SOME PROBLEMS ON RANDOM WALK IN SPACE

A. DVORETZKY AND P. ERDÖS
NATIONAL BUREAU OF STANDARDS

1. Introduction

Consider the lattice formed by all points whose coordinates are integers in d-dimensional Euclidean space, and let a point $S_d(n)$ perform a move randomly on this lattice according to the following rules: At time zero it is at the origin and if at any time $n - 1$ ($n = 1, 2, \ldots$) it is at some point S then at time t it will be at one of the $2d$ lattice points nearest S, the probability of it being at any specified one of those being $1/(2d)$. In 1921 G. Pólya [7] discovered the remarkable fact that a point moving randomly according to the rules explained above will, with probability 1, return infinitely often to the origin if $d \leqq 2$ while if $d > 2$ then it will, again with probability 1, wander off to infinity.

While the random walk on the line has been very extensively studied there were relatively few studies of random walk in the plane or in the space. In particular many problems arising in connection with the above mentioned results of Pólya have been completely neglected. This is somewhat unfortunate since these questions, besides being of intrinsic interest, also arise in certain physical and statistical investigations. In the present paper we study two asymptotic problems concerning random walk.

The first problem is concerned with the number of different lattice points through which the random walk path passes; it is studied in sections 2–5. The other problem is that of the rate with which a point walking randomly in d-space ($d \geqq 3$) escapes to infinity and is studied in section 6. The treatments of the two problems are independent.

We find that during the first n steps a random path in the plane passes in the average through approximately $\pi n/\log n$ different points, while in d-space ($d \geqq 3$) it passes through approximately $n\gamma_d$ different points (with $0 < \gamma_d < 1$). We also estimate the variance of the number of different points covered and show that not only weak but even strong laws of large numbers hold. The proof of the strong law in the plane (section 5) is considerably more difficult than in d-space for $d \geqq 3$ (section 4). We deliberately refrain from applying our methods to the same problem in one dimensional random walk (for some remarks on this problem see Erdös [2]).

In section 6 we characterize all monotone functions $g(n)$ having the property that, with probability 1,[1]

$$g(n)\sqrt{n} = o\,[\|S_d(n)\|] \qquad d = 3, 4, \ldots \tag{1.1}$$

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[1] o and O always refer to the relevant variable (usually n) tending to infinity.

with $\|S\|$ denoting the distance of S from the origin. A typical result of this kind is that in ordinary 3-space $g(n) = \log^{-1-\epsilon} n$ satisfies (1.1) for every $\epsilon > 0$ but not for $\epsilon = 0$.

In section 7 we make some remarks concerning generalizations and extensions of the preceding results.

2. Definitions and preliminary results

For $d = 2, 3, 4, \ldots$, let $P_1, \ldots, P_d$ be d mutually orthogonal unit vectors in d-dimensional Euclidean space. We denote by X_d the random vector which can assume any of the values $\pm P_1, \ldots, \pm P_d$ all with the same probability $1/(2d)$.

Let

$$X_d(1), X_d(2), \ldots, X_d(n), \ldots$$

be an infinite sequence of mutually independent random vectors each of which has the same distribution as X_d. We put

$$S_d(n) = X_d(1) + X_d(2) + \ldots + X_d(n), \qquad n = 1, 2, \ldots.$$

The sequence

$$S_d(1), S_d(2), \ldots, S_d(n), \ldots \tag{2.1}$$

is called a *random walk* or *path* in d-space.

Let $L_d(n)$, $n = 1, 2, \ldots$, be the number of *different* vectors among the first n terms of the sequence (2.1). $L_d(n)$ is a random variable representing the number of different points through which the path (2.1) passed during the first n steps. The random variable $L_d(n)$ is studied in this and the following three sections.

Let $\gamma_d(n)$, $n = 1, 2, \ldots$, be the probability that the n-th step take the path to a point through which it did not pass in any of the preceding $n - 1$ steps. Due to considerations of symmetry and independence we have[2]

$$\begin{aligned}
\gamma_d(n) &= P\{X_d(1) + \ldots + X_d(i) \neq X_d(1) + \ldots + X_d(n) \\
&\qquad\qquad \text{for } i = 1, \ldots, n-1\} \\
&= P\{X_d(n) + X_d(n-1) + \ldots + X_d(i+1) \neq \mathbf{0} \\
&\qquad\qquad \text{for } i = 1, \ldots, n-1\} \\
&= P\{X_d(1) + X_d(2) + \ldots + X_d(n-i) \neq \mathbf{0} \\
&\qquad\qquad \text{for } i = 1, \ldots, n-1\} \\
&= P\{X_d(1) + X_d(2) + \ldots + X_d(j) \neq \mathbf{0} \text{ for } j = 1, \ldots, n-1\} \\
&= P\{S_d(j) \neq \mathbf{0} \text{ for } j = 1, \ldots, n-1\}.
\end{aligned}$$

That is, $\gamma_d(n)$ is also the probability that the path does not pass during the first $n - 1$ steps through the origin. From this we immediately have

$$1 = \gamma_d(1) \geqq \gamma_d(2) \geqq \ldots \geqq \gamma_d(n) \geqq \gamma_d(n+1) \geqq \ldots > 0, \tag{2.2}$$

[2] Throughout the paper $P\{\ \}$ denotes the probability of the event within the braces. Similarly $E\{\ \}$ is the expected value of the random variable within the braces.

