

STATIONARITY, BOUNDEDNESS, ALMOST PERIODICITY OF RANDOM- VALUED FUNCTIONS

S. BOCHNER
PRINCETON UNIVERSITY

1. Introduction

The present paper is an enlarged version of the previous paper [1], to which we shall refer occasionally, but the notation has been changed in places.

The problem is as follows. Let $x(t)$ on $-\infty < t < \infty$ be a random-valued function, and let

$$(1.1.1) \quad y = \Lambda x$$

be a linear operation which is commutative with translations on the t -axis ("linear smoothing"). If $x(t)$ is stationary in the sense of Khintchine then $y(t)$ is likewise so. But the converse does not hold. The function $y(t) \equiv 0$ is certainly stationary, but there may exist solutions $\xi(t)$ of the homogeneous equation $0 = \Lambda\xi$ which are not stationary themselves. Also, if the inhomogeneous equation does happen to have a stationary solution $x^0(t)$, then by forming a sum $x(t) = x^0(t) + \xi(t)$ a new solution of it will result which will not again be stationary, frequently.

Now, in the present paper we shall establish a result, in several nonequivalent versions, to the following effect. If $y(t)$ is stationary, and if there exists a solution $x(t)$ which is bounded in t in a suitable sense, then some other solution $x^0(t)$, which need not be $x(t)$ itself, will again be stationary. More information must not be expected, because for any bounded $\xi(t)$ the sum $x^0(t) + \xi(t)$ will be bounded too.

We emphasize that the "boundedness" of $x(t)$ required will in no case be the naïve boundedness in norm (that is "in second moment"), but always a more refined notion, as it were. We shall introduce three types of it, but more composite ones could be devised and utilized.

The principal notion is what we call " V -boundedness." It has come up in analysis before, as we shall describe. The next one in importance is termed " CT -boundedness," and it is suited to the needs of the simplest version of almost periodicity for Banach-valued functions. We also note that stationary functions which are almost periodic were first studied by Slutsky [2]. The last one, finally, is what we have termed " L_2 , z -boundedness," and introduced not only in [1] but prior to it (see p. 154 in [3]). We think it is a promising concept (it subsumes the so-called "Wiener process" more readily than any other stochastic process sufficiently general), but not necessarily so for the present context, and we shall deal with it but briefly here.

This research was supported by the United States Air Force through the Office of Scientific Research of the Air Force Research and Development Command.

We shall also pay some attention to a multiple time variable $t = (t^1, \dots, t^k)$, and in this connection we note that the space variable of statistical turbulence would be the three-dimensional time variable of the present paper, and the time variable of the equation of turbulence from which the inertia force has been omitted would be the parameter ϵ of section 2.6 of the present paper and of the corresponding section of our earlier paper [4].

Most of our statements could be upheld for many group spaces other than the Euclidean (t^1, \dots, t^k) , but we will not formulate them for such, not even for discrete time points of the familiar kind, for which it could be done easily.

The problem we are dealing with is not new. Wold [5] already has it in some form, and it may be older than that. But our results are apparently novel, and we do not think that the papers [6] and [7], for instance, have anticipated our paper [1] and the present one to an appreciable extent.

2. Stationarity

2.1. We shall consider in $-\infty < t < \infty$ functions $x(t)$ with values in a Banach space B . Such a function will be called "Banach-valued," or "vector-valued" or "random-valued" indiscriminately. If B is the space of complex numbers we shall call the function "numerical," or "number-valued," or "complex." *Unless otherwise stated, the space B will be a Hilbert space*, and denoted by H . In keeping with the stochastic view, the inner product of elements x, y in H will be denoted as the "expectation"

$$(2.1.1) \quad E\{x\bar{y}\}.$$

But we shall have no occasion to view a random-valued function as a random function, that is, as a randomized family of numerical functions.

For any B the function $x(t)$ will be always assumed to be measurable. The norm in B will be denoted by $\| \cdot \|$, and bounds and limits are always meant to be in norm. Thus $x(t)$ is bounded if

$$(2.1.2) \quad \|x(t)\| \leq M, \quad -\infty < t < \infty.$$

It is continuous, if $\lim_{h \rightarrow 0} \|x(t+h) - x(t)\| = 0$, and differentiable if

$$(2.1.3) \quad \lim_{h \rightarrow 0} \left\| \frac{x(t+h) - x(t)}{h} - \frac{dx(t)}{dt} \right\| = 0.$$

2.2. We denote by Δx a difference-differential operator

$$(2.2.1) \quad \sum_{\rho=0}^r \sum_{\sigma=0}^{\rho} c_{\rho\sigma} \frac{d^{\rho} x(t - \tau_{\sigma})}{dt^{\rho}}$$

with constant complex coefficients $c_{\rho\sigma}$ and arbitrary (real) spans τ_{σ} , for some $r \geq 0$; or the more general operator

$$(2.2.2) \quad \sum_{\rho=0}^r \int_{-\infty}^{\infty} x^{(\rho)}(t - \tau) dC_{\rho}(\tau)$$

in which $C_0(\tau), \dots, C_r(\tau)$ are complex functions of bounded variation: $\int_{-\infty}^{\infty} |dC_{\rho}(\tau)| < \infty$, $\rho = 0, \dots, r$. We note that it would be possible to characterize (2.2.2) as a linear operator commutative with translations $t \rightarrow t - \tau$, subject to suitable closure properties,

but doing so would not be enlightening to our problems.

We denote by $T(a)$ the functions

$$(2.2.3) \quad \sum_{\rho=0}^r \sum_{\sigma=0}^s c_{\rho\sigma} (2\pi i a)^\rho e^{2\pi i a \tau_\sigma}$$

or more generally

$$(2.2.4) \quad \sum_{\rho=0}^r (2\pi i a)^\rho \int_{-\infty}^{\infty} e^{2\pi i a \tau} dC_\rho(\tau),$$

and call it the *generator* of Λx . Obviously

$$(2.2.5) \quad \Lambda_t e^{2\pi i a t} = T(a) \cdot e^{2\pi i a t}.$$

We consider $T(a)$ always for real values of a only, even if it is analytic in the complex a -plane as is (2.2.3). However some vestiges of analyticity will be assumed present.

The function $T(a)$ as defined by (2.2.4) is bounded and continuous, but we are adding the assumption that $T(a) \in C^{(2)}$, that is, has two continuous derivatives in $-\infty < a < \infty$. (Somewhat less differentiability would suffice.)

To this we shall frequently add the assumption that there are at most countably many points $\{a_m\}$ at which $T(a)$ is 0: $T(a_m) = 0$. We shall then usually single out the case in which the set $\{a_m\}$ has finitely many points at most.

We also note that for a pure differential operator

$$(2.2.6) \quad \Lambda x = \sum_{\rho=0}^r c_\rho x^{(\rho)}(t) \quad T(a) = \sum_{\rho=0}^r c_\rho (2\pi i a)^\rho$$

we have

$$(2.2.7) \quad T(a)^{-1} = O(|a|^{-r}), \quad |a| \rightarrow \infty,$$

and that this is also so for the general operator (2.2.1) provided

$$(2.2.8) \quad \inf_{-\infty < a < \infty} \left| \sum_{\sigma=0}^s c_{r\sigma} e^{2\pi i a \tau_\sigma} \right| > 0.$$

2.3. We say that a function $x(t)$ is *K-stationary* ("K" for "Khintchine") if it is "stationary wide sense," that is, if it is continuous and its covariance function

$$(2.3.1) \quad R_x(u, v) = E\{x(u) \bar{x}(v)\}$$

has the invariance property

$$(2.3.2) \quad R_x(u+h, v+h) = R_x(u, v), \quad -\infty < h < \infty.$$

A function $x(t)$ is *K-stationary* if and only if there exists a continuous function of one variable

$$(2.3.3) \quad R(t) \equiv R_x(t), \quad -\infty < t < \infty,$$

so that $R_x(u, v) = R(u - v)$ for all u, v .

