.5}$$

is a lower PWB function for f_n. But then the function

$$\max\left[v, \sum_{1}^{\infty} v_j\right] \tag{3.6}$$

is a lower PWB function for f_n, for every n, so that this maximum is a lower PWB function for f. Hence

$$u_f(z_0) \geqq \sum_{1}^{\infty} v_j(z_0) \geqq u(z_0) - \epsilon . \tag{3.7}$$

Thus $u_f(z_0) \geq u(z_0)$, and it follows that $u_f \geqq u$ on R. Since, as we have already noted, $u^f \leqq u$ on R, it follows that $u_f = u^f = u$ on R, so that f is PWB resolutive. Finally, the sequence $\{u_n, n \geqq 1\}$ converges uniformly on every compact subset of R because (Dini's theorem) it is a monotone sequence of continuous functions with a continuous limit.

In the classical cases, it is proved that every bounded continuous boundary function is PWB resolutive. In these cases $R \cup R'$ is a complete (and usually even compact) metric space, and it will then follow from the properties listed above that, using the Riesz representation theorem or the Daniell integral definition, there is a transition measure $\{\mu(z, R, \cdot), z \in R\}$ such that the function f on R' is PWB resolutive if, and only if, extending our previous notation in the obvious way, $f \in L(R)$, and such that, if f is PWB resolutive, the corresponding solution is the average

$$u(z) = \int_{R'} f(\zeta)\, \mu(z, R, d\zeta) . \tag{3.8}$$

The key problem in such an analysis is to prove that the bounded continuous functions on R' are PWB resolutive, and the proof of the assertion has required specific facts on the character of regular functions and the properties of boundaries. We shall approach the analysis of the characteristics of the PWB resolutive class of boundary functions entirely differently in this paper.

In our discussion of the first boundary value problem, R can be replaced by any of its open subsets D, under the hypothesis $\mathrm{M}(D, D')$. In particular, if D is regular, $\mathrm{M}(D, D')$ is automatically satisfied, and every continuous boundary function f is resolutive. In fact, in view of RS2, the regular average of f on D' relative to D is in both lower and upper PWB classes for f, and is therefore the PWB solution for f. The full details of the classical case, as stated in the preceding paragraph, are valid in this case, and in fact the definition of a regular set was made with this possibility in mind.

Going back to the general case, we shall call R *weakly* PWB *resolutive* if $\mathrm{M}(R, R')$ is satisfied and if there is a transition measure $\{\mu(z, R, \cdot), z \in R\}$ such that, with the obvious extension of our customary notation, whenever $f \in L(R)$, the regular average of f on R' relative to R, as defined by this transition measure, is regular on R, and such that, if f is PWB resolutive, then $f \in L(R)$ and f has as solution the regular average just described. If, in addition, every function in the class $L(R)$ is PWB resolutive, R will be called PWB *resolutive*. If R is PWB resolutive, and if, in addition, the domain of $\mu(z, R, \cdot)$ includes all the Borel subsets of R', for every point z of R, R will be called *strongly* PWB *resolutive*. We have already remarked that, in the classical cases, all open sets are strongly PWB resolutive, and that, in our case, a regular set is always strongly PWB resolutive. We shall prove below that, under our usual hypothesis $\mathrm{M}(R, R')$ as slightly strengthened below to $M'(R, R')$, R is always weakly PWB resolutive, and that, if $f \in L(R)$, the corresponding regular average is the solution of the first boundary value problem as obtained by a natural extension of the PWB method.

We shall now extend slightly the validity of a defining inequality of subregular functions. Suppose that D is an open set, whose closure is a compact subset of R, and suppose that u is a function defined and subregular on a set including the closure of D. Let u on D' define the boundary function f. Then, if D is regular, and if $z \in D$, $u(z)$ is less than or equal to the regular average of f relative to D, at z. This inequality remains true if $M(D, D')$ is satisfied and if D is merely strongly PWB resolutive, where the transition measure determining the regular average is that involved in the definition of a strongly PWB resolutive set. To see this, suppose first that u is bounded on the closure of D. Then f is PWB resolutive, and u on D is a lower PWB function for f. Hence, if f has PWB solution u', $u \leqq u'$ on D, that is, u on D is at most the regular average of f on D' relative to D. In the general case, u is bounded from above on the closure of D, and the result to be proved is obtained by applying the result in the bounded case to the function $\max[u, n]$, and letting $n \to -\infty$.

It is now not difficult to prove that theorem 2.3 of section 2 is valid for D_1 and D_2 not necessarily regular, but satisfying $M(D_1, D_1')$, $M(D_2, D_2')$ and strongly PWB resolutive, if we suppose that D_1, D_2 have compact closures, with $D_1 \cup D_1' \subset D_2$. In fact then, since u is subregular on a set including the closure of D_2,

$$(3.9) \qquad \int_{D_1'} u(\zeta_1)\,\mu(z, D_1, d\zeta_1) \leqq \int_{D_1'} \mu(z, D_1, d\zeta_1) \int_{D_2'} u(\zeta_2)\,\mu(\zeta_1, D_2, d\zeta_2)\,.$$

Now if A is a Borel subset of D_2', $\mu(\zeta_1, D_2, A)$ defines a regular function of ζ_1 on D_2, and the latter set includes the closure of D_1. Hence

$$(3.10) \qquad \mu(z, D_2, A) = \int_{D_1'} \mu(\zeta_1, D_2, A)\,\mu(z, D_1, d\zeta_1)\,,$$

so that, reversing the order of integration in the above iterated integral, the integral becomes

$$(3.11) \qquad \int_{D_2'} u(\zeta_2)\,\mu(z, D_2, d\zeta_2)\,,$$

and we have thus obtained the desired inequality between the regular averages of u on D_1' and D_2'. If D_1 is allowed to be regular, we no longer need suppose that $D_1' \subset D_2$, since then the classical proof used in the theory of subharmonic functions (involving the replacement of u on D_1 by its regular average on D_1' relative to D_1, to obtain a new subregular function) can be used.

The following remark is useful in analyzing the character of a set D relative to the PWB method. If a measure family $\{\mu(z, D, \cdot), z \in D\}$ can be shown to satisfy conditions TM1, TM2, TM3, where the function u in TM3 is regular, then TM4 must also be satisfied. In fact if f is nonnegative, defined on R', and measurable with respect to $\mu(z, R, \cdot)$, for every z, define $f_n = \min[f, n]$, $n \geqq 1$, and let u_n $[u]$ be the regular average of f_n $[f]$ on R' relative to R. Then u_n is regular, and $u_n \to u$ monotonely, so that u is regular on an open subset of R if it is finite valued on a dense subset of this set. This remark shows, for example, that, if every bounded Borel measurable function on D' is PWB resolutive, then D is strongly PWB resolutive. In fact, each PWB solution can be expressed as the regular average of a measure family satisfying TM1, TM2, TM3, using the Daniell integral definition, and then TM4 is automatically satisfied, so that the measure family is a transition measure.

4. The classes H, D

We suppose in this section that R is *strongly* PWB *resolutive from below*, by which we mean that R can be expressed as the union of a sequence $\{R_n, n \geqq 1\}$ of strongly PWB resolutive sets, with compact closures, for which $R_n \cup R_n' \subset R_{n+1}$. Let z be a point of R, and let u be a function defined and subregular on R. We shall denote by $H(R)$ the class of functions u with the property that the regular average of $|u|$ on R_n' at z, relative to R_n, and considering only values of n so large that $z \in R_n$, defines a bounded sequence of numbers, for every point z of R. If u is nonnegative, these regular averages determine a monotone sequence, so that the boundedness is obviously independent of the choice of the R_n sequence. In the general case, it is easily seen that the boundedness is equivalent to the same condition for the nonnegative subregular function $\max [u, 0]$, so that $u \in H(R)$ if and only if $\max [u, 0] \in H(R)$, and again the choice of the R_n sequence is irrelevant.

We denote by $D(R)$ the class of functions u, regular on R, such that, for each point z of R, the sequence of functions obtained by considering u on R_n' for n so large, $n \geqq N_z$, that the specified point z is in R_n, is uniformly integrable relative to the corresponding sequence $\{\mu(z, R_n, \cdot), n \geqq N_z\}$, that is, that

$$\lim_{a \to \infty} \int_{\{|u(\zeta)| \geqq a\}} |u(\zeta)| \, \mu(z, R_n, d\zeta) = 0 \tag{4.1}$$

uniformly for $n \geqq N_z$. Then clearly $D(R) \subset H(R)$. In the following we shall omit the set R from the notation here if there is no possible ambiguity. We shall use the following lemma to prove that the class D is independent of the choice of R_n sequence.

LEMMA 4.1. *For each point t in the index set T, let μ_t be a totally finite measure on the space X_t. Let $\{g_t, t \in T\}$ be a family of measurable functions, where g_t is defined on X_t. If there is a positive function ψ, defined and monotone nondecreasing on* $[0, \infty)$, *with* $\lim_{\xi \to \infty} \psi(\xi)/\xi = \infty$, *and such that*

$$\sup_t \int_{X_t} \psi(|g_t|) \, d\mu_t < \infty, \tag{4.2}$$

then the family is uniformly integrable. Conversely, if the family is uniformly integrable, there is a function ψ with these properties, which is convex.

This result, with slightly different conditions on ψ, is due to de la Vallée Poussin [13]. The fact that in his discussion the spaces, and measures, are identical is irrelevant to the proof. His method of proof, with minor modifications, yields the result as stated here.