and

$$\gamma_d(2n-1) = \gamma_d(2n), \qquad n = 1, 2, \ldots . \tag{2.3}$$

Let $u_d(n) = P\{S_d(n) = \mathbf{0}\}$, that is, the probability that the n-th step take the path to the origin. Clearly

$$u_d(1) = u_d(3) = \ldots = u_d(2n+1) = \ldots = 0,$$

and it is not difficult to compute $u_d(n)$ for even n. We shall use the following easy estimate [7],

$$u_d(2n) = 2\left(\frac{d}{4n\pi}\right)^{d/2} + o\left(\frac{1}{n^{d/2}}\right) = O\left(\frac{1}{n^{d/2}}\right). \tag{2.4}$$

Classifying the paths according to the last return to the origin, we have

$$\sum_{i=0}^{n-1} P\{S_d(i) = \mathbf{0}, S_d(j) \neq \mathbf{0} \quad \text{for} \quad j = i+1, \ldots, n-1\} = 1. \tag{2.5}$$

Here and in the sequel we make the convention $S_d(0) = \mathbf{0}$ [similarly below $u_d(0) = 1$]. Since the summands in (2.5) may be written as

$$P\{S_d(i) = \mathbf{0}\} \cdot P\{S_d(j) - S_d(i) \neq \mathbf{0} \text{ for } j = i+1, \ldots, n-1\}$$
$$= P\{S_d(i) = \mathbf{0}\} \cdot P\{S_d(j) \neq \mathbf{0} \text{ for } j = 1, \ldots, n-i-1\},$$

we obtain

$$u_d(0)\gamma_d(n) + u_d(2)\gamma_d(n-2) + \ldots + u_d(2m)\gamma_d(n-2m) = 1, \tag{2.6}$$

with $m = n/2 - 1$ for even n and $m = (n-1)/2$ for odd n.

Next we estimate $\gamma_d(n)$. The method of estimation for $d \geqq 3$ differs somewhat from the one for $d = 2$.

When $d \geqq 3$ we have from (2.4)

$$U_d = \sum_{n=0}^{\infty} u_d(2n) < \infty, \qquad d = 3, 4, \ldots . \tag{2.7}$$

Let us put

$$\gamma_d = \lim_{n=\infty} \gamma_d(n), \tag{2.8}$$

because of (2.2) this limit exists and, indeed, $\gamma_d \leqq \gamma_d(n)$ for $n = 1, 2, \ldots$. From (2.6) and (2.2) we have for $1 \leqq k \leqq m$

$$\gamma_d(n-2k)\sum_{i=0}^{k} u_d(2i) + \sum_{i=k+1}^{m} u_d(2n) \geqq 1.$$

Letting $k \to \infty$ so that $n - 2k \to \infty$ we get

$$\gamma_d(n-2k) \cdot U_d \geqq 1 + o(1)$$

which gives

$$\gamma_d \geqq \frac{1}{U_d}. \tag{2.9}$$

[It then follows from (2.2) that $\gamma_d(n) \geqq 1/U_d$ for all n.] Subtracting

$$\frac{1}{U_d}[u_d(0)+u_d(2)+\ldots+u_d(2m)]$$

from both sides of (2.6), we have from (2.9)

$$u_d(0)\left(\gamma_d(n)-\frac{1}{U_d}\right)\leqq 1-\frac{1}{U_d}\sum_{i\leqq m}u_d(2i),$$

whence by (2.4) and (2.9), we have

$$\gamma_d=\frac{1}{U_d}, \qquad \gamma_d<\gamma_d(n)<\gamma_d+O(n^{1-d/2}) \tag{2.10}$$

for $d \geqq 3$.

When $d = 2$, (2.4) yields

$$u_2(0)+\ldots+u_2(2m)=\frac{1+o(1)}{\pi}\log m \tag{2.11}$$

hence, by (2.2),

$$\frac{\gamma_2(n)}{\pi}\log n \leqq 1+o(1). \tag{2.12}$$

Similarly, for $0 < k < m$, we have

$$\begin{aligned}\gamma_2(n-2k)[u_2(0)+\ldots+u_2(2k)]+&\\ u_2(2k+2)+\ldots+u_2(2m)&\geqq 1.\end{aligned} \tag{2.13}$$

Thus, if k tends to infinity together with n, (2.13) yields

$$\gamma_2(n-2k)\cdot\frac{1+o(1)}{\pi}\log k+\frac{1+o(1)}{\pi}\log\frac{m}{k}\geqq 1. \tag{2.14}$$

Taking[3] $k = m - [m/\log m]$, we have

$$\gamma_2(n-2k)\frac{1+o(1)}{\pi}\log(n-2k)+o(1)\geqq 1.$$

Together with (2.12) this gives

$$\gamma_2(n)=\frac{\pi+o(1)}{\log n}. \tag{2.15}$$

The estimate (2.15) is not good enough for our purposes. In order to improve it we use instead of (2.4) an estimate of (see, for example, [7] or [3, p. 298])

$$u_2(2n)=\frac{1}{4^{2n}}\sum_{k=0}^{2n}\frac{(2n)!}{k!k!(n-k)!(n-k)!}=\frac{1}{4^{2n}}\binom{2n}{n}^2$$

by Stirling's formula, that is,

$$u_2(2n)=\frac{1}{\pi n}+O\left(\frac{1}{n^2}\right). \tag{2.16}$$

Then the right side of (2.11) can be replaced by $\pi^{-1}\log cm + 0(1/n)$ with a suitable positive constant c. This immediately improves (2.12) to

$$\gamma_2(n)\leqq\frac{\pi}{\log cm}+O\left(\frac{1}{n\log^2 n}\right). \tag{2.17}$$

[3] Throughout the paper [ ] denotes the integral part of the number within the square brackets.