A very particular property of a *K-stationary* function is that it is bounded, since in fact

$$(2.3.4) \quad \|x(t)\|^2 = R(t-t) = R(0) = \|x(0)\|^2.$$

As known, a K -stationary function and its covariance function are characterized one by Fourier transformations

$$(2.3.5) \quad x(t) \sim \int_{-\infty}^{\infty} e^{2\pi i t a} dE(a)$$

$$(2.3.6) \quad R(t) = \int_{-\infty}^{\infty} e^{2\pi i t a} dA(a)$$

$$(2.3.7) \quad dA(a) \geq 0, \quad \int_{-\infty}^{\infty} dA(a) \equiv R(0) < \infty$$

of the following description. There is defined on all Borel sets $\{S\}$ of $(-\infty, \infty)$ a finitely additive function $E(S)$ with values in H , having the orthogonality property

$$(2.3.8) \quad E\{E(S_1) \cdot \overline{E(S_2)}\} = 0 \quad \text{if} \quad S_1 \cap S_2 = 0.$$

The nonnegative function on $\{S\}$,

$$(2.3.9) \quad A(S) \equiv E\{|E(S)|^2\},$$

which by (2.3.8) is finitely additive, is then even σ -additive, and for any two complex continuous functions $\psi_1(a)$, $\psi_2(a)$ with compact support we have

$$(2.3.10) \quad E\left\{\int_{-\infty}^{\infty} \psi_1(\beta) dE(\beta) \cdot \int_{-\infty}^{\infty} \overline{\psi_2(\gamma)} d\overline{E(\gamma)}\right\} = \int_{-\infty}^{\infty} \psi_1(a) \overline{\psi_2(a)} dA(a).$$

Also, if

$$(2.3.11) \quad \varphi(t) = \int_{-\infty}^{\infty} e^{2\pi i t a} \psi(a) da,$$

then (2.3.5) implies

$$(2.3.12) \quad \int_{-\infty}^{\infty} \varphi(-\tau) x(\tau) d\tau = \int_{-\infty}^{\infty} \psi(a) dE(a),$$

in the sense that if, for instance, $\psi(a)$ is $C^{(1)}$ and has compact support, then the right-hand integral has the value

$$(2.3.13) \quad - \int_{-\infty}^{\infty} \frac{d\psi(a)}{dt} E(a) da$$

precisely.

2.4. Relation (2.3.12), with the interpretation (2.3.13), can be considerably expanded (compare [8] and p. 536 in [9]).

THEOREM 2.4.1. *If $x(t)$ is K -stationary and has r continuous derivatives, then these derivatives are also K -stationary, and so is the function $y = \Delta x$. Also*

$$(2.4.1) \quad y(t) \sim \int_{-\infty}^{\infty} e^{2\pi i a t} T(a) dE(a)$$

$$(2.4.2) \quad R_y(t) = \int_{-\infty}^{\infty} e^{2\pi i a t} |T(a)|^2 dA(a).$$

The meaning of the last two formulas is as follows. Since the function y is K -stationary there is a representation

$$(2.4.3) \quad y(t) \sim \int_{-\infty}^{\infty} e^{2\pi i a t} dF(a)$$

$$(2.4.4) \quad R_y(t) = \int_{-\infty}^{\infty} e^{2\pi i a t} dB(a),$$

and the assertion is that we have

$$(2.4.5) \quad dF(\alpha) = T(\alpha) dE(\alpha), \quad dB(\alpha) = |T(\alpha)|^2 dA(\alpha),$$

in the sense that, for any Borel set S ,

$$(2.4.6) \quad F(S) = \int_S T(\alpha) dE(\alpha), \quad B(S) = \int_S |T(\alpha)|^2 dA(\alpha)$$

A particular case of theorem 2.4.1 is the following.

THEOREM 2.4.2. *If $x(t)$ is K -stationary, and $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$, say, then the function*

$$(2.4.7) \quad x_{\varphi}(t) = \int_{-\infty}^{\infty} \varphi(t - \tau) x(\tau) d\tau$$

is also K -stationary, and

$$(2.4.8) \quad x_{\varphi}(t) \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} \psi(\alpha) dF(\alpha)$$

where

$$(2.4.9) \quad \psi(\alpha) = \int_{-\infty}^{\infty} e^{-2\pi i \alpha t} \varphi(t) dt.$$

Next a lemma (compare pp. 536–540 in [9]).

THEOREM 2.4.3. *If for a K -stationary function (2.3.5) we have*

$$(2.4.10) \quad \int_{-\infty}^{\infty} \alpha^{2r} dA(\alpha) < \infty,$$

then $x(t)$ has r continuous derivatives (all K -stationary).

We shall denote by W the Euclidean space: $-\infty < \alpha < \infty$, and later its k -dimensional generalization: $-\infty < \alpha_j < \infty$, $j = 1, \dots, k$; and by Q the point set on which $T(\alpha) = 0$, in one or k variables, no matter how small or large the point set.

THEOREM 2.4.4. *If $x(t)$ and $y(t)$ are as in theorem 2.4.1, then “ $dF(\alpha)$ ” vanishes on Q , meaning that*

$$(2.4.11) \quad F(S \cap Q) = 0,$$

for any Borel set S . Also

$$(2.4.12) \quad \int_{W-Q} \frac{dB(\alpha)}{|T(\alpha)|^2} < \infty,$$

and also

$$(2.4.13) \quad \int_{W-Q} \alpha^{2\rho} \frac{dB}{|T(\alpha)|^2} < \infty, \quad \rho = 1, \dots, r,$$

for $r \geq 1$.

Remark. If (2.2.7) holds, then (2.4.13) is a formal consequence of (2.4.12) and $B(W) < \infty$.

THEOREM 2.4.5. *If for a K -stationary function $y(t)$ the spectrum has the property (2.4.11) and (2.4.12), and also (2.4.13), then there exists a K -stationary solution of the equation*

$$(2.4.14) \quad \Delta x = y.$$

One such solution is the function

$$(2.4.15) \quad x^0(t) \sim \int_{W-Q} e^{2\pi i \alpha t} \frac{dF(\alpha)}{T(\alpha)}$$

$$(2.4.16) \quad R_{x^0}(t) = \int_{W-Q} e^{2\pi i \alpha t} \frac{dB(\alpha)}{|T(\alpha)|^2},$$

that is to say, the function

$$(2.4.17) \quad x^0(t) \sim \int_w e^{2\pi i a t} dE^0(a)$$

for which

$$(2.4.18) \quad E^0(S \cap Q) = 0, \quad E^0(S) = \int_{S-Q} \frac{dE(a)}{T(a)}.$$

2.5. Assumption (2.4.13) in theorem 2.4.4 was necessary in order to insure that the function (2.4.17) have r continuous derivatives with which to form the expression Λx literally. Now, in this and related theorems condition (2.4.13) and the mention of derivatives of $x(t)$ can be omitted by introducing so-called *weak solutions* of the equation (2.4.14) instead of "strong" ones as tacitly envisaged till now (see [10], and also p. 158 in [11]). Weak solutions are very well suited to linear operators $\Lambda x(t)$ which are commutative with translations on the t -axis as ours are. Every strong solution is a weak solution, and "many" weak solutions are strong solutions automatically. But whether they are so or not, the weak solutions are conceptually very pertinent to our context and we will work with them exclusively.

The definition is as follows. If a numerical $\varphi(t)$ in $C^{(r)}$ has compact support, then

$$(2.5.1) \quad \int_{-\infty}^{\infty} \varphi(t) \cdot \Lambda x(t) dt = \int_{-\infty}^{\infty} \Lambda' \varphi(t) \cdot x(t) dt,$$

where

$$(2.5.2) \quad \Lambda' \varphi(t) = \sum_{\rho=0}^r (-1)^\rho \int_{-\infty}^{\infty} \varphi^{(\rho)}(t-\tau) dC_\rho(\tau),$$

so that equation (2.4.14) is equivalent to the system of equations

$$(2.5.3) \quad \int_{-\infty}^{\infty} \Lambda' \varphi(t) \cdot x(t) dt = \int_{-\infty}^{\infty} \varphi(t) \cdot y(t) dt,$$

the latter formed with all such functions $\varphi(t)$. However this system of equations (2.5.3) can be set up without regard to differentiability of $x(t)$, and a *weak* solution of (2.4.14) is by definition a solution of the system (2.5.3), subject to such assumptions on $x(t)$ and $y(t)$ as may be called for.

Now, the following statement can be readily established.

