HYDRODYNAMICAL DESCRIPTION OF STELLAR MOTIONS

L. G. HENYEY
UNIVERSITY OF CALIFORNIA

1. Introduction

In the study of the structure of our galaxy and of the stars contained in it one encounters the problem of the space and velocity distributions of stars belonging to various sets. Of greatest theoretical interest are those investigations which deal with sets whose members are selected on the basis of such intrinsic attributes as luminosity or spectral characteristics. The general concern is with the properties of a function

$$\psi\,(x,\,y,\,z;\,\dot{x},\,\dot{y},\,\dot{z};\,t)$$

giving the number of stars, at time t, in the neighborhood of point x, y, z and velocity $\dot{x}, \dot{y}, \dot{z}$, reckoned per unit volume and per unit range of velocities, which belong to a particular set. The function ψ can and does depend on the set criteria.

Here we are not directly concerned with the methods that have been used to determine the nature of such distributions from statistical treatments of observed characteristics. It is sufficient to point out that our present knowledge is largely confined to the neighborhood of the sun and has been derived only after involved discussions of the effects of observational and sampling errors. Moreover the time interval over which observations have been made is so brief as to be hopelessly inadequate for revealing any significant changes.

In this paper we shall be concerned with the dynamical theory used in the study of the function ψ and, particularly, with the specific methods that this theory may employ. We consider first a brief summary of the main kinematical features that observations reveal, secondly a statement of the basic theoretical formulation of the problems to be considered, thirdly an account of methods that have been utilized, and finally a proposed modification in the mode of attack on the basic problem. This final section uses a representation of the statistics of stellar motions in what is substantially a hydrodynamical scheme.

Before proceeding with these points we must note that the problems encountered in the study of our galaxy are paralleled by similar problems for other galaxies. While the observational techniques are different, the theoretical problems are similar. Moreover, facts discovered for these objects bear directly on the study and interpretation of our own system.

2. Summary of kinematical characteristics

The general form of the function $\psi(x, y, z; \dot{x}, \dot{y}, \dot{z}; t)$ implies the possibility of defining, for each point in space and time, an average velocity. Let

$$(1)\qquad n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi\,(x,\,y,\,z;\,\dot{x},\,\dot{y},\,\dot{z};\,t)\,d\dot{x}\,d\dot{y}\,d\dot{z}$$

be the number of stars per unit volume. Then the averages U, V, W, are defined by the typical expression

$$nU = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{x}\psi \, d\dot{x} \, d\dot{y} \, d\dot{z} \,. \tag{2}$$

That the observed velocities of stars in the solar neighborhood permit of the evaluation of such an average is an important verification of the whole concept of a distribution function. The velocity with components U, V, W we shall term the local velocity centroid.

It is convenient to refer all the velocities of the stars under consideration to this centroid. Let

$$\dot{x} = U + u, \quad \dot{y} = V + v, \quad \dot{z} = W + w \,. \tag{3}$$

The velocity (u, v, w) defined in this fashion for a star is known as its peculiar motion with respect to the local centroid.

As an example we may consider the sun's peculiar motion. Since in the first instance we measure motions with respect to a coordinate system in which the sun is at rest, the sun's peculiar motion is simply the reflex of the centroid motion in the coordinate system.

It is convenient to investigate the statistical nature of stellar motions in terms of peculiar motions. Accordingly we define a new function φ giving the distribution in this scheme:

$$\varphi(x, y, z; u, v, w; t) = \psi(x, y, z; U + u, V + v, W + w; t) \,. \tag{4}$$

The observational results may now be summarized in terms of facts (a) about U, V, W and (b) about the distribution of u, v, w.

(a) Observed stellar motions are not sufficiently extensive to provide the means for mapping U, V, W over the entire galaxy but they are sufficient to reveal first order trends in the solar neighborhood. If ΔU, ΔV, ΔW are components of the centroidal motion at a point, not too far from the sun, as referred to the local centroid, then we have

$$\Delta U = \frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta y + \frac{\partial U}{\partial z}\Delta z \tag{5}$$

with similar expressions for the other two components. Here Δx, Δy, Δz are the displacements of the point in question from the sun. We do not consider second order terms because the observations do not provide any definite information about them.