The condition that the regular function u be a member of the class D is independent of the choice of the R_n sequence, according to the following argument. If $u \in D$, let ψ be the convex monotone function that exists, according to the lemma, with the property that

$$\sup_n \int_{R_n'} \psi[|u(\zeta)|] \, \mu(z_0, R_n, d\zeta) < \infty. \tag{4.3}$$

Here z_0 is fixed, and ψ may depend on the choice of z_0. Now $|u|$ is subregular, so that, applying Jensen's inequality, $\psi(|u|)$ is also subregular. Thus the condition that $u \in \boldsymbol{D}$ becomes the condition that the sequence of regular averages at z_0 in (4.3), a monotone sequence, is bounded. The monotoneity implies that the boundedness is independent of the R_n sequence.

A familiar argument, going back to F. Riesz in the context of subharmonic functions, shows that $u \in \boldsymbol{H}$ if and only if there is a nonnegative regular function v on R, such that $u \leqq v$. If $u \in \boldsymbol{H}$, v can be taken as the best regular majorant of max $[u, 0]$, which necessarily exists. This property can be used to define the class $\boldsymbol{H}$ even if R cannot be expressed in the way presupposed throughout this section.

We shall use below without further comment the fact that, if $u \in \boldsymbol{D}$, and if u_n is the best regular majorant of the subregular function max $[u, n]$, then $u \leqq u_n \leqq u_{n+1}$, and $\lim_{n\to-\infty} u_n = u$ on R. The fact that the limit here is u follows at once from the uniform integrability definition of the class $\boldsymbol{D}$, together with our construction of the best regular majorant, as given in section 2, if in that construction strongly PWB resolutive rather than regular sets are used.

The following theorem gives a simple but important condition necessary and sufficient that a function belong to the class $\boldsymbol{D}$.

THEOREM 4.1. *If u is a function defined and regular on R, $u \in \boldsymbol{D}$ if and only if, to every positive ϵ and every point z of R, there correspond two functions u_1 and u_2, with u_1 and $-u_2$ subregular and bounded from above on R, and satisfying*

$$\begin{aligned} &u_1 \leqq u \leqq u_2\,, \\ &u_2(z) - u_1(z) \leqq \epsilon\,. \end{aligned} \tag{4.4}$$

To prove the theorem, suppose first that $u \in \boldsymbol{D}$, and define v_n as the best regular majorant of max $[u, n]$. Then

$$n \leqq v_n\,, \qquad u \leqq \cdots \leqq v_{-2} \leqq v_{-1}\,, \qquad \lim_{n\to-\infty} v_n = u\,. \tag{4.5}$$

For any ϵ, z, we can choose $-n$ so large that $v_n(z) - u(z) \leqq \epsilon/2$, and with such a choice of n, define

$$u_2 = v_n\,. \tag{4.6}$$

If u_1 is defined in the obvious analogous way, the conditions of the theorem on u_1 and u_2 are now satisfied. Conversely, if u satisfies the conditions of the theorem, choose any ϵ, z, and let u_1, u_2 be the corresponding pair of functions. Let K be an upper bound for both u_1 and $-u_2$. Then, for every $a \geqq 0$, using the fact that u_1 $[u_2]$ is subregular [superregular], we find that

$$\begin{aligned} \int_{\{|u(\zeta)|\geqq a\}} |u(\zeta)|\,\mu(z,R_n,d\zeta) &\leqq \int_{\{u(\zeta)\geqq a\}} u_2(\zeta)\,\mu(z,R_n,d\zeta) \\ - \int_{\{u(\zeta)\leqq -a\}} u_1(\zeta)\,\mu(z,R_n,d\zeta) &= \int_{\{|u(\zeta)|\geqq a\}} [u_2(\zeta) - u_1(\zeta)]\,\mu(z,R_n,d\zeta) \\ &\quad - \int_{\{u(\zeta)\leqq -a\}} u_2(\zeta)\,\mu(z,R_n,d\zeta) + \int_{\{u(\zeta)\geqq a\}} u_1(\zeta)\,\mu(z,R_n,d\zeta) \\ &\leqq [u_2(z) - u_1(z)] + K\mu(z,R_n,\{|u(\zeta)|\geqq a\})\,. \end{aligned} \tag{4.7}$$

Setting $a = 0$, we find that $u \in \boldsymbol{H}$. Let K' be the supremum of the integral on the left, when n varies and $a = 0$. Then, for every positive value of a,

$$\int_{\{|u(\zeta)| \geqq a\}} |u(\zeta)| \mu(z, R_n, d\zeta) \leqq [u_2(z) - u_1(z)] + \frac{KK'}{a}. \tag{4.8}$$

Here z is fixed, and K' depends on the choice of z, whereas K depends only on the choices of u_1 and u_2. When $a \to \infty$, the integral on the left in (4.8) approaches zero uniformly (as n varies), for each z, because the bracketed difference on the right can be made arbitrarily small by proper choices of u_1, u_2, whereupon the last term goes to zero with $1/a$.

The defining conditions for the classes $\boldsymbol{H}$ and $\boldsymbol{D}$, as originally stated, are conditions depending on a point z of R, which are to hold for all z. In some applications, for example if R is an open set of a finite dimensional Euclidean space, or if R is a Riemann surface, and if regularity means harmonicity, the conditions are valid for all points z if they are valid for one. It is easily seen to be sufficient, for example, if there is a valid Harnack theorem, stating that whenever u is positive and regular on R, $u(z_1)/u(z_2)$ is bounded from above, for z_1 and z_2 on any specified compact subset of R, by a constant depending only on the compact set, but not on u. In this case a function u is in the class $\boldsymbol{D}$ if and only if there is a function ψ, positive, monotone nondecreasing and convex on $[0, \infty)$, with $\lim_{\xi \to \infty} \psi(\xi)/\xi = \infty$, such that $\psi(|u|) \in \boldsymbol{H}$.

The importance of the classes $\boldsymbol{H}$ and $\boldsymbol{D}$ in the special case when R is a finite dimensional open sphere and regularity means harmonicity is well known. In this case, $u \in \boldsymbol{D}$ if and only if u can be expressed by the Poisson integral, with a Lebesgue integrable boundary function, also if and only if u is a PWB solution. We shall see that there are corresponding characterizations in the general case. We shall also see that the classical boundary value theorems for the classes $\boldsymbol{H}$ and $\boldsymbol{D}$ in the above special case remain valid in full detail in the general case, if the concept of approach to R' is suitably defined, generalizing radial approach if R is a sphere. Parreau [11] has recently studied the classes $\boldsymbol{H}$ and $\boldsymbol{D}$ and related ones when R is a Riemann surface and regularity means harmonicity. Our results are applicable to this case, giving theorems in a somewhat different direction from those of Parreau.

If $\mathrm{M}(R, R')$ is satisfied, and if f is a PWB resolutive boundary function on R', the corresponding PWB solution u satisfies the conditions of theorem 4.1, and is therefore in the class $\boldsymbol{D}$. In fact the functions u_1 and u_2 in theorem 4.1 can be taken as functions in the lower and upper classes for f, respectively. The result is closely related to the following one, but neither seems to imply the other. Note that we do not suppose in the following theorem that f is PWB resolutive.

THEOREM 4.2. *Suppose that R is weakly PWB resolutive, and strongly PWB resolutive from below. Then, if $f \in L(R)$, the regular average of f on R', relative to R, is a member of the class* $\boldsymbol{D}$.

If $f \in L(R)$, and if $z_0 \in R$, there is, according to a rather special case of lemma 4.1, a positive monotone convex function ψ on $[0, \infty)$ such that $\lim_{\xi \to \infty} \psi(\xi)/\xi = \infty$, and that $\psi(|f|) \in L(z_0, R)$. Let u [v] be the regular average of f [$\psi(f)$] on R', relative to R. Then, applying Jensen's inequality,

$$\int_D \psi[|u(\zeta)|] \mu(z_0, D, d\zeta) \leqq \int_{D'} \mu(z_0, D, d\zeta) \cdot \int_{R'} \psi[|f(\zeta')|] \mu(\zeta, D, d\zeta') = \int_{D'} v(\zeta) \mu(z_0, D, d\zeta) = v(z_0), \tag{4.9}$$

Thus the integral on the left is bounded independently of D, where we take D to be any strongly PWB resolutive open set containing z_0, whose closure is a compact subset of R. It follows that $u \in \boldsymbol{D}$.

The following simple remark gives further insight into the importance of the class $\boldsymbol{D}$ for the first boundary value problem. Let R, R' satisfy the condition $\mathrm{M}(R, R')$, so that the PWB method can be applied. Let f be a not necessarily finite-valued function on R', and suppose that there is a function u, regular on R, with limit $f(z)$ at each point z of R'. We now consider the problem of determining the properties of f and u necessary and sufficient that f be PWB resolutive, with PWB solution u. We suppose also that R is strongly PWB resolutive from below. Then we have seen that, if u is a PWB solution, $u \in \boldsymbol{D}$. Conversely, we shall now show that, if $u \in \boldsymbol{D}$, f is PWB resolutive, with PWB solution u. This is obvious if u is bounded (equivalently, if f is bounded) because in that case u is in both lower and upper PWB classes for f. In general, if $u \in \boldsymbol{D}$, let u_n be the best regular majorant on R of the subregular function max $[u, n]$. Then u_n is in the PWB upper class relative to f, and $\lim_{n \to -\infty} u_n = u$. Hence the upper PWB solution is not greater than u. Similarly the lower PWB solution is not less than u, so these two solutions are both u, as was to be proved.

