AN EXTENSION OF AN ALGORITHM OF HOTELLING

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1. Introduction

The algorithm in question is one for the rapid approximate computation of the inverse to a linear operator. It has become known to statisticians and others through the work of Hotelling [**1**],[1] who used it for inverting finite matrices—linear operators in finite-dimensional vector spaces—and found a bound for the error. Somewhat earlier, Ostrowski [**4**] had proposed its use in a rather general class of problems, with special emphasis on integral equations of second kind and Volterra type. He did not give a bound for the error. More recently Rademacher [**5**] has applied the same idea to calculating Laurent expansions of algebraic functions.

The object of this paper is to show how the error can be limited in a rather general problem, and to suggest specific procedures for applying it to linear integral equations of second kind and Fredholm type.

2. Vector spaces

For this purpose it may be useful to state some facts about normed linear vector spaces. Suppose L is such a space (which will for definiteness be assumed real): that is, if x_1 and x_2 are vectors in L, so is $ax_1 + bx_2$, where a and b are real numbers; with every x in L is associated a non-negative real number $\|x\|$, the norm of x; $\|x\| > 0$ unless x is the zero-element of L, whose norm is zero. Furthermore, $\|x_1 + x_2\| \leqq \|x_1\| + \|x_2\|$, and $\|ax\| = |a| \cdot \|x\|$.

We consider an operator K which transforms any element x of L into an element Kx of L; the special operator I is that which carries each x of L into itself. We suppose K to be additive, homogeneous, and continuous (i.e., linear); there then exists a constant

$$M(K) = \underset{x \in L}{\text{l.u.b.}} \frac{\|Kx\|}{\|x\|}, \tag{2.1}$$

called the (upper) bound, or norm, of K. Clearly, $M(I) = 1$. If

$$M(K) < 1, \tag{2.2}$$

the equation

$$x - Kx \equiv (I - K)x = y \tag{2.3}$$

[1] Boldface numbers in brackets refer to references at the end of the paper (see p. 358).

possesses a unique solution x in L for any given y in L, and

$$x = \sum_{k=0}^{\infty} K^k y , \tag{2.4}$$

where $K^0 = I$, $K^{k+1} = K \cdot K^k$ for $k \geqq 0$. That is, (2.2) guarantees that operator $I - K$ has the inverse $(I - K)^{-1}$ given by the Liouville-Neumann expansion

$$(I - K)^{-1} = \sum_{k=0}^{\infty} K^k . \tag{2.5}$$

If $(I - K)^{-1}$ is approximated by the nth partial sum of expansion (2.5), it is easily seen that the error $E_n = (I - K)^{-1} - \sum_{k=0}^{n-1} K^k$ satisfies the inequality

$$M(E_n) \leqq \sum_{k=n}^{\infty} \{M(K)\}^k = \{M(K)\}^n/\{1 - M(K)\} . \tag{2.6}$$

This follows from (2.2) and the facts that the sum $A + B$ and the product AB of two linear operators are bounded, and $M(A + B) \leqq M(A) + M(B)$, $M(AB) \leqq M(A)M(B)$.

3. Limitation of error

The error committed in using the nth partial sum of (2.5) is thus $0(\{M(K)\}^n)$. By using [4] an identity of Euler

$$(1 - z)^{-1} = \prod_{k=0}^{\infty} (1 + z^{2^k}) , \qquad |z| < 1 ,$$

we can write the inverse as an infinite product:

$$(I - K)^{-1} = \prod_{k=0}^{\infty} (I + K^{2^k}) , \qquad M(K) < 1 . \tag{3.1}$$

It will be shown that the error committed in using the nth partial product of (3.1) is $0(\{M(K)\}^{2^n})$.

