8. POWER ESTIMATES.

In this chapter we study the behavior of the percolation probability and the expected size of an occupied cluster in a one-paremeter problem. As defined in Ch. 3 this means that we consider probability measures for which

$$P_p\{v \text{ is occupied}\} = p$$

is the same for all vertices v of the studied graph $\mathcal{G}$, and the occupancies of all vertices are independent. We want to know the asymptotic behavior of

$$\theta(p) = \theta(p,z_0) = P_p\{\#W(z_0) = \infty\}$$

and of [1]

$$E_p\{\#W(z_0);\ \#W(z_0) < \infty\}$$

as p approaches the critical probability p_H (see Sect. 3.4). By analogy with results in statistical mechanics, and on the basis of numerical evidence (see Stauffer (1979) and Essam (1980)) it is generally believed that

$$\text{(8.1)} \qquad \theta(p) \sim C_0(p-p_H)^{\beta} \quad , \quad p \downarrow p_H ,$$

$$\text{(8.2)} \qquad E_p\{\#W(z_0);\ \#W(z_0) < \infty\} \sim C_+(p-p_H)^{-\gamma+} , \ P \downarrow p_H$$

and

$$\text{(8.3)} \qquad E_p\{\#W(z_0)\} \sim C_-(p_H-p)^{-\gamma-} , \ p \uparrow p_H$$

for suitable constants C_0 $C_\pm$ and $0 < \beta, \gamma_\pm < \infty$. Similar power laws are conjectured for other quantities. It is also conjectured that the so-called critical exponents $\beta, \gamma_\pm$ do not depend (or

1) $E\{X;A\}$ stands for $E\{X I_A\}$, i.e., the integral of X over the set A.

depend very little) on the detailed structure of $\mathcal{G}$, but depend (almost) entirely on the dimension of $\mathcal{G}$ only. In other words, these exponents should be (almost) the same for all periodic graphs $\mathcal{G}$ imbedded in $\mathbb{R}^d$ with one particular d . As far as the author knows powerlaws like (8.1)-(8.3) have been established mathematically for very few models, and not at all for any of the percolation models discussed in this monograph. It is not even clear how strictly (8.1) should be interpreted. Does it mean

$$\lim_{p \downarrow p_H} \frac{\theta(p)}{(p-p_H)^\beta} = c_0 \neq 0 ,$$

or

$$\frac{\theta(p)}{(p-p_H)^\beta} \text{ is a slowly varying function of } p-p_H \text{ as } p \downarrow p_H,$$

or perhaps only

$$\lim_{p \downarrow p_H} \frac{\log \theta(p)}{\log (p-p_H)} = \beta \quad ?$$

A similar comment applies to (8.2), (8.3) and other conjectured power laws. The best we can prove so far is that the left hand sides of (8.1)-(8.3) are bounded above and below by suitable powers of $|p-p_H|$ for percolation problems on certain two dimensional graphs $\mathcal{G}$. We believe that the method of proof will work for many graphs in the plane in which the horizontal and vertical direction play symmetric roles, but to simplify matters somewhat we restrict ourselves here to site - and bond - percolation on the simple quadratic lattice. The graph for site percolation on $\mathbb{Z}^2$ is $\mathcal{G}_0$ of Ex. 2.1 (i). In keeping with the tenor of these notes we treat bond percolation on $\mathbb{Z}^2$ in its equivalent version as site percolation on the graph $\mathcal{G}_1$ of Ex. 2.1 (ii) (see Sect. 2.5, especially Ex. 2.5 (ii)). We also deal with the matching graph $\mathcal{G}_0^*$ of $\mathcal{G}_0$ described in Ex. 2.2 (i) and the matching graph $\mathcal{G}_1^*$ of $\mathcal{G}_1$. $\mathcal{G}_1^*$ is isomorphic to $\mathcal{G}_1$ (see Ex. 2.2 (ii)). When $\mathcal{G} = \mathcal{G}_i^*$, $i = 0$ or 1, then $\mathcal{G}^*$ will be the graph $\mathcal{G}_i$ itself, in accordance with Comment 2.2 (v) .

The principal result of this chapter is the following theorem.

<u>Theorem 8.1</u>. <u>For one-parameter site-percolation on</u> $\mathcal{G} = \mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_0^*$ <u>or</u> $\mathcal{G}_1^*$ <u>there exist constants</u> $0 < C_i, \beta_i, \gamma_i < \infty$ <u>such that for</u> $p_H = p_H(\mathcal{G})$ <u>one has</u>

$$(8.4)\qquad C_3(p-p_H)^{\beta_1} \le \theta(p) \le C_4(p-p_H)^{\beta_2} \quad , \quad p \ge p_H \ ,$$

$$(8.5)\qquad C_5(p_H-p)^{-\gamma_1} \le E_p\{\#W\} \le C_6(p_H-p)^{-\gamma_2} \quad , \quad p \le p_H \quad ,$$

and

$$(8.6)\qquad C_7(p-p_H)^{-\gamma_3} \le E_p\{\#W;\ \#W < \infty\} \le C_8(p-p_H)^{-\gamma_4} \quad , \ p > p_H \ .$$

In the course of proving this theorem we derive the following estimates, some of which will be used in the next chapter.

Theorem 8.2. For one-parameter site-percolation on $\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_0^*$, or $\mathcal{G}_1^*$ there exist constants $0 < C_i, \gamma_i < \infty$ such that uniformly for $0 \le p \le 1$

$$(8.7)\qquad P_p\{n \le \#W < \infty\} \le C_9 n^{-\gamma_5}$$

and

$$(8.8)\qquad E_p\{(\#W)^{\frac{1}{2}\gamma_5}\ ;\ \#W < \infty\} \le C_{10} \quad .$$

Also, at $p = p_H = p_H(\mathcal{G})$

$$(8.9)\qquad C_{11} n^{\gamma_6^{-1}} \le P_{p_H}\{\#W \ge n\} = P_{p_H}\{n \le \#W < \infty\} \le C_9 n^{-\gamma_5} \ .$$

Remark .

For $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G}_1^*$ $p_H(\mathcal{G}) = \frac{1}{2}$ by Application 3.4 (ii). Also, by Application 3.4 (iv) (see also Russo (1981))

$$p_H(\mathcal{G}_1) = 1 - p_H(\mathcal{G}_1^*) \ .$$

In the graphs considered here all vertices play the same role so that $\theta(p,z_0)$ and the distribution of $\#W(z_0)$ are the same for all vertices z_0 . Therefore no reference to z_0 is necessary in the theorem. Finally, for $p \le p_H$ $\#W < \infty$ with P_p-probability one (see Theorem 3.1 (ii)). Therefore, for $p \le p_H$ (8.8) simply becomes

$$E_p\{(\#W)^{\frac{1}{2}\gamma_5}\} \le C_{10} \ . \qquad ///$$

In each of (8.4)-(8.6) one of the inequalities is much easier to prove than the other one. In (8.4) the first inequality is the difficult one. To motivate our principal lemma we shall work backwards from this inequality. Assume then that we want to prove

$$\theta(p) \ge C_3(p-p_H)^{\beta_1} \quad , \quad p \ge p_H \ .$$

First we fix a vertex z_0 . If $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_0^*$ we take $z_0 = (0,0)$, the origin, and if $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G}_1^*$ we take $z_0 = (\frac{1}{2},0)$. We introduce the following notation. For any vertex $v = (v(1),v(2))$ of $\mathcal{G}$

$$(8.10) \qquad S(v,M) = [v(1)-M,\ v(1)+M] \times [v(2)-M,\ v(2)+M]$$

(a square around v). The topological boundary of $S(v,M)$ is denoted by

$$(8.11) \qquad \Delta S(v,M) = \{x = (x(1),x(2)) : |x(1)-v(1)| = M \ , \ |x(2)-v(2)| \le M \text{ or } |x(1)-v(1)| \le M \ , \ |x(2)-v(2)| = M\} \ .$$