There is some interest in carrying further the discussion of the preceding paragraph. In the following, R_0' will denote the one point boundary of R obtained if initially R is not compact, and has no boundary points, and if then a single point is adjoined in the usual way to make the extended space compact. The weakest possible condition $\mathrm{M}(R, R')$ is the case when $R' = R_0'$. The condition $\mathrm{M}(R, R_0')$ means that, if u is a subregular function on R, bounded from above, then its supremum is the limiting value of u along some sequence of points of R, only finitely many of which are in any compact subset of R. Under hypothesis $\mathrm{M}(R, R_0')$, we now introduce a new boundary R' as follows. Let $\boldsymbol{D}_0$ be any class of functions, defined on R, including every function in the class $\boldsymbol{H}$, and hence including every subregular function that is bounded from above. Choose R' as the minimal boundary under which $R \cup R'$ is compact, and under which every function in the class $\boldsymbol{D}_0$ has a finite or infinite limit at every point of R'. In other words, the sequence $\{z_n, n \geqq 1\}$ of points of R is convergent to a point z of R' if and only if only finitely many points of the sequence are in any compact subset of R, the sequence $\{u(z_n), n \geqq 1\}$ has a finite or infinite limit whenever $u \in \boldsymbol{D}_0$, and a second sequence of points of R has a different limit on R' if and only if one of the functions u has a different limit along the second sequence. With this definition of R', $\mathrm{M}(R, R')$ is necessarily satisfied, since we have already supposed that $\mathrm{M}(R, R_0')$ is satisfied. Moreover, every function in the class $\boldsymbol{D}$ has a boundary function, defined by continuity at every point of R'. Thus, according to the preceding paragraph, the class of PWB solutions in this case is precisely $\boldsymbol{D}$, and the corresponding boundary functions are those defined by continuity. In this case, the subregular lower PWB functions for any boundary function f are simply those subregular functions, bounded from above, whose boundary limit functions are less than or equal to f.

5. Trajectories to the boundary

In the following, we shall say that a sequence of points of R approaches R' if only finitely many points of the sequence are in any compact subset of R. This definition is applicable even if R' is empty. Unless $R \cup R'$ is compact, such a sequence need have no limit point on R'. Suppose that one wishes to define a systematic way for sequences of points of R to approach R', to obtain a family of sequences approaching the boundary and playing the same role in boundary limit theorems of Fatou type as the family of radii of a plane disc when one considers harmonic functions. What is desired is a family of sequences, depending on a parameter, where the parameter has a measure space as domain. In probability language, which is convenient in such situations, if we suppose that the total measure of the parameter space is 1, one can then speak of the probability of a set of sequences, meaning the measure of the corresponding parameter set. If the approach to the boundary is not to refer to the specific properties of an individual boundary R', but is to be intrinsically defined by the properties of the regular functions, as is a natural requirement if the approach is not to be changed with each change of R', the approach must be defined in terms of the properties of the regular functions, that is, basically, in terms of the given transition measures on the boundaries of regular sets, perhaps as extended to the boundaries of strongly PWB resolutive sets. In this way, one is led to the trajectories to be defined now.

We shall suppose throughout this section that R is strongly PWB resolutive from below, so that R is the union of a sequence $\{R_n, n \geqq 1\}$ of strongly PWB resolutive sets, with $R_n \cup R_n' \subset R_{n+1}$, whose closures are compact subsets of R. Let z_0 be any point of R. Since R_n is strongly PWB resolutive, there is a certain transition measure $\{\mu(z, R_n, \cdot), z \in R_n\}$, as required by the definition of strong PWB resolutivity. In particular, if R_n is regular, the family of measures is the one specified in the definition of regularity of a set. We define a trajectory system from z_0 as follows. A trajectory in the system is a sequence of points $\{z_n(\omega), n \geqq 0\}$, where $z_0(\omega) \equiv z_0$, depending on the parameter ω, a point in a measure space $\Omega(z_0)$. As the notation indicates, the measure space may depend on the initial point z_0. Let N be the smallest subscript satisfying the relation $z_0 \in R_N$, and define $z_0 = \cdots = z_{N-1}$, if $N > 1$. Using probability language, we can now define the trajectories as follows. We choose z_N at random on R_N' in accordance with the probability distribution $\mu(z_{N-1}, R_N, \cdot)$; we then choose z_{N+1} at random on R_{N+1}', in accordance with the probability distribution $\mu(z_N, R_{N+1}, \cdot)$, and so on, obtaining a random walk starting at z_0 and proceeding from one boundary of a set in the nested sequence of sets to the next boundary. In more formal language, we define a stochastic process on a measure space $\Omega(z_0)$, with probability measure P. The integral of a random variable (function) x on $\Omega(z_0)$ is denoted by $E\{x\}$. In particular $P\{\Omega(z_0)\} = 1$. The range of the random variable z_n is a subset of R_n', for $n \geqq N$, and z_n is measurable in the sense that the inverse image under z_n of a Borel subset of R_n' is a measurable $\Omega(z_0)$ set. The z_n process is to be a Markov process, so that only the initial value and transition probabilities need be specified to ensure the existence of the process and to determine the joint distributions of its random variables. We have already prescribed that $z_0, \cdots, z_{N-1}$ be identical, and have defined the transition probability measures otherwise by setting, in the usual notation, aside from the ambiguities inherent in conditional probabilities,

$$P\{ z_{n+1}(\omega) \in A \mid z_n\} = \mu(z_n, R_{n+1}, A), \qquad n \geqq N-1 . \tag{5.1}$$

It is of fundamental importance that, in view of the fact that $\mu(\cdot, R_n, A)$ is a regular function on R_n, and that $R_m \cup R_m' \subset R_n$ when $m < n$, we have

$$P\{ z_n(\omega) \in A \mid z_m\} = \mu(z_m, R_n, A), \qquad N-1 \leqq m < n , \tag{5.2}$$

with probability 1.

The conditions we have imposed determine the distribution of $z_0, \cdots, z_n$ for every value of n, and any process with these joint distributions will be called a trajectory process from z_0. The point z_0 will be called the *initial* point of the trajectories. For a specified point ω of $\Omega(z_0)$, the trajectory $\{z_n(\omega), n \geqq 0\}$ will be called the *trajectory determined by* ω, or the ω *trajectory*. The set of limit points of the ω trajectory is a possibly empty closed subset of R', and will be denoted by $R(\omega)$.

THEOREM 5.1. *Let R be strongly resolutive from below, and let $\{z_n, n \geqq 0\}$ be a trajectory process. If u is subregular [regular] on R, the process $\{u(z_n), n \geqq 1\}$ is a semimartingale [martingale]. If $u(z_0) > -\infty$, the parameter value 0 can be adjoined to the process in this assertion.*

For a detailed discussion of semimartingales and martingales, see Doob [4]. We have remarked in section 3 that u on R_n is less than or equal to its regular average on R_n', relative to R_n, if u is subregular. If this inequality is translated into probability language, we obtain, if $m < n$ and if we use the customary notation for conditional expectations,

$$(5.3)\quad u(z_m) \leqq \int_{R_n'} u(\zeta)\,\mu(z_m, R_n, d\zeta) = E\{u(z_n) \mid z_m\} = E\{u(z_n) \mid z_0, \cdots, z_m\},$$

with probability 1. Hence, if $u(z_0) > -\infty$, the $u(z_n)$ process is a semimartingale, and in fact is a semimartingale relative to the family of Borel fields $\{\mathcal{G}_n, n \geqq 0\}$, where $\mathcal{G}_n$ is the smallest Borel field of $\Omega(z_0)$ sets relative to which $z_0, \cdots, z_n$ are measurable. If $u(z_0) = -\infty$, the parameter value 0 must be omitted from this argument. If u is regular, there is equality in (5.3), and the $u(z_n)$ process is therefore a martingale with respect to the same family of Borel fields.

The following theorem is our generalization of Fatou's boundary value theorem. Note that no condition whatever has been imposed on R'.

THEOREM 5.2. *If R is strongly PWB resolutive from below, if u is subregular on R, and if $u \in \boldsymbol{H}$, then u has a finite limit among almost every ω trajectory of any trajectory process from a point of R, and the expectation of the limit is finite.*

If $u \in \boldsymbol{H}$, then $E\{|u(z_n)|\}$ is bounded independently of $n \geqq 1$ in the preceding theorem, because this expectation is the regular average of $|u|$ on R_n' relative to R_n, at z_0. The convergence result is now simply an application of a standard semimartingale convergence theorem to the semimartingale obtained in the preceding theorem, and, in view of Fatou's lemma on integration to the limit, the expectation of the limit is finite.

In the following, we shall hold the nested sequence $\{R_n, n \geqq 1\}$ fast, but allow z_0 to vary, defining z_n as before. The limit in theorem 5.2 is a random variable $x(z_0)$, depending on z_0 and in fact defined on $\Omega(z_0)$.

If u is subregular on R, and if u is bounded from above, $u \in \boldsymbol{H}$. More generally, if u is bounded from above by a function $v \in \boldsymbol{D}$, then $u \in \boldsymbol{H}$ again, because $\boldsymbol{H}$ is a linear class, and $v \in \boldsymbol{D} \subset \boldsymbol{H}$, $u - v \in \boldsymbol{H}$. Under such an added restriction on u, theorem 5.2 can be strengthened as follows.

THEOREM 5.3. *Let R be strongly PWB resolutive from below, and let u be subregular on R, and bounded from above by a function $v \in \boldsymbol{D}$. Then if the limit of u along ω trajectories from z defines the random variable $x(z)$, $E\{x(z)\}$ is finite, and*

$$(5.4)\qquad u(z) \leqq E\{x(z)\}.$$

In particular, if $u \in \boldsymbol{D}$, there is equality here.

If u is subregular on R, and if $\{z_n, n \geqq 0\}$ is a trajectory process from $z_0 = z$, then, taking the expectation of both sides of (5.3), with $m = 0$, we find that

$$(5.5)\qquad u(z) \leqq E\{u(z_n)\},$$

and, in particular, if v is regular on R, we find that

$$v(z) = E\{v(z_n)\}. \tag{5.6}$$

Now if $v \in D$, the sequence $\{v(z_n), n \geqq 0\}$ is uniformly integrable on $\Omega(z_0)$, by definition of D, so that, when $n \to \infty$ in (5.6), we find that

$$v(z) = E\{\lim_{n\to\infty} v(z_n)\}. \tag{5.7}$$

This proves the last assertion of the theorem. If now $n \to \infty$ in the inequality

$$u(z) - v(z) \leqq E\{u(z_n) - v(z_n)\}, \tag{5.8}$$

we find, in view of Fatou's integration lemma, that, if $u - v \leqq 0$,

$$u(z) - v(z) \leqq E\{x(z) - \lim_{n\to\infty} v(z_n)\}, \tag{5.9}$$

so that $u(z) \leqq E\{x(z)\}$, as was to be proved.