Following Hotelling's discussion of algebraic linear systems, we consider the equation

$$Ax = y , \tag{3.2}$$

where A is a linear transformation in L. We suppose that $A = (I - G)A_0$, where A_0 has the known inverse $C_0 = A_0{}^{-1}$ and $M(G) < 1$. Then

$$\begin{aligned} A^{-1} &= C_0(I - G)^{-1} \\ &= C_0 \sum_{k=0}^{\infty} G^k , \end{aligned}$$

so by (3.1)

$$A^{-1} = C_0 \prod_{k=0}^{\infty} (I + G^{2^k}). \tag{3.3}$$

For $n \geqq 1$, write

$$C_n = C_0 \prod_{k=0}^{n-1} (I + G^{2^k}). \tag{3.4}$$

Then C_n is an approximation to A^{-1}. The error $A^{-1} - C_n$ is bounded by

$$M(A^{-1} - C_n) \leqq M(C_0)\{M(G)\}^{2^n}/\{1 - M(G)\}. \tag{3.5}$$

This follows from (3.4); for $A_0 = (I - G)^{-1}A$, so

$$C_n = A^{-1}(I - G) \prod_{k=0}^{n-1} (I + G^{2^k}) = A^{-1}(I - G^{2^n}),$$

whence

$$A^{-1} - C_n = A^{-1}G^{2^n} = C_0(I - G)^{-1}G^{2^n}.$$

From (3.5) it follows that $M(A^{-1} - C_n) < \epsilon$ if n exceeds the complicated-looking quantity

$$(\log 2)^{-1} \log \frac{\log \epsilon[1 - M(G)] - \log M(C_0)}{\log M(G)}.$$

The approximations C_n satisfy the recursion relation

$$C_{n+1} = C_n(2I - AC_n), \tag{3.6}$$

which may have advantages in practice.

4. The algebraic case

The case where $Ax = y$ is a system of linear algebraic equations was treated rather thoroughly by Hotelling [1], [2]. In our terminology, he adopted the norm $\|x\| = (\Sigma x_i^2)^{\frac{1}{2}}$, where $x = (x_1, \cdots, x_n)$; if A has the matrix (a_{ij}) $M(A) \leqq (\Sigma\Sigma a_{ij}^2)^{\frac{1}{2}}$ by the Lagrange-Cauchy inequality. Analogously, if $p > 1$ one can define $\|x\| = (\Sigma |x_i|^p)^{1/p}$; and by the Hölder inequality $M(A) \leqq (\Sigma\Sigma |a_{ij}|^q)^{1/q}$, where $q = p/(p-1)$. Another norm is $\|x\| = \Sigma\, |x_i|$, for which $M(A) \leqq \Sigma A_i$, where $A_i = \max\limits_j |a_{ij}|$. Still another is $\|x\| = \max\, |x_i|$, for which $M(A) \leqq \max B_i$, where $B_i = \sum_{j=1}^{n} |a_{ij}|$.

An extension to corresponding types of infinite linear systems (in Hilbert space, spaces $l^{(p)}$, etc.) is suggested, provided that the norms and bounds converge.

5. Volterra integral equations

The Volterra-type linear integral equation of second kind

$$x(s) - \int_0^s K(s, t)\, x(t)\, dt = y(s)\,, \qquad 0 \leqq s \leqq t \leqq 1\,, \tag{5.1}$$

occupies a special position, as Ostrowski remarked, in that the Liouville-Neumann expansion converges to give the solution, even though no condition of form (2.2) holds. For fixed $s \leqq 1$ we suppose $y(t)$ and $K(s, t)$ to be continous on $0 \leqq t \leqq s$. Then $|y(t)|$ and $|K(s, t)|$ have (finite) maxima on $(0, s)$, which we denote by $\|y\|_s$ and M_s respectively. Application of the Liouville-Neumann iterative scheme, with the customary estimates, yields a continuous $x(s)$ for which

$$|x(s)| \leqq \|y\|_s\, e^{sM_s}\,.$$

Since the inequality holds for all s in $(0, 1)$, and since the right member is an increasing function of s,