Similarly (2.13) gives, instead of (2.14), the inequality

$$\gamma_2(n-2k)\left\{\frac{\log ck}{\pi}+O\left(\frac{1}{k}\right)\right\}+\frac{1}{\pi}\log\frac{m}{k}+O\left(\frac{1}{k}\right)\geqq 1\,.$$

Hence

$$\gamma_2(n-2k)\log(n-2k)\geqq\left\{\pi-\log\frac{m}{k}+O\left(\frac{1}{k}\right)\right\}\frac{\log(n-2k)}{\log k+O(1)}\,.$$

Taking for k the same value as led to (2.15), this and (2.17) yield

$$\gamma_2(n)=\frac{\pi}{\log n}+O\left(\frac{\log\log n}{\log^2 n}\right). \tag{2.18}$$

The expected number of different points covered in the first n steps is given by

$$E\{L_d(n)\}=\gamma_d(1)+\gamma_d(2)+\ldots+\gamma_d(n)\,. \tag{2.19}$$

Thus (2.18) and (2.10) give the following result.

THEOREM 1. *The expected number $E_d(n)$ of different points encountered during n steps by a random path in d-dimensional space satisfies*

$$E_2(n)=\frac{\pi n}{\log n}+O\left(\frac{n\log\log n}{\log^2 n}\right), \tag{2.20}$$

$$E_3(n)=n\gamma_3+O(\sqrt{n}), \tag{2.21}$$

$$E_4(n)=n\gamma_4+O(\log n) \tag{2.22}$$

and

$$E_d(n)=n\gamma_d+\beta_d+O(n^{2-d/2})\quad \text{for}\quad d=5,6,\ldots \tag{2.23}$$

with positive constants $\beta_d(d=5,6,\ldots)$.

Remark. (2.10) and (2.17) imply the result of Pólya that the probability of returning to the origin infinitely often is 1 in the plane and 0 in d-space ($d\geqq 3$). Our derivation of this fact is similar to that of Feller [3, p. 298] who has to rely on the theory of recurrent events.

3. Weak laws for $L_d(n)$

Our next aim is to estimate the variance

$$V_d(n)=E\{L_d^2(n)\}-E_d^2(n) \tag{3.1}$$

of $L_d(n)$. Clearly

$$E\{L_d^2(n)\}=\sum_{i,j=1}^{n}\gamma_d(i,j) \tag{3.2}$$

where $\gamma_d(i,j)$ is the probability that the d-dimensional path pass at the i-th step through a point not passed during the preceding $i-1$ steps *and* also pass at the j-th step through a point not passed during the preceding $j-1$ steps.

For integers m, n with $1 \leqq m \leqq n$ we have

$$\begin{aligned}\gamma_d(m, n) &= P\{S_d(i) \neq S_d(m) \text{ for } i = 1, \ldots, m-1; \\ &\qquad\qquad S_d(j) \neq S_d(n) \text{ for } j = 1, \ldots, n-1\} \\ &\leqq P\{S_d(i) \neq S_d(m) \text{ for } i = 1, \ldots, m-1; \\ &\qquad\qquad S_d(j) \neq S_d(n) \text{ for } j = m, \ldots, n-1\} \\ &= P\{S_d(i) \neq S_d(m) \text{ for } i = 1, \ldots, m-1\} \\ &\qquad\qquad \times P\{S_d(j) \neq S_d(n) \text{ for } j = m, \ldots, n-1\} \\ &= P\{S_d(i) \neq S_d(m) \text{ for } i = 1, \ldots, m-1\} \\ &\qquad\qquad \times P\{S_d(j) \neq S_d(n-m+1) \text{ for } j = 1, \ldots, n-m\},\end{aligned}$$

or

$$\gamma_d(m, n) \leqq \gamma_d(m)\,\gamma_d(n-m+1)\,, \qquad 1 \leqq m \leqq n\,. \tag{3.3}$$

This relation merely expresses the fact that to pass through new points at both the m-th and n-th step it is necessary (a) to pass through a new point at the m-th step and (b) that if $S_d(m)$ is identified with the initial step of a second path that this second passes through a new point in the $n - m + 1$-st step. To obtain (3.3) it is only necessary to remark that the events (a) and (b) are independent.

From (3.1), (3.2), (3.3) and (2.19) we have

$$\begin{aligned}V_d(n) &= \sum_{i,j=1}^{n} \gamma_d(i, j) - \sum_{i=1}^{n} \gamma_d(i) \sum_{j=1}^{n} \gamma_d(j) = \sum_{i,j=1}^{n} \{\gamma_d(i, j) - \gamma_d(i)\,\gamma_d(j)\} \\ &\leqq 2 \sum_{1 \leqq i \leqq j \leqq n} \{\gamma_d(i, j) - \gamma_d(i)\,\gamma_d(j)\} \\ &\leqq 2 \sum_{1 \leqq i \leqq j \leqq n} \{\gamma_d(i)\,\gamma_d(j-i+1) - \gamma_d(i)\,\gamma_d(j)\} \\ &= 2 \sum_{i=1}^{n} \Big\{\gamma_d(i) \Big(\sum_{j=i}^{n} \gamma_d(j-i+1) - \sum_{j=i}^{n} \gamma_d(j)\Big)\Big\} \\ &\leqq 2 \sum_{i=1}^{n} \gamma_d(i) \max_{1 \leqq i \leqq n} \Big(\sum_{j=i}^{n} \gamma_d(j-i+1) - \sum_{j=i}^{n} \gamma_d(j)\Big).\end{aligned}$$