The conclusion of the theorem can be interpreted to mean that the stochastic process $\{u(z_n), 0 \leqq n \leqq \infty\}$ is a semimartingale (or martingale when $u \in D$), if we interpret $u(z_\infty)$ as $x(z)$, and omit the parameter value 0 if $u(z_0) = -\infty$.

In discussing the first boundary value problem on R, in anything like its usual form, one must restrict oneself in some way so that functions regular on R are conditioned by their limit values at the individual points of R'. This fact explains why the additional restriction $\mathrm{M}(R, R')$ was made in discussing the PWB method, and indeed why some restriction of this sort is in fact necessary. Similarly here, in theorems 5.2 and 5.3, if the limit of u along an ω trajectory is to be related to the limits of u at points of R', we must ensure that the trajectories approach points of R', in some loose sense, at least. For this reason, in applying the convergence theorems of this section to the first boundary value problem, we shall be forced to make an additional hypothesis, as follows. (We shall always state it explicitly when it is needed.)

$\mathrm{M}'(R, R')$. *R is strongly PWB resolutive from below, and each trajectory process, from each point of R, has the property that almost no set $R(\omega)$ is empty.*

The hypothesis on $R(\omega)$ is of course automatically fulfilled in the most important case, when $R \cup R'$ is compact. To exhibit the significance of $\mathrm{M}'(R, R')$, we discuss the maximum principle. Let u be defined and subregular on R, and bounded from above, with supremum c, and let c' be the supremum of the limiting values of u at the individual points of R'. Then $c' \leqq c$, and condition $\mathrm{M}(R, R')$ forces equality. If now condition $\mathrm{M}'(R, R')$ is valid, theorem 5.3 can be applied, and we find that $u \leqq c'$, so that again $c' = c$. Thus the validity of $\mathrm{M}'(R, R')$ implies that of $\mathrm{M}(R, R')$.

We shall prove in the next section that, if $\mathrm{M}'(R, R')$ is satisfied, if $R \cup R'$ is metrizable, and if R is strongly PWB resolutive, then, for each trajectory process from a point of R, almost every set $R(\omega)$ contains exactly one point. In particular, if $R \cup R'$ is compact, and if R is strongly PWB resolutive and strongly PWB resolutive from below, these hypotheses are satisfied, and it follows that almost every trajectory of any trajectory process converges to a unique point of R'.

In some studies, one deals with a first boundary value problem, and then refines it by ramifying the boundary. For our purposes, ramification can be defined as follows. Instead of the space $R \cup R'$, one considers the space $R_1 \cup R_1'$. Here R and R_1 are homeomorphic, and the regular sets and functions of the two spaces go into each other under the given homeomorphism. That is, R and R_1 can be identified and only the boundaries

R' and R_1' are different. The concept of ramification means that, whenever a sequence on R_1 converges to a point of R_1', the corresponding sequence on R has a unique limit on R'. That is, the given map from R_1 onto R can be extended to be a continuous map from $R_1 \cup R_1'$ into $R \cup R'$. The condition $\mathrm{M}'(R, R')$ is invariant under ramification in important cases. For example, if it is known that whenever a sequence of points of R has a limit point on R' the sequence of images on R_1 has a limit point on R_1', then the validity of $\mathrm{M}'(R, R')$ implies that of $\mathrm{M}'(R_1, R_1')$.

Finally, we make the following remark. Suppose that R is strongly PWB resolutive from below, but that $\mathrm{M}'(R, R')$ is not necessarily true. Then the limits of regular functions along our trajectories must still play an essential role in any solution of the first boundary value problem following the PWB method, or in fact any method which leads to solutions given by integral averages of the type we have discussed. In fact, we have proved that, in all such cases, if u is a solution, $u \in D$, so that, according to theorem 5.3, $u(z)$ is the expected value of the limit of u along ω trajectories from z. In the language of theorem 5.3, $u = E\{x(z)\}$. It would indeed be surprising if $x(z)$ could not be represented in some simple way in terms of the given boundary function leading to the solution u, but such a representation has not been found, except when condition $\mathrm{M}'(R, R')$ is satisfied (see the next section). An alternative approach now seems not unreasonable. If the limit $x(z)$ is so fundamental, we can drop the classical concept of a preassigned boundary function entirely, and demand instead that $x(z)$ be prescribed, for each z. This has been done in a special case in [7], and will be carried through in the general case in section 7.

6. The first boundary value problem on R

Throughout this section, R is strongly PWB resolutive from below, $\{R_n, n \geqq 1\}$ is a sequence of strongly PWB resolutive sets, with compact closures, and $R_n \cup R_n' \subset R_{n+1}$. A trajectory process from a point z_0 of R is defined as explained in the preceding section, and $R(\omega)$ is defined as explained in that section.

In section 4 we outlined the PWB method, imposing the condition $\mathrm{M}(R, R')$, and we shall now find boundary limit properties of PWB solutions which make more intuitive the characterization of a PWB solution as a generalized first boundary value problem solution. The price is the strengthening of the old condition $\mathrm{M}(R, R')$ to $\mathrm{M}'(R, R')$.

THEOREM 6.1. *Let f be a function on R', suppose that condition* $\mathrm{M}'(R, R')$ *is satisfied, and let v (not identically $\pm\infty$) be in the lower [upper] PWB class for f. Then, for almost all ω, v has a finite limit $y(\omega)$ along the ω trajectory from any point of R, and $f \geqq y(\omega)$ $[f \leqq y(\omega)]$ on $R(\omega)$. In particular, if f is PWB resolutive, with solution u, then, for almost all ω, f is constant on $R(\omega)$, and u has this constant as a limit on the ω trajectory.*

Thus, according to this theorem, a PWB solution takes on the assigned boundary function value as a limit along our trajectories even if the trajectories do not approach individual points of R'. In fact, as will be seen from the proof, the theorem remains true if the hypothesis that almost no set $R(\omega)$ is empty is deleted from $\mathrm{M}'(R, R')$. The assertion of the theorem is of course not very useful for a point ω for which $R(\omega)$ is empty!

If v is a lower PWB function for f, v is subregular and bounded from above (excluding the function $-\infty$), so that $v \in H$, and therefore v has a finite limit $y(\omega)$ along almost every ω trajectory from z_0. By definition of lower PWB functions, $f \geqq y(\omega)$ on $R(\omega)$. If v is an upper PWB function for f, $-v$ is a lower PWB function for $-f$, and we can apply the result just proved to derive the corresponding result. Finally, if f is PWB resolutive, with solution u, we have seen in section 4 that $u \in D$, and hence u has a finite limit $x(\omega)$ on almost every ω trajectory from z_0. Let u_n [v_n] be a function in the lower [upper] PWB class for f, with

$$(6.1)\qquad u_1 \leqq u_2 \leqq \cdots, \qquad \lim_{n\to\infty} u_n = u \; [v_1 \geqq v_2 \geqq \cdots, \; \lim_{n\to\infty} v_n = v] .$$

We can define u_n, for example, as $\max_{j\leqq n} u_j'$, where $\{u_j', j \geqq 1\}$ is a sequence of lower PWB functions for f, whose upper limit at each point of a dense denumerable set on R is the value of u at the point. Let u_n [v_n] have the limit $x_n(\omega)$ [$y_n(\omega)$] on an ω trajectory. Then

$$(6.2)\qquad u_n \leqq u \leqq v_n , \qquad x_n \leqq x \leqq y_n$$

(the second inequality is true with probability 1). Applying theorem 5.3, we find that

$$(6.3)\qquad u_n(z_0) \leqq E\{x_n\} \leqq E\{x\} \leqq E\{y_n\} \leqq v_n(z_0) .$$

Since the extremes in this inequality both have limit $u(z_0)$ when $n \to \infty$, it follows that, with probability 1,

$$(6.4)\qquad \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x .$$

Now if ω is chosen so that the limits $x_n(\omega)$, $y_n(\omega)$ exist for all n, it follows from the first part of the theorem that

$$(6.5)\qquad x_n(\omega) \leqq f(\zeta) \leqq y_n(\omega) \qquad \text{if} \qquad \zeta \in R(\omega) , \qquad n \geqq 1 .$$

Hence, in view of the preceding equation, we find that $x(\omega) = f(\zeta)$ for $\zeta \in R(\omega)$, for almost all ω, as was to be proved.

This theorem completes the PWB theory in a satisfactory way, by showing that, in an appropriate limit sense, a PWB resolutive boundary function is really the boundary function of its PWB solution. Without such a result, these solutions are only linked to the specified boundary functions which determine them by the PWB method itself, and this method does not give a very satisfactory connection in intuitive terms. In particular, if R is itself a regular set of a larger space in which this theory is discussed, and in this case we shall see below that almost every set $R(\omega)$ contains exactly one point, the given transition measure $\{\mu(z, R, \cdot), z \in R\}$ defines, for each Borel subset A of R', a PWB solution $\mu(\cdot, R, A)$, and theorem 6.1 thus gives us information on the limiting behavior of $\mu(z, R, A)$ for z near R'.

We shall now extend the notion of a PWB solution by enlarging the lower and upper PWB classes for a given boundary function. The class of PWB solutions determined by the PWB resolutive boundary functions is rather unnatural. In fact this class of PWB resolutive boundary functions is clumsy, because, for example, it is not known that max $[f, 0]$ is PWB resolutive whenever f is. This difficulty will be overcome by our extension, which will make it possible to prove that R' is weakly PWB resolutive under the hypothesis $\mathrm{M}'(R, R')$.