$$\|x\|_s \leqq \|y\|_s\, e^{sM_s}\,.$$

We consequently have a norm in terms of which

$$M\{(I - K)^{-1}\} \leqq e^{sM_s}\,. \tag{5.2}$$

The process of section 3 can be applied to speed up convergence, with $C_0 = I$ and $G = K$. We have

$$M(K^{2^n}) \leqq \frac{(sM_s)^{2^n}}{(2^n)\,!}\,, \tag{5.3}$$

which, when substituted with (5.2) into (3.5), gives

$$M(A^{-1} - C_n) \leqq e^{sM_s}(sM_s)^{2^n}/(2^n)\,!\,. \tag{5.4}$$

Inequality (5.4) can be simplified—with loss of precision—by replacing s and M_s by 1 and $M = \max M_s$ respectively.

6. Fredholm integral equations

For the linear integral equation of second kind and Fredholm type, matters are less simple. Success of the iterative scheme depends on some such restriction as (2.2).

Let the equation be

$$x(s) - \int_0^1 K(s, t)\, x(t) dt = y(s)\,, \qquad 0 \leqq s \leqq 1\,. \tag{6.1}$$

Choice of a norm depends on the class of functions considered. Thus if $y(s)$ is continuous [L the space of functions continuous on (0, 1)], one may take $\|y\| = \max |y(s)|$; now $M(K) \leqq \max k(s)$, where $k(s) = \int_0^1 |K(s,t)|dt$. Another familiar norm is the quadratic:

$$\|y\| = \left\{\int_0^1 y^2(t)dt\right\}^{1/2}.$$

For this,

$$M(K) \leqq \left\{\int_0^1 \int_0^1 [K(s,t)]^2 ds\, dt\right\}^{1/2}.$$

It may now happen that $M(K) \geqq 1$, so that convergence of the direct Liouville-Neumann process is not assured. It will then be desirable to find an approximate inverse C_0, as in section 3. One method for doing this is that of "kernel-splitting" [**6**], which reduces the problem to an algebraic one. An error limit which will give a bound for $M(G)$ is to be found in [**3**]. Other processes for determining a C_0 are also discussed in [**3**]; namely, the method of "segments," as applied to an infinite linear system equivalent to (6.1), and the method of least squares.

7. Integral equations of first kind

A linear integral equation of first kind and Fredholm type

$$\int_0^1 K(s,t)x(t)dt = y(s), \qquad 0 \leqq s \leqq 1,$$

may also be amenable to the algorithm of section 3. For, as with (6.1), it may be possible by "kernel-splitting" to find a sufficiently good C_0.

REFERENCES

1. Hotelling, Harold. "Some new methods in matrix calculation," *Annals of Math. Stat.*, vol. 14 (1943), pp. 1–34.
2. ———. "Further points on matrix calculation and simultaneous equations," *ibid.*, pp. 440–441.
3. Lonseth, A. T. "Numerical solutions of integral equations" (address delivered in a Symposium on Recent Developments in Numerical Methods, at New Brunswick, N.J., September, 1945).
4. Ostrowski, Alexandre. "Sur une transformation de la série de Liouville-Neumann," *Comptes Rendus de l'académie des Sciences de Paris*, t. 203 (1936), p. 602.
5. Rademacher, H. A. "Laurent expansions of algebraic functions" (address delivered in a Symposium on Recent Developments in Numerical Methods, at New Brunswick, N.J., September, 1945).
6. Schmidt, Erhard. "Zur Theorie der linearen und nichtlinearen Integralgleichungen (II)," *Mathematische Annalen*, Bd. 64 (1907), pp. 161–174.

Added in proof: Part of [3] has now appeared in the paper, Lonseth, A. T., "The propagation of error in linear problems," *Trans. Amer. Math. Soc.*, vol. 62 (1947), pp. 193–212.

For [5], see Rademacher, H. A., and I. J. Schoenberg, "An iteration method for calculation with Laurent series," *Quart. Appl. Math.*, vol. 4 (1946), pp. 142–159.