THEOREM 2.5.1. (i) *If $x(t)$ and $y(t)$ are two K -stationary functions which satisfy (2.4.14) weakly then (2.4.11) and (2.4.12) hold.*

(ii) *If $y(t)$ is K -stationary and if (2.4.11) and (2.4.12) hold, then there exists a K -stationary solution $x^0(t)$ weakly, and (2.4.15) is such.*

(iii) *If $T(a)$ has property (2.2.7) then the latter solution is also a strong solution, and if in addition to that $T(a)$ has only finitely many zeros $\{a_m\}$ then (by theorem 3.3.2 to follow) every weak solution of (2.4.14) is a strong solution.*

2.6. Weak solutions can be set up in terms of objects $x(t)$, $y(t)$ which are more general than point functions, for instance, "distributions" in the terminology of L. Schwartz, which, however, we prefer to call "symbolic functions" following P. Lévy. The adaptation, especially to random-valuedness, can be done in different ways. An adaptation closely following Schwartz was recently done by K. Itô [12], and he defined K -stationarity in such a manner that an object $x(t)$ is K -stationary if and only if one can associate with it spectra $dE(a)$, and $dA(a)$ of the following kind. As in section 2.3 there is a finitely additive set function $E(S)$, with values in H , which is defined over Borel sets S , and has

the orthogonality property (2.3.8) there. However it is defined for *bounded* Borel sets only. The resulting nonnegative set function $A(S)$ is definable for all Borel sets $\{S\}$ and is σ -additive on them, but its value may be $+\infty$ if the Borel set S is unbounded. However, there is some (nonfixed) integer g such that

$$(2.6.1) \quad \int_{-\infty}^{\infty} \frac{dA(a)}{1+a^{2g}} < \infty.$$

A little earlier than Itô, we ourselves (see [4], also [3], p. 166) introduced a class of K -stationary symbolic functions in a somewhat different manner, and our class does in fact include all Itô's functions and others as well. We introduce $x(t)$ as the (would-be but nonexistent perhaps) limit as $\epsilon \downarrow 0$ of "ordinary" K -stationary functions $x^\epsilon(t)$, $0 < \epsilon < \infty$, whose expansion is of the form

$$(2.6.2) \quad x^\epsilon(t) \sim \int_{-\infty}^{\infty} e^{2\pi i a t} e^{-\epsilon \pi a^2} dE(a)$$

$$(2.6.3) \quad R_{x^\epsilon}(t) = \int_{-\infty}^{\infty} e^{2\pi i a t} e^{-2\epsilon \pi a^2} dA(a).$$

The "common" function $E(S)$ is defined as just described, and (2.6.1) is replaced by

$$(2.6.4) \quad \int_{-\infty}^{\infty} e^{-2\epsilon \pi a^2} dA(a) < \infty, \quad 0 < \epsilon < \infty,$$

so that our class does indeed include the class which Itô introduced afterwards.

In either case it is possible to define weak solutions of (2.4.14) and the result is as follows.

THEOREM 2.6.1. *Parts (i) and (ii) of theorem 2.5.1 also hold for symbolic functions of either kind.*

2.7. Our operators and the statements about them can be extended from the line $-\infty < t < \infty$ to group spaces, more or less, but we will deal with multidimensional Euclidean space only.

Let t, τ, h , etc. be the vector points (t^1, \dots, t^k) , (τ^1, \dots, τ^k) , (h^1, \dots, h^k) , etc., for any $k \geq 1$; but $x(t)$, $y(t)$, $\varphi(t)$, etc. shall continue to be functions of one component. The operator Δx is an expression

$$(2.7.1) \quad \sum_{\rho_1=0}^{\tau_1} \cdots \sum_{\rho_k=0}^{\tau_k} \int_W \frac{\partial^{\rho_1+\dots+\rho_k} x(t-\tau)}{\partial (t^1)^{\rho_1} \cdots \partial (t^k)^{\rho_k}} dC_{\rho_1 \dots \rho_k}(\tau)$$

where " $dC_{\rho_1 \dots \rho_k}(\tau)$ " indicates integration with respect to a σ -additive complex set function over the Borel sets of W . We replace $2\pi i t a$ by

$$(2.7.2) \quad 2\pi i(t, a) \equiv 2\pi i(t^1 a^1 + \cdots + t^k a^k),$$

and the generator $T(a)$ is of course

$$(2.7.3) \quad \sum_{\rho_1=0}^{\tau_1} \cdots \sum_{\rho_k=0}^{\tau_k} (2\pi i a^2)^{\rho_1} \cdots (2\pi i a^k)^{\rho_k} \int_W e^{2\pi i(a, \tau)} dC_{\rho_1 \dots \rho_k}(\tau).$$

The "one-dimensional" assumption that $T(a)$ is always $C^{(2)}$ has to be generalized to mean that it is always $C^{(k+1)}$.

Next, K -stationarity of functions $X(t)$ in several variables is readily defined and very

well known (as for instance in the theory of turbulence). For any $k \geq 1$ there is a representation

$$(2.7.4) \quad x(t) \sim \int_W e^{2\pi i(a, t)} dE(a), \quad R(t) = \int_W e^{2\pi i(a, t)} dA(a)$$

in which “ $dE(a)$ ” signifies integration with respect to an additive set function $E(S)$ with values in H over the Borel sets of the Euclidean $W: (a^1, \dots, a^k)$, and the property (2.3.8) and the definition (2.3.9) of $A(S)$ remain literally as before.

Weak solutions carry over easily, and parts (i), (ii) of theorem 2.5.1 remain. “Symbolic functions” also carry over, although in places their Fourier analytic treatment requires more sophistication, and theorem 2.6.1 remains.

But there are occasions on which multidimensionality calls for special comment, and in order to avoid misunderstandings we shall be thinking of the one-dimensional situation as a rule, and deal with multidimensionality expressly.

3. Boundedness

3.1. We shall denote by Φ^n , $n = 0, 1, 2, \dots, \infty$, the class of complex functions $\varphi(t)$ presentable as an integral

$$(3.1.1) \quad \varphi(t) = \int_{-\infty}^{\infty} e^{2\pi i a t} \psi(a) da$$

in which $\psi(a)$ belongs to C^n and is 0 outside a finite interval.

For each n , $\varphi(t) \in C^\infty$, and, for $n \neq \infty$,

$$(3.1.2) \quad \frac{d^p \varphi}{dt^p} = O(|t|^{-n}), \quad |t| \rightarrow \infty,$$

for $p = 0, 1, 2, \dots$.

If $\varphi(t) \in \Phi^2$, then $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$, and hence the following statement.

THEOREM 3.1.1. *For any Banach space B , if $x(t)$ is bounded and $\varphi \in \Phi^2$ we can form the smoothing (2.4.7), and if two such bounded functions $x(t)$, $y(t)$ satisfy the equation $\Delta x = y$ weakly, then their simultaneous smoothings satisfy it strongly,*

$$(3.1.3) \quad \Delta x_\varphi(t) = y_\varphi(t).$$

3.2. If a complex function $x(t)$ is defined a.e. and $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$, then by Plancherel theory the integrals

$$(3.2.1) \quad E_0(a) \sim \int_{-\infty}^{\infty} e^{-2\pi i t a} x(t) dt, \quad x(t) \sim \int_{-\infty}^{\infty} e^{2\pi i a t} E_0(a) da$$

exist a.e. in a certain sense, and

$$(3.2.2) \quad \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |E_0(a)|^2 da.$$

This statement can be easily generalized to functions $x(t)$ with values in H . If in H we introduce a fixed unitary coordinate system and apply the theorem just stated to each component separately then we conclude as follows. If $\int_{-\infty}^{\infty} \|x(t)\|^2 dt < \infty$, then there exists a function $E_0(a)$, with values in H , which is linked to $x(t)$ by (3.2.1) and also

$$(3.2.3) \quad \int_{-\infty}^{\infty} \|x(t)\|^2 dt = \int_{-\infty}^{\infty} \|E_0(a)\|^2 da.$$

Next the formula

$$(3.2.4) \quad E(\alpha) = \int_{-1}^1 x(t) \frac{e^{-2\pi i \alpha t} - 1}{-2\pi i t} dt + \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) x(t) \frac{e^{-2\pi i \alpha t}}{-2\pi i t} dt$$

defines an indefinite integral

$$(3.2.5) \quad E(\alpha) = \int^{\alpha} E_0(\beta) d\beta,$$

but, by the very theorem of Planchèrel, not only the first but also the decisive second term in the formula (3.2.4) defines a function a.e. if only

$$(3.2.6) \quad \int_{-\infty}^{\infty} \frac{\|x(t)\|^2}{1+t^2} dt < \infty.$$

Now if $x(t)$ is bounded, $\|x(t)\| \leq M$, this function $E(\alpha)$ need no longer be an indefinite integral, but if we introduce the association

$$(3.2.7) \quad x(t) \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} dE(\alpha),$$

then a Fourier representation ensues whose properties are known (see [13]) and which we shall name as the occasion arises.