An analysis of stellar motions reveals that the pattern of these differential effects has a very special form. The results are compatible with, and in fact suggest, the notion that the stars are moving with a rotational motion about the center of the galaxy with the rotational speed decreasing outward. If certain other data are used to implement these results, we may summarize in numerical form as follows:

Linear speed of rotation.	275 km/sec,
Gradient in rotational speed.	0.005 km/sec/parsec[1],
Distance of sun from center.	9000 parsecs.

[1] A parsec is the distance at which the semimajor axis of the earth's orbit around the sun subtends one second of arc ($\approx 3.1 \times 10^{13}$ km).

(b) The nature of the function φ, giving the distribution of stellar peculiar velocities, may be studied observationally only for the immediate neighborhood of the sun. This ranges from a few hundred to a few thousand parsecs depending on the class of objects considered.

In a general way the velocities exhibit a high degree of randomness with, however, some preference for a particular direction in space. A number of analytic representations have been proposed for the frequency function. Of these the most satisfactory appears to be that of a generalized three dimensional normal distribution. For this case the function defined by (4) assumes the following form:

$$\varphi = n \left(\frac{\operatorname{Det} Q}{8\pi^3}\right)^{1/2} e^{-Q/2} \tag{6}$$

where Q is a quadratic function of the variables u, v, w. For numerical purposes it is preferable to consider the inverse quadratic form Q^{-1} which gives the mobility in various directions directly. It is represented by an ellipsoid whose semiaxes are the standard deviations of the velocities from the centroid.

Since the numerical results depend on the classes of stars that may be considered, any summary that we may consider here must necessarily be limited to such as gives the main features only. It is found that the axis of greatest mobility has a direction falling in the galactic plane and pointing nearly in the direction of the galactic center, deviating only by about 15 degrees. It is believed that this deviation is probably real and not a result of random or systematic errors. The deviation is in the sense that stars moving in toward the center tend slightly to prefer peculiar motions in the sense of the galactic rotation rather than the reverse. The standard deviation in this direction has an approximate value of 24 km/sec. The other two axes are nearly equal although most recent investigations attempt to distinguish between them. While, as a result of the near equality, the directions are somewhat indeterminate it may be said that the statistical results are not incompatible with the picture of one axis in the galactic plane and one perpendicular to it. The deviations are around 15 km/sec with the one in the galactic plane, possibly larger by one or two kilometers per second.

To the extent that the law given by (6) may be extrapolated to other regions of the galaxy and into the past and future, the parameters entering it must be functions of position and time.

The closeness with which the law represents the observations may be considered as satisfactory to the extent that we may interpret the deviations from it as arising from the relative smallness of each of the samples used in its study. This belief is probably justified for the majority of the stars, namely those possessing peculiar velocities not excessively large. On the other hand the law fails for the stars of high velocity in that they display a pronounced asymmetry amounting to a complete avoidance of one hemisphere of directions at the very highest speeds. A qualitative interpretation of this phenomenon is apparent as soon as one notes that the avoided directions are precisely those that would add numerically to the speed of galactic rotation. Hence with respect to the galaxy as a whole the speeds would be very large and, above a certain level, would exceed the critical speed for escape from the system. Stars possessing such velocities could only have come from outside or have attained their status through a close encounter with another object. The remoteness of either possibility provides the explanation of the avoidance.

In the other case, namely that in which the peculiar velocity numerically subtracts from the speed of galactic rotation, we have stars that are in reality slow moving. These must have come from the region of the galactic center and are presumably destined to fall back into that neighborhood. One of the chief sources of interest in these objects is the indication that they possess physical characteristics, probably basically connected with their chemical composition, distinguishing them from stars of low peculiar velocities. These facts make their study much more significant than their relatively low numerical frequency in the solar vicinity might seem to indicate. Some investigators have suggested that their motions should be represented statistically by considering that they constitute a separate stream and that a significant point of view is to regard the function ψ as consisting of two separable parts: one for stars of ordinary velocities and one for these objects. All this is in the spirit of Baade's suggestion of the existence of two distinct types of stellar population in our galaxy. Recent results have tended to minimize the complete distinctness of the two extremes, providing some evidence of a more or less continuous transition between them.

3. The dynamical problem

The dynamical study of the structure and motions of our galaxy, and of other similar systems, proceeds mainly with a consideration of the limitations imposed by general dynamical laws on the statistical scheme used in the representation of the significant features suggested by observations. One of its primary tasks is to identify the nature of the forces acting and to clarify their main characteristics.