Suppose then that $\mathrm{M}'(R, R')$ is satisfied. We define the stochastic lower [upper] PWB class of functions corresponding to an arbitrary boundary function f in the way suggested by theorem 6.1. That is, the lower [upper] SPWB class consists, in addition to the function which is identically $-\infty$ [$+\infty$], of all subregular [superregular] functions on R, bounded from above [below], and having the following additional property. If the function is in the lower [upper] SPWB class it has a limit $y(\omega) \leqq f(\zeta)$ [$y(\omega) \geqq f(\zeta)$] for

$\zeta \in R(\omega)$, for almost all ω. According to theorem 6.1, the SPWB classes for f include the corresponding PWB classes for f. The existence of the limit $y(\omega)$ for almost all ω is assured by theorem 5.2. The lower [upper] SPWB solution is the supremum [infimum] of the functions in the lower [upper] SPWB class, and is regular, if finite on a dense set. Moreover, it is in the lower [upper] SPWB class, if bounded from above [below], and is greater [less] than or equal to the lower [upper] PWB solution. Finally, f will be called SPWB resolutive if the lower and upper SPWB solutions are regular and equal, and the common solution will then be called the SPWB solution for f. This solution is in both lower and upper SPWB classes for f, if it is bounded. We now see that, always under $M'(R, R')$, if f is PWB resolutive, it is SPWB resolutive. Theorem 4.1 is applicable, and shows that every SPWB solution is in the class $\boldsymbol{D}$. Theorem 6.1 and its proof remain valid for the SPWB method.

The properties RF1 to RF5 of the class of PWB resolutive boundary functions remain valid for the class of SPWB resolutive boundary functions, and in this version will be denoted by SRF1 to SRF5. In addition, however, the new class has the following property.

SRF6. *If f is SPWB resolutive, and if ϕ is a convex monotone nondecreasing function, bounded from below, defined on $(-\infty, \infty)$, then $\phi(f)$ is SPWB resolutive if its upper SPWB solution is finite.*

The most important application of this property, whose validity we shall prove in the next paragraph, is to prove that, if f and g are SPWB resolutive, then $\max [f, g]$ is also. In fact this assertion for $g = 0$ follows directly from SRF6, and follows in general from the equality

$$\max [f, g] = \max [f - g, 0] + g . \tag{6.6}$$

PROOF OF SRF6. Let u be the SPWB solution for f. If u_0 is a function in the SPWB lower class for f, $\phi(u_0)$ is one in that for $\phi(f)$. Hence, if $u'[u'']$ is the lower [upper] SPWB solution for $\phi(f)$, $u' \geqq \phi(u)$. Moreover, if $\hat{u}$ is the best regular majorant of $\phi(u)$,

$$\phi (u) \leqq \hat{u} \leqq u' \leqq u'' . \tag{6.7}$$

Now $\hat{u}$ is bounded from below, because ϕ is, and has a limit $\geqq f$ on almost every ω trajectory of any specified trajectory process, since $\phi(u)$ has the limit $\phi(f)$ along almost every such trajectory. Hence $\hat{u}$ is in the upper SPWB class for $\phi(f)$, so that $\hat{u} \geqq u''$. Combining this inequality with (6.7), we find that $u' = u''$. Thus $\phi(f)$ is resolutive, as was to be proved. In addition, we have found the SPWB solution for $\phi(f)$ explicitly.

Let z be any point of R. Then, if f is an SPWB resolutive boundary function, with solution u, $u(z)$ defines a function on the class of SPWB resolutive boundary functions. Because of the properties of this class, $u(z)$ can be represented as a Daniell integral (see [9]),

$$u (z) = \int_{R'} f (\zeta) \mu (z, R, d\zeta) , \tag{6.8}$$

where $\mu(z, R, \cdot)$ is a measure of subsets of R', defined on the sets whose characteristic functions are SPWB resolutive. In particular, the measure of R' itself is 1. The domain of $\mu(z, R, \cdot)$ may depend on z. The class of functions measurable and integrable with respect to every one of these measures, that is, for every point z of R, is the class of SPWB resolutive boundary functions. Property SRF4 implies that each measure is the completed measure of itself restricted to the Borel subsets of its domain. The measure family is a transition measure, as defined in section 2. We now make the obvious definitions of weak SPWB resolutivity, SPWB resolutivity, and strong SPWB resolutivity of a set, corresponding to the definitions for the PWB method. In these terms, R is necessarily SPWB resolutive, and this fact implies that R is weakly PWB resolutive. Note that we have not proved that the bounded continuous functions on R' are SPWB resolutive, and in fact they may not be.

A question that commonly arises in these studies is the following. If a boundary function f on R' is specified, would the corresponding lower and upper SPWB solutions for f be changed if (aside from the identically infinite members) the discontinuous members of the lower and upper SPWB classes for f were excluded? Actually the answer is "no" even if the subregular and superregular functions which are not regular are excluded from these classes. To see this, say for the lower solution, we use a result to be proved in section 7, that to a subregular function v which is bounded from above there corresponds a regular function $v_1 \geqq v$, also bounded from above, with the same boundary limit along almost every trajectory from any point of R. The omission of the subregular but not regular functions from the lower SPWB class does not change the lower SPWB solution, because, if v is in the class, v_1 is also, and remains there.

The SPWB method is applied after a specific choice of the sequence $\{R_n, n \geqq 1\}$ and of the trajectory processes. We omit the trivial proof that the SPWB resolutivity of a boundary function, and the identity of its corresponding solution if there is SPWB resolutivity, are actually independent of these choices.

Let $\mathcal{F}$ be the Borel field of those subsets of R' for which $\mu(z, R, \cdot)$ is defined for all z. According to our discussion, every function which is the characteristic function of a set A in $\mathcal{F}$ is a SPWB resolutive boundary function. Applying theorem 6.1 (stated for the SPWB method), we find that, for any trajectory process from a point of R, either $R(\omega) \subset A$, or $R(\omega) \subset R' - A$, for almost all ω. The transition measure has the usual elegant probability interpretation. That is, if $A \in \mathcal{F}$, and if $z_0 \in R$, $\mu(z_0, R, A)$ is the probability that an ω trajectory, of any trajectory process from z_0, has the property that $R(\omega) \subset A$. In fact the regular function $\mu(\cdot, R, A)$ is the SPWB solution corresponding to the boundary function which is the characteristic function of A, and therefore this solution has the limit 1 on almost every ω trajectory for which $R(\omega) \subset A$, the limit 0 on almost every other. More generally, if f is SPWB resolutive, with corresponding solution u, $u(z_0)$ is the expected value of the limit of u along ω trajectories from z_0, that is, the average value of f over the sets $R(\omega)$ on which the trajectories approach points of R'.

In the most important special cases, R is strongly SPWB resolutive, so that every bounded continuous boundary function on R' is SPWB resolutive, and in fact in most cases R is even strongly PWB resolutive. But in any event we have shown in this study how far one can go with no specific hypotheses of regularity of boundary points, capacity, and associated ideas commonly used to develop the subject.

The following theorems give insight into the class of SPWB resolutive boundary functions. Let $\mathcal{F}' \supset \mathcal{F}$ be the Borel field consisting of those Borel subsets A of R' with the property that, for every trajectory process from a point of R, almost every set $R(\omega)$ lies either in A or $R' - A$. Let $\mathbf{F}'$ be the class of boundary functions which are measurable with respect to $\mathbf{F}'$. Then every Borel measurable SPWB resolutive boundary function is in the class $\mathbf{F}'$.

THEOREM 6.2. *Suppose that* $R \cup R'$ *is metrizable, and that* $\mathrm{M}'(R, R')$ *is satisfied. Then, if* $f \in \mathbf{F}'$, *and if* f *is continuous and bounded, it follows that* f *is SPWB resolutive.*

Let f^* be a function, bounded and continuous on $R \cup R'$, equal to f on R'. There is such an extension of f, because $R \cup R'$ is metrizable. Let $\{R_n, n \geqq 1\}$ be as above, and let $\{z_n, n \geqq 0\}$ be a corresponding trajectory process, with initial point $z = z_0$. In the following, n is to be taken so large that $z_0 \in R_n$. Define u_n as the regular average of f^* on R_n', relative to R_n,

$$u_n(z) = E\{f^*(z_n)\}. \tag{6.9}$$

When $n \to \infty$, the sequence $\{f^*[z_n(\omega)], n \geqq 1\}$ converges, for almost all ω, to the constant value f has on each set $R(\omega)$. Stretching the notation, we denote this value by $f[R(\omega)]$, so that

$$\lim_{n\to\infty} u_n(z) = u(z) = E\{f[R(\omega)]\}. \tag{6.10}$$

The function u is regular and bounded on R. According to our definition, if $m < n$,

$$E\{f^*(z_n) \mid z_0, \cdots, z_m\} = E\{f^*(z_n) \mid z_m\} = u_n(z_m) \tag{6.11}$$

with probability 1. When $n \to \infty$ we find that, with probability 1,

$$E\{\lim_{n\to\infty} f^*(z_n) \mid z_0, \cdots, z_m\} = u(z_m). \tag{6.12}$$

Hence, when $m \to \infty$, we find that, with probability 1,

$$\lim_{m\to\infty} u(z_m) = E\{\lim_{n\to\infty} f^*(z_n) \mid z_0, z_1, \cdots\} = \lim_{n\to\infty} f^*(z_n), \tag{6.13}$$

that is, u has the limit $f[R(\omega)]$ along almost every ω trajectory. Since u is bounded, it must be both a lower and an upper SPWB function for f, so that f is SPWB resolutive, with solution u.