First of all, (3.2.4) implies

$$(3.2.8) \quad E(\beta) - E(\alpha) = \int_{-\infty}^{\infty} x(t) \frac{e^{-2\pi i \beta t} - e^{-2\pi i \alpha t}}{-2\pi i t} dt,$$

for almost all α and almost all β , but it is known in the theory of K -stationary processes that for a K -stationary $x(t)$, this in substance defines the spectral function $E(\alpha)$ introduced by (2.3.5), except that the latter function is defined precisely everywhere and not only almost everywhere, and that the additive constant in it might not be the same as in (3.2.4). Thus our representation (3.2.7) may be used as a generalization of (2.3.5) from K -stationary functions $x(t)$ to all bounded ones.

Next, if $\varphi \in \Phi^2$, then for the smoothed function (2.4.7) we have in a well defined sense

$$(3.2.9) \quad x_{\varphi}(t) \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} \psi(\alpha) dE(\alpha)$$

and the value of the integral is

$$(3.2.10) \quad - \int_{-\infty}^{\infty} \frac{d}{d\alpha} [e^{2\pi i \alpha t} \psi(\alpha)] E(\alpha) d\alpha,$$

literally. For any Λ [whose generator $T(\alpha)$ is in $C^{(2)}$] we also have

$$(3.2.11) \quad \Lambda x_{\varphi}(t) \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} \psi(\alpha) T(\alpha) dE(\alpha),$$

and if we introduce

$$(3.2.12) \quad y(t) \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} dF(\alpha),$$

then (3.1.3) leads to the equations

$$(3.2.13) \quad \psi(\alpha) T(\alpha) dE(\alpha) = \psi(\alpha) dF(\alpha)$$

$$(3.2.14) \quad T(\alpha) dE(\alpha) = dF(\alpha)$$

from which certain specific conclusions can be drawn.

3.3. The first statement which concerns only a single function is as follows.

THEOREM 3.3.1. (i) If the transform $E(\alpha)$ of a bounded function $x(t)$ is piecewise constant with saltuses a_1, \dots, a_n at finitely many points $\alpha_1, \dots, \alpha_n$, then

$$(3.3.1) \quad x(t) = \sum_{m=1}^n a_m e^{2\pi i a_m t}.$$

(ii) If there are countably many points of discontinuity $\alpha_1, \alpha_2, \dots$, with $|\alpha_n| \rightarrow \infty$, and $x(t)$ is uniformly continuous, then $x(t)$ is a uniform limit of finite sums

$$(3.3.2) \quad \sum_{m=1}^n b_m e^{2\pi i a_m t}.$$

Also $x(t)$ is then an almost periodic function in a well-defined sense (compare definition 5.1.2) and the series

$$(3.3.3) \quad x(t) \sim \sum_{m=1}^{\infty} a_m e^{2\pi i a_m t}$$

with the given saltuses $\{a_m\}$ is its Fourier series.

This will lead to the following theorem.

THEOREM 3.3.2. If a bounded function $x(t)$ is a weak solution of

$$(3.3.4) \quad \Delta x = 0,$$

and if $T(\alpha)$ has at most finitely many zeros $\{a_m\}$, then $x(t)$ is a finite sum (3.1.1).

If $T(\alpha)$ has countably many zeros and $x(t)$ is known to be also uniformly continuous, then $x(t)$ is almost periodic with an expansion (3.3.3).

PROOF. Since $y = 0$ we have $dF = 0$, and (3.2.14) is therefore

$$(3.3.5) \quad T(\alpha) dE(\alpha) = 0.$$

In any interval $\beta < \alpha < \gamma$ in which $T(\alpha) \neq 0$ it is permissible to divide through by $T(\alpha)$, so that $dE = 0$, the interpretation being that E is constant in such an interval. Now apply theorem 3.3.1.

THEOREM 3.3.3. If bounded functions $x(t), y(t)$ are weakly connected by

$$(3.3.6) \quad \Delta x = y,$$

and if in a neighborhood of an isolated zero α_0 of $T(\alpha)$ the transform $F(\alpha)$ of $y(t)$ is constant except for a saltus b_0 at α_0 then this saltus is zero.

PROOF. Let $\epsilon > 0$ be such that $T(\alpha)$ is $\neq 0$ in the two intervals

$$(3.3.7) \quad \alpha_0 - 2\epsilon < \alpha < \alpha_0, \quad \alpha_0 < \alpha < \alpha_0 + 2\epsilon$$

and that $F(\alpha)$ is constant in each. We replace the equation (3.3.6) by the "smoothed" equation (3.1.3) where $\varphi(t)$ has a transform $\psi(\alpha)$ which is $+1$ at α_0 and 0 outside the interval $\alpha_0 - \epsilon < \alpha < \alpha_0 + \epsilon$. If now we denote the transforms of x_φ and y_φ by $E^0(\alpha)$ and $F^0(\alpha)$ respectively, so that

$$(3.3.8) \quad dE^0(\alpha) = \psi(\alpha) dE(\alpha), \quad dF^0(\alpha) = \psi(\alpha) dF(\alpha), \quad T(\alpha) dE^0 = dF^0(\alpha),$$

then our assumptions together with the choice of $\psi(\alpha)$ have the following implications. $F^0(\alpha)$ is constant both in $(-\infty, \alpha_0)$ and (α_0, ∞) with the same saltus b_0 as $F(\alpha)$ itself; and

in these half-lines we can also divide through (3.3.8) by $T(a)$ so that $dE^0(a) = 0$ there. Thus E^0 is also constant, except for a saltus a_0 at a_0 . Therefore by theorem 3.3.1 we have

$$(3.3.9) \quad x_\varphi = a_0 e^{2\pi i a_0 t}, \quad y_\varphi = b_0 e^{2\pi i a_0 t},$$

and (3.1.3) gives $b_0 = T(a_0)a_0$. Therefore $b_0 = 0$ as claimed.

The preceding theorems lead to the following conclusion which is the one we shall require.

THEOREM 3.3.4. *Let $T(a)$ have finitely many zeros $\{a_n\}$ at most. If we are given an equation*

$$(3.3.10) \quad \Lambda x^1 = y^1,$$

in which $y^1(t)$ is a finite sum (3.3.2) and x^1 is bounded, then $y^1 = 0$ and x^1 is a finite sum (3.3.1).

For a $T(a)$ with countably many zeros, if $y^1(t)$ is an almost periodic function with exponents $\{a_n\}$, and x^1 is bounded and uniformly continuous, then $y^1 = 0$ and x^1 is almost periodic with an expansion (3.3.3).

3.4. The next theorem has a different trend, and the link with theorem 3.3.4 will be established in section 4.

THEOREM 3.4.1. *For any $T(a)$, let $y(t)$ be K -stationary and let $x(t)$ be a bounded function which is a weak solution of (3.3.6).*

If $\varphi \in \Phi^2$ is such that its transform $\psi(a)$ is zero outside a compact subset of the open set $W - Q$ on which $T(a) \neq 0$, then $x_\varphi(t)$ is uniquely determined by $\{\Lambda, y, \varphi\}$ and is also K -stationary:

$$(3.4.1) \quad x_\varphi(t) \sim \int_{W-Q} e^{2\pi i a t} \frac{\psi(a)}{T(a)} dF(a)$$

$$(3.4.2) \quad \int_{W-Q} \frac{|\psi(a)|^2}{|T(a)|^2} dB(a) = \|x_\varphi(0)\|^2 = \left\| \int_{-\infty}^{\infty} \varphi(-t) x(t) dt \right\|^2.$$

PROOF. The function

$$(3.4.3) \quad \psi(a) = \begin{cases} \psi(a) T(a)^{-1}, & a \in W - Q, \\ 0, & a \in Q, \end{cases}$$

is such that

$$(3.4.4) \quad \tilde{\varphi}(t) = \int_{-\infty}^{\infty} e^{2\pi i a t} \psi(a) da$$

belongs to Φ^2 , and that (3.2.13) implies

$$(3.4.5) \quad \psi(a) dE(a) = (a) dF(a)$$

in the sense that

$$(3.4.6) \quad x_\varphi(t) = y_{\tilde{\varphi}}(t).$$

But this readily implies all parts of theorem 3.4.1.