Existing evidence indicates that the nature of the forces is basically gravitational. The source of the force modifying the motion of a single star is to be found in the gravitational fields contributed by the other stars and by the interstellar gas and dust permeating large parts of many galactic systems. In the main, the effects produced by most of this matter can be regarded as those arising from a smoothed distribution of matter, since the precise disposition of a star out of 10^{11} is of little consequence. An exception must be noted for the chance encounters of stars in which a close approach may contribute, at least cumulatively, to a significant alteration in the state of motion of the individuals. Actually these effects are of negligible importance. Various ways of representing this fact are available. The time of relaxation for the observed density of stars near the sun is of the order of 10^{14} years while the period of rotation is of the order of 10^{8} years. Moreover the life of the galaxy is thought to be something like 3×10^{9} years. The mean free path of a star is a very large multiple ($\sim$ several thousand) of the diameter of the system [1].

As a consequence of the primary importance of the smoothed gravitational field we may regard the force acting on a star as being given by a general potential field, depending possibly on time. The components of the forces derived from this we shall represent by (X, Y, Z), all functions of x, y, z and t. It may then be shown

[1] that the function ψ is subject to the condition:

$$\frac{\partial \psi}{\partial t}+\dot{x}\,\frac{\partial \psi}{\partial x}+\dot{y}\,\frac{\partial \psi}{\partial y}+\dot{z}\,\frac{\partial \psi}{\partial z}+X\,\frac{\partial \psi}{\partial \dot{x}}+Y\,\frac{\partial \psi}{\partial \dot{y}}+Z\,\frac{\partial \psi}{\partial \dot{z}}=0\,. \tag{7}$$

This expression differs from the Maxwell-Boltzmann equation of gas kinetic theory in the absence of the collisional term.

To complete the basic requirements of the theory we must next include Poisson's equation or its equivalent to provide the basis for computing the force. This requires a knowledge of the distribution of stellar masses and of the density of interstellar matter. This latter in turn requires a suitable dynamic theory of interstellar matter. Since the properties of interstellar matter are only partially understood at present, the complete theory is yet to be formulated. In the present paper we shall confine our attention to a discussion of (7).

4. Theoretical methods

We consider here one important type of solution of (7), which has been studied in various forms and degrees of generality or rigor since the early part of the present century. We present it in its most general form, indicate important deductions from it in special cases, and finally discuss its deficiencies.

Observations of stellar motions indicate that the distribution of stellar motions is nearly normal in the vicinity of the sun at least. This fact suggests that we should inquire concerning the general conditions under which (7) admits of a solution of such a type. In effect we are regarding (7) as an equation for determining the potential function. As has been pointed out [1], after substituting the appropriate partial derivatives of the potential for the components of the force we obtain a first order linear nonhomogeneous equation for the potential.

An alternative, but equivalent, approach to the problem can be presented. The differential equations of the characteristics of (7), regarded as an equation for ψ, are precisely the equations of motion of a star in the force field given by X, Y, Z. Hence ψ must be a first integral of these equations of motion, or, more generally, any function of ψ must be such a first integral. For example, 2 log ψ is an integral. In this case it is readily seen that we are demanding that the motion admit of a first integral which is a second degree polynomial in the velocities. The integral may contain linear terms and a term independent of the velocities, in addition to the quadratic ones. The most general formulation of the problem requires the determination of the conditions on the potential such that the motions admit of a quadratic first integral. It should be noted that if either, or both, of the integrals of energy and angular momentum exist then a solution can readily be constructed from them. However, it would be too special and we desire more general ones.

It is not our present intention to consider the details of the subsidiary conditions to which the general formulation leads. For these, reference should be made to Chandrasekhar's monograph [1]. The most general form of the problem has not been completely solved although some phases of it have. With additional assumptions concerning either the integral or the geometrical symmetry of the potential even more progress has been possible, with complete solutions achieved for special cases.

Two important applications of special aspects of the problem have been made to galactic structure. The first of these deals with the observed ratio of the axes of the ellipsoid which fall in the galactic plane, while the second provides a basis for studying the density of matter in the galaxy from the distribution of velocities in the direction perpendicular to the galactic plane.