If some point $y \in \Delta S(z_0,M)$ belongs to an edge in $W = W(z_0)$, then W can be finite only if there exists a vacant circuit on $\mathcal{G}^*$ surrounding z_0 and y , by virtue of Cor. 2.2. Such a circuit c^* must contain at least M vertices. (e.g. if y is on the top edge of ΔS , then c^* must contain a vertex below the horizontal line $x(2) = 0$ and a vertex above the horizontal line $x(2) = M$.) Consequently, for any M

$$\theta(p) \ge P_p\{ \exists \text{ occupied path on } \mathcal{G} \text{ from } z_0 \text{ to some } y \text{ on } \Delta S(v_0,M) \text{ but there does not exist a vacant circuit on } \mathcal{G}^* \text{ surrounding } z_0 \text{ and containing at least } M \text{ vertices}\} \ .$$

By the FKG inequality this implies

$$(8.12) \qquad \theta(p) \ge P_p\{ \exists \text{ occupied path on } \mathcal{G} \text{ from } z_0 \text{ to some } y \text{ on } \Delta S(z_0,M)\} \ . \ P_p\{\text{there does not exist a vacant circuit on } \mathcal{G}^* \text{ surrounding } 0 \text{ and containing at least } M \text{ vertices}\}.$$

It is not hard to prove (see Smythe and Wierman (1978), formula (3.34); a better estimate is in Lemma 8.4 below) that the first factor in the right hand side of (8.12) is at least C_{12}/M for any $p \ge p_H$. To estimate the second factor, observe that any circuit on $\mathcal{G}^*$ surrounding z_0 and containing $\ell \ge M$ vertices must intersect the first coordinate axis in one of the vertices $(j,0)$ $((j + \frac{1}{2}, 0))$, $1 \le j \le \ell$, if $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_0^*(\mathcal{G}_1$ or $\mathcal{G}_1^*)$. If $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_0^*$ and there exists a vacant circuit on $\mathcal{G}^*$ through $(j,0)$ which contains ℓ vertices, then $W^*(j,0)$, the vacant component of $(j,0)$ on $\mathcal{G}^*$ contains at least ℓ vertices. Consequently

$$(8.13)\qquad \theta(p) \geq P_p\{\ \exists \text{ occupied path on } \mathcal{G} \text{ from } z_0 \text{ to some } y \text{ on } \Delta S(z_0,M)\}\ \{1 - \sum_{\ell=M}^{\infty} \sum_{j=1}^{\ell} P_p\{\#W^*(j,0) \geq \ell\}\}$$

$$= P_p\{\ \exists \text{ occupied path on } \mathcal{G} \text{ from } z_0 \text{ to some } y \text{ on } \Delta S(z_0,M)\}\{1 - \sum_{\ell=M}^{\infty} \ell\, P_p\{\#W^*(z_0) \geq \ell\}\}$$

(8.13) remains valid when $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G}_1^*$. The difficult part is now to find a good upper bound for

$$P_p\{\#W^*(z_0) \geq \ell\}$$

when $p > p_H$, but p close to p_H . For this we use Theorem 5.1, applied to $\mathcal{G}^*$. By Theorem 3.1 (iii) , for $p > p_H$

$$E_p\{\#W^*(z_0)\} < \infty$$

and thus, by Theorem 5.1 (with the role of occupied and vacant interchanged)

$$(8.14)\qquad P_p\{\#W^*(z_0) \geq \ell\} \leq C_1\, e^{-C_2 \ell}$$

and the problem is reduced to getting a grip on C_1, C_2 . Lemma 5.3 shows that we can take

$$(8.15)\qquad C_1 = (\tfrac{7}{5})^2\ (50e)\quad ,\quad e^{-C_2} = 2^{-A}\quad ,$$

where

$$(8.16)\qquad A = \frac{1}{49}\ \frac{1}{N^2}$$

as soon as N is so large that

$$(8.17)\qquad \tau^*((N,N);\ i,\ p,\ \mathcal{G}) \leq \kappa = \frac{1}{4}\,(50e)^{-121}\ ,\ i = 1,2\quad .$$

Actually (5.42) still contains the quantity

$$(8.18)\qquad \tau^*((N,N);\ 1,\ p,\ \mathcal{G}) + \tau^*((N,N);\ 2,\ p,\ \mathcal{G})$$

but one easily sees from (5.46) that our upper bound (8.14) is increasing in the quantity (8.18), and we may therefore substitute 2κ for the quantity (8.18), as long as (8.17) holds. On the graphs which we consider here the horizontal and vertical directions are equivalent so that it suffices to choose N such that (8.17) holds for $i = 1$. For any such N (8.14)-(8.16) yield

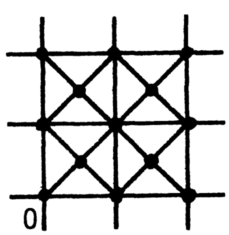
$$(8.19) \qquad P_p\{\#W^*(z_0) \geq \ell\} \leq C_1 \exp - \left(\frac{\log 2}{49} \frac{1}{N^2} \ell\right) .$$

To find an N for which (8.17) holds we reexamine the proof of Theorem 3.1. Specifically we shall go back to Russo's formula in the form (7.44). We write $E^* = E^*(N)$ for the event

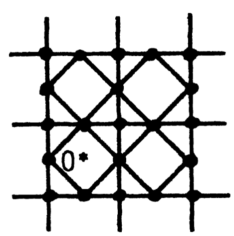
$$(8.20) \qquad E^* = \{ \exists \text{ vacant horizontal crossing on } \mathcal{G}^*_{p\ell} \text{ of } [0,N] \times [0,3N]\}$$

and N_0^* for the number of pivotal sites on $\mathcal{M}$ for E^*. (Here and in the sequel, we view $(\mathcal{G},\mathcal{G}^*)$ as a matching pair, based on $(\mathcal{M},\mathcal{F})$ as described in Sect. 2.2. The planar modifications $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ were defined in Sect. 2.3.)