3.5. In contrast to theorem 3.4.1 the preceding theorems 3.3.1 to 3.3.4 do not involve K -stationarity, and can, as a matter of fact, be upheld for an arbitrary Banach space. However, if B is not H , Plancherel's theorem is not available, and the second integral of (3.2.4) cannot be readily defined for a bounded $x(t)$. However we can operate with a "higher" Fourier transform

$$(3.5.1) \quad x(t) \sim \int_{-\infty}^{\infty} e^{2\pi i a t} \frac{d^2 E_2(a)}{da}$$

in which $E_2(a)$ is an indefinite integral of $E(a)$ whenever the latter exists, and otherwise is defined by

$$(3.5.2) \quad \int_{-1}^1 x(t) \frac{e^{-2\pi i a t} - 1 + 2\pi i a t}{(-2\pi i t)^2} dt + \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) x(t) \frac{e^{-2\pi i a t}}{(-2\pi i t)^2} dt.$$

Piecewise constancy of $E(a)$ is to be replaced by piecewise linearity of $E_2(a)$, and the saltus at a point is a difference of slopes.

Actually the integrals defining $E_2(a)$ are easier to handle and $E_2(a)$ is continuous. In our book [14] it is this type of generalized Fourier transform that was elaborated whereas in the earlier paper [13] the more complicated theory based on Plancherel's theorem had been developed. We note in passing that one can also introduce generalized transforms using local L_p -norms, and an interesting statement on such has recently been made by Blackman [15].

3.6. If we turn to several variables $t = (t^1, \dots, t^k)$ the syllogistic discrepancy between theorems 3.4.1 and 3.3.4 widens considerably.

Theorem 3.4.1 carries over easily in both wording and proof. If $\psi(a_1, \dots, a_k) \in C^{(k+1)}$, and if it is 0 outside a compact subset of the open set $W - Q$ over which $T(a^1, \dots, a^k) \neq 0$, then the function

$$(3.6.1) \quad x_\varphi(t) \equiv \int_W \varphi(t^1 - \tau^1, \dots, t^k - \tau^k) x(\tau^1, \dots, \tau^k) d v_\tau$$

is again K -stationary, and analogues to (3.4.1) and (3.4.2) arise.

However, theorem 3.3.4 is much less easily adjusted. It continues to hold, even for an arbitrary B (compare L. Schwartz [16], text beginning with line 5 from the bottom of p. 509), but the analysis required is rather more recondite than in the one-dimensional case, and generalizations to locally compact commutative groups other than Euclidean ones are even in doubt.

4. V -boundedness

4.1. DEFINITION 4.1.1. For any B , we say that $x(t)$ is V -bounded if

(i) it is (measurable and) bounded in the ordinary sense,

$$(4.1.1) \quad \|x(t)\| \leq M,$$

and, what is decisive, if also

(ii) there is a finite constant $N > 0$, such that

$$(4.1.2) \quad \left\| \int_{-\infty}^{\infty} \varphi(-t) x(t) dt \right\| \leq N \cdot \sup_{-\infty < a < \infty} |\psi(a)|$$

where $\varphi(t)$ is a complex function a.e. for which $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$, and

$$(4.1.3) \quad \psi(a) = \int_{-\infty}^{\infty} e^{-2\pi i a t} \varphi(t) dt$$

is its Fourier transform.

We note that for complex-valued functions $x(t)$, that is, if B is one-dimensional, this concept of boundedness has been first introduced in our paper [17] where the following theorem has been featured.

In order that a function $x(t)$ have a representation

$$(4.1.4) \quad x(t) = \int_{-\infty}^{\infty} e^{2\pi i a t} dV(a)$$

with

$$(4.1.5) \quad \int_{-\infty}^{\infty} |dV(a)| < \infty$$

it is (necessary and) sufficient that for any points t_1, \dots, t_n , $n \geq 1$, and any numbers c_1, \dots, c_n we have

$$(4.1.6) \quad \left| \sum_{m=1}^n c_m x(\tau_m) \right| \leq N \cdot \sup_a \left| \sum_{m=1}^n c_m e^{-2\pi i a \tau_m} \right|.$$

Soon afterwards, I. J. Schoenberg [18] replaced the “discrete” condition (4.1.6) by the present condition (4.1.2) and showed that the theorem holds likewise. Also, Schoenberg required locally only Lebesgue integrability of $x(t)$, whereas in [17] continuity of $x(t)$ was demanded. This point attracted the attention of R. S. Phillips [19] and he proved that even in the discrete version (4.1.6) the measurability of $x(t)$ is sufficient. He also considered Banach-valued functions $x(t)$ and he could then validate the assertion under a certain requirement of conditional compactness explicitly added.

Recently, V -boundedness was introduced into abstract harmonic analysis in general, first by H. Helson [20], and then more systematically and directly by S. Helgason [21], and interesting properties were revealed. Our own paper [1] has reintroduced it then from its own context, and with a result which had apparently not been obtained in the papers [6] and [7] that had come closest to it.

4.2. We note that the “discrete” condition (4.1.6) immediately implies the condition (4.1.1) of ordinary boundedness, but for our present purpose it is necessary, or at any rate suitable, to demand (4.1.1) explicitly for the following reason.

Suppose we do not require (4.1.2) for all functions $\varphi(t)$ which are integrable over $(-\infty, \infty)$ but only those among them which are 0 outside a finite interval each. For such a function $\varphi(t)$ the expression

$$(4.2.1) \quad \int_{-\infty}^{\infty} \varphi(-t) x(t) dt$$

can be set up for any function $x(t)$, with values in any B , which is Lebesgue integrable over any finite interval ($a < t < b$) but is not restricted at all as $|t| \rightarrow \infty$. Now, for any a and any h the function: $\varphi(t) = 1/2h$ in $a \leq t \leq a + 2h$, and = 0 at other points is such that $|\psi(a)| \leq 1$, so that (4.1.2) implies

$$(4.2.2) \quad \left\| \frac{1}{2h} \int_{-h}^h x(t + \tau) d\tau \right\| \leq N$$

for all $h > 0$ and all t , and this implies $\|x(t)\| \leq N$, in almost all t . Actually, it would suffice to stipulate (4.1.2) for functions $\varphi(t)$ with compact support which are C^∞ and this would have the added advantage that, to start with, condition (4.1.2) could be set up for “symbolic” functions in general. The result would still be that it is bounded a.e., and thus a “concrete” function automatically.

However, in our applications, the functions $\varphi(t)$ available are those belonging to Φ^2 [whose transform $\psi(a)$ has compact support] and such a function $\varphi(t)$, being analytic, can never be 0 outside a finite interval. Of course, if the boundedness condition (4.1.1) is already known, then the condition (4.1.2) can be extended from such $\varphi(t)$ to all integrable $\varphi(t)$ —this extension will not even be needed—but the boundedness (4.1.1) itself cannot be easily inferred from it at first.

As a comment on what actually can be inferred we will state a certain criterion, which however will not be needed as such.

THEOREM 4.2.1. If $x(t)$ is Lebesgue integrable on every finite interval and

$$(4.2.3) \quad \int_{-\infty}^{\infty} \frac{\|x(t)\|}{1+|t|^s} dt < \infty \quad \text{for some } s > 0,$$

and if (4.1.2) holds for all φ in Φ^2 (or only in Φ^∞) then (4.1.1) holds a.e. (with $M = N$, for instance).