Because of (2.2) the max is attained for $i = [n/2] + 1$. Hence we obtain

$$V_d(n) \leqq 2E_d(n) \Big\{E_d\Big(n - \Big[\frac{n}{2}\Big]\Big) - E_d(n) + E_d\Big(\Big[\frac{n}{2}\Big]\Big)\Big\}\,. \tag{3.4}$$

From this and theorem 1, we easily obtain:

THEOREM 2. *The variance of $L_d(n)$ satisfies*

$$V_2(n) = O\Big(\frac{n^2 \log\log n}{\log^3 n}\Big), \tag{3.5}$$

$$V_3(n) = O(n^{3/2})\,, \tag{3.6}$$

$$V_4(n) = O(n \log n) \tag{3.7}$$

and

$$V_d(n) = O(n) \qquad \text{for } d = 5, 6, \ldots . \tag{3.8}$$

This immediately leads to

THEOREM 3. *The random variable* $L_d(n)$ *obeys the weak law of large numbers, that is,*

$$\underset{n=\infty}{\text{im}}\, P\{\, |L_d(n) - E_d(n)| > \epsilon E_d(n) \} = 0 \tag{3.9}$$

for every $\epsilon > 0$.

Indeed, applying Chebyshev's inequality to $L_d(n)$ we obtain from theorems 1 and 2 not only (3.9) but also that

$$\lim_{n=\infty} P\left\{ \left| L_2(n) - \frac{\pi n}{\log n} \right| > \phi_n \frac{n \log^{1/2} \log n}{\log^{3/2} n} \right\} = 0 ,$$

$$\lim_{n=\infty} P\{\, |L_3(n) - n\gamma_3| > \phi_n n^{3/4} \} = 0 ,$$

$$\lim_{n=\infty} P\{\, |L_4(n) - n\gamma_4| > \phi_n (n \log n)^{1/2} \} = 0 ,$$

$$\lim_{n=\infty} P\{\, |L_d(n) - n\gamma_d| > \phi_n n^{1/2} \} = 0 , \qquad d = 5, 6, \ldots$$

whenever the sequence of numbers ϕ_n satisfies

$$\lim_{n=\infty} \phi_n = \infty .$$

4. Strong laws for $L_d(n)$ when $d \geqq 3$

We proceed to improve on the last theorem by proving

THEOREM 4. *The random variable* $L_d(n)$ *obeys the strong law of large numbers, that is,*

$$P\left\{ \lim_{n=\infty} \frac{L_d(n)}{E_d(n)} = 1 \right\} = 1 . \tag{4.1}$$

The estimates of theorem 2 are sufficient to prove (4.1) for $d \geqq 3$, (3.5) is, however, not strong enough to imply (4.1) for $d = 2$. We therefore postpone the treatment of the plane case to the next section.

We thus assume $d \geqq 3$ and shall prove a somewhat stronger result than (4.1).

Let α be any number satisfying

$$\tfrac{5}{6} < \alpha < 1 , \tag{4.2}$$

and take for β any number with

$$\frac{2}{4\alpha - 3} < \beta < \frac{1}{1 - \alpha} , \tag{4.3}$$

such a choice of β is possible because of (4.2).

Put

$$n_k = [k^\beta] , \qquad k = 1, 2, \ldots . \tag{4.4}$$

Using $V_d(n) = O(n^{3/2})$ for $d \geqq 3$ and applying Chebyshev's inequality we have

$$P\{|L_d(n_k) - n_k\gamma_d| > n_k^a\} = O(n_k^{3/2-2a}) = O(k^{(3-4a)\beta/2}).$$

Since $(3 - 4a)\beta/2 < -1$ by (4.3) it follows that

$$\sum_{k=1}^{\infty} P\{|L_d(n_k) - n_k\gamma_d| > n_k^a\} < \infty .$$

Hence, by the Borel-Cantelli lemma, there is probability 1 that

$$|L_d(n_k) - n_k\gamma_d| \leqq n_k^a \tag{4.5}$$

hold for all sufficiently large k. But (4.5) implies

$$|L_d(n) - n\gamma_d| < n_k^a + n_{k+1} - n_k \tag{4.6}$$

for $n_k \leqq n < n_{k+1}$. By (4.4), $n_{k+1} - n_k = O(k^{\beta-1})$; since $\beta - 1 < a\beta$ we have also $k^{\beta-1} = O(n_k^a)$. Thus the right side of (4.6) is $O(n_k^a)$ and hence, *a fortiori*, $0(n^a)$ for $n_k \leqq n < n_{k+1}$. We have thus proved:

If $d \geqq 3$ then for almost all paths

$$L_d(n) = n\gamma_d + O(n^a) \tag{4.7}$$

for every $a > 5/6$.

This proves theorem 4 for $d \geqq 3$.