THEOREM 6.3. *If $R \cup R'$ is metrizable, if $M'(R, R')$ is satisfied, and if R is strongly SPWB resolutive, then almost every ω trajectory of any trajectory process determines a limit set $R(\omega)$ containing exactly one point.*

To prove this theorem, suppose that $\zeta \in R'$, and define

$$f(z) = \arctan d(z, \zeta), \qquad z \in R', \tag{6.14}$$

where $d(z_1, z_2)$ is the distance between z_1 and z_2 in some metrization of $R \cup R'$. Then f is a continuous bounded function, so is SPWB resolutive, and, in our usual notation, is constant on each set $R(\omega)$, for almost every point ω as specified by any trajectory process from z_0. Hence, excluding a set of points ω of probability 0, all the points of each set $R(\omega)$ are at the same distance from ζ. Since R, and therefore R', is separable, the assertion just made is true for a sequence of values of ζ dense on R' (with the same ω set excluded for each value), and this means that almost every $R(\omega)$ contains exactly one point.

Note that, if $R \cup R'$ is compact in theorem 6.3, the conclusion can be stated more simply: almost every trajectory of any specified trajectory process converges to a point of R'.

To illustrate the possibilities, we give an example in which $R \cup R'$ is a complete metric space, $M(R, R')$ is satisfied, but in which almost no $R(\omega)$ contains only a single point. The bounded continuous functions are not necessarily SPWB resolutive. Let R be the upper half-plane of the complex plane, let the regular functions be the harmonic functions, and let the regular sets be the open circular discs. According to a theorem of Brelot [1], every bounded open set is strongly PWB resolutive, if its closure is a compact subset of the half-plane. Hence R is strongly PWB resolutive from below. For any two points z_1 and z_2 of R, let $\delta(z_1, z_2)$ be the maximum angle subtended by these points from a point of the real axis. Metrize R by setting the distance between z_1 and z_2 equal to the sum of Euclidean distance and $\delta(z_1, z_2)$. Completing R in this metric, we obtain a boundary R' which is, roughly, the real axis with each point replaced by an interval. A sequence $\{w_n, n \geqq 1\}$ converges to a point of R' if and only if it converges to a point w of the real line in the usual sense, and if $\arg (w_n - w)$ (defined to be between $-\pi/2$ and $\pi/2$) converges to a limit a, $-\pi/2 \leqq a \leqq \pi/2$. Each value of a determines a boundary point. Thus R' is the set of pairs (w, a). In this case $\mathrm{M}'(R, R')$ is satisfied, and it is easily seen that almost every $R(\omega)$ consists of a boundary set of the form $\{(w, a), -\pi/2 \leqq a \leqq \pi/2\}$, with fixed w. Thus the high degree of ramification is simply irrelevant to the solution of the first boundary value problem.

7. Stochastic boundary functions

Throughout this section, R is strongly PWB resolutive from below, and we use the notation of section 6 for a specified nested sequence of strongly PWB resolutive sets which cover R, and for the corresponding trajectory processes. We shall use the same nested sequence, regardless of the initial point of the trajectories.

The approach we shall make to the first boundary value problem in this section is different from that in preceding sections, in that the boundary R' will not be involved explicitly. Thus no hypothesis like $\mathrm{M}(R, R')$ will be necessary, and, in fact, only the Hausdorff space R itself will appear in the discussion. What we shall do is to allow the limiting properties of a subregular function, as described in theorems 5.2 and 5.3, to determine the significance of the first boundary value problem. Note that the trajectories from a point of R always converge to R' in the sense we have been using, but whether a trajectory converges to an individual point of R', or whether it has any limit points on R', or even whether R' is empty or not, will be immaterial to the discussion.

Suppose that u is a function defined on R, which, considered on each boundary R_n', is Borel measurable. If $\{z_n, n \geqq 0\}$ is the specified trajectory process from $z = z_0$, suppose that $\lim_{n\to\infty} u(z_n) = x(z)$ exists, as a finite or infinite limit, with probability 1, for each z. The family of random variables $\{x(z), z \in R\}$ obtained in this way has the following properties.

SBF1. *The random variable $x(z)$ is defined on the measure space $\Omega(z)$ of the trajectory process from z, and, for each m, $x(z)$ is defined in terms of $z_m, z_{m+1}, \cdots$.* That is, if $\mathcal{G}_n$ (which depends on z) is the smallest Borel field of $\Omega(z)$ sets containing the sets of measure 0, and such that z_k is measurable with respect to $\mathcal{G}_n$ for every $k \geqq n$, then $x(z)$ is measurable with respect to $\bigcap_1^\infty \mathcal{G}_n$.

SBF2. *If A is any linear Borel set, the probability that the value of $x(z)$ lies in A defines a regular function of z on R.*

The first property is obviously valid. Before proving the validity of the second, we make some preliminary remarks. The joint distributions of the z_n's, once the initial point of the trajectory process and the R_n sequence are chosen, are uniquely determined. Thus the joint distributions of the $u(z_n)$'s are also uniquely determined by these choices, whatever the choice of the measure space $\Omega(z)$, or, more precisely, whatever the actual choice of the trajectory process from z. Thus the existence of the limit $x(z)$ does not depend on the process chosen, and such quantities as $P_z\{x(z, \omega) \in A\}$, $E_z\{x(z)\}$ have unique meanings. Here ω is a point of $\Omega(z)$, and the subscript z indicates that the space involved, and therefore the probability measure and expectation, depend on the initial trajectory point z. From now on, we drop this subscript in such probability and expectation evaluations, since its omission should cause no confusion.

In proving that SBF2 is valid, we can and shall assume that u and each $x(z)$ is finite valued, since u can be replaced by arctan u without changing the force of SBF2. In the following, a complex-valued function will be called regular if its real and imaginary parts are regular. Let $\Phi(\cdot, z)$ be the characteristic function of $x(z)$, and let $\Phi_n(\cdot, z)$ be the characteristic function of $u(z_n)$. Then, since $u(z_n)$ has the limit $x(z)$, with probability 1,

$$(7.1) \qquad \lim_{n\to\infty} \Phi_n(t, z) = \Phi(t, z)$$

for fixed z, uniformly on every finite t interval. Since $\Phi_n(t, \cdot)$ is regular on R_n (it is by definition a regular average on R_n' relative to R_n), and since there is bounded convergence, the limit $\Phi(t, \cdot)$ is regular on R. The integral expression for Φ_n shows that this function is a Borel measurable function of the pair (t, z), for $z \in R_n$. Hence the limit Φ is also a Borel measurable function of the pair, for $z \in R$. Now if $a < b$, the Lévy inversion formula expresses the quantity

$$(7.2) \qquad P\{a < x(z, \omega) < b\} + \tfrac{1}{2}P\{x(z, \omega) = a\} + \tfrac{1}{2}P\{x(z, \omega) = b\}$$

as the bounded limit of a sequence of integrals involving the characteristic function $\Phi(\cdot, z)$. Since each integral is clearly a regular function of z, the limit is also regular, so that the above sum is regular. Finally, if $b \downarrow b_0$ and if $a \downarrow a_0$, we find that

$$(7.3) \qquad P\{a_0 < x(z, \omega) \leqq b_0\}$$

defines a regular function of z. Thus SBF2 is true if A is a right semiclosed interval, and therefore A is a finite union of such intervals. The class of linear Borel sets A for which SBF2 is true is monotone, and therefore includes all Borel sets, as was to be proved.

In the following, we shall call the family of random variables $\{x(z), z \in R\}$ that we have been discussing the *stochastic boundary function* of u, noting that it depends on the choice of trajectory processes, even after the nested sequence of strongly PWB resolutive sets in terms of which the trajectory processes are defined has been specified. In theorem 5.2 we have proved that every subregular function in the class H has a stochastic boundary function, whatever the specified nested sequence, and that, if u is regular and in the class D, it can be expressed in a simple way in terms of its stochastic boundary function, $u(z) = E\{x(z)\}$.

Going in the other direction, suppose that to each point of R, nested sequence $\{R_n, n \geqq 1\}$ (the same for all initial points) and measure space $\Omega(z)$ on which a corresponding trajectory process from z is defined, there is assigned a not necessarily finite-valued random variable $x(z)$. Then the family of random variables $\{x(z), z \in R\}$ will be called a stochastic boundary function if conditions SBF1 and SBF2 are satisfied. If there is a constant K such that

$$(7.4) \qquad P\{|x(z, \omega)| \leqq K\} = 1, \qquad z \in R,$$

the stochastic boundary function will be said to be bounded. It is natural, in the present discussion, to denote by $L(R)$ the class of stochastic boundary functions for which $E\{x(z)\}$ exists and is finite, for all z on R. Note that, if ϕ is a Baire function on the line, and if $\{x(z), z \in R\}$ is a stochastic boundary function, then $\{\phi[x(z)], z \in R\}$ is also a stochastic boundary function.

We complete the preliminary discussion by proving that every stochastic boundary function is the stochastic boundary function of some function u on R. Suppose first that the boundary function is in the class $L(R)$, and define

$$u(z) = E\{x(z)\}. \tag{7.5}$$

Then it follows readily from SBF2 that u is regular. Now consider the trajectory process from z. It is clear from the Markov property of the process that, with probability 1,

$$E\{x(z) \mid z_n\} = u(z_n) = E\{x(z) \mid z_0, \cdots, z_n\}, \tag{7.6}$$

and hence that, with probability 1,

$$\lim_{n\to\infty} u(z_n) = E\{x(z) \mid z_0, z_1, \cdots\}, \tag{7.7}$$

and, by SBF1, the conditional expectation on the right reduces to $x(z)$, with probability 1. Thus u is a regular function, with $\{x(z), z \in R\}$ as its stochastic boundary function, and, for each z, the stochastic process $\{u(z_n), 0 \leqq n \leqq \infty\}$ is a martingale, if $u(z_\infty)$ is interpreted as $x(z)$. Finally, even if the given stochastic boundary function is not in the class $L(R)$, there is such a corresponding function u because, if we replace $x(z)$ by arctan $x(z)$, we obtain a bounded stochastic boundary function, and if the latter is the stochastic boundary function of u, as just described, the given stochastic boundary function is the stochastic boundary function of $u' = \tan u$. Here u' may be infinite valued, but, if infinite-valued functions are not desired, we can replace u' by u'', finite valued and with the same stochastic boundary function, where u'' is obtained from u', for example, by bounding u' in successively larger ring regions, with successively larger bounds.