PROOF. We take an element $\varphi(t)$ in Φ^2 for which $\int_{-\infty}^{\infty} \varphi(t) dt = 1$. The function $\varphi_n(t) = n\varphi(nt)$ likewise belongs to Φ^2 , and if we put

$$(4.2.4) \quad x_n(t) = n \int_{-\infty}^{\infty} \varphi_n(t-\tau) x(\tau) d\tau,$$

then (4.1.2) implies

$$(4.2.5) \quad \|x_n(t)\| \leq N, \quad n = 1, 2, \dots$$

Now, we have

$$(4.2.6) \quad x_n(t) - x(t) = n \int_{-\infty}^{\infty} \varphi_n(\tau) [x(t+\tau) - x(t)] d\tau,$$

and our assertion will follow if we ascertain that

$$(4.2.7) \quad \lim_{n \rightarrow \infty} [x_n(t) - x(t)] = 0$$

in almost all t . It follows from (3.1.2) that given t and for any $\delta > 0$ we have

$$(4.2.8) \quad \int_{\delta}^{\infty} \varphi_n(\tau) [x(t+\tau) - x(t)] d\tau = O\left(\frac{1}{n^{s-1}}\right) \cdot \int_{\delta}^{\infty} \frac{\|x(t+\tau)\| + \|x(t)\|}{|t|^s} d\tau$$

and for $s > 1$ this tends to 0 as $n \rightarrow \infty$. The same holds for $\int_{-\infty}^{-\delta}$ and it now suffices to estimate the integral

$$(4.2.9) \quad \int_0^{\delta} \varphi_n(\tau) [x(t+\tau) - x(t)] d\tau$$

and the analogous integral $\int_{-\delta}^0$.

Now, if for fixed t we introduce the function

$$(4.2.10) \quad X(\tau) = \int_0^{\tau} [x(t+u) - x(t)] du$$

then (4.2.9) is

$$(4.2.11) \quad n\varphi(n\delta) X(\delta) - n^2 \int_0^{\delta} \varphi'(n\tau) X(\tau) d\tau$$

and this tends to 0 as $\delta \rightarrow 0$, uniformly for $n \geq 1$, provided $X(\tau)/\tau \rightarrow 0$ as $\tau \rightarrow 0$. But the latter takes place for almost all values of t , and this completes the proof of the theorem.

4.3. If a complex function $\xi(t)$ is V -bounded, then so is the function $c\xi(t)$ for any element c in B . Also, the sum of V -bounded functions is again V -bounded. Hence the first part of the following theorem.

THEOREM 4.3.1. For any B , a finite sum

$$(4.3.1) \quad \sum_{m=1}^n c_m e^{2\pi i a_m t}$$

is V -bounded for any real a_m and any c_m in B .

Also, if $x(t)$ is V -bounded then so is also any smoothing $x_\varphi(t)$ as defined by (2.4.7).

The second half follows immediately from the fact the Fourier transform of $\varphi_2(t) = \int_{-\infty}^{\infty} \varphi_1(t - \tau)\varphi_2(\tau)d\tau$ is $\psi_2(\alpha) = \psi_1(\alpha)\psi(\alpha)$.

THEOREM 4.3.2. *If $x(t)$ is such that for its covariance function $R(u, v) = E\{x(u)\overline{x(v)}\}$ there exists an absolutely convergent Fourier representation*

$$(4.3.2) \quad R(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(\alpha u - \beta v)} d\Gamma(\alpha, \beta),$$

$$(4.3.3) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |d\Gamma(\alpha, \beta)| = N^2 < \infty,$$

then $x(t)$ is V -bounded.

PROOF. We have

$$(4.3.4) \quad \left\| \int_{-\infty}^{\infty} \varphi(-t)x(t)dt \right\| = E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(-u)\overline{\varphi(-v)}x(u)\overline{x(v)}dudv \right\} \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(-u)\overline{\varphi(-v)}R(u, v)dudv,$$

and if we insert (4.3.2) we obtain for this

$$(4.3.5) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\alpha)\overline{\psi(\beta)}d\Gamma(\alpha, \beta) \leq N^2 \cdot \sup_{\alpha} |\psi(\alpha)|^2,$$

as claimed. The functions of theorem 4.3.2 were introduced by M. Loève, under the name "harmonizable."

THEOREM 4.3.3. *Every K -stationary function is V -bounded.*

PROOF. Every K -stationary function is harmonizable, and the set function $\Gamma(S^a, S^b)$ occurring in (4.3.2) is concentrated on the diagonal line $\alpha = \beta$ for it.

4.4. THEOREM 4.4.1. (i) *If we are given a K -stationary function $y(t)$ with an expansion (2.4.3), and if there exists a weak solution $x(t)$ of*

$$(4.4.1) \quad \Lambda x = y$$

which is V -bounded, then we have

$$(4.4.2) \quad \int_{W-Q} \frac{dB(\alpha)}{|T(\alpha)|^2} < \infty,$$

whatever the point set Q on which $T(\alpha)$ is zero. Therefore there exists a K -stationary function with the expansion

$$(4.4.3) \quad x^0(t) \sim \int_{W-Q} e^{2\pi i \alpha t} \frac{dF(\alpha)}{T(\alpha)}$$

and we have

$$(4.4.4) \quad \Lambda x^0 = y^0$$

weakly, where

$$(4.4.5) \quad y^0 \sim \int_{W-Q} e^{2\pi i \alpha t} dF(\alpha).$$

In other words, if in the equation (4.4.1) for a K -stationary right side $y(t)$ there exists a solution $x(t)$ which is V -bounded, then after omitting from the spectrum of $y(t)$ those frequencies which are in resonance with the zeros of $T(\alpha)$ there exists a solution x^0 which is likewise K -stationary.

PROOF. The theorem is essentially a remark to theorem 3.4.1. In relation (3.4.2) we restrict the function $\psi(\alpha)$ there occurring to be $|\psi(\alpha)| \leq 1$, and we take the maximum for all $\psi(\alpha)$ thus admissible. It is easily seen that we have

$$(4.4.6) \quad \sup_{|\psi(\alpha)| \leq 1} \int_{W-Q} \frac{|\psi(\alpha)|^2}{|T(\alpha)|^2} dB(\alpha) = \int_{W-Q} \frac{dB(\alpha)}{|T(\alpha)|^2}.$$

On the other hand, by the very definition 4.1.1 we have for $|\psi(\alpha)| \leq 1$,

$$(4.4.7) \quad \left\| \int_{-\infty}^{\infty} \varphi(-t) x(t) dt \right\|^2 \leq N^2,$$

and thus (4.4.6) is $\leq N^2$, which verifies (4.4.2). The second part of the theorem is then obvious.

In theorem 4.4.1, the point set Q on which $T(\alpha)$ vanishes was unrestricted, but if Q consists of isolated points only then y^0 must of necessity be y itself. In fact, if we put $y^1 = y - y^0$, $x^1 = x - x^0$, then theorem 3.3.4 can be applied, and the following conclusion ensues.

THEOREM 4.4.2. *If $T(\alpha)$ has countably many zeros at most, and if for a K -stationary function $y(t)$ there exists a weak V -bounded solution $x(t)$ of $\Delta x(t) = y(t)$, then for the same $y(t)$ there exists another solution $x^0(t)$ which is K -stationary itself.*

If $x(t)$ is uniformly continuous also, then the difference $x^1(t) = x(t) - x^0(t)$ is almost periodic with an expansion (3.3.3), and if $T(\alpha)$ has finitely many zeros only, then the assumption of uniform continuity need not be made.

We remark that $x(t)$ may indeed fail to be K -stationary itself. In fact, if to a K -stationary $x^0(t)$ we add a finite trigonometric sum as introduced in theorem 4.3.1, then V -boundedness is preserved, but the K -stationarity may be disturbed indeed.

Furthermore, it follows from what we have stated in section 3.5 that theorems 4.4.1 and 4.4.2 both carry over to several variables. But theorem 4.4.1 does it easily and universally, whereas theorem 4.4.2 does it more reluctantly, and because of the Euclidean character only.

5. CT-boundedness

5.1. DEFINITION 5.1.1. For any B , a function $x(t)$ is *CT-bounded* ("CT" for "continuous and totally") if it is continuous and if every infinite sequence of real numbers $\{h'_m\}$ contains an infinite subsequence $\{h_n\}$ such that the sequence of functions

$$(5.1.1) \quad x_n(t) \equiv x(t + h_n)$$

is convergent towards a function $x^*(t)$ uniformly in every finite interval $-t_0 \leq t \leq t_0$.

We note that a complex function $x(t)$ is *CT-bounded* if (and only if) it is uniformly continuous and bounded in the ordinary sense, as is well known. For any B , the ordinary boundedness is to be replaced by the requirement that the range of values in B has a compact closure, but this characterization will not enter our context.