Under certain conditions, particularly involving the assumption of axial symmetry for the potential function, it is found that the ratio of the axes mentioned must be equal to a quantity computed from the local characteristics of the galactic rotation. If r is the distance from the galactic center and Θ is the linear speed of rotation, it turns out that the ratio squared is given by

$$\frac{1}{2}\left(\frac{r}{\Theta}\frac{d\Theta}{dr}+1\right). \tag{8}$$

Using the observed values, one obtains a reasonably good check. However, one difficulty remains. An observed characteristic is the slight tilt of the long axis with respect to the direction to the galactic center. The theory can explain this provided a gradual expansion of the galaxy is admissible, and requires that a slow systematic recession of stars from the sun, in amount linear with distance, be present. There is, at present, no clear verification of this requirement. Admittedly the observational result is also open to some doubt.

The second application is based on the supposition that the motions perpendicular to the galactic plane are governed chiefly by the local concentration of matter in the system. Under the assumption that these velocities have a normal distribution Oort [2] has been able to demonstrate a connection with the density and uses the theory to get a numerical estimate of the latter. Recently [3] the assumption of normality has been criticized and some evidence for the existence of preferential perpendicular velocities has been presented.

The desirability of a complete exploration of the theoretical problem discussed in the preceding paragraphs is, no doubt, apparent. Nevertheless certain criticisms of the whole approach must be made on the basis of its inherent inadequacy as a physical theory.

In the first place it must be noted that its strict adherence to a normal distribution does not enable it to cope with the problem of the so called high velocity stars. There is good reason to believe that these objects represent a basically important aspect of the galactic system and that their consideration is decisive to a profound study of galactic structure.

From a theoretical point of view one must note the implausibility of the supposition that the complex phenomena represented by the behavior of interstellar matter, and the determination of the gravitational field by the actual distribution of matter, conveniently adjust themselves to lead to just the right force field for precisely maintaining an exact normal distribution of velocities. It does not appear unreasonable to expect that even slight deviations from such ideal conditions may be in the essence of the over all behavior of the galaxy. The very least that one may ask is: why do we have a nearly normal distribution of velocities in the solar vicinity?

In view of these remarks the desirability for some extension in the theoretical mode of treating the statistical representation of stellar motions becomes apparent. We examine one possibility here and propose its further consideration in the study of galactic structure.

Statistical investigations of distributions of various types often revolve around a consideration of the various moments that are defined by the properties investigated. One uses the moments to describe the implied distribution function, at least in an approximate way. A natural basis for the treatment of equation (7) is, therefore, through the limitations it imposes on the various moments for velocity as defined by the distribution function ψ. Such an approach provides a convenient method for developing an approximate theory in which any required deviation for normalness may be handled. In particular we wish to allow for some skewness but to neglect excess. The method may be extended to include coefficients of excess, with, however, considerable increase in complexity.

Negligible excess may be expressed numerically as the vanishing of the fourth order semi-invariants. It appears desirable, therefore, to transform equation (7) into one involving the logarithm of the characteristic function.

Let $T(x, y, z; t; \xi, \eta, \zeta)$ be defined by the expression:

$$(9) \qquad T = \log \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\xi\dot{x}+\eta\dot{y}+\zeta\dot{z}}\psi(x, y, z; \dot{x}, \dot{y}, \dot{z}; t)\, d\dot{x}\, d\dot{y}\, d\dot{z}\,.$$

It may be shown, formally, that equation (7) implies that T must satisfy the differential equation:

$$(10) \qquad \frac{\partial T}{\partial t} + \sum\left(\frac{\partial^2 T}{\partial \xi\, \partial x} + \frac{\partial T}{\partial \xi}\frac{\partial T}{\partial x} - \xi X\right) = 0\,,$$

where the summation is carried simultaneously over all corresponding pairs of the variables x, y, z and ξ, η, ζ respectively.

We consider next the expansion of T into a series in powers of ξ, η, ζ. It assumes the following form:

$$(11) \qquad T = \log n + \sum_{\mu} U_\mu \xi_\mu + \sum_{\mu,\nu} b_{\mu\nu}\xi_\mu\xi_\nu + \frac{1}{6}\sum c_{\mu\nu\sigma}\xi_\mu\xi_\nu\xi_\sigma + \ldots\,,$$

where μ, ν and σ run over the set x, y, z or the set ξ, η, ζ. The presence of $\log n$ is necessitated by the fact that ψ, regarded as representing a distribution in the velocities conditional to definite values of x, y, z and t, is not a frequency function but contains n as an added factor. The first order terms contain the components of the velocity centroid. The quantities $b_{\mu\nu}$ and $c_{\mu\nu\sigma}$ are the second and third order moments respectively. The terms not explicitly shown are of order five and higher, in accordance with our supposition that the excesses are negligible.