Note that we are dealing with crossings on $\mathcal{G}^*_{p\ell}$ rather than $\mathcal{G}^*$ in (8.20). The planar modifications of the graphs are useful whenever we want to use the RSW theorem (Theorem 6.1) as we shall have to do repeatedly here. For the present graphs we always choose the central vertices in $\mathcal{G}^*_{0,p\ell}$ and $\mathcal{G}^*_{1,p\ell}$ at points of the form $(k_1+\frac{1}{2}, k_2+\frac{1}{2})$, $k_i \in \mathbb{Z}$, and in $\mathcal{G}_{1,p\ell}$ at points (k_1,k_2), $k_i \in \mathbb{Z}$. The edges incident to the central vertices are straight line segments as illustrated in Fig. 8.1. Of course $\mathcal{G}_{0,p\ell} = \mathcal{G}_0$, and $\mathcal{G}^*_{1,p\ell}$ is obtained by translating $\mathcal{G}_{1,p\ell}$ by $(\frac{1}{2},\frac{1}{2})$. One of the simplifications



$\mathcal{G}^*_{0,p\ell}$



$\mathcal{G}^*_{1,p\ell}$

Figure 8.1 Illustration of $\mathcal{G}^*_{0,p\ell}$ and $\mathcal{G}^*_{1,p\ell}$. Each black circle represents a vertex. In $\mathcal{G}^*_{1,p\ell}$ the origin is marked by *; it is not a vertex of $\mathcal{G}^*_{1,p\ell}$.

obtained by this construction is that for each of the graphs $\mathcal{G}_0$ and $\mathcal{G}_1$ an edge of $\mathcal{G}_{p\ell}$ or of $\mathcal{G}^*_{p\ell}$ intersects each vertical line $x(1) = n$ or horizontal line $x(2) = n$, $n \in \mathbb{Z}$, in a vertex of $\mathcal{G}_{p\ell}$ or $\mathcal{G}^*_{p\ell}$, respectively. It is not hard to check the proof of Lemma 2.1b and to verify that for the special graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}^*_0$ and $\mathcal{G}^*_1$ the existence of a vacant horizontal crossing on $\mathcal{G}^*$ of $[0,N] \times [0,3N]$ implies the existence of a vacant horizontal crossing on $\mathcal{G}^*_{p\ell}$ of $[0,N] \times [0,3N]$ [1]. (One just inserts central sites of $\mathcal{G}^*_{p\ell}$ wherever necessary; these are vacant by our convention (7.3)). Therefore

$$\begin{aligned}(8.21)\qquad \tau^*((N,N);\,1,p,\mathcal{G}) &= \tau((N,N);\,1,q,\mathcal{G}^*)\\ &\le \tau((N,N);\,1,q,\mathcal{G}^*_{p\ell}) = P_p\{E^*(N)\} \quad (q = 1-p)\ .\end{aligned}$$

We now apply Russo's formula (4.22) as we did in (7.42)-(7.44). $\mathcal{G}^*_{p\ell}$, E^* are substituted for $\mathcal{G}$ and E, respectively. We also have to replace p by $q = P_p\{v$ is vacant$\}$. The quantity α in (7.43) is to be replaced by

$$\inf_{v\in\mathcal{M}} (P_{p_H}\{v \text{ is vacant}\} - P_p\{v \text{ is vacant}\}) = p - p_H\ ,$$

and if we take $p(t) = tp_H + (1-t)p$ then we find exactly as in (7.44)

$$\begin{aligned}(8.22)\qquad \tau^*((N,N);\,1,p,\mathcal{G}) &\le P_p\{E^*(N)\}\\ &\le \exp - (p-p_H) \int_0^1 E_{p(t)}\{N^*_0 | E^*\}dt\ ,\ p > p_H\ .\end{aligned}$$

In the present setup a vacant horizontal crossing on $\mathcal{G}^*_{p\ell}$ of $[0,N] \times [0,3N]$ is a vacant path $r^* = (v^*_0, e^*_1, \dots, e^*_\nu, v^*_\nu)$ on $\mathcal{G}^*_{p\ell}$ with the properties (8.23)-(8.25) :

$$(8.23)\qquad (v^*_1, e^*_2, \dots, e^*_{\nu-1}, v^*_{\nu-1}) \subset \text{int}\,(J)\ ,$$

where J is the perimeter of $[0,N] \times [0,3N]$ viewed as a Jordan curve.

(8.24) e^*_1 intersects J only in the point v^*_0 which belongs to $\{0\} \times [0,3N]$.

[1] This fact is not at all crucial; it merely allows us to do away with Λ on most places in the proof. Also we do not have to construct J laboriously as in Lemmas 7.1 and 7.4.

(8.25) e^*_ν intersects J only in the point v^*_ν which belongs to $\{N\} \times [0,3N]$.

Note that $v^*_1 \subset \text{int}(J)$ and (8.24) together imply $e^*_1 \setminus \{v^*_0\} \subset \text{int}(J)$. Similarly (8.23) and (8.25) imply $e^*_\nu \setminus \{v^*_\nu\} \subset \text{int}(J)$. By Prop. 2.3, whenever E^* occurs, then there exists a unique vacant horizontal crossing r^* of $[0,N] \times [0,3N]$ for which the component of $\text{int}(J) \setminus r^*$ below r^* (i.e., with the lower edge of $J = [0,N] \times \{0\}$ in its boundary) is minimal. We shall denote this lowest vacant horizontal crossing by R^*. As in (7.46)

$$E^* = \cup\{R^* = r^*\} ,$$

where the union is over all paths $r^* = (v^*_0, e^*_1, \dots, e^*_\nu, v^*_\nu)$ on $\mathcal{G}^*_{p\ell}$ which satisfy (8.23)-(8.25). Also, when $\{R^* = r^*\}$ occurs, and v^* is a vertex of $\mathcal{M}$ on $r^* \cap \text{int}(J)$, then v^* is pivotal for E^* whenever it has an occupied connection to $\mathring{C} := (0,N) \times \{3N\}$ above r^*. Analogously to Lemma 7.4 we mean by this that there exists a path $s = (v_0, e_1, \dots, e_\rho, v_\rho)$ on $\mathcal{G}_{p\ell}$ such that

(8.26) there exists an edge e of $\mathcal{G}_{p\ell}$ between v^* and v_0 such that $\mathring{e} \subset J^+(r^*)$

(8.27) $$v_\rho \in \mathring{C} ,$$

(8.28) $$(v_0, e_1, \dots, e_\rho \setminus \{v_\rho\}) \subset J^+(r^*) ,$$

and

(8.29) all vertices of s are occupied,

where $J^+(r^*)$ is the component of $\text{int}(J) \setminus r^*$ with $C = [0,N] \times \{3N\}$ in its boundary (compare (7.47)-(7.49)).

Still following the proof of Lemma 7.4 we now set

(8.30) $N^*(r^*) = N^*(r^*,N) = \#$ of vertices v^* of $\mathcal{M}$ on $r^* \cap \text{int}(J)$ which have an occupied connection to $\mathring{C}$ above r^*.

Exactly as in (7.54) we then have

$$E_{p(t)}\{N^*_0 | E^*\} \geq \min_{r^*} E_{p_H}\{N^*(r^*)\} ,$$

where the minimum is over all paths r^* on $\mathcal{G}^*_{p\ell}$ satisfying (8.23)-(8.25). Together with (8.22) this yields

$$(8.31)\qquad \tau^*((N,N);1,p,\mathcal{G}) \le \exp - (p-p_H) \min_{r^*} E_{p_H}(N^*(r^*,N)) ,$$

and we finally must estimate how fast

$$(8.32)\qquad \min_{r^*} E_{p_H}(N^*(r^*,N))$$

grows with N.

