9. THE NATURE OF THE SINGULARITY AT p_H.

The arguments of Sykes and Essam (1964) which led them (not quite rigorously) to values for $p_H(\mathcal{G})$ for certain graphs were based on "the average number of clusters per site". In a one parameter problem with

$$(9.1) \qquad P_p\{v \text{ is occupied}\} = p$$

for all vertices v this average is, of course, a function, $\Delta(p,\mathcal{G})$ say, of p. Sykes and Essam's motivation for introducing this function lay in analogies with statistical mechanics, and on the basis of such analogies they assumed that $\Delta(p)$ has exactly one singularity as a function of p, and that this singularity is located at $p = p_H$. This assumption was actually their only non rigorous step. They then proved that for a matching pair of graphs $(\mathcal{G},\mathcal{G}^*)$ one has the remarkable relationship

$$(9.2) \qquad \Delta(p,\mathcal{G}) - \Delta(1-p,\mathcal{G}^*) = \text{a polynomial in } p.$$

It was for this relation that Sykes and Essam introduced matching pairs of graphs. They then proceeded to locate p_H, which was presumably the singularity of Δ, by means of (9.2) for certain matching pairs in which $\mathcal{G}$ and $\mathcal{G}^*$ have a close relation. E.g. for bond percolation on $\mathbb{Z}^2$, $\mathcal{G}_1^*$ is isomorphic to $\mathcal{G}_1$ and hence $\Delta(\cdot,\mathcal{G}_1) = \Delta(\cdot,\mathcal{G}_1^*)$.

In this chapter we shall first give the precise definition and show the existence of $\Delta(p)$, following Grimmett (1976) and Wierman (1978). We then derive the Sykes-Essam relation (9.2) and show that for the matching pairs $(\mathcal{G},\mathcal{G}^*)$ to which Theorem 3.1 applies $\Delta(p,\mathcal{G})$ is analytic in p for $p \neq p_H(\mathcal{G})$. This justifies part of the Sykes-Essam assumption: For various matching pairs $\Delta(\cdot,\mathcal{G})$ has at most one singularity, and if there is one it must be at $p_H(\mathcal{G})$. Unfortunately we have been unable to show that $\Delta(\cdot,\mathcal{G})$ has any singularity at p_H as a function of p only. (There is an obvious singularity if one brings in additional variables; compare the study of the function $f(h)$ in Kunz and Souillard (1978) or Remark 9.3 (iv) below.) We shall prove that for site- or bond-percolation on $\mathbb{Z}^2$ $\Delta(p)$ is twice continuously differentiable at all $p \in [0,1]$, including at p_H. The belief is (see Stauffer (1979), formula (6a) and Essam (1980), Formula (2.22)) that $\Delta(p)$ also satisfies a power law, i.e.,

$$\Delta(p,\mathcal{G}) \sim C_0|p-p_H|^{2-\alpha_\pm} \quad , \; p \to p_H,$$

for some $0 < \alpha_\pm < 1$ (α_+ corresponding to $p \downarrow p_H$ and α_- to $p \uparrow p_H$). In particular $(\frac{d}{dp})^3\Delta(p)$ should blow up as $p \to p_H$. So far we have been unable to show that any derivative of Δ fails to exist at p_H.

9.1 The existence of $\Delta(p)$.

Intuitively, the average number of clusters per site should be the limit (in some sense) of

$$\text{(9.3)} \qquad (\# \text{ of sites in } B_n)^{-1}(\# \text{ of occupied clusters in } B_n)$$

as B_n runs through a sequence of blocks which increase to the whole space. Sykes and Essam (1964) did not show that such a limit exists. This was first done by Grimmett (1976), and an expression for the limit was given by Wierman (1978). Their results follow quickly from the ergodic theorem and are reproduced in Theorem 9.1. Of course we must first define the expression in (9.3) properly.

Def. 9.1. For a block $B = \prod_1^d [a_i,b_i]$ and a vertex v of $\mathcal{G}$ in B, the occupied cluster of v (on $\mathcal{G}$) in B is the union of all edges and vertices of $\mathcal{G}$ which belong to an occupied path on $\mathcal{G}$ contained in B and with initial point v. ///

This is the obvious analogue of Def. 2.7. Note that two vertices v_1 and v_2 may belong to different occupied clusters in B, even though they belong to the same occupied cluster on the graph as a whole. This will happen if and only if there exists one or more occupied paths from v_1 to v_2, but all such paths go outside of B. When counting the number of occupied clusters in B the clusters of v_1 and v_2 will be counted as two separate clusters in this situation.

We also need the following notation. For any block $B = \prod[a_i,b_i]$ and v a vertex of $\mathcal{G}$ in B we set

$$(9.4)\qquad \Gamma(v,B) = \begin{cases} 0 \text{ if } v \text{ is vacant,} \\ (\text{\# of vertices of } \mathcal{G} \text{ in the occupied} \\ \text{cluster of } v \text{ in } B)^{-1} \text{ if } v \text{ is occupied.} \end{cases}$$

Also

$$(9.5)\qquad \Gamma(v) = \begin{cases} 0 \text{ if } v \text{ is vacant,} \\ (\#W(v))^{-1} \text{ if } v \text{ is occupied.} \end{cases}$$

(If $\#W(v) = \infty$, then $\Gamma(v) = 0$.)

Theorem 9.1. *Let $\mathcal{G}$ be a periodic graph imbedded in $\mathbb{R}^d$ with μ = number of vertices of $\mathcal{G}$ in $[0,1)^d$. Let P_p be the one-parameter probability distribution on the occupancy configurations of $\mathcal{G}$ determined by (9.1) and let $B(n_1,\dots,n_d) = [0,n_1]\times\dots\times[0,n_d]$. Then*

$$(9.6)\qquad \frac{\text{\# of occupied clusters in } B(n_1,\dots,n_d)}{\text{\# of vertices of } \mathcal{G} \text{ in } B(n_1,\dots,n_d)}$$

$$\to \frac{1}{\mu}\sum_{v\in[0,1)^d} E_p\{\frac{1}{\#W(v)}; \#W(v) \geq 1\}$$

$$= \frac{1}{\mu}\sum_{v\in[0,1)^d}\sum_{n=1}^{\infty}\frac{1}{n}P_p\{\#W(v) = n\} \quad ,$$

as $n_1 \to \infty,\dots,n_d \to \infty$ independently. The convergence in (9.6) holds a.e. $[P_p]$ and in every $L_r(P_p)$, $r > 0$.