Remark. The variance estimates of theorem 2 suffice, of course, to obtain much sharper results than (4.7). Using more sophisticated methods it is possible to deduce that, with probability 1, $L_3(n) = n\gamma_3 + O(n^{3/4+\delta})$ and $L_d(n) = n\gamma_d + O(n^{1/2+\delta})$ for $d \geqq 4$ (δ being an arbitrarily small positive number). It is even possible to replace n^δ in these estimates by a suitable power of log n. It is, however, possible that much stronger results hold. Hence it would seem reasonable to gather further information about the variance and higher moments before applying the known but complicated arguments leading to the sharper estimates.

5. Proof of the strong law in the plane

If we could replace $\log^3 n$ in the denominator in (3.5) by any larger power of $\log n$, then we could deduce from Chebyshev's inequality an estimate of

$$P\left\{\left|L_2(n) - \frac{\pi n}{\log n}\right| > \epsilon \frac{n}{\log n}\right\}$$

sharp enough to prove (4.1) for $d = 2$. Since we cannot improve the estimate of the variance, we use a more laborious method to sharpen the probability estimate.

Let us first assume the validity of the estimate

$$P\left\{\left|L_2(n) - \frac{\pi n}{\log n}\right| > \epsilon \frac{n}{\log n}\right\} = O\left(\frac{1}{\log^{1+\delta} n}\right), \qquad \delta > 0, \tag{5.1}$$

for every $\epsilon > 0$ (the constant involved in O may, of course, depend on ϵ but δ is fixed).

Let θ be any number satisfying

$$0 < \theta < \frac{1}{1+\delta}, \tag{5.2}$$

and put

$$n_k = [\exp k^\theta], \qquad k = 1, 2, \ldots, \tag{5.3}$$

the brackets denoting again the integral part.

Because of (5.1) and (5.2) we have

$$\sum_{k=1}^{\infty} P\left\{ \left| L_2(n_k) - \frac{\pi n_k}{\log n_k} \right| > \frac{\epsilon n_k}{2 \log n_k} \right\} < \infty .$$

Hence it follows from the Borel-Cantelli lemma that for almost all paths

$$\left| L_2(n_k) - \frac{\pi n_k}{\log n_k} \right| \leqq \frac{\epsilon n_k}{2 \log n_k} \tag{5.4}$$

for all sufficiently large k.

But for $n_k \leqq n < n_{k+1}$ we have

$$L_2(n_k) - \frac{\pi n_{k+1}}{\log n_{k+1}} < L_2(n) - \frac{\pi n}{\log n} \leqq L_2(n_{k+1}) - \frac{\pi n_k}{\log n_k}. \tag{5.5}$$

(5.4) together with the same equation with k replaced by $k+1$ implies that the absolute value of the extreme members of (5.5) is at most

$$\frac{\epsilon n_{k+1}}{2 \log n_{k+1}} + \pi \left(\frac{n_{k+1}}{\log n_{k+1}} - \frac{n_k}{\log n_k} \right). \tag{5.6}$$

Since $\theta < 1$ it follows from (5.3) that, for sufficiently large k, (5.6) is smaller than

$$\frac{\epsilon n_k}{\log n_k} \leqq \frac{\epsilon n}{\log n}. \tag{5.7}$$

Combining (5.4), (5.5) and (5.7) we see that for every $\epsilon > 0$ and for almost all paths

$$\left| L_2(n) - \frac{\pi n}{\log n} \right| \leqq \frac{\epsilon n}{\log n}$$

for all sufficiently large n. But this is equivalent to (3.9) with $d = 2$. Thus the proof of theorem 4 will be achieved as soon as (5.1) is established.

Proof of (5.1). Let $N = N(n)$ be a positive integer satisfying

$$\lim_{n=\infty} N = \infty, \qquad \log N = O(\log n). \tag{5.8}$$

Put now

$$n_i = \left[\frac{ni}{N}\right] \qquad \text{for } i = 0, 1, 2, \ldots, N.$$

For $i = 1, \ldots, N$ let A_i denote the event

$$L_2(n_i) - L_2(n_{i-1}) > \left(1 + \frac{\epsilon}{2}\right) \frac{\pi n}{N \log n}$$

and B_i the event

$$L_2(n_i) - L_2(n_{i-1}) > \frac{\epsilon \pi n}{2 \log n}.$$

From (3.5) and Chebyshev's inequality we have by (5.8)[4]

$$P\{A_i\} < c_1 \frac{\log \log n}{\log n} \tag{5.9}$$

where c_1 (and c_2, c_3, c_4 below) are finite positive constants which may depend on ϵ (but not on n, N or i). Also, since $N \to \infty$, the variance estimate gives

$$P\{B_i\} < c_2 \frac{\log \log n}{N^2 \log n}. \tag{5.10}$$

The inequality $L_2(n) > (1+\epsilon)\pi n/\log n$ cannot occur unless either at least two of the events $A_1, \ldots, A_N$ occur or at least one of the events $B_1, \ldots, B_N$. The probability that at least one of the events $A_1, \ldots, A_N$ occur cannot exceed $NP\{A_i\}$, and, since these events are mutually independent, the probability that at least two of these occur is smaller than $(NP\{A_i\})^2$. Similarly the probability that at least one of the events B_i occurs is not greater than $NP\{B_i\}$. Thus (5.9) and (5.10) give

$$P\left\{L_2(n) > (1+\epsilon)\frac{\pi n}{\log n}\right\} < c_1^2\left(\frac{N \log \log n}{\log n}\right)^2 + c_2 \frac{\log \log n}{N \log n}. \tag{5.11}$$