The argument so far has been largely a repetition of the proof of Lemma 7.4 with $\mathcal{G}^*_{p\ell}$ and "vacant" taking the place of $\mathcal{G}_{p\ell}$ and "occupied". One could continue to imitate the proof of Lemma 7.4. For this we would first show that there exists $\delta_k > 0$ such that

$$(8.33)\qquad \sigma((k\ell,\ell);\ 1,p_H(\mathcal{G}),\ \mathcal{G}_{p\ell}) \ge \delta_k > 0 \quad , \quad \ell \ge 1 ,$$

and

$$(8.34)\qquad P_{p_H(\mathcal{G})}\{\ \exists \text{ occupied circuit on } \mathcal{G}_{p\ell} \text{ surrounding } 0 \text{ and inside the annulus } [-2\ell,2\ell]\times[-2\ell,2\ell]\setminus(-\ell,\ell) \times (-\ell,\ell) \ge \delta_4^4 \ , \ N \ge 1 .$$

((8.33) and (8.34) will be needed in any case and will be demonstrated below; see Lemma 8.1). (8.33) and (8.34) are just (7.18) and (7.20) for the present graphs; they show that we may take $\overline{N}_\ell = (\ell,\ell)$ in (7.18) and (7.20) . However, for Lemma 7.4 we wanted disjoint annuli so that we rather take $\overline{N}_\ell = (2^{2\ell},2^{2\ell})$. The analogues of (7.62), (7.70) and (7.72) then yield

$$(8.35)\qquad \min_{r^*} E_{p_H}\{N^*(r^*,N)\} \ge \delta'(\#\text{ of } k \text{ with } N_k < \tfrac{1}{2}N) \ge \delta'' \log N$$

for some δ', $\delta'' > 0$. When this estimate is substituted in (8.31) one obtains

$$\tau^*((N,N);\ 1,p,\mathcal{G}) \le N^{-\delta''(p-p_H)} ,$$

so that (8.17) holds when $N \ge \exp C(p-p_H)^{-1}$ for some constant C. Next, by (8.19) this would give

$$\sum_{\ell=M}^{\infty} \ell\ P_p\{\#W^*(v_0) \ge \ell\} \le \frac{1}{2}$$

whenever

$$M \ge \exp\{(2+\varepsilon)\ C\ (p-p_H)^{-1}\}$$

for fixed $\varepsilon > 0$ and p close to p_H. Finally, (8.13) with an estimate of order M^{-1} for the first factor in the right hand side would show

$$\theta(p) \geq \exp - \{2+\varepsilon)C((p-p_H)^{-1}\} \quad , \quad p \downarrow p_H$$

Obviously this is much weaker than the left hand inequality in (8.4). The reason for this is the poor lower bound (8.35) for (8.32). Retracing the above steps we see that we will obtain the first inequality of (8.4) when we improve (8.35) to

$$\min_{r^*} E_{p_H}\{N^*(r^*,N)\} \geq CN^{(2+\varepsilon)/\beta_1} \quad .$$

The principal step of our proof is therefore to obtain this improvement on Lemma 7.4. It is established in Lemma 8.3.

<u>Lemma 8.1</u>. <u>For</u> $\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_0^*$, or $\mathcal{G}_1^*$ <u>and integral</u> k <u>there exists</u> <u>a</u> $\delta_k > 0$ <u>such that</u>

(8.36) $$\sigma((k\ell,6\ell);\ 1,\ p_H(\mathcal{G}),\mathcal{G}_{p\ell}) = \sigma((6\ell,k\ell);\ 2,\ p_H(\mathcal{G}),\mathcal{G}_{p\ell}) \geq \delta_k$$

<u>and</u>

(8.37) $$P_{p_H(\mathcal{G})}\{\ \exists \text{ occupied circuit on } \mathcal{G}_{p\ell} \text{ surrounding the origin in } [-12\ell,\ 12\ell] \times [-12\ell,12\ell] \setminus (-6\ell,6\ell) \times (-6\ell,6\ell)\} \geq \delta_4^4\ ,\ell \geq 1\ .$$

<u>Remarks</u> .

(i) This lemma proves (8.33) and (8.34) with ℓ replaced by 6ℓ. Using monotonicity properties such as in Comment 3.3 (v) we could obtain (8.33) and (8.34) for all ℓ, but this will not be needed.

(ii) For $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_0^*$ this lemma was proved by Russo (1981) by means of Theorem 5.4. For $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G}_1^*$ the lemma was proved by Seymour and Welsh (1978), but formulated for bond percolation. Their argument for $\mathcal{G}_1$ runs roughly as follows in our notation: By a simple variant of Prop. 2.2

(8.38) $$\sigma((\ell,\ell);\ 1,p,\mathcal{G}_{1,p\ell}) + P_p\{\ \exists \text{ vacant vertical crossing on } \mathcal{G}_{1,p\ell}^* \text{ of } [\tfrac{1}{2}\ ,\ \ell-\tfrac{1}{2}] \times [\tfrac{1}{2},\ell-\tfrac{1}{2}]\} \geq 1$$

(compare (7.14). Now use the fact that $\mathcal{G}^*_{1,p\ell}$ is just $\mathcal{G}_{1,p\ell}$ shifted by $(\frac{1}{2}, \frac{1}{2})$ and the fact that the horizontal and vertical direction play identical roles on $\mathcal{G}_{1,p\ell}$ to obtain

$$P_p\{\exists \text{ vacant vertical crossing of } [\tfrac{1}{2}, \ell-\tfrac{1}{2}] \times [\tfrac{1}{2}, \ell-\tfrac{1}{2}] \text{ on } \mathcal{G}^*_{1,p\ell}\} = P_{1-p}\{\exists \text{ occupied vertical crossing of } [0,\ell-1] \times [0,\ell-1] \text{ on } \mathcal{G}_{1,p\ell}\} = \sigma((\ell-1,\ell-1);1,1-p,\mathcal{G}_{1,p\ell}) .$$

Together with (8.38) this gives for $p = \frac{1}{2}$

$$\sigma((\ell,\ell);\ 1,\ \tfrac{1}{2},\ \mathcal{G}_{1,p\ell}) + \sigma((\ell-1,\ell-1);\ 1,\ \tfrac{1}{2},\ \mathcal{G}_{1,p\ell}) \geq 1$$

so that for each N

$$\sigma((m,m);\ 1,\ \tfrac{1}{2},\ \mathcal{G}_1) \geq \tfrac{1}{2} \quad \text{for } m = \ell-1 \text{ or } m = \ell .$$

This is essentially (8.36) for $k = 6$, since $p_H(\mathcal{G}_1) = \frac{1}{2}$ (see Application 3.4 (ii)). From this one can easily obtain Lemma 8.1 by means of the RSW theorem.

Our proof below is essentially as in Russo (1981), and works simultaneously for all the $\mathcal{G}$ under consideration. The only difference is that we use Theorem 5.1 instead of Theorem 5.4.