Special case: When all vertices of $\mathcal{G}$ play the same role such as on the graphs $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_0^*$ and $\mathcal{G}_1^*$ considered in the last chapter, then the right hand side of (9.6) reduces to

$$\sum_{n=1}^{\infty}\frac{1}{n}P_p\{\#W(v) = n\} .$$

Remarks.

(i) Theorem 9.1 remains valid if $B(n_1,\dots,n_d)$ is replaced by the box $[-n_1,n_1]\times\dots\times[-n_d,n_d]$ which is symmetric with respect to the origin. This follows easily by writing $[-n_1,n_1]\times\dots\times[-n_d,n_d]$ as the union of 2^d boxes, to each of which one can apply Theorem 9.1 after an interchange of the positive and negative direction along a number of coordinate axes.

(ii) One can easily generalize Theorem 9.1 to λ-parameter periodic probability measures P_p, but we shall have no use for this generalization.

Proof of Theorem 9.1. By the periodicity of $\mathcal{G}$

$$\lim \frac{1}{n_1 \cdots n_d} (\text{\# of vertices of } \mathcal{G} \text{ in } B(n_1,\ldots,n_d)) = \mu .$$

It is also clear that

$$(9.7) \qquad \text{\# of occupied clusters in } B(n_1,\ldots,n_d)$$
$$= \sum_{v \in B(n_1,\ldots,n_d)} \Gamma(v, B(n_1,\ldots,n_d)),$$

since for any occupied cluster W in B, containing exactly the n vertices $v_1,\ldots,v_n \in B$, the right hand side of (9.7) contains

$$\Gamma(v_1,B) + \Gamma(v_1,B) + \ldots + \Gamma(v_n,B) = n \times \frac{1}{n} = 1.$$

It is clear from the definitions (9.4) and (9.5) that

$$(9.8) \qquad \Gamma(v,B) \geq \Gamma(v), \quad v \in B.$$

Moreover, the ergodic theorem (Dunford and Schwartz (1958), Theorem VIII.6.9 or Tempel'man (1972), Theorem 6.1 and Cor. 6.2; see also Harris (1960), Lemma 3.1) applied to the bounded function Γ shows that

$$\frac{1}{n_1 n_2 \cdots n_d} \sum_{0 \leq k_i < n_i} \Gamma(v + \sum_1^d k_i \xi_i) \to E_p\{\Gamma(v)\} \quad \text{a.e. } [P_p]$$

as $n_1,\ldots,n_d \to \infty$ for every $v \in [0,1)^d$. Since

$$E_p\{\Gamma(v)\} = \sum_{n=1}^{\infty} \frac{1}{n} P_p\{\#W(v) = n\}$$

it follows that

$$(9.9) \qquad (\text{\# of vertices in } B(n_1,\ldots,n_d))^{-1} \sum_{v \in B(n_1,\ldots,n_d)} \Gamma(v)$$
$$\to \frac{1}{\mu} \sum_{v \in [0,1)^d} \sum_{n=1}^{\infty} \frac{1}{n} P_p\{\#W(v) = n\} \quad \text{a.e. } [P_p].$$

This together with (9.7) and (9.8) also shows that

$$(9.10)\qquad \liminf_{n_i\to\infty} \frac{\#\text{ of occupied clusters in } B(n_1,\dots,n_d)}{\#\text{ of vertices of } \mathcal{G} \text{ in } B(n_1,\dots,n_d)}$$

$$\geq \frac{1}{\mu} \sum_{v\in[0,1)^d} \sum_{n=1}^{\infty} \frac{1}{n} P_p\{\#W(v) = n\}\ ,\ \text{a.e. } [P_p].$$

To obtain a bound in the other direction we note that $\Gamma(v,B) = \Gamma(v)$ whenever $W(v)$ is contained entirely in B. Consequently

$$(9.11)\qquad \#\text{ of occupied clusters in } B(n_1,\dots,n_d)$$

$$= \sum_{v\in B(n_1,\dots,n_d)} \Gamma(v,B(n_1,\dots,n_d))$$

$$\leq \sum_{v\in B(n_1,\dots,n_d)} \Gamma(v) + \#\text{ of occupied clusters in } B(n_1,\dots,n_d)$$

which are part of an occupied cluster on $\mathcal{G}$ which contains vertices outside $B(n_1,\dots,n_d)$.

The last term in the right hand side of (9.11) is bounded by $z \times$ the number of vertices of $\mathcal{G}$ in $\partial(B(n_1,\dots,n_d))$, i.e., $z \times$ the number of vertices outside $B(n_1,\dots,n_d)$ but adjacent to a vertex in $B(n_1,\dots,n_d)$. This is so because each occupied component which contains vertices inside and outside B must contain a vertex in ∂B (cf. (2.3) for z). If $\Lambda \geq$ diameter of any edge of $\mathcal{G}$, then any $v \in \partial B$ satisfies

$$-\Lambda \leq v(j) \leq n_j+\Lambda \quad \text{for} \quad 1 \leq j \leq d \quad \text{and}$$

$$-\Lambda \leq v(i) < 0 \quad \text{or} \quad n_i < v(i) \leq n_i+\Lambda \quad \text{for some} \quad 1 \leq i \leq d.$$

Thus the last term in (9.11) is bounded by

$$\#\partial(B(n_1,\dots,n_d)) \leq 2\mu(\Lambda+1)\left(\frac{1}{n_1}+\frac{1}{n_2}+\dots+\frac{1}{n_d}\right) \prod_1^d (n_j+2\Lambda+1).$$

This together with (9.11) and (9.9) shows

$$\limsup_{n_i\to\infty} \frac{\#\text{ of occupied clusters in } B(n_1,\dots,n_d)}{\#\text{ of vertices of } \mathcal{G} \text{ in } B(n_1,\dots,n_d)}$$

$$\leq \frac{1}{\mu} \sum_{v\in[0,1)^d} \sum_{n=1}^{\infty} \frac{1}{n} P_p\{\#W(v) = n\},\ \text{a.e. } [P_p].$$