We have just observed that, to each stochastic boundary function $\{x(z), z \in R\}$ in the class $L(R)$ there corresponds a regular function u, defined by (7.5). The function u is in the class H, since it is bounded from above by the nonnegative regular function obtained on replacing $x(z)$ by $|x(z)|$ in the definition of u. Actually u is even in the class D. In fact, for each z, we need only show that the sequence of random variables $\{u(z_n), n \geqq 0\}$ is uniformly integrable on $\Omega(z)$. Now we have seen that this sequence, augmented by $x(z)$ at the end, is a martingale. Thus the uniform integrability is a consequence of the known uniform integrability of the random variables of any martingale with a last random variable.

In view of the close relations we have already found between members of the class D and solutions of generalized first boundary value problems, it is natural to describe u here as the solution of the first boundary value problem corresponding to the specified stochastic boundary function. This is justified by the fact that, as we showed above, this stochastic boundary function is in fact the stochastic boundary function of u. In more intuitive terms, the use of stochastic boundary functions amounts to the use of a stochastically ramified boundary, and our new solutions will accordingly be called *SR* solutions. We stress that we have imposed no restrictions whatever on the relation between R and R', and that R' may be empty. The class D now appears as the class of *SR* solutions. This class appears to be the maximal class of functions obtained as solutions for properly formulated first boundary value problems. It is instructive to see how the PWB and SPWB solutions fit into the new theory. Suppose for simplicity that R is strongly PWB resolutive. Then, if f is a resolutive boundary function, that is, if f is in the class $L(R)$ of (ordinary) boundary functions, there is a PWB solution u, with the assigned boundary values, specified by f, as limits along almost all trajectories from any point of R. That is, in the language of section 5, if we define $x(z)$ as the constant value of f on each set $R(\omega)$ determined by the ω trajectory from z, f thereby determines a stochastic boundary function $\{x(z), z \in R\}$, and u is the corresponding SR solution. On the other hand, there may be regular functions in the class D which are neither PWB nor SPWB solutions, although they are necessarily solutions in the present sense.

Finally we remark that the approach using stochastic boundary functions can be put in a form analogous to the PWB and SPWB forms, but it does not appear that there is anything to be gained by doing this.

As an application of some of these ideas, we consider the best regular majorant v of a subregular function u on R, as defined at the end of section 2, under the hypothesis that R is strongly PWB resolutive from below. If $u \in H$, v exists. Let $\{x(z), z \in R\}$ be the stochastic boundary function of u. According to theorem 5.2, this stochastic boundary function is in the class $L(R)$. Let u' be the corresponding SR solution. If, in particular, u is bounded from above by a regular function in the class D, then, according to theorem 5.3,

$$u(z) \leqq u'(z) = E\{x(z)\}. \tag{7.8}$$

Since $u \leqq v \leqq u'$, and since u, u' have the same stochastic boundary function, v must also have this stochastic boundary function. It is natural to suppose that $u' = v$, but this is not true in general, as is easily seen by examples from harmonic function theory. If, however, enough conditions are imposed to ensure that v be a function in the class D, then $u' = v$, since functions in this class are determined by their stochastic boundary functions. For example, if u on R_n' as n varies defines a uniformly integrable sequence relative to the measure sequence $\{\mu(z, R_n, \cdot), n \geqq 1\}$, for each point z of R, then $u \in H$, so that v exists. Moreover, since $u \leqq v \leqq u'$, v has the same uniform integrability property as u, so that $v \in D$, and therefore $v = u'$.

8. General trajectory systems

We have defined trajectory systems in a very special way. This way has the advantage that it is intrinsic to the class of regular functions, but more can be done in some applications. For example, continuous trajectories can be used in the theory of harmonic functions. We now, therefore, detail a set of properties of trajectory systems sufficient for our results to remain valid. We suppose that there is a certain linear set T, the parameter set. This set is a subset of $[0, \infty)$, includes the point 0, does not contain its supremum, and contains every point which is a limit point of T from the right. To each point z of R we suppose that there corresponds a measure space $\Omega(z)$ of total measure 1. This is the space on which the trajectory process with initial point z is defined. All the following concepts depend on this initial point and $\Omega(z)$, but this dependence will be omitted from the notation. There is supposed to be a family $\{\mathcal{F}(t), t \in T\}$ of Borel fields of measurable $\Omega(z)$ sets, with $\mathcal{F}(s) \subset \mathcal{F}(t)$ when $s < t$. There is a random variable τ, the trajectory *stopping time*, defined on $\Omega(z)$, satisfying the following conditions.

ST1. *The range of* τ *is a subset of the closure of* T, *augmented by* $+\infty$ *if* T *is unbounded.*

ST2. $\{\tau(\omega) \leqq c\} \in \mathcal{F}(c)$ *for all* c.

Finally, it is supposed that, to each value of $t \in T$ there corresponds a function z_t, defined on the subset $\{\tau(\omega) > t\}$ of $\Omega(z)$, and measurable relative to $\mathcal{F}(t)$.

In the following, $T(\omega)$ will be used to denote the linear set $T \cap [0, \tau(\omega)]$. For each point ω of $\Omega(z)$, the curve $\{z_t(\omega),\ t \in T(\omega)\}$ will be called the ω trajectory, and these trajectories from z are those we shall consider.

In the simplest cases, the trajectory system from z is changed, when R is decreased to an open subset R_0, merely by considering only initial points in R_0, and replacing τ by the first parameter value in which a trajectory meets the complement of R_0. The following hypotheses are satisfied in the most important applications, and will be satisfied for subsets R_0, with the convention just made, if they are satisfied for R.

TP1. *Almost every ω trajectory is a continuous function of the parameter on $T(\omega)$.*

In view of this condition, almost every trajectory from a point will either never meet a specified closed set or will meet it at a first parameter value.

TP2. *If u is subregular on an open set D, u is a finite-valued right continuous function of the parameter on almost every trajectory from a point of D, as long as it remains in D, and disregarding the initial trajectory point if it is an infinity of u.*

TP3. *Let z be a point of R, and consider a trajectory process from z, with stopping time τ. Let τ_1 be a random variable on $\Omega(z)$, with range a subset of T plus possibly the supremum b of T, satisfying condition ST2. Suppose that $\tau_1 \leqq \tau$, and that equality is only allowed at a point ω of $\Omega(z)$ if $\tau_1(\omega) = \tau(\omega) = b$. Let D be an open subset of R, containing z and almost every trajectory from z, for parameter values $\leqq \tau_1$. Then, if u is a function defined, bounded from above, and subregular on D, the $u(z_t)$ process stopped at time τ_1 is a semimartingale except that parameter value 0 is omitted if z is an infinity of u.*

By the stopped process we mean the process with parameter set T augmented by the point b, defined by

$$x_t(\omega) = \begin{cases} u[z_t(\omega)] & \text{if} \quad t < \tau_1(\omega) \\ u[z_{\tau_1(\omega)}(\omega)] & \text{if} \quad \tau_1(\omega) \leqq t\,. \end{cases} \tag{8.1}$$

The property TP3 implies the fundamental property of our special trajectories defined in section 5. Let $\{R_n,\ 1 \leqq n \leqq N\}$ be strongly PWB resolutive sets, with $R_n \cup R_n' \subset R_{n+1}$, and suppose that the closure of R_N is a compact subset of R. Let z be a point of R_1, and, under the above hypotheses on the trajectories, let τ_1 be the first time (parameter value) that a trajectory from z meets R_1', if ever; let τ_2 be the first time after τ_1 that a trajectory from z meets R_2', if ever; and so on. We impose as an additional hypothesis, usually trivially satisfied, that $\tau_1, \cdots, \tau_N$ are defined with probability 1 and that $z_t(\omega) \in R_N$ if $t < \tau_N(\omega)$. Let $w_j = z_{\tau_j}$. Then, if u is a function defined and subregular on an open set containing the closure of R_N, the process $\{u(w_n),\ 0 \leqq n \leqq N\}$, with $w_0 = z$, is a semimartingale, except that the parameter value 0 must be omitted if z is an infinity of u. To see this suppose first that $u(z) > -\infty$. The $u(z_t)$ process stopped at time τ_N is then a semimartingale, according to TP3, and if this process is stopped successively at time 0, $\tau_1, \cdots, \tau_N$, we find, from standard semimartingale theorems, that the resulting $u(w_n)$ process is a semimartingale. If $u(z) = -\infty$, we can reduce the theorem to be proved to the special case just considered by replacing u, in a small regular neighborhood of z, by its regular average on the boundary of the neighborhood, relative to the neighborhood.

The argument of the preceding paragraph is applicable without change if, instead of a finite number of sets R_n, there is any class of these sets, ordered by the same inclusion relations.

In the following, we shall call the trajectory systems defined in section 5 *special trajectory systems*. It is readily checked that these special systems satisfy the conditions TP1–TP3, with the stopping time identically $+\infty$ and T the set of nonnegative integers.