Proof of Lemma 8.1. Theorem 5.1 implies

$$\tau((\ell,\ell);\ i,p_H(\mathcal{G}),\ \mathcal{G}) \geq \kappa = \kappa(2) \text{ for } i=1 \text{ and } i=2, \tag{8.39}$$

for otherwise by (5.11)

$$E_{p_H}(\#W) < \infty .$$

But this is impossible since $p_H \geq p_T$ and by (5.17)

$$E_{p_T}(\#W) = \infty ,$$

while $E_p\{\#W\}$ increases with p (Lemma 4.1). Thus (8.39) holds. However, for the graphs $\mathcal{G}$ considered in this lemma the horizontal and vertical direction play identical roles so that

$$\sigma((\ell_1,\ell_2);1,\ p,\ \mathcal{G}) = \sigma((\ell_2,\ell_1);\ 2,\ p,\ \mathcal{G}) . \tag{8.40}$$

In particular

$$\sigma((\ell,3\ell);\ 1,\ p_H,\mathcal{G}) = \sigma((3\ell,\ell);\ 2,\ p_H,\ \mathcal{G}) = \tau((\ell,\ell);\ 1,\ p_H,\mathcal{G}) = \tau((\ell,\ell);\ 2,\ p_H,\ \mathcal{G}) \geq \kappa$$

We also saw above (see the argument before (8.21)) that the existence of an occupied horizontal (vertical) crossing of a rectangle $[0,\ell_1]\times[0,\ell_2]$ on $\mathcal{G}$ implies the existence of such a crossing on $\mathcal{G}_{p\ell}$ (ℓ_1, ℓ_2 integral). Thus

$$\sigma((\ell,3\ell);1,p_H(\mathcal{G}),\mathcal{G}_{p\ell}) \geq \sigma((\ell,3\ell);1,p_H(\mathcal{G}),\mathcal{G}) \geq \kappa \ ,$$

$$\sigma((3\ell,\ell);2,p_H(\mathcal{G}),\mathcal{G}_{p\ell}) \geq \sigma((3\ell,\ell);2,p_H(\mathcal{G}),\mathcal{G}) \geq \kappa \ .$$

(8.36) and (8.37) now follow from Theorem 6.1 and (the proof of) Corollary 6.1. □

We need some preparation for Lemma 8.2. Let a and Θ each be a vertex of $\mathcal{G}_{p\ell}$ or of $\mathcal{G}^*_{p\ell}$, and ℓ, N integers ≥ 0 such that

$$S(a,3.2^{\ell}) \subset S(\Theta,N) \tag{8.41}$$

(see (8.10) for S and (8.11) for ΔS). Let $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ and $s = (w_0,f_1,\dots,f_\sigma,w_\sigma)$ be two paths on $\mathcal{G}_{p\ell}$ with the following properties:

$$v_0 = a, \quad v_\nu \in \Delta S(\Theta,N), \tag{8.42}$$

$$(v_0,e_1,\dots,v_{\nu-1},e_\nu \setminus \{v_\nu\}) = e \setminus \{v_\nu\} \subset \mathring{S}(\Theta,N), \tag{8.43}$$

$$w_0 = a, \quad w_\sigma \in \Delta S(\Theta,N), \tag{8.44}$$

$$(w_0,f_1,\dots,w_{\sigma-1},f_\sigma \setminus \{w_\sigma\}) = s \setminus \{w_\sigma\} \subset \mathring{S}(\Theta,N), \tag{8.45}$$

$$r \cap s = \{a\} \ . \tag{8.46}$$

If (8.42)-(8.46) hold then r and s are two paths in $S(\Theta,N)$ from a to $\Delta S(\Theta,N)$ which intersect only in a. (This can only happen if a is a vertex of $\mathcal{G}_{p\ell}$.) The reverse of r followed by s is a simple curve which divides $\mathring{S}(\Theta,N)$ into two components, each of which is bounded by this simple curve and one of the arcs of $\Delta S(\Theta,N)$ between w_σ and v_ν (see Fig. 8.2). Denote these components in arbitrary order $S' = S'(\Theta,r,s)$ and $S'' = S''(\Theta,r,s)$.

Def. 8.1. For any subset R of $S(\Theta,N)$ and vertex v_i on r we say that v_i is <u>connected to s in R</u> if $v_i = a$ or if there exists a path $t = (u_0,g_1,\dots,g_\tau,u_\tau)$ on $\mathcal{G}_{p\ell}$ which satisfies

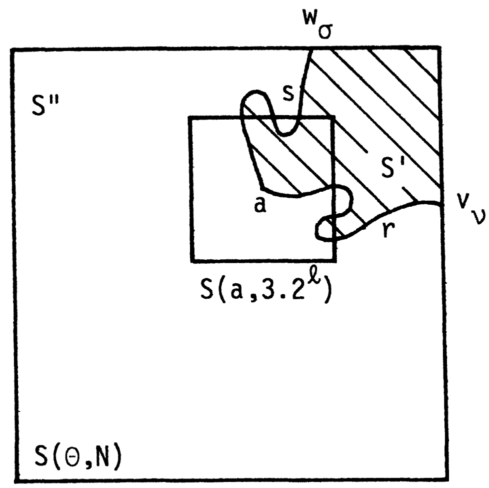


Figure 8.2 The outer square is $S(\Theta,N)$; the inner square is $S(a,3.2^\ell)$. S' is the hatched region.

$$(8.47)\qquad (g_1\setminus\{u_0\},u_1,\dots,u_{\tau-1},g_\tau\setminus\{u_\tau\}) = t\setminus\{u_0,u_\tau\} \subset R,$$

$$(8.48)\qquad u_0 = v_i \text{ and } u_\tau = w_j \text{ on } s \text{ for some } 1 \le j \le \sigma ,$$

and

$$(8.49)\qquad u_1,\dots,u_{\tau-1} \text{ are occupied.} \qquad ///$$

Next we set

$$(8.50)\qquad Y(v_i,a,\ell,r,s) = \begin{cases} 1 & \text{if } v_i \text{ is connected to } s \text{ in} \\ & \quad S'(\Theta,r,s) \cap S(a,3.2^\ell), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(8.51)\qquad Z(\ell) = \min E_{p_H(\mathcal{G})}\Big\{ \sum_{\substack{v_i \varepsilon r \cap S(a,3.2^\ell) \\ v_i \text{ a vertex of } \mathcal{M}}} Y(v_i,a,\ell,r,s) \Big| \ \omega(v) = \varepsilon(v),\ v \,\varepsilon\, \overline{S}''(\Theta,r,s)\Big\},$$

where the min in (8.51) is over all a, Θ, N, r, s which satisfy (8.41)-(8.46) and over all choices of $+1$ or -1 for $\varepsilon(v)$ with v

a vertex of $\mathcal{G}_{p\ell}$ in $\overline{S}''$. Note that the sum (8.51) is simply the number of vertices of $\mathcal{M}$ or r inside $S(a,3.2^{\ell})$ which are connected to s in $S' \cap S(a,3.2^{\ell})$. This sum depends only on occupancies of vertices in S', and hence is independent of $\{\omega(v): v \in \overline{S}''\}$. Thus, the conditioning in the expectation in the right hand side of (8.51) does not influence the expectation. It is nevertheless useful for the proof of the next lemma to introduce this conditioning.