Thus the convergence in (9.6) holds a.e. $[P_p]$. The convergence in

$L_r(P_p)$ follows from this since the left hand side of (9.6) lies between zero and one. $\square$

9.2 The Sykes-Essam relation for matching graphs.

In view of Theorem 9.1 we define for any periodic graph $\mathcal{G}$ and one-parameter probability measure P_p the "average number of occupied clusters per site" as

$$\Delta(p) = \Delta(p,\mathcal{G}) = \frac{1}{\mu} \sum_{v\in[0,1)^d} \sum_{n=1}^{\infty} \frac{1}{n} P_p\{\#W(v) = n\} \tag{9.12}$$

$$= \frac{1}{\mu} \sum_{v\in[0,1)^d} E_p\{\frac{1}{\#W(v)} ; \#W(v) \geq 1\} \quad .$$

We now prove (9.2).

Theorem 9.2. Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair of periodic groups in $\mathbb{R}^2$. Then there exists a polynomial $\Phi(p) = \Phi(p,\mathcal{G})$ for which

$$\Delta(p,\mathcal{G}) - \Delta(1-p,\mathcal{G}^*) = \Phi(p,\mathcal{G}),\ 0 \leq p \leq 1. \tag{9.13}$$

Remarks.

(iii) The pair $(\mathcal{G}^*,\mathcal{G})$ is also a matching pair (Comment 2.2 (v)). It is obvious from (9.13) that the corresponding $\Phi(p,\mathcal{G}^*)$ is given by

$$\Phi(p,\mathcal{G}^*) = -\Phi(1-p,\mathcal{G}). \tag{9.14}$$

(iv) The proof below will give an explicit expression for Φ:

$$\Phi(p,\mathcal{G}) = C_1' + (1-C_2')p + (C_3-C_2'')p^2 \tag{9.15}$$

$$-\frac{1}{\mu} \sum_{n=1}^{\infty} (1-p)^n \gamma_n(\mathcal{M},\mathcal{F}) + \frac{1}{\mu} \sum_{n=1}^{\infty} p^n \gamma_n^*(\mathcal{M},\mathcal{F}),$$

where C_i and C_i' are given in (9.21)-(9.25) and

> $\gamma_n(\mathcal{M},\mathcal{F})(\gamma^*(\mathcal{M},\mathcal{F}))$ = # of central vertices in $[0,1)\times[0,1)$ of a face F of $\mathcal{M}$ in $\mathcal{F}$ (not in $\mathcal{F}$) with exactly n vertices of $\mathcal{M}$ on the perimeter of F.

For example, when $\mathcal{G} = \mathcal{G}_0$, the simple quadratic lattice (see Ex. 2.1(i) and Ex. 2.2 (i)) one has $\mu = 1$, $\gamma_n = 0$ for all n, $\gamma_m^* = 0$ for $m \neq 4$

while $\gamma_4^* = 1$, and

$$\Phi(p,\mathcal{G}_0) = p - 2p^2 + p^4 ,$$

$$\Phi(p,\mathcal{G}_0^*) = 1 + (1-4)p + (4-2)p^2 - (1-p)^4 = p - 4p^2 + 4p^3 - p^4 . \qquad ///$$

Proof: Sykes and Essam's proof works with $\mathcal{G}$ directly. We find it easier to work with $\mathcal{G}_{p\ell}$ and so we shall first prove

(9.16) $\quad \Delta(p,\mathcal{G}_{p\ell}) - \Delta(1-p,\mathcal{G}_{p\ell}^*)$ is a polynomial in p, $\Phi(p,\mathcal{G}_{p\ell})$ say.

Following Sykes and Essam (1964) we first define the occupied and vacant graphs. For some mosaic $\mathcal{M}^\circ$ and subset $\mathcal{F}^\circ$ of its faces, let $(\mathcal{G}^\circ,\mathcal{G}^{\circ*})$ be the matching pair based on $(\mathcal{M}^\circ,\mathcal{F}^\circ)$, and let $\mathcal{G}_{p\ell}^\circ$, $\mathcal{G}_{p\ell}^{\circ*}$ and $\mathcal{M}_{p\ell}^\circ$ be the corresponding planar modifications as in Sect. 2.3. Any occupancy configuration ω of $\mathcal{M}^\circ$ is also an occupancy configuration of $\mathcal{G}^\circ$ and $\mathcal{G}^{\circ*}$, and can be extended to an occupancy configuration of $\mathcal{M}_{p\ell}^\circ$, $\mathcal{G}_{p\ell}^\circ$ and $\mathcal{G}_{p\ell}^{\circ*}$ by taking all central vertices of a face of $\mathcal{M}^\circ$ in $\mathcal{F}^\circ$ (not in $\mathcal{F}^\circ$) as occupied (vacant) as we did in (2.15), (2.16). For a fixed configuration ω we define $\mathcal{G}^\circ$ (ω, occupied) as the graph whose vertex set consists of the occupied vertices of $\mathcal{G}^\circ$ and whose edge set consists of all edges of $\mathcal{G}^\circ$ connecting two occupied vertices of $\mathcal{G}^\circ$. $\mathcal{G}_{p\ell}^\circ$ (ω, occupied) is defined in the same way by replacing $\mathcal{G}^\circ$ by $\mathcal{G}_{p\ell}^\circ$. Similarly $\mathcal{G}^{\circ*}$ (ω, vacant) and $\mathcal{G}_{p\ell}^{\circ*}$ (ω, vacant) are defined by replacing $\mathcal{G}^\circ$ by $\mathcal{G}^{\circ*}$ and $\mathcal{G}_{p\ell}^{\circ*}$, respectively, and "occupied" by "vacant". Note that the components of $\mathcal{G}^\circ$ (ω, occupied) are precisely the occupied clusters of $\mathcal{G}^\circ$, and similarly the components of $\mathcal{G}^{\circ*}$ (ω, vacant) are the vacant clusters of $\mathcal{G}^{\circ*}$.