Taking $N = [(\log n/\log \log n)^{1/3}]$ the preceding inequality gives

$$P\left\{L_2(n) > (1+\epsilon)\frac{\pi n}{\log n}\right\} < c_3\left(\frac{\log \log n}{\log n}\right)^{4/3}. \tag{5.12}$$

For $i \neq j$ and $1 \leqq i, j \leqq N$ let M_{ij} be the number of points which are common to both part paths M_i and M_j where M_i denotes the sequence of points

$$S_d(n_{i-1}+1), \quad S_d(n_{i-1}+2), \ldots, S_d(n_i),$$

that is, the part covered by the path between the n_{i-1}-th and n_i-th steps. (N is again a function of n satisfying (5.8) which will be specified later.) It is not difficult to deduce from (2.20) that the expected value of M_{ij} satisfies

$$E\{M_{ij}\} = O\left(\frac{n \log \log n}{N \log^2 n}\right). \tag{5.13}$$

Hence, for every fixed η with $1 > \eta > 0$ we have

$$P\left\{M_{ij} > \frac{n \log \log n}{N \log^{1+\eta} n}\right\} = O\left(\frac{1}{\log^{1-\eta} n}\right). \tag{5.14}$$

Let C_{ij} denote the event whose probability is evaluated in (5.14). Then the probability that at least one of the events C_{ij} $(1 \leqq i < j \leqq N)$ occurs is $O(N^2/\log^{1-\eta} n)$, while the probability that two such events C_{ij} and $C_{i'j'}$ with i, i', j, j' all different from one another occur, is

$$O\left(\frac{N^4}{\log^{2-2\eta} n}\right). \tag{5.15}$$

[4] (5.9) and all other similar assertions in this section involving $\log \log n$ are assumed to apply only for $n \geqq 3$.

Let D_i ($i = 1, \ldots, N$) denote the event that M_i covers less than $(1 - \epsilon/2)\pi n/(N \log n)$ different points. Then, exactly as in (5.9), we have

$$P\{D_i\} < c_4 \frac{\log \log n}{\log n},$$

and thus the probability that at least two of the events D_i occur is smaller than

$$\left(c_4 \frac{N \log \log n}{\log n}\right)^2. \tag{5.16}$$

But if not more than one of the events D_i occurs and if there exist i', j' with $1 \leqq i' \leqq j' \leqq N$ such that no C_{ij} occurs unless at least one of the pair i, j coincides with either i' or j', then it is easily seen that

$$L_2(n) > (N-3)\left(1 - \frac{\epsilon}{2}\right)\frac{\pi n}{N \log n} - \frac{(N-3)(N-2)}{2} \frac{n \log \log n}{N \log^{1+\eta} n}.$$

Taking $N = [\log \log n]$ this gives for large n

$$L_2(n) > (1-\epsilon)\frac{\pi n}{\log n}.$$

Hence (5.15) and (5.16) yield

$$P\left\{L_2(n) < (1-\epsilon)\frac{\pi n}{\log n}\right\} = O\left(\frac{\log^4 \log n}{\log^{2-2\eta} n}\right) = O\left(\frac{1}{\log^{2-3\eta} n}\right). \tag{5.17}$$

Together with (5.12) this establishes (5.1) and thus completes the proof of theorem 4.

Remarks. 1. (5.12) can be improved by repeating the argument which led to it. Indeed, if we estimate (5.9) using (5.12) the first summand on the right in (5.11) will become a constant multiple of

$$N^2 \left(\frac{\log \log n}{\log n}\right)^{8/3}.$$

Taking $N = [(\log n/\log \log n)^{5/9}]$ this improves the exponent in (5.12) to 14/9. Generally, it is easily seen that if we estimate (5.9) by $c\,(\log \log n/\log n)^{\alpha}$ and then choose in (5.11) $N = [(\log n/\log \log n)^{\beta}]$ with $\beta = (2\alpha - 1)/3$ then the exponent in (5.12) becomes $2(\alpha + 1)/3$. Putting $\alpha_0 = 1$, $\alpha_i = 2(\alpha_{i-1} + 1)/3$ for $i = 1, 2, \ldots$ it is easily seen that α_i is increasing and tends to 2 as $i \to \infty$. Combining this with (5.17) we obtain

$$P\left\{\left|L_2(n) - \frac{\pi n}{\log n}\right| > \epsilon \frac{n}{\log n}\right\} = O\left(\frac{1}{\log^{2-\delta} n}\right) \tag{5.18}$$

for every $\epsilon > 0$ and $\delta > 0$.

2. The proof of this section can be adapted to yield $L_2(n) = \pi n/\log n + O(n/\log^a n)$ with suitable $a > 1$ for almost all paths.

6. Rate of escape

It was mentioned in the introduction that for $d \geqq 3$ there is probability 1 that $S_d(n)$ tend to infinity as n increases. In this section we study some quantitative aspects of this fact.

Adopting a terminology of P. Lévy we say that a positive function $g(n)$ of n

belongs to the *lower d-class*, in symbols $g \in \mathcal{L}_d$ if[5]

$$P\{\|S_d(n)\| < g(n)\sqrt{n} \quad \text{for infinitely many } n\} = 0 . \tag{6.1}$$

Similarly $g(n)$ is said to belong to the *upper d-class*, or $g \in \mathcal{U}_d$, if

$$P\{\|S_d(n)\| < g(n)\sqrt{n} \quad \text{for infinitely many } n\} = 1 . \tag{6.2}$$

We shall give a characterization of lower and upper classes for monotone $g(n)$. [Since increasing $g(n)$ obviously belong to the upper d-class for all d, we may restrict ourselves to decreasing functions.]