We shall also need an analogue of Z when r is replaced by a path on $\mathcal{G}^*_{p\ell}$ (instead of on $\mathcal{G}_{p\ell}$). In other words $r = (v^*_0, e^*_1, \dots, e^*_\nu, v^*_\nu)$ will now be a path on $\mathcal{G}^*_{p\ell}$ which satisfies

$$(8.52) \qquad v^*_0 = a,\ v^*_\nu \in \Delta S(\Theta,N)$$

and

$$(8.53) \qquad (v^*_0, e^*_1, \dots, v^*_{\nu-1}, e^*_1 \setminus \{v^*_\nu\}) = r^* \setminus \{v^*_\nu\} \subset \mathring{S}(\Theta,N).$$

$s = (w_0, f_1, \dots, f_\sigma, w_\sigma)$ will again be a path on $\mathcal{G}_{p\ell}$ which satisfies (8.44) and (8.45). Analogously to (8.46) we require

$$(8.54) \qquad r^* \cap s = \{a\}\ ,$$

which can happen only if a is a vertex of $\mathcal{G}$ and $\mathcal{G}^*$, i.e., a vertex of $\mathcal{M}$. Def. 8.1 can be copied without change for vertices v^*_i on r^* instead of vertices v_i on r. Finally $Y^*(v^*_i,a,\ell,r^*,s)$ is defined as in (8.50) by replacing v_i and r by v^*_i and r^*. Similarly $Z^*(\ell)$ is defined by replacing $Y(v_i,a,\ell,r,s)$ by $Y^*(v^*_i,a,\ell,r^*,s)$ and $\overline{S}''(\Theta,r,s)$ by $\overline{S}''(\Theta,r^*,s)$ in (8.51).

<u>Lemma 8.2</u>. <u>There exist constants</u> $0 < C_{12}, \alpha_1 < \infty$, <u>such that for</u> $\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}^*_0$ <u>or</u> $\mathcal{G}^*_1$

$$(8.55) \qquad Z(\ell) \geq C_{12} 2^{\alpha_1 \ell}\ , \quad \ell \geq 0,$$

<u>and</u>

$$(8.56) \qquad Z^*(\ell) \geq C_{12} 2^{\alpha_1 \ell}\ , \quad \ell \geq 0\ .$$

<u>Proof</u>: We restrict ourselves to (8.55) since the proof of (8.56) is practically the same. Throughout we fix $\mathcal{G}$ as one of the four graphs $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}^*_0$ or $\mathcal{G}^*_1$. If $\mathcal{G} = \mathcal{G}^*_i$, then $\mathcal{G}^* = \mathcal{G}_i$, $i = 0,1$.

The idea of the proof is very similar to the last part of the proof of Lemma 7.4 (but s^* is now replaced by s). Let r and s be as in (8.42)-(8.46). Again we start with considering disjoint annuli centered at $\lfloor a \rfloor := (\lfloor a(1) \rfloor, \lfloor a(2) \rfloor)$ $(a = (a(1),a(2)) = v_0 = w_0)$. In view of (8.37) suitable annuli to take now are

$$V_k := S(\lfloor a \rfloor, 6.2^{2k}) \setminus \mathring{S}(\lfloor a \rfloor, 6.2^{2k-1}).$$

As in the argument for (7.72) we estimate the probability of some v_i being connected in V_k to s by the probability of there existing an occupied circuit surrounding $\lfloor a \rfloor$ in V_k. Assume now that there is a vertex v_{i_k} of r in V_k which is connected by a path $t = (u_0, g_1, \dots, g_\tau, u_\tau)$ in $V_k \cap S'(\Theta,r,s)$ to s. In Lemma 7.4 we only used the estimate that V_k contains at least the one vertex v_{i_k} of r connected to s, in this situation. Here we shall be less casual with our estimation. Let u_τ, the final point of t, equal w_j on s. Consider the path s_k consisting of t followed by $(w_j, f_{j+1}, \dots, f_\sigma, w_\sigma)$ (a tail piece of s). Just as s itself, s_k is

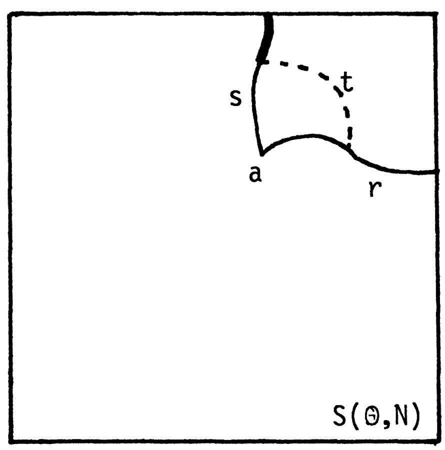


Figure 8.3 r and s are drawn as solid curves. t is dashed. s_k is the composition of t and the boldly drawn part of s.

a path on $S(\Theta,N)$ from a point of r to $\Delta S(\Theta,N)$, and s_k intersects r only in the initial point v_{i_k} of s_k. Therefore s_k can take over the role of s. If some v_i is connected to s_k in $S'(\Theta,r,s)$

then v_i is connected to s itself in S'. We can therefore obtain new points connected to s, by looking for points which are connected to s_k. This is done by considering the annuli

$$S(\lfloor v_{i_k} \rfloor, 6.2^{2m}) \setminus \mathring{S}(\lfloor v_{i_k} \rfloor, 6.2^{2m-1})$$

centered at $\lfloor v_{i_k} \rfloor := (\lfloor v_{i_k}(1) \rfloor, \lfloor v_{i_k}(2) \rfloor)$. This procedure can be repeated and we obtain something resembling a branching process in which the first generation consists of the v_{i_k}. Each vertex v which in some generation has been found to be connected to s by a path $\tilde{s}$ "produces" a next generation of vertices connected to s, namely the ones connected to $\tilde{s}$ inside suitable annuli centered at $\lfloor v \rfloor$. Closer scrutiny would show that we are dealing with a supercritical branching process with mean number of offspring per individual equal to $m > 1$, say. Estimating $Z(\ell)$ would then amount to estimating the total number of individuals in the branching process after $\varepsilon\ell$ generations, for some $\varepsilon > 0$. This number would have an expectation

$$m^{\varepsilon\ell} = 2^{\ell(\varepsilon \log m/\log 2)}.$$

This is precisely the kind of estimate claimed in (8.55). Rather than follow the above outline in detail we shall prove the recursion relation

$$Z(\ell) \geq (1 + \delta_{30}^4) Z(\ell-3), \quad \ell \geq 4, \tag{8.57}$$

with δ_{30} as in (8.36). This corresponds roughly to decomposing the branching process into the separate branching processes generated by the individuals of the first generation of the original branching process. The $(\varepsilon\ell)$-th generation of the former branching process is the sum of the $(\varepsilon\ell-1)$-th generation for the latter branching process.