Now, let our periodic pair $(\mathcal{G},\mathcal{G}^*)$ be based on $(\mathcal{M},\mathcal{F})$, a periodic mosaic and periodic subset of its faces. We shall apply Euler's relation to the planar graph $\mathcal{G}_{p\ell}$ (ω, occupied), or rather to a "truncated modification" of this graph, which we construct as follows. Let J^n be a circuit made up of edges of $\mathcal{M}_{p\ell}$, surrounding $(\Lambda_3,n-\Lambda_3) \times (\Lambda_3,n-\Lambda_3)$, and contained in the annulus

(9.17) $$[0,n] \times [0,n] \setminus (\Lambda_3,n-\Lambda_3) \times (\Lambda_3,n-\Lambda_3).$$

Here Λ_3 is a suitably large constant depending on $\mathcal{M}_{p\ell}$ only; we constructed this kind of circuit already in the proof of Lemma 7.1. Let $\mathcal{M}_{p\ell}^n$ be the graph obtained by removing from $\mathcal{M}_{p\ell}$ all edges and vertices which are not contained in $\bar{J}^n = J^n \cup \mathrm{int}(J^n)$. Thus $\mathcal{M}_{p\ell}^n$ has exactly one unbounded face, namely $ext(J^n)$, and the other faces of $\mathcal{M}^n_{p\ell}$ are exactly the faces of $\mathcal{M}_{p\ell}$ in $int(J^n)$. The unbounded face of $\mathcal{M}^n_{p\ell}$ contains no vertices and does not intersect any edges of $\mathcal{M}^n_{p\ell}$. An occupancy configuration ω of $\mathcal{M}_{p\ell}$ can be restricted to an occupancy configuration on $\mathcal{M}^n_{p\ell}$ and the corresponding graph $\mathcal{G}^n_{p\ell}$ (ω, occupied) is then defined as above. It is a planar graph since it is a subgraph of the planar graph $\mathcal{G}_{p\ell}$. We therefore have Euler's relation

$$V^n_{p\ell} - E^n_{p\ell} + F^n_{p\ell} = C^n_{p\ell} + 1 \ , \tag{9.18}$$

where $V^n_{p\ell}$, $E^n_{p\ell}$, $F^n_{p\ell}$ and $C^n_{p\ell}$ are the number of vertices, edges, faces and components of $\mathcal{G}^n_{p\ell}$ (ω, occupied), respectively. (Cf. Bollobás (1979), Theorem I.11 if $C^n_{p\ell} = 1$; the general case follows easily by induction on $C^n_{p\ell}$.) We need to look closer at $F^n_{p\ell}$. Note first that each vacant vertex of $\mathcal{M}_{p\ell}$ must be a vertex of $\mathcal{G}^*_{p\ell}$, since the only vertices of $\mathcal{M}_{p\ell}$ which do not belong to $\mathcal{G}^*_{p\ell}$ are central vertices of some face in $\mathcal{F}$, and these have all been taken as occupied. Therefore, if a face F of $\mathcal{G}^n_{p\ell}$ (ω, occupied) contains a vacant vertex of $\mathcal{M}^n_{p\ell}$, then it belongs to $\mathcal{G}^{n*}_{p\ell}$ (ω, vacant), and in this case F contains at least one component of $\mathcal{G}^*_{p\ell}$ (ω,vacant). Some examples will convince the reader that in this case F contains exactly one component of $\mathcal{G}^{n*}_{p\ell}$ (ω, vacant). A formal statement and proof of this fact is given in Prop. A.1 in the Appendix. Thus, if $C^{n*}_{p\ell}$ denotes the number of components of $\mathcal{G}^{n*}_{p\ell}$ (ω, vacant)

$$F^n_{p\ell} = C^{n*}_{p\ell} + \#\text{ of faces of } \mathcal{G}^n_{p\ell}\ (\omega,\text{ occupied}) \text{ which contain no vacant vertex of } \mathcal{M}^n_{p\ell}. \tag{9.19}$$

Let us call the faces of $\mathcal{G}^n_{p\ell}$ (ω, occupied) which contain no vacant vertex of $\mathcal{M}^n_{p\ell}$ empty faces. Recall now that $\mathcal{M}_{p\ell}$ is completely triangulated (Comment 2.3 (vi)). In other words each face of $\mathcal{M}_{p\ell}$ is a "triangle", bounded by three edges of $\mathcal{M}_{p\ell}$, and containing exactly three vertices of $\mathcal{M}_{p\ell}$ on its perimeter. We claim that the bounded empty faces of $\mathcal{G}^n_{p\ell}$ (ω, occupied) are precisely those triangular faces of $\mathcal{M}_{p\ell}$ in $int(J_n)$ with all three of its boundary vertices belonging to $\mathcal{G}_{p\ell}$ and occupied. Such faces are therefore also faces of $\mathcal{G}_{p\ell}$. To see this consider a face G of $\mathcal{G}^n_{p\ell}$ (ω, occupied) and let e be an edge of $\mathcal{M}_{p\ell}$ in $Fr(G)$. e necessarily is an edge of $\mathcal{G}_{p\ell}$ in $\bar{J}_n$, and its endpoints, v_1 and v_2, say, are necessarily occupied. $\mathring{e}$ belongs to the boundary of exactly two triangular faces, F_1 and F_2 say, of $\mathcal{M}_{p\ell}$. Each F_i belongs to a unique face, G_i say, of $\mathcal{G}^n_{p\ell}$ (ω, occupied). $\mathring{e}$ belongs only to the boundary of G_1 and G_2, but not to the boundary of any other face of $\mathcal{G}^n_{p\ell}$ (ω, occupied), so that G is one of G_1 or G_2 ($G_1 = G_2$ is possible, though). Let w_i the third vertex of $\mathcal{M}_{p\ell}$ on the perimeter of F_i (in addition to v_1 and v_2). If F_i lies in $\text{ext}(J_n)$ then F_i is contained in the unbounded face of $\mathcal{G}^n_{p\ell}$, and hence also in the unbounded face of $\mathcal{G}^n_{p\ell}$ (ω, occupied). In this case G_i equals the unbounded face of $\mathcal{G}^n_{p\ell}$ (ω, occupied). If $F_i \subset \text{int}(J_n)$ and w_i is occupied, then in particular w_i cannot be a central vertex of $\mathcal{G}^*_{p\ell}$, since these are taken vacant. Therefore all vertices on the perimeter of F_i belong to $\mathcal{G}^n_{p\ell}$ and are occupied, and consequently belong to $\mathcal{G}^n_{p\ell}$ (ω, occupied). In this case F_i is itself a face of $\mathcal{G}^n_{p\ell}$ (ω, occupied) and $G_i = F_i$. Finally if $F_i \subset \text{int}(J_n)$ but w_i is vacant, then G_i contains the vacant vertex w_i. (Since no edges of $\mathcal{G}^n_{p\ell}$ (ω, occupied) are incident to w_i, so that a full neighborhood of w_i belongs to one face of $\mathcal{G}^n_{p\ell}$ (ω, occupied); this face must therefore contain F_i and cannot be any other face than G_i.) The only bounded empty faces of $\mathcal{G}^n_{p\ell}$ (ω, occupied) which we encountered in the above list was the triangle F_i, in the case where w_i was occupied and $F_i \subset \text{int}(J_n)$. This proves our claim. As a consequence (9.19) can be written as