DEFINITION 5.1.2. For any B , a function $x(t)$ is *almost periodic* (in the sense of H. Bohr) if it is continuous and if every infinite sequence $\{h'_m\}$ contains an infinite subsequence $\{h_n\}$ for which the sequence of functions (5.1.1) is convergent uniformly in the entire infinite interval $-\infty < t < \infty$.

This is not Bohr's original definition but an equivalent one due to ourselves. The following statement is a principal theorem which however for certain conclusions need not be invoked.

THEOREM 5.1.1. *A function $x(t)$ satisfies definition 5.1.2 if and only if it is a limit, as $p \rightarrow \infty$, uniformly in $-\infty < t < \infty$, of finite exponential sums*

$$(5.1.2) \quad \sum_n a_n^p e^{2\pi i a_n t}$$

for some real $\{a_n\}$.

Definitions 5.1.1 and 5.1.2 readily generalize to several variables, and theorem 5.1.1 holds then also, if, as usual, $a_n t$ is replaced by (a_n, t) .

5.2. We recall the definition of Λx and $T(a)$, and we also recall that $T(a)$ is always $C^{(2)}$, and, for k variables $a = (a^1, \dots, a^k)$, more generally $C^{(k+1)}$.

THEOREM 5.2.1. *For any B , in one or several variables $t = (t^1, \dots, t^k)$, if Λx is such that $T(a)$ has countably many zeros $\{a_n\}$ at most; if $y(t)$ is almost periodic, and $x(t)$ is CT-bounded; and if we have*

$$(5.2.1) \quad \Lambda x = y$$

weakly; then $x(t)$ is likewise almost periodic.

For complex numbers and one variable we proved this theorem in [22] for an operator Λ which was there specified to be a difference-differential expression of the form (2.2.1). On the other hand, for any B and several variables we have proved in [23] a theorem which after minor adjustments of wording subsumes the present theorem for a Λx having the property that any weak CT-bounded solution of the homogeneous equation $\Lambda x = 0$ is of necessity a constant, $x(t) = x(t_0)$. For one variable this applies then in particular, by a theorem of P. Bohl, a forerunner of H. Bohr, and H. Bohr himself, to the operator

$$(5.2.2) \quad \Lambda x \equiv \frac{dx}{dt},$$

and more generally to $\Lambda x = d^p x/dt^p$, $p = 1, 2, \dots$, say. And for several variables it can be shown that this is so for the Laplacian

$$(5.2.3) \quad \Lambda x \equiv \frac{\partial^2 x}{\partial (t^1)^2} + \dots + \frac{\partial^2 x}{\partial (t^k)^2}.$$

All these are very interesting and satisfactory particular cases of theorem 5.2.1, and we shall not give here a proof of the entire theorem in the general version stated. But we note that the subsequent theorem which is based on the previous one is not rendered uncertain by this, because we may assume added to theorem 5.2.2 the restriction that the operator Λx occurring in it shall be one to which theorem 5.2.1 is known to apply.

THEOREM 5.2.2. *In one or several variables, if $T(a)$ has a finite number of zeros $\{a_n\}$ at most; if $y(t)$ is almost periodic and K -stationary, and $x(t)$ is CT-bounded, and if we have (5.2.1) weakly, then there is another solution $x^0(t)$ which is both almost periodic and K -stationary.*

Also, if we introduce the Fourier expansion

$$(5.2.4) \quad y(t) \sim \sum b_m e^{2\pi i (\beta_m, t)}, \quad b_m \neq 0,$$

then each β_m must be different from each a_n , and

$$(5.2.5) \quad x^0(t) \sim \sum \frac{b_m}{T(\beta_m)} e^{2\pi i (\beta_m, t)},$$

for instance.

PROOF. For general B , and countably many $\{a_n\}$, $x(t)$ is almost periodic by theorem 5.2.1 and if we introduce the expansion

$$(5.2.6) \quad x(t) \sim \sum c_m e^{2\pi i(\gamma_m, t)}$$

then we have

$$(5.2.7) \quad y(t) \sim \sum T(\gamma_m) c_m e^{2\pi i(\gamma_m, t)}.$$

But since $T(\gamma_m) = 0$ if γ_m is an a_n , it follows that each γ_m must be different from each a_n .

Next, if there is only a finite number of zeros $\{a_n\}$, it follows that we can put

$$(5.2.8) \quad x(t) = x^0(t) + \sum d_n e^{2\pi i(a_n, t)}$$

where $x^0(t)$ is an almost periodic solution of $\Delta x^0 = y$ whose expansion is (5.2.5).

A function $y(t)$ is K -stationary and almost periodic if it is a K -stationary function whose spectrum is purely discontinuous; and if (5.2.4) is its expansion then

$$(5.2.9) \quad E\{b_m \bar{b}_n\} = 0 \text{ if } m \neq n, \quad \text{and} \quad \sum_m \|b_m\|^2 < \infty.$$

Conversely any expansion (5.2.4) whose coefficients are of this kind is the Fourier series of a function which is K -stationary and almost periodic.

Therefore, in order to prove our theorem we only have to verify that we have

$$(5.2.10) \quad \|x^0(0)\|^2 \equiv \sum_m \frac{\|b_m\|^2}{|T(\beta_m)|^2} < \infty.$$

For a finite expansion (5.2.4) this is trivial. However, we must extend this to an infinite expansion (5.2.4), but without the benefit of V -boundedness by means of which this was achieved in section 4.4.

By the theory of approximation for almost periodic functions, for any sequence of exponents $\{\beta_m\}$ there exists a matrix of real numbers $\{\lambda_{rm}\}$, $r = 1, 2, 3, \dots$; $m = 1, 2, 3, \dots$, having the following properties: for each r only finitely many λ_{rm} are $\neq 0$; $|\lambda_{rm}| \leq 1$; $\lim_{m \rightarrow \infty} \lambda_{rm} = 1$; and the functions

$$(5.2.11) \quad x_r^0(t) = \sum_m \lambda_{rm} \frac{b_m}{T(\beta_m)} e^{2\pi i(\beta_m, t)},$$

$$(5.2.12) \quad y_r(t) = \sum \lambda_{rm} b_m e^{2\pi i(\beta_m, t)}$$

are uniformly convergent to $x^0(t)$ and $y(t)$ respectively. In particular therefore

$$(5.2.13) \quad \lim_{r \rightarrow \infty} x_r^0(0) = x^0(0).$$

Now, by what we have already stated we have

$$(5.2.14) \quad \sum_m |\lambda_{rm}|^2 \frac{\|b_m\|^2}{|T(\beta_m)|^2} = \|x_r^0(0)\|^2,$$

and by letting $r \rightarrow \infty$ we obtain (5.2.10), as claimed.

6. $L_2, 2$ -boundedness

6.1. If $X(\Delta)$, $\Delta \equiv \Delta_{\rho\sigma}: \{\rho < t \leq \sigma\}$, $-\infty < \rho < \sigma < \infty$, is a finitely additive interval function with values in any vector space then for a numerical function $\varphi(t)$ the Stieltjes integral

$$(6.1.1) \quad \int_{-\infty}^{\infty} \varphi(t) dX(t)$$

can always be set up, as a finite sum

$$(6.1.2) \quad \sum_{m=1}^n \varphi_m X(\Delta^m),$$

for any function $\varphi(t)$ which has constant values $\{\varphi_m\}$ on finitely many nonoverlapping intervals $\{\Delta^m\}$, and is 0 on the complementary set. Now, if the vector space is H , then as in [3] (see p. 153), and in [1], we now define as follows.

DEFINITION 6.1.1. We say that $X(\Delta)$ is $L_2, 2$ -finite if there is a finite constant M , such that for the functions $\varphi(t)$ just mentioned we have

$$(6.1.3) \quad \left\| \int_{-\infty}^{\infty} \varphi(t) dX(t) \right\|^2 \leq M^2 \cdot \int_{-\infty}^{\infty} |\varphi(t)|^2 dt.$$

If this is so, then the very inequality permits us to extend the definition of the integral (6.1.1) to all functions $\varphi(t)$ in $L_2(-\infty, \infty)$ and the inequality (6.1.3) remains in force then.