The series may now be substituted into (10) and corresponding powers of ξ, η, and ζ isolated. This operation leads finally to the following set of expressions:

$$(12) \qquad \frac{1}{n}\frac{\partial n}{\partial t} + \sum_{\tau}\frac{1}{n}\frac{\partial}{\partial x_\tau}(n U_\tau) = 0\,,$$

$$(13) \qquad \frac{\partial U_\mu}{\partial t} + \sum_{\tau}\left(U_\tau \frac{\partial U_\mu}{\partial x_\tau} + \frac{1}{n}\frac{\partial}{\partial x_\tau}[n\, b_{\mu\tau}]\right) = X_\mu\,,$$

$$(14) \qquad \frac{\partial b_{\mu\nu}}{\partial t} + \sum_{\tau}\left(U_\tau \frac{\partial b_{\mu\nu}}{\partial x_\tau} + b_{\mu\tau}\frac{\partial U_\nu}{\partial x_\tau} + b_{\nu\tau}\frac{\partial U_\mu}{\partial x_\tau} + \frac{1}{n}\frac{\partial}{\partial x_\tau}[n\, c_{\mu\nu\sigma}]\right) = 0\,,$$

$$\frac{\partial c_{\mu\nu\sigma}}{\partial t}+\sum_{\tau}\left(U_{\tau}\frac{\partial c_{\mu\nu\sigma}}{\partial x_{\tau}}+b_{\mu\tau}\frac{\partial b_{\nu\sigma}}{\partial x_{\tau}}+b_{\nu\tau}\frac{\partial b_{\mu\sigma}}{\partial x_{\tau}}+b_{\sigma\tau}\frac{\partial b_{\mu\nu}}{\partial x_{\tau}}\right.$$
$$\left.+c_{\mu\nu\tau}\frac{\partial U_{\sigma}}{\partial x_{\tau}}+c_{\mu\sigma\tau}\frac{\partial U_{\nu}}{\partial x_{\tau}}+c_{\nu\sigma\tau}\frac{\partial U_{\mu}}{\partial x_{\tau}}\right)=0\,. \tag{15}$$

It is apparent that (12) and (13) represent the equations of continuity and of motion respectively, in a hydrodynamical sense. The tensor $(nb_{\mu\nu})$ represents the general stress. Equations (14) and (15) serve to determine the behavior of the remaining variables. The first three are rigorous while the last lacks terms involving the coefficients of excess.

These formulae constitute a generalization of the scheme defined by the requirement of a normal velocity distribution. This may appear intuitively obvious but may be verified by substituting $c_{\mu\nu\sigma} = 0$. The resultant expressions may then, after suitable transformation, be proved equivalent to the equations expressing the requirement of normalness in conjunction with the fundamental equation (7).

While a close parallel appears here with hydrodynamical concepts, too much emphasis on this point may be misleading. The relatively long mean free path of a star in the galaxy is indicative of the basic difference to be expected. Nevertheless some quasihydrodynamic phenomena may be expected to play a role. In a fundamental way the content of expression (14) and (15) is completely analogous to the concepts underlying the gas kinetic theories of viscosity, heat conduction, etc. There is here also considerable similarity to recent developments in the theory of turbulence, which involve consideration of moments.

It remains to consider a program for the study of these differential equations and their application to the problem of galactic structure. Probably the initial efforts should be directed to a study of systems possessing axial symmetry with primary attention to the case of a steady state, that is, no dependence on time.

REFERENCES

[1] S. Chandrasekhar, *Principles of Stellar Dynamics*, University of Chicago Press, Chicago, 1942.

[2] J. H. Oort, "The force exerted by the stellar system in the direction perpendicular to the galactic plane and some related problems," *Bull. Astr. Soc. Netherlands*, Vol. 6 (1932), pp. 249–287.

[3] J. Schilt, "The gravitational galactic force and the density of interstellar matter," *Astr. Jour.*, Vol. 55 (1950), pp. 97–111.

[4] W. M. Smart, *Stellar Dynamics*, Cambridge University Press, Cambridge, 1938.