$$F^n_{p\ell} = C^{n*}_{p\ell} + T^n_{p\ell} + \varepsilon_n \,, \tag{9.20}$$

where

$T^n_{p\ell}$ = # of triangular faces of $\mathcal{M}_{p\ell}$ in $\bar{J}_n$ with all three vertices on their perimeter occupied

and

$$\varepsilon_n = \begin{cases} 1 & \text{if the unbounded face of } \mathcal{G}^n_{p\ell} \text{ } (\omega, \text{ occupied}) \\ & \text{is an empty face,} \\ 0 & \text{otherwise.} \end{cases}$$

We substitute (9.20) into (9.18), divide by μn^2 and take limits as $n \to \infty$. This gives

$$\lim_{n\to\infty} \frac{1}{\mu n^2}[V^n_{p\ell} - E^n_{p\ell} + T^n_{p\ell} + C^{n*}_{p\ell} - C^n_{p\ell}] = 0 \quad \text{a.e. } [P_p].$$

It therefore suffices for (9.16) to show that a.e. $[P_p]$

$$\lim_{n\to\infty} \frac{1}{\mu n^2} V^n_{p\ell} = C_1' + C_1'' p ,$$

$$\lim_{n\to\infty} \frac{1}{\mu n^2} E^n_{p\ell} = C_2' p + C_2'' p^2 ,$$

$$\lim_{n\to\infty} \frac{1}{\mu n^2} T^n_{p\ell} = C_3 p^2 ,$$

$$\lim_{n\to\infty} \frac{1}{\mu n^2} C^n_{p\ell} = \Delta(p,\mathcal{G}) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\mu n^2} C^{*n}_{p\ell} = \Delta(1-p,\mathcal{G}^*)$$

for suitable $C_1', C_1'', \dots, C_j$. In fact these relations are easily proved from the ergodic theorem, with the constants C_i and C_i' determined as follows: Order the vertices of $\mathcal{G}_{p\ell}$ lexicographically, i.e., $v = (v(1), v(2))$ precedes $w = (w(1), w(2))$ iff $v(1) < w(1)$ or $v(1) = w(1)$ and $v(2) < w(2)$. Then

(9.21) $\quad C_1' = \frac{1}{\mu}\{\#$ of central vertices of $\mathcal{G}_{p\ell}$ in $[0,1)\times[0,1)\}$,

(9.22) $\quad C_1'' = \frac{1}{\mu}\{\#$ of vertices of $\mathcal{G}$ in $[0,1)\times[0,1)\} = 1$,

(9.23) $\quad C_2' = \frac{1}{\mu}\{\#$ of edges of $\mathcal{G}_{p\ell}$ between two vertices, v_1 and v_2 say, such that v_1 precedes v_2, $v_1 \in [0,1)\times[0,1)$ and such that v_1 or v_2 is a central vertex of $\mathcal{G}_{p\ell}\}$

(9.24) $\quad C_2'' = \frac{1}{\mu}\{\#$ of edges of $\mathcal{G}_{p\ell}$ between two vertices, v_1 and v_2 say, such that v_1 precedes v_2, $v_1 \in [0,1)\times[0,1)$ and such that v_1 and v_2 are both non-central vertices of $\mathcal{G}_{p\ell}\}$

(9.25) $\quad C_3 = \frac{1}{\mu}\{\#$ of triangular faces of $\mathcal{G}_{p\ell}$ with vertices v, v_1 and v_2, say, on its perimeter, with v preceding v_1 and v_2 and $v \in [0,1)\times[0,1))$.

We only prove

(9.26) $$\lim_{n\to\infty} \frac{1}{\mu n^2} T^n_{p\ell} = C_3 p^2 .$$

The other relations are proved in a similar way. Now, J_n is contained in the annulus (9.17). Therefore

(9.27) $T^n_{p\ell} \leq$ number of triangular faces of $\mathcal{M}_{p\ell}$ contained in $[0,n] \times [0,n]$ with all three vertices on their perimeter occupied,

while the inequality has to be reversed if $[0,n] \times [0,n]$ is replaced by $(\Lambda_3, n-\Lambda_3) \times (\Lambda_3 - n - \Lambda_3)$. Now let

$N(v)$ = # of triangular faces of $\mathcal{M}_{p\ell}$ with vertices v, w_1 and w_2, say, on their perimeter such that v precedes w_1 and w_2 and such that v, w_1 and w_2 are occupied .