Trajectory systems satisfying our hypotheses (for which we shall use the adjective *general* below) can be of the most varied nature. In all cases, however, the theorems proved for special trajectory systems remain valid for the general ones, so that, for example, the theorems on the existence of boundary limits for subregular functions can take on quite different forms, depending on whether the trajectories have continuous parameters or not. Note that we have not imposed any condition that the trajectories from a point of R approach R' in any sense, and in fact such a hypothesis is not necessary for the validity of our theorems. In the most important applications, however, it is shown that, depending on R, almost every trajectory from a point of R is either dense on R for parameter values arbitrarily near the stopping time, or approaches R' as the parameter value approaches the stopping time. The second case is that best adapted to the first boundary value problem. The limiting character of the trajectories for parameter values near the stopping time may be implicit in the trajectory definitions, as in the case of the special trajectories, or may be deeper, and then must be derived from the known properties of R, the trajectories, and the class of regular functions. Such derivations are necessary, for example, in considering the continuous Brownian trajectories which are natural in the study of harmonic functions.

The theorems we have proved, involving special trajectories, require no change in the general case, but the proof of theorem 6.1 may depend on a slight generalization of the standard martingale convergence theorem. This generalization (theorem 1.2 of [6]) has already been used for this purpose in a special case [7].

There is some interest in the fact that trajectory systems can be defined in a way quite similar to that in section 5, without the hypothesis that the space is the union of a monotone sequence of strongly PWB resolutive sets whose closures are compact subsets of the space. Even without this hypothesis, we can write $R = \bigcup_1^\infty R_n$, where each R_n is regular. (The following argument would need no change if R_n were only supposed strongly PWB resolutive, with closure a compact subset of R.) We can then define trajectories from a point z as follows. The parameter set T is the set of nonnegative integers and $+\infty$ is the stopping time. Let $z_0 = z$; let n_1 be the smallest value of j with $z_0 \in R_j$. Choose z_1 at random on D'_{n_1}, with probability distribution $\mu(z_0, R_{n_1}, \cdot)$. Given z_1, let n_2 be the smallest value of j with $z_1 \in R_j$ and choose z_2 at random on R'_{n_2}, with probability distribution $\mu(z_1, R_{n_2}, \cdot)$, and so on. This is precisely the construction in section 5, if the sequence of regular sets is monotone. The properties of the trajectories defined in this way depend on the sequence of regular sets chosen, and some variation could be made by different choices of n_k at each stage. The properties TP1–TP3 are readily checked to hold for these trajectories.

9. Applications

The applications are too well known to require much explanation, but we describe some to clarify the theory.

Let R be an open nonnull set in N-dimensional Euclidean space. If the regular functions are to be the harmonic functions, the class of regular sets can be chosen in a variety of ways. For example, the regular sets can be chosen as the open spheres whose closures lie in R. If this is done, a transition measure is given by a density function, the kernel of the classical Poisson integral which expresses a harmonic function in a sphere in terms of its boundary function. For simplicity, suppose that R is bounded, and that R' is the ordinary boundary. It is checked trivially that all our hypotheses, including $M'(R, R')$, are satisfied. Thus the PWB method can be applied, and Brelot [1] proved that R (and *a fortiori* all its nonempty open subsets) is strongly resolutive. The proof involves the notion of capacity and irregular boundary point. Then property $M(R, R')$ is also valid, and the special trajectory systems of section 5 can be defined. Actually, continuous (Brownian) trajectories can be introduced in this case [5]. Since open subsets of R with sufficiently smooth boundaries have the property that the classical first boundary value problem can be solved for them, that is, sufficiently smooth boundaries have no irregular points, the class of regular sets can be enlarged, if desired. The maximal class is the class containing every nonempty open subset of R whose boundary has no irregular point.

The most natural generalization of this case, from the point of view of the domain, is to drop the hypothesis that R be bounded, and that its boundary be the ordinary relative boundary. For a discussion of the ramified Dirichlet problem in extended Euclidean space, see [2]. The results are not essentially changed. We stress that our theorems on the limits of PWB solutions at the boundary hold whether the boundary has been ramified or not.

The most natural generalization of the Dirichlet problem, as just considered, from the point of view of differential equations, is to go from the solutions of Laplace's equation to those of more general linear elliptic equations of the second order. Here we must omit any term in the unknown function, because we require that the boundary function 1 have the corresponding solution 1. Tautz [12] has discussed the first boundary value problem for such functions (without however making the special hypothesis just described) using the PWB method, generalizing the usual results for harmonic functions. Our results are applicable to his solutions. In this case, it would be expected that, as in the case of harmonic functions, it would be possible to use continuous trajectories. In fact it would be expected that these trajectories would be the usual diffusion trajectories which generalize Brownian trajectories, but this case has never been treated.

A generalization of the first boundary value problem for harmonic functions in a slightly different direction is that to the same problem for solutions of the heat equation in an open nonempty set of the Euclidean space of points $(\xi_1, \cdots, \xi_N, t)$. This has been done from the present point of view in [7]. The regular sets can be taken, for example, as the class of those bounded open sets, with compact closures in R, each of which has as boundary pieces of finitely many hyperplanes, none perpendicular to the t axis. Then every open set is strongly PWB resolutive from below. It is not known whether all bounded nonempty open sets, with the usual relative boundaries, are strongly PWB resolutive, but all are strongly SPWB resolutive. A new feature of this case is the following. Let R be bounded, and let R' be its (relative) boundary. Then the concept of a regular boundary point is defined as usual. A regular boundary point is a boundary point at which a PWB solution has the assigned boundary value as a limit whenever the assigned boundary function is PWB resolutive, bounded, and continuous at that point. The corresponding definition for the SPWB method leads to S-regular boundary points. Every S-regular boundary point is regular. In discussing the first boundary value problem for harmonic functions, it is proved that the irregular boundary points of an open set R have harmonic measure, that is, $\mu(z, R, \cdot)$ measure, 0, for all z in R. The irregular boundary points are thus exceptional, in this sense at least, and the PWB solution corresponding to a PWB resolutive boundary function is unaffected by the values of the boundary function at the irregular boundary points. Since the SPWB method leads to no more than the PWB method in this case, the regular and S-regular boundary points are the same. In discussing the first boundary value problem for the solutions of the heat equation, however, there is some difficulty because it is not known which sets are strongly PWB resolutive, so that it is not known whether the regular and S-regular boundary points are the same. However, there are simple examples of strongly PWB resolutive bounded open sets whose irregular boundary points constitute a set of positive $\mu(z, R, \cdot)$ measures for some points z of R. For example, in the above notation, if $N = 1$ (heat equation in one space variable) the interior of a disc less one radius perpendicular to the t axis is such an open set. The points of the radius other than the point on the perimeter are all irregular, and the set of these irregular boundary points has positive $\mu(z, R, \cdot)$ measure, if z has a larger t coordinate than the points of the radius. In this example, and in fact for a general open set, it is possible to ramify the boundary in a very simple way in order that the set of S-irregular boundary points on the new boundary will have $\mu(z, R, \cdot)$ measure 0, identically on R, according to [7].

The most natural generalization of the first boundary value problem for harmonic functions on Euclidean sets, from still another point of view, is this problem for harmonic functions on an open (that is, noncompact) Riemann surface. In this case, R has no intrinsic relative boundary, so that a boundary must be defined before any PWB discussion is possible, although of course such a discussion can be carried out on open subsets of R, with compact closures, using relative boundaries. See Ohtsuka [10] for a discussion of the PWB method in this case. In this case, as well as in the more general case of harmonic functions on Green spaces, treated by Brelot and Choquet [3], all open subsets of R whose closures are compact subsets of R are strongly PWB resolutive, if boundaries are taken as relative boundaries. In these cases, the special trajectories of section 5 can be replaced by continuous trajectories.

It is unfortunate that the discussion in this paper is not general enough to be applicable to preharmonic functions, but it was thought advisable to formulate definitions in terms of topological concepts which are not adapted to the study of functions on a discrete lattice. Let R be the set of points of N-dimensional Euclidean space with integer coordinates. A preharmonic function defined on a subset D of R is one which has the property that its value at any point is the average of its values at its $2N$ nearest neighbors, if these belong to D. If the boundary of D is defined as the set of points of D whose $2N$ nearest neighbors do not all belong to D, the first boundary value problem can be formulated in the obvious way, and was solved long ago for bounded sets D. If probabilistic methods are used, the simplest trajectories are the classical random walk trajectories, for which each step is from a point to one of its nearest neighbors, with $1/(2N)$ as the probability of going to each neighbor. From our point of view it is more interesting to consider the problem when D is unbounded, and a boundary point may be either a point of D, as described above, or an ideal point obtained, for example, by completing D in a suitable metric. The methods we have developed in this paper are applicable with no significant change. Our results, such as the existence of limits along trajectories of subregular, in this case subpreharmonic, functions in the class $\boldsymbol{H}$, and so on, are still valid, and remain so even if the lattice is an abstract lattice.

We conclude with a remark made in a missionary spirit, which classical analysts may find more irritating than illuminating. In any of the above situations, there is a certain natural class of maps of R, under which the basic results should be invariant. The classical results on the Dirichlet and associated problems do not commonly have this invariant form. One example is Fatou's boundary value theorem for harmonic functions on a disc. This theorem involves the disc boundary and the class of radii. Neither of these constructs means anything on a simply connected Riemann surface not homeomorphic to the sphere or conformally equivalent to the finite plane, although such a Riemann surface is the "natural" domain involved here. Obtaining invariant forms for classical theorems is not only a rather unexpected generalization of these theorems, but increases insight into their true significance, separating the essential from the accidental. For example, it now appears from our generalization of Fatou's theorem that even the simple connectedness of the given domain is really irrelevant. It is gratifying that the invariant forms we have obtained have not lost the elegance of the classical theorems, but of course certain features have become less concrete. It is inevitable that the trajectories used here are more abstract than the radii of a disc. It is typical that the classical approach uses curves depending on the smoothness of the given domain; it is just as typical that the invariant approach, which does not require domain smoothness, uses curves defined by intrinsic properties of the function class under examination.

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