THEOREM 5. *A monotone function $g(n)$ belongs to the lower class $\mathcal{L}_d$ or to the upper class $\mathcal{U}_d (d \geqq 3)$ according as the series*

$$\sum_{m=1}^{\infty} g^{d-2}(2^m) \tag{6.3}$$

converges or diverges.

It is a well known consequence of the central limit theorem (see, for example, Kac [4]) that if the unit step in the random walk is replaced by h and the unit time by h^2, then as $h \to 0$ the random walk approximates more and more the Brownian motion. Thus, if $x_d(t)$ denotes the position at time t of a point moving in Brownian motion in d-space, if $g(t)$ is defined for all $t > 0$ and if we put $g \in \mathcal{L}_d$ or $g \in \mathcal{U}_d$ according as the probability of the event $\|x_d(t)\| < g(t)\sqrt{t}$ for some arbitrarily large t is 0 or 1, then theorem 5 is equivalent to:

THEOREM 6. *For monotone functions $g(t)$ we have*

$$g \in \begin{Bmatrix} \mathcal{L}_d \\ \mathcal{U}_d \end{Bmatrix} \quad \text{if} \quad \sum_{m=1}^{\infty} g^{d-2}(2^m) \begin{Bmatrix} < \\ = \end{Bmatrix} \infty \tag{6.4}$$

for $d = 3, 4, \ldots$.

Remark. Since $g(t)$ is monotone the convergence or divergence of the series in (6.4) is not affected if 2^m is replaced by λ^m with any $\lambda > 1$. A similar observation applies to (6.3).

It is more convenient to prove theorem 6. For the definition and relevant properties of Brownian motion, see, for example, [6], [5], [1]. First we give some lemmas.

LEMMA 1. *Let $C(S, r)$ denote the sphere with center S and radius r ($r \leqq R = \|S\|$) in d-space ($d \geqq 3$). Then the probability that $x_d(t)$ ever enter $C(S, r)$ is given by*

$$P\{x_d(t) \in C(S, r) \quad \text{for some} \quad t > 0\} = \left(\frac{r}{R}\right)^{d-2} . \tag{6.5}$$

For $r > R$ the probability in question obviously equals 1. For $d = 3$ this lemma is stated in Kakutani [5]. For a proof of the general case, see Dvoretzky [1].

LEMMA 2. *For every $T \geqq 0$ and $r \geqq 0$ put*

$$Q_d(r, T) = P\{\|x_d(t)\| \leqq r \quad \text{for some } t > T\}$$

then we have for $d \geqq 3$

$$q_d\left(\frac{r}{\sqrt{T}}\right)^{d-2} e^{-r^2/(2T)} \leqq Q_d(r, T) \leqq q_d\left(\frac{r}{\sqrt{T}}\right)^{d-2} , \tag{6.6}$$

[5] For any point S in Euclidean d-space we denote by $\|S\|$ the distance of S from the origin.

with

$$q_d = \frac{d}{2^{d/2}\Gamma(d/2+1)}. \tag{6.7}$$

PROOF. Let $u_1(t), \ldots, u_d(t)$ be the components of $x_d(t)$, then the distribution of $x_d(T)$ has the density

$$p_d(x, T) = \left(\frac{1}{2\pi T}\right)^{d/2} \exp\left(-\frac{u_1^2 + \ldots + u_d^2}{2T}\right). \tag{6.8}$$

On the other hand it follows from lemma 1, the temporal homogeneity of the Brownian motion and its Markovian character that if $x(T) = S$ then the conditional probability that $\|x(t)\| < r$ for some $t > T$ is given by

$$\min\left[1, \left(\frac{r}{\|S\|}\right)^{d-2}\right].$$

Therefore,

$$\int_{\|x\|>r} \ldots \int \left(\frac{r}{\|x\|}\right)^{d-2} p_d(x, T)\, du_1 \ldots du_d \leqq Q_d(r, T)$$

$$\leqq \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \left(\frac{r}{\|x\|}\right)^{d-2} p_d(x, t)\, du_1 \ldots du_d.$$

Putting σ_d for the $(d-1)$-dimensional area of the unit sphere in d-space we have from (6.8)

$$\frac{\sigma_d r^{d-2}}{(2\pi T)^{d/2}} \int_r^{\infty} \rho e^{-\rho^2/(2T)} d\rho \leqq Q_d(r, T) \leqq \frac{\sigma_d r^{d-2}}{(2\pi T)^{d/2}} \int_0^{\infty} \rho e^{-\rho^2/(2T)} d\rho.$$

Integrating and substituting $\sigma_d = d\pi^{d/2}/\Gamma(d/2+1)$ we obtain (6.6).

We also need the following

LEMMA 3. *For every $T \geqq 0$, $r > 0$ and $K > 1$ put*

$$P_d(r, T, K) = P\{\|x_d(t)\| \leqq r \text{ for some } T \leqq t < KT\}.$$

Then we have for $d \geqq 3$

$$P_d(r, T, K) > \frac{q_d}{10} \cdot \left(\frac{r}{\sqrt{T}}\right)^{d-2}, \tag{6.9}$$

provided that $r \leqq \sqrt{T}$ and $K \geqq 4$.