Now for the detailed proof of (8.57). Fix a, Θ, N, ℓ, r and s such that (8.41)-(8.46) hold. Obviously (8.41)-(8.46) continue to hold when ℓ is replaced by $\ell-3$ in (8.41), and also

$$Y(v_i, a, \ell, r, s) \geq Y(v_i, a, \ell-3, r, s) \quad \text{for} \quad v_i \in S(a, 3.2^{\ell-3}).$$

Consequently

$$(8.58)\qquad E_{P_H}\{\sum_{\substack{v_i\in r\cap S(a,3.2^{\ell-3})\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell,r,s)\,|\,\omega(v)=\varepsilon(v), v\in \bar{S}''\}$$

$$\geq E_{P_H}\{\sum_{\substack{v_i\in r\cap S(a,3.2^{\ell-3})\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell-3,r,s)\,|\,\omega(v)=\varepsilon(v), v\in \bar{S}''\}$$

$$\geq Z(\ell-3).$$

Next define the closed annulus

$$(8.59)\qquad V := S(\lfloor a\rfloor, 3(2^{\ell-1}+2^{\ell-3}))\setminus \mathring{S}(\lfloor a\rfloor, 3(2^{\ell-1}-2^{\ell-3}))$$

$$\subset \mathring{S}(a,3.2^{\ell}) \subset \mathring{S}(\Theta,N),$$

where as above $a=(a(1),a(2))$, $\lfloor a\rfloor = (\lfloor a(1)\rfloor, \lfloor a(2)\rfloor)$. We shall apply Prop. 2.3 with S taken equal to V. For J we take the perimeter of $S'(\Theta,r,s)$ and for the arcs B_1, A, B_2, C making up J we take

$$(8.60)\qquad \begin{aligned} &B_1 = \text{reverse of } r,\ A=\{a\}\ \text{(a single point)},\ B_2 = s \text{ and}\\ &C = \text{arc of } \Delta S(\Theta,N) \text{ from } w_\sigma \text{ to } v_\nu \text{ in the boundary of } S'(\Theta,r,s). \end{aligned}$$

We shall be concerned with the collection of paths $t=(u_0,g_1,\dots,g_\tau,u_\tau)$ on $\mathcal{G}_{p\ell}$ which satisfy

$$(8.61)\qquad t\subset V,$$

$$(8.62)\qquad (g_1\setminus\{u_0\},u_1,\dots,u_{\tau-1},g_\tau\setminus\{u_\tau\}) = t\setminus\{u_0,u_\tau\}\subset S'(\Theta,r,s),$$

$$(8.63)\qquad u_0 \text{ is some vertex } v_i \text{ on } r \text{ with } 0<i<\nu,$$

and

$$(8.64)\qquad u_\tau \text{ is some vertex } w_j \text{ on } s \text{ with } 0<j<\sigma.$$

Let $G(r,s)$ be the event

$$G(r,s) = \{\exists \text{ occupied path } t \text{ on } \mathcal{G}_{p\ell} \text{ which satisfies (8.61)-(8.64)}\}.$$

The properties (8.62)-(8.64) are the properties (2.23)-(2.25) in the present set up. Therefore Prop. 2.3 can be applied, and on the event $G(r,s)$ there exists a unique occupied path t satisfying (8.61) -(8.64) for which

$$J^-(t) = \text{component of } \operatorname{int}(J)\setminus t \text{ with } \{a\} \text{ in its boundary}$$
$$= \text{component of } S'(\Theta,r,s)\setminus t \text{ with } \{a\} \text{ in its boundary}$$

is minimal. We shall denote this path with minimal $J^-(t)$ by T whenever it exists. Then Prop. 2.3 implies

$$G(r,s) = \cup \{T = t\}, \tag{8.65}$$

where the union is over all t satisfying (8.61)-(8.64) (compare (7.46)). Also, as in (7.72), any occupied circuit c on $\mathcal{G}_{p\ell}$ and in V, surrounding $\lfloor a \rfloor$, contains an occupied path t satisfying (8.61) -(8.64). (This time apply Lemma A.2 with $J_1 = J$, $J_2 = c$ and note that, by virtue of (8.59) and $\ell \geq 4$

$$c \subset V \subset \mathring{S}(\Theta,N), \quad a = v_0 = w_0 \notin V;$$

consequently any vertex v_i of r on c must have $0 < i < \nu$ and any vertex w_j of s one must have $0 < j < \sigma$ and hence the requirements $0 < i < \nu$ in (8.63) and $0 < j < \sigma$ in (8.64) are automatically fulfilled.) It follows that

$$P_p\{G(r,s)\} \geq P_p\{\exists \text{ occupied circuit on } \mathcal{G}_{p\ell} \text{ in } V \text{ and surrounding } \lfloor a \rfloor\}. \tag{8.66}$$

As in the proof of Cor. 6.1 from Theorem 6.1 one can find an occupied circuit in V surrounding $\lfloor a \rfloor$ as soon as there exist occupied vertical crossings on $\mathcal{G}_{p\ell}$ of the two rectangles (one corresponding to the plus signs and one to the minus signs)

$$[\lfloor a(1) \rfloor \pm 3.2^{\ell-1} - 3.2^{\ell-3}, \lfloor a(1) \rfloor, \pm 3.2^{\ell-1} + 3.2^{\ell-3}] \times$$
$$[\lfloor a(2) \rfloor - 3(2^{\ell-1} + 2^{\ell-3}), \lfloor a(2) \rfloor + 3(2^{\ell-1} + 2^{\ell-3})],$$

as well as occupied horizontal crossings on $\mathcal{G}_{p\ell}$ of the two rectangles

$$[\lfloor a(1)\rfloor - 3(2^{\ell-1}+2^{\ell-3}), \lfloor a(1)\rfloor + 3(2^{\ell-1}+2^{\ell-3})] \times$$
$$[\lfloor a(2)\rfloor \pm 3.2^{\ell-1} - 3.2^{\ell-3}, \lfloor a(2)\rfloor \pm 3.2^{\ell-1} + 3.2^{\ell-3}].$$

By the FKG inequality and (8.36) the right hand side of (8.66) is therefore at least δ_{30}^4, when $p = p_H(\mathcal{G})$. In other words

(8.67) $$P_{p_H(\mathcal{G})}\{G(r,s)\} \geq \delta_{30}^4 .$$

Now let $t = (u_0, g_1, \dots, g_\tau, u_\tau)$ be a fixed path satisfying (8.61)-(8.64) (no reference to the occupancies of the u_i is made at the moment). Then t is a crosscut of J and divides $\text{int}(J) = S'(\Theta,r,s)$ into two components. The one which contains $\{a\}$ in its boundary we denoted above as $J^-(t)$, while the one with C in its boundary is denoted as usual by $J^+(t)$ (see Fig. 8.4). It is important to give