Then the right hand side of (9.27) clearly equals

$$\sum_{v \varepsilon [0,1) \times [0,1)} \sum_{\substack{0 \leq k_1 < n \\ 0 \leq k_2 < n}} N(v + k_1 \xi_1 + k_2 \xi_2) + O(n) \tag{9.28}$$

where $\xi_1 = (1,0)$, $\xi_2 = (0,1)$, and the $O(n)$ term is at most equal to the number of triangular faces of $\mathcal{M}_{p\ell}$ whose closure intersects $\mathrm{Fr}([0,n] \times [0,n])$. Thus, by (9.27) and the ergodic theorem (Dunford and Schwartz (1958) Theorem VIII.6.9 or Tempel'man (1972), Theorem 6.1 and Cor. 6.2)

$$\limsup \frac{1}{\mu n^2} T^n_{p\ell} \leq \frac{1}{\mu} \sum_{v \varepsilon [0,1) \times [0,1)} E_p N(v) \quad \text{a.e. } [P_p]. \tag{9.29}$$

To calculate $\sum E_p N(v)$ we have to recall that $\mathcal{M}_{p\ell}$ is constructed by inserting a central vertex in each face F of $\mathcal{M}$, and by connecting this central vertex v say by an edge to each vertex of $\mathcal{M}$ on the perimeter of F. This means that the triangular faces of $\mathcal{M}_{p\ell}$ all have one central vertex w and two non-central vertices v_1 and v_2 say on their perimeter. If w is a central vertex of $\mathcal{G}^*_{p\ell}$, i.e., lies in a face $F \notin \mathfrak{F}$, then it is vacant and the triangle with vertices w, v_1 and v_2 cannot contribute to any $N(v)$. If w is a vertex of $\mathcal{G}_{p\ell}$, i.e., lies in a face $F \varepsilon \mathfrak{F}$, then it is occupied with probability one, the triangle with vertices w, v_1 and v_2 is a face of $\mathcal{G}_{p\ell}$, and all three vertices w, v_1 and v_2 are occupied with probability p^2.

Consequently

$$\sum_{v\varepsilon[0,1)\times[0,1)} E_pN(v) = p^2C_3 .$$

Together with (9.29) this shows that

$$\limsup \frac{1}{\mu n^2} T^n_{p\ell} \le C_3p^2 \quad \text{a.e. } [P_p].$$

It follows similarly from the lower bound given after (9.27) that

$$\liminf \frac{1}{\mu n^2} T^n_{p\ell} \ge C_3p^2 \quad \text{a.e. } [P_p].$$

This proves (9.26) and (9.16).

To obtain (9.13) from (9.16) we merely have to show that

(9.30) $$\Delta(p,\mathcal{G}_{p\ell}) - \Delta(p,\mathcal{G}) = \frac{1}{\mu} \sum_{v\varepsilon[0,1)\times[0,1)} \sum_{n=1}^{\infty} (1-p)^n I[v \text{ is a central vertex of a face of } \mathcal{M} \text{ in } \mathfrak{F} \text{ with } n \text{ vertices on its perimeter}].$$

Indeed the right hand side of (9.30) is only a finite sum by (2.3), (2.4), hence a polynomial in p. Also, interchanging the roles of $\mathcal{G}$ and $\mathcal{G}^*$,

$$\Delta(1-p,\mathcal{G}^*_{p\ell}) - \Delta(1-p,\mathcal{G}^*) \text{ is a polynomial in } p.$$

To prove (9.30) we use Cor. 2.1. This corollary shows that each occupied cluster on $\mathcal{G}$ belongs to a unique occupied cluster on $\mathcal{G}_{p\ell}$. Moreover, if $W(v_1)$ and $W(v_2)$ are two distinct occupied clusters on $\mathcal{G}$, then the occupied clusters $W_{p\ell}(v_1)$ and $W_{p\ell}(v_2)$ on $\mathcal{G}_{p\ell}$ to which they belong are also disjoint, since by (2.20) any vertex w of $W_{p\ell}(v_1) \cap W_{p\ell}(v_2)$ would have to be a central vertex of $\mathcal{G}$, adjacent to some $w_i \varepsilon W(v_i)$ for $i = 1,2$. But then w_1 and w_2 would lie on the perimeter of a close-packed face of $\mathcal{G}$ (cf. Comment 2.3 (iv)) and would be adjacent on $\mathcal{G}$ and hence belong to the same cluster. On the other hand it is possible to have an occupied cluster on $\mathcal{G}_{p\ell}$ which does not contain an occupied cluster on $\mathcal{G}$. Again by (2.20), this can occur only if the cluster on $\mathcal{G}_{p\ell}$ contains no vertex v of $\mathcal{G}$ - otherwise it equals $W_{p\ell}(v)$ which contains $W(v)$. Since two central vertices are never adjacent on $\mathcal{G}_{p\ell}$ (Comment 2.3 (iv)) this means that the only occupied clusters on $\mathcal{G}_{p\ell}$ which do not contain a cluster on $\mathcal{G}$ are isolated central vertices, i.e., central vertices of a face $F \varepsilon \mathfrak{F}$ with all vertices on the perimeter of F vacant (the central vertex is automatically occupied by (2.15)). From the above observations it follows that

$$\begin{aligned}&|(\#\text{ of occupied clusters on } \mathcal{G}_{p\ell} \text{ on } B(n,n))\\ &-(\#\text{ of occupied clusters on } \mathcal{G} \text{ in } B(n,n))\\ &-(\#\text{ of central vertices of } \mathcal{G}_{p\ell} \text{ in } B(n,n) \text{ which belong to}\\ &\quad \text{a face with only vacant vertices on its perimeter})|\\ &\le z\cdot(\#\text{ of vertices of } \mathcal{G}_{p\ell} \text{ in } \partial(B(n,n))\end{aligned}$$

(compare with the estimate for the last term in (9.11)). (9.30) now follows from Theorem 9.1 and another application of the ergodic theorem. □

9.3 Smoothness of $\Delta(p)$.

Theorem 9.3. Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair of periodic graphs in $\mathbb{R}^2$. Then $\Delta(p,\mathcal{G})$ is an analytic function of p outside the interval $[p_T(\mathcal{G}),1-p_T(\mathcal{G}^*)]$ (see (3.63) for p_T). If the conditions of Theorem 3.1 are fulfilled for $\lambda = 1$ (i.e., in the one-parameter problem) and some $0 < p_0 < 1$, then $\Delta(p,\mathcal{G})$ is analytic for $p \neq p_H(\mathcal{G}) = p_0$.

Remarks.