PROOF. We obviously have

$$P_d(r, T, K) \geqq Q_d(r, T) - Q_d(r, TK). \tag{6.10}$$

Applying (6.6) to (6.10) we obtain

$$P_d(r, T, K) \geqq q_d \left(\frac{r}{\sqrt{T}}\right)^{d-2} \left\{ e^{-r^2/(2T)} - \left(\frac{1}{\sqrt{K}}\right)^{d-2} \right\}. \tag{6.11}$$

Since $r \leqq \sqrt{T}$, $K \geqq 4$ and $d \geqq 3$ the expression within the braces is at least

$$e^{-1/2} - \tfrac{1}{2} > \tfrac{1}{10}.$$

Thus (6.9) follows from (6.11).

PROOF OF THEOREM 6. First we assume the convergence of the series (6.3) and show that $g \in \mathcal{L}_d$.

Indeed,

$$P\{\|x_d(t)\| \leqq g(t)\sqrt{t} \quad \text{for some } 2^m < t \leqq 2^{m+1}\} \tag{6.12}$$

is, because of the monotonicity of $g(t)$ and lemma 2, smaller than

$$P\{\|x_d(t)\| \leqq g(2^m)\,2^{(m+1)/2} \text{ for some } t > 2^m\} \leqq q_d[\sqrt{2}\,g(2^m)]^{d-2}.$$

Thus the series whose general term is given by (6.12), is convergent and it follows from the Borel-Cantelli lemma that $g \in \mathcal{L}_d$.

Let us now assume the divergence of the series (6.3) and show that $g \in \mathcal{U}_d$. We assume,

$$\lim_{t=\infty} g(t) = 0 \tag{6.13}$$

this obviously entails no loss of generality.

Let

$$m_1, m_2, \ldots, m_i, \ldots \tag{6.14}$$

be an increasing sequence of integers satisfying

$$\lim_{i=\infty}(m_{i+1} - m_i) = \infty. \tag{6.15}$$

Denote by A_i the event

$$\|x_d(t)\| \leqq g(t)\sqrt{t} \quad \text{for some } 4^{m_i} \leqq t < 4^{m_i+1}.$$

The events A_i $(i = 1, 2, \ldots)$ are of course not independent. However, it can be shown by studying $P\{A_iA_j\} - P\{A_i\}P\{A_j\}$ that the correlations between the different events are so small that one may apply an extension of the Borel lemma and deduce from

$$\sum_{i=1}^{\infty} P\{A_i\} = \infty \tag{6.16}$$

the occurrence, with probability 1, of infinitely many events A_i.

Thus the proof of the theorem will be achieved provided we can find a sequence (6.14) satisfying (6.15) for which (6.16) holds.

By assumption (compare remark following theorem 6) $\sum g^{d-2}(4^m)$ is a divergent series. Hence, by a general result on infinite series, there exists a sequence (6.14) satisfying (6.15) for which

$$\sum_{i=1}^{\infty} g^{d-2}(4^{m_i+1}) = \infty. \tag{6.17}$$

In view of (6.13), we may furthermore assume that m_1 has been chosen so large that

$$g(4^{m_1}) \leqq 1. \tag{6.18}$$

Since $g(t)$ is a monotone function we clearly have

$$P\{A_i\} \geqq P\{\|x_d(t)\| \leqq g(4^{m_i+1})\,2^{m_i} \quad \text{for some } 4^{m_i} \leqq t < 4^{m_i+1}\}.$$

Thus, we have from (6.18) and lemma 3

$$P\{A_i\} > \frac{q_d}{10} g^{d-2}(4^{m_i+1}).$$

Hence (6.16) follows from (6.17) and the proof is completed.

7. Remarks

1. Our methods obviously extend to more general types of random walk. In particular, all that was needed in section 6 was the fact that the random walk approximates, in a suitable sense, the Brownian motion. Thus the result on the rate of escape applies, roughly speaking, whenever the central limit theorem holds. It is even unnecessary to restrict the consideration to the case of identically distributed individual steps.

The problem of sections 1–5 concerns only random walk in a lattice. But here again the result can be generalized to the case of identically distributed individual steps with zero mean and finite variance (and also to still more extensive cases). In simple special cases it is also easy to find the exact constants involved, for example, for the triangular (6 possibilities for every step) and the hexagonal (3 possibilities for every step) random walk in the plane.

2. Pólya [7] has shown that two points starting simultaneously and moving randomly (as defined in section 1 or 2) in the plane will meet (that is, be at the same place at the same time) infinitely often with probability 1. On the other hand he has shown that in 3-dimensional space the probability of infinitely many meetings is 0. Similarly it can be shown that there is probability 1 for three points moving on the line meeting (all three together) infinitely often, but that in the plane the probability of this event is zero. We can also show that the probability of four points meeting together infinitely often is 0 even in the 1-dimensional case.

3. The problem of sections 2–5 is the discrete analogue of that of the Hausdorff dimensionality of Brownian paths in d-space ($d \geqq 2$).

4. For Brownian motion there is in the plane a problem similar to that treated in section 6 for $d \geqq 3$. It is known that, with probability 1, $x_2(t) \neq \mathbf{0}$ for $t > 0$ but that $\liminf_{t=\infty} \|x_2(t)\| = 0$. It would be interesting to characterize the rate with which the "small" values of $\|x_2(t)\|$ approach zero.

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