If there is a numerical finitely additive interval function $A(\Delta)$ such that $X(\Delta) = x_0 \cdot A(\Delta)$, for some $x_0 \in H$, then (6.1.3) holds if and only if $A(\Delta)$ is absolutely continuous, so that $A(\Delta) = \int_{\rho}^{\sigma} a(t) dt$, and the function $a(t)$ belongs to $L_2(-\infty, \infty)$ itself. But in general the function $X(\Delta)$ need be neither an indefinite integral nor, in any manner, small at infinity, a so-called "Wiener process" being a case in point (see p. 153 in [3]).

The Plancherel theorem for numerical functions $\varphi(t)$ dualizes into a theorem for $L_2, 2$ -bounded interval functions $X(\Delta)$ in the following manner. With each such function $X(\Delta)$ on the t -axis there is associable one-one a such-like function $E(\Delta)$ on the α -axis, in symbols

$$(6.1.4) \quad \frac{dX(t)}{dt} \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} dE(\alpha),$$

such that for any numerical L_2 -function $\varphi(t)$ and its Plancherel transform (4.1.3) we have

$$(6.1.5) \quad \int_{-\infty}^{\infty} \varphi(-t) dX(t) = \int_{-\infty}^{\infty} \psi(\alpha) dE(\alpha),$$

the integrals being as just defined.

We now take another such function

$$(6.1.6) \quad \frac{dY(t)}{dt} \sim \int_{-\infty}^{\infty} e^{2\pi i \alpha t} dF(\alpha),$$

and we assume that $X(\Delta)$ and $Y(\Delta)$ are connected by the equation

$$(6.1.7) \quad \Lambda X = Y$$

in the following sense, suited to the occasion. If $T(\alpha)$ is the generator of the operator Δ then we have

$$(6.1.8) \quad \int_{-\infty}^{\infty} \psi(\alpha) T(\alpha) dE(\alpha) = \int_{-\infty}^{\infty} \psi(\alpha) dF(\alpha),$$

the integrals being as previously defined. We wish to point out that $T(\alpha)$ is bounded and that therefore $\psi(\alpha)T(\alpha)$ is an L_2 -function if $\psi(\alpha)$ is one.

6.2. We are now introducing K -stationarity into the context, the definition being as follows [3], [1].

DEFINITION 6.2.1. An L_2, \mathfrak{z} -bounded function $X(\Delta)$ is K -stationary if, for any L_2 -function $\varphi(t)$, the "second moment"

$$(6.2.1) \quad E \left[\int_{-\infty}^{\infty} \varphi(u+t) dX(u) \int_{-\infty}^{\infty} \overline{\varphi(v+t)} d\overline{X(v)} \right]$$

is independent of t .

If $E(\Delta)$ is the transform of $X(\Delta)$ then this is equivalent with the following properties. We again have

$$(6.2.2) \quad E\{E(\Delta^1) \cdot \overline{E(\Delta^2)}\} = 0 \quad \text{for } \Delta_1 \cap \Delta_2 = 0$$

and the nonnegative (additive) interval function

$$(6.2.3) \quad A(\Delta) = E\{|E(\Delta)|^2\}$$

is the indefinite integral of a density which is likewise bounded,

$$(6.2.4) \quad A(\Delta) = \int_{\rho}^{\sigma} b(\alpha) d\alpha, \quad |b(\alpha)| \leq N.$$

We note that $E(\Delta)$ is continuous in the endpoints of the interval Δ , among other properties.

Now, property (6.2.4) produces a great simplification of the equation (6.1.7) or the equation (6.1.8) alternately. If we write (6.1.8) in the form

$$(6.2.5) \quad T(\alpha) dE(\alpha) = dF(\alpha),$$

then this can be inverted to

$$(6.2.6) \quad dE(\alpha) = \frac{1}{T(\alpha)} dF(\alpha)$$

in the following sweeping manner.

THEOREM 6.2.1. *If in equation (6.1.7) the function $Y(\Delta)$ is L_2, \mathfrak{z} -bounded and K -stationary and $X(\Delta)$ is L_2, \mathfrak{z} -bounded, then $X(\Delta)$ is also K -stationary.*

Note that the assertion of the theorem refers to $X(\Delta)$ itself and not to some other L_2, \mathfrak{z} -bounded solution $X^0(\Delta)$ of the equation. In fact, if $X^1(\Delta)$ is any L_2, \mathfrak{z} -bounded solution of the homogeneous equation $X^1(\Delta) = 0$ then $X^1(\Delta) \equiv 0$.

REFERENCES

- [1] S. BOCHNER, "Boundedness and stationarity of time series," *Proc. Nat. Acad. Sci.*, Vol. 40 (1954), pp. 289-294.
- [2] E. SLUTSKY, "Sur les fonctions aléatoires presque périodiques et sur la décomposition des fonctions aléatoires stationnaires en composantes," *Actualités Scientifiques et Industrielles*, No. 738 (1938), pp. 33-55.
- [3] S. BOCHNER, *Harmonic Analysis and the Theory of Probability*, Berkeley and Los Angeles, University of California Press, 1955.

- [4] ———, "Fourier transforms of time series," *Proc. Nat. Acad. Sci.*, Vol. 39 (1953), pp. 302–307.
- [5] H. WOLD, *A Study in the Analysis of Stationary Time Series*, Uppsala, Almqvist and Wiksells, 1938.
- [6] K. KARHUNEN, "Lineare Transformationen stationärer stochastischer Prozesse," *Den 10. Skandinaviske Matematiker Kongres i København, 1946*, København, Jul. Gjellerups, 1947, pp. 320–324.
- [7] K. NAGABHUSHANAM, "The primary process of a smoothing relation," *Arkiv för Matematik*, Vol. 1 (1951), pp. 421–488.
- [8] K. KARHUNEN, "Über lineare Methoden in der Wahrscheinlichkeitsrechnung," *Ann. Acad. Scient. Fennicae*, Ser. A, I, No. 37 (1947), pp. 1–79.
- [9] J. L. DOOB, *Stochastic Processes*, New York, John Wiley and Sons, 1953.
- [10] S. BOCHNER, "Linear partial differential equations with constant coefficients," *Annals of Math.*, Vol. 47 (1946), pp. 202–212.
- [11] S. BOCHNER and W. T. MARTIN, *Several Complex Variables*, Princeton, Princeton University Press, 1948.
- [12] K. ITÔ, "Stationary random distributions," *Mem. Coll. Sci., Univ. of Kyoto*, Ser. A, Vol. 28 (1953), pp. 209–223.
- [13] S. BOCHNER, "Darstellung reellvariabler und analytischer Funktionen durch verallgemeinerte Fourier- und Laplace-Integrale," *Math. Annalen*, Vol. 97 (1927), pp. 635–662.
- [14] ———, *Vorlesungen über Fouriersche Integrale*, New York, Chelsea Publishing Company, 1948.
- [15] J. BLACKMAN, "The inversion of the generalized Fourier transform by Abelian summability," *Trans. Amer. Math. Soc.*, Vol. 78 (1955), pp. 19–29.
- [16] L. SCHWARTZ, "Analyse et synthèse harmoniques dans les espaces de distributions," *Can. Jour. Math.*, Vol. 3 (1951), pp. 503–512.
- [17] S. BOCHNER, "A theorem on Fourier-Stieltjes integrals," *Bull. Amer. Math. Soc.*, Vol. 40 (1934), pp. 271–276.
- [18] I. J. SCHOENBERG, "A remark on the preceding note by Bochner," *Bull. Amer. Math. Soc.*, Vol. 40 (1934), pp. 277–278.
- [19] R. S. PHILLIPS, "On Fourier-Stieltjes integrals," *Trans. Amer. Math. Soc.*, Vol. 69 (1950), pp. 312–323.
- [20] H. HELSON, "Isomorphisms of Abelian group algebras," *Arkiv för Matematik*, Vol. 2 (1953), pp. 475–487.
- [21] S. HELGASON, "The derived algebra of a Banach algebra," *Proc. Nat. Acad. Sci.*, Vol. 40 (1954), pp. 994–995.
- [22] S. BOCHNER, "Über gewisse Differential- und allgemeinere Gleichungen deren Lösungen fastperiodisch sind. I, II, III," *Math. Annalen*, Vol. 102 (1929), pp. 489–504; Vol. 103 (1930), pp. 588–597; Vol. 104 (1931), pp. 579–587.
- [23] ———, "Abstrakte fastperiodische Funktionen," *Acta Math.*, Vol. 61 (1933), pp. 149–184.