(i) In particular if $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G} = \mathcal{G}_1$, then $\Delta(p,\mathcal{G})$ is analytic, except possibly at $p_H(\mathcal{G})$.

(ii) The proof will also show that $E_p\{\pi(\#W(z_0))\}$ is an analytic function of p on $0 \le p < p_T(\mathcal{G})$, for any polynomial π. Theorem 5.3 shows that the function $p \to E_p\{\pi(\#W(z_0));\#W(z_0) < \infty\}$ is infinitely often differentiable on $p_H(\mathcal{G}) < p \le 1$ (cf. Russo (1978)). ///

Proof: This theorem is immediate from Theorems 5.1, 9.1 and 9.2. Indeed for $p \le p_1 < p_T(\mathcal{G})$ we have by (5.11) and Lemma 4.1

$$P_p\{\#W(z_0) \ge n\} \le P_{p_1}\{\#W(z_0) \ge n\} \le C_1e^{-C_2n} \tag{9.31}$$

for each vertex z_0 and some constants C_1, C_2 depending on p_1 and $\mathcal{G}$ only. Now take $a(n,\ell) = a(n,\ell,z_0)$ as in (5.18), (5.19). By (5.24) (with $q = 1-p$)

$$P_p\{\#W(z_0) = n\} = \sum_\ell a(n,\ell)p^nq^\ell = \sum_\ell a(n,\ell)p^n(1-p)^\ell \quad , \tag{9.32}$$

and by (5.25) the sum over ℓ may be restricted to $\ell = 1,\dots,zn$. Thus (9.12) can be written as

$$\Delta(p) = \frac{1}{\mu} \sum_{v\varepsilon[0,1)\times[0,1)} \sum_{n=1}^{\infty} \sum_{\ell=1}^{zn} \frac{1}{n} a(n,\ell,v)p^n(1-p)^\ell .$$

It therefore suffices to prove for fixed z_0 that

$$\sum_{n=1}^{\infty} \sum_{\ell=1}^{zn} \frac{1}{n} a(n,\ell)p^n(1-p)^\ell$$

is analytic in p on $[0,p_1]$, whenever $p_1 < p_T(\mathcal{G})$. But for any such $p \neq 0$ and a complex number ζ with

$$|\zeta-p| \leq \delta \tag{9.33}$$

we have for $\ell \leq zn$

$$\begin{aligned}|a(n,\ell)\zeta^n(1-\zeta)^\ell| &\leq (\tfrac{p+\delta}{p})^n(\tfrac{1-p+\delta}{1-p})^\ell \; a(n,\ell)p^n(1-p)^\ell \\ &\leq (\tfrac{p+\delta}{p})^n(\tfrac{1-p+\delta}{1-p})^\ell \; P_p\{\#W(z_0) \geq n\} \\ &\leq C_1\{e^{-C_2}(\tfrac{p+\delta}{p})(\tfrac{1-p+\delta}{1-p})^z\}^n .\end{aligned}$$

Thus for $0 < p \leq p_1$, we can choose δ such that

$$\sum_{n=1}^{\infty} \sum_{\ell=1}^{zn} \frac{1}{n} a(n,\ell)\zeta^n(1-\zeta)^\ell$$

converges uniformly in the disc defined by (9.33). For p close to zero we have the estimate

$$|a(n,\ell)\zeta^n(1-\zeta)^\ell| \leq a(n,\ell)|\zeta|^n \leq \{z^{-z}(z+1)^{z+1}|\zeta|\}^n ,$$

by virtue of (5.22), so that analyticity holds on $|\zeta| < z^z(z+1)^{-z-1}$. A slightly improved version of this last argument already appears in Kunz and Souillard (1978). This proves the analyticity of $\Delta(p,\mathcal{G})$ on $[0,p_T(\mathcal{G}))$ and consequently also of $\Delta(p,\mathcal{G}^*)$ on $[0,p_T(\mathcal{G}^*))$. But then $\Delta(p,\mathcal{G})$ is also analytic on $(1-p_T(\mathcal{G}^*),1]$, by virtue of (9.13). This proves the first statement in the theorem.

If for some $p_0 \varepsilon (0,1)$ Condition A or B of Sect. 3.2 holds, and $\mathcal{G}$ has an axis of symmetry as required in Theorem 3.1, then Theorem 3.1 shows that

$$p_T(\mathcal{G}) = p_H(\mathcal{G}) = p_0 = 1-p_T(\mathcal{G}^*) = 1-p_H(\mathcal{G}^*).$$

In such a case we obtain that $\Delta(p,\mathcal{G})$ is analytic for all $p \neq p_H(\mathcal{G})$ as claimed. $\square$

Theorem 9.4. *Let $\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_0^*$ or $\mathcal{G}_1^*$ (see Ex. 2.1 (i), 2.1 (ii), 2.2 (i), 2.2 (ii) for these graphs). Then $\Delta(p,\mathcal{G})$ is twice continuously differentiable in p on all of $[0,1]$.*

Proof: In view of Theorem 9.3 and its proof it suffices to show that

$$\sum_{n=N}^{\infty} \sum_{\ell=1}^{zn} \frac{1}{n} a(n,\ell) \left|\left(\frac{d}{dp}\right)^r p^n(1-p)^\ell\right| \to 0 \quad (N \to \infty)$$

uniformly for p in some neighborhood of $p_H(\mathcal{G})$, and $r = 1,2$. Now, with $q = 1-p$,

$$\frac{d}{dp} p^n(1-p)^\ell = \left(\frac{n}{p} - \frac{\ell}{q}\right)p^n q^\ell ,$$

$$\frac{d^2}{dp^2} p^n(1-p)^\ell = \left(\frac{n}{p} - \frac{\ell}{q}\right)^2 p^n q^\ell - \left(\frac{n}{p^2} + \frac{\ell}{q^2}\right)p^n q^\ell .$$

We shall only prove that

$$\sum_{n=N}^{\infty} \sum_{\ell=1}^{zn} \frac{1}{n} a(n,\ell)\left(\frac{n}{p} - \frac{\ell}{q}\right)^2 p^n q^\ell \to 0 \quad (N \to \infty) \tag{9.34}$$

uniformly in a neighborhood of p_H. The other terms can all be handled in the same way. To estimate (9.34) we split the sum over ℓ into two pieces: the ℓ with

$$\left|\frac{n}{p} - \frac{\ell}{q}\right| \leq n^{\frac{1}{2} + \frac{1}{8}\gamma_5} , \tag{9.35}$$

and the ℓ with

$$\left|\frac{n}{p} - \frac{\ell}{q}\right| > n^{\frac{1}{2} + \frac{1}{8}\gamma_5} , \tag{9.36}$$

where γ_5 is as in Theorem 8.2. The sum over the ℓ satisfying (9.35) contributes at most

$$\sum_{n=N}^{\infty} \sum_{\ell=1}^{zn} n^{\frac{1}{4}\gamma_5} a(n,\ell)p^n q^\ell = \sum_{n=N}^{\infty} n^{\frac{1}{4}\gamma_5} P_p\{\#W = n\} \quad \text{(see (9.32))} \tag{9.37}$$

$$\leq N^{-\frac{1}{4}\gamma_5} E_p\{(\#W)^{\frac{1}{2}\gamma_5}; \#W < \infty\} \leq C_{10} N^{-\frac{1}{4}\gamma_5} ,$$

by Theorem 8.2. For the sum over the ℓ satisfying (9.36) we use Lemma 5.1. We take

$$x = n^{\frac{1}{8}\gamma_5 - \frac{1}{2}}$$

in (5.23). We then find that the sum over the ℓ satisfying (9.36) contributes at most

$$\sum_{n=N}^{\infty} \frac{1}{n}\left(\frac{n}{p}+\frac{zn}{q}\right)^2 zn \exp -\frac{1}{3} n^{\frac{1}{4}\gamma_5} p^2 q \ ,$$

which obviously tends to zero as $N \to \infty$, uniformly for p in some neighborhood of $p_H(\mathcal{G}) \in (0,1)$. $\square$

<u>Remarks.</u>

(iii) Since we only know that $\gamma_5 > 0$ we cannot push the argument above further to obtain a third derivative of $\Delta(\cdot)$. As observed in the introduction to this Chapter it is assumed that $(\frac{d}{dp})^3 \Delta(p)$ blows up at p_H. It should be noted that one needs none of the difficult estimates of Ch. 8 for the present proof if $p \le p_H(\mathcal{G})$. Indeed, for such p one obtains

$$P_p\{\#W \ge n\} \le P_{p_H}\{\#W \ge n\} \le C_{22} n^{-\alpha_2}$$

from the very simple Lemma 8.5 (cf. (8.113)). This is enough to make the above estimates go through for $p \le p_H(\mathcal{G})$ and to conclude that $\Delta(\cdot)$ has two continuous derivatives on $[0,p_H(\mathcal{G}))$ and these have finite limits as $p \uparrow p_H(\mathcal{G})$. Applying this to $\mathcal{G}^*$ and using Theorem 9.2 we see that there also exist two continuous derivatives on $(p_H(\mathcal{G}),1]$ and that these have finite limits as $p \downarrow p_H(\mathcal{G})$. Thus the hard part of the above theorem is that Δ' and Δ'' do not have a jump at p_H. In fact Grimmett (1981) already gave a simple proof of this for the first derivative. For $\mathcal{G} = \mathcal{G}_1$ we can use the fact that $\mathcal{G}_1^*$ is isomorphic to $\mathcal{G}_1$ whence $\Delta(\cdot,\mathcal{G}_1^*) = \Delta(\cdot,\mathcal{G}_1)$ and

$$\Delta(p,\mathcal{G}_1) = \Delta(1-p,\mathcal{G}_1) + \Phi(p,\mathcal{G}_1) \ .$$

The polynomial Φ must be an odd function of $p - \frac{1}{2}$ therefore, and $\Phi''(\frac{1}{2},\mathcal{G}) = 0$ is then automatic. This shows that for $\mathcal{G} = \mathcal{G}_1$ even the second derivative of Δ must be continuous at $p_H(\mathcal{G}_1) = \frac{1}{2}$. It does not seem possible to handle $\Delta''(p,\mathcal{G}_0)$ in the same simple way.

(iv) Kunz and Souillard (1978) discuss the series

$$\sum_{n=1}^{\infty} e^{-nh}\pi(n)P_p\{\#W = n\} = E_p\{\pi(\#W)e^{-h\#W}\}$$

for a polynomial π or $\pi(n) = \frac{1}{n}$. The series converges for all $p \in [0,1]$, $h \geq 0$. It is not analytic in h at $h = 0$, $p > p_H(\mathcal{G})$, whenever π is always nonnegative. In fact, if we write ζ for e^{-h}, then

$$\sum e^{-nh}n^{\delta} P_p\{\#W = n\} = \sum \zeta^n n^{\delta} P_p\{\#W = n\}$$

is a power series with positive coefficients in ζ, whose radius of convergence equals 1 whenever $p > p_H(\mathcal{G})$ (by Theorem 5.2). The same is true for $p = p_H(\mathcal{G})$ if $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$ by (8.9). Such a power series has a singularity at $\zeta = 1$ by Pringsheim's theorem (Hille (1959), Theorem 5.7.1).

We also point out that if we view

$$\Delta(p) = \sum_{n=1}^{\infty} \sum_{\ell} \frac{1}{n} a(n,\ell)p^n q^{\ell}$$

as a function of two independent variables p and q, then

$$\frac{\partial^2}{\partial p^2} \sum_{n=1}^{\infty} \sum_{\ell} \frac{1}{n} a(n,\ell)p^n q^{\ell} \tag{9.38}$$

$$= \frac{1}{p^2} \sum_{n=1}^{\infty} \sum_{\ell} (n-1)a(n,\ell)p^n q^{\ell} = \frac{1}{p^2} E_p\{(\#W-1);\#W < \infty\}$$

on the set $\{q = 1-p\}$. By (5.17) the right hand side of (9.38) blows up as $p \to p_T(\mathcal{G})$. Despite these facts we could not show that $\Delta(p)$ has a singularity at $p = p_H$ when viewed as a function of the single variable p.