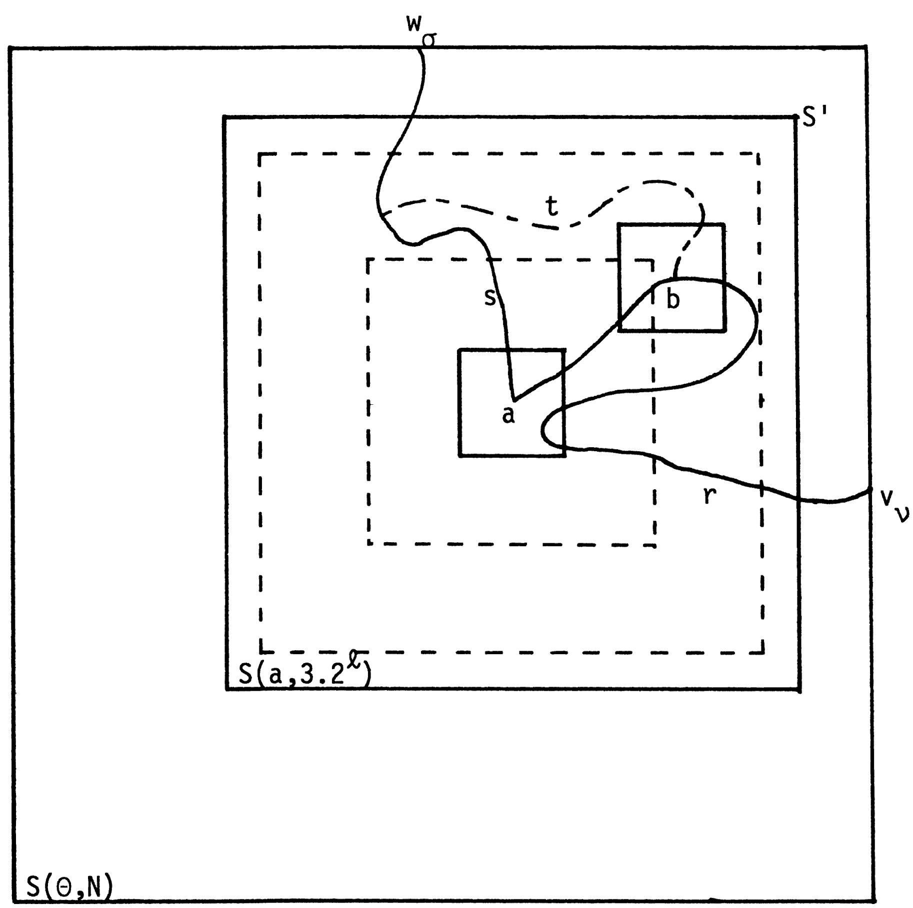


Figure 8.4 V is the annulus between the dashed lines. The small squares centered at a and b are $S(a,3.2^{\ell-3})$ and $S(b,3.2^{\ell-3})$. r and s are drawn solidily; t is indicated by — - —— - —— .

another equivalent description of $J^+(t)$. Let $u_0 = v_{i_0}$ and denote by r_0 the subpath $(v_{i_0}, e_{i_0+1}, \dots, e_\nu, v_\nu)$ of r. Let $u_\tau = w_{j_0}$ and denote by s_0 the following path, consisting of t and the piece of s from w_{j_0} on:

$$s_0 = (u_0, g_1, \dots, g_\tau, u_\tau = w_{j_0}, f_{j_0+1}, w_{j_0+1}, \dots, f_\sigma, w_\sigma).$$

Finally, we write b for $u_0 = v_{i_0}$. Then the paths r_0 and s_0 on $\mathcal{G}_{p\ell}$ satisfy the following analogues of (8.42)-(8.46)

$$v_{i_0} = b, \; v_\nu \in \Delta S(\Theta, N),$$

$$(v_{i_0}, e_{i_0+1}, \dots, e_\nu \setminus \{v_\nu\}) = r_0 \setminus \{v_\nu\} \subset \mathring{S}(\Theta, N),$$

$$u_0 = b, \; w_\sigma \in \Delta S(\Theta, N),$$

$$(u_0, g_1, \dots, g_\tau, u_\tau = w_{j_0}, f_{j_0+1}, \dots, w_{\sigma-1}, f_\sigma \setminus \{w_\sigma\})$$

$$= s_0 \setminus \{w_\sigma\} \subset \mathring{S}(\Theta, N) \quad (\text{use } \; t \subset V \subset \mathring{S}(\Theta, N)),$$

$$r_0 \cap s_0 = \{b\} .$$

In addition, since $b = u_{i_0} \in V$ (8.41) implies

$$S(b, 3.2^{\ell-3}) \subset S(a, 3.2^{\ell}) \subset S(\Theta, N). \tag{8.68}$$

Therefore r_0, s_0 and b can take over the roles of r, s and a, respectively. In particular the simple curve consisting of the reverse of r_0 followed by s_0 divides $\mathring{S}(\Theta, N)$ into two components $S'(\Theta, v_0, s_0)$ and $S''(\Theta, r_0, s_0)$, where we now choose $S'(\Theta, r_0, s_0)$ to be that component with the arc C between w_σ and v_ν in its boundary. (This arc is also in the boundary of $S'(\Theta, r, s)$; see (8.60).) Also

$$E_{p_H}\Big\{ \sum_{\substack{v_i \in r_0 \cap S(b, 3.2^{\ell-3}) \\ v_i \in \mathcal{M}}} Y(v_i, b, \ell-3, r_0, s_0) \,|\, \omega(v) = \varepsilon(v), \; v \in \overline{S}''(\Theta, r_0, s_0)\Big\} \geq Z(\ell-3). \tag{8.69}$$

We claim that $S'(\Theta,r_0,s_0)$ is the same as $J^+(t)$. This follows immediately from the fact that these two Jordan domains have the same boundary. Indeed, the boundary of $J^+(t)$ consists of t, the piece of s between its intersection with t, i.e., w_{j_0}, and its endpoint w_σ (these two pieces make up s_0), the arc C and the piece of r from u_ν to the intersection u_{i_0} of r and t. These same curves make up the boundary of $S'(\Theta,r_0,s_0)$. It immediately follows from this claim that

(8.70) $$S'(\Theta,r_0,s_0) = J^+(t) \subset \text{int}(J) = S'(\Theta,r,s)$$

and

(8.71) $$\bar{S}''(\Theta,r,s) \cup \bar{J}^-(t) \subset \bar{S}''(\Theta,r_0,s_0) \; .$$

Let us now assume that in addition to (8.61)-(8.64) t satisfies

(8.72) $$u_1,\dots,u_{\tau-1} \text{ are occupied .}$$

Then t satisfies (8.47)-(8.49) with $i = i_0$ and

$$R = V \cap S'(\Theta,r,s) \subset \mathring{S}(a,3.2^\ell) \cap S'(\Theta,r,s)$$

(see (8.62) and (8.59)). Thus v_{i_0} is connected to s in $S'(\Theta,r,s) \cap S(a,3.2^\ell)$ and $Y(v_{i_0},a,\ell,r,s) = 1$. However, more is true. We claim that if t satisfies (8.61)-(8.64) and (8.72) and $v_i \in r$ is connected to s_0 in $S'(\Theta,r_0,s_0) \cap S(b,3.2^{\ell-3})$, then v_i is also connected to s in $S'(\Theta,r,s) \cap S(a,3.2^\ell)$. In formulas

(8.73) $$Y(v_i,b,\ell-3,r_0,s_0) = 1 \text{ implies } Y(v_i,a,\ell,r,s) = 1.$$

To prove (8.73) assume that $v_i \in r$, and that $t_0 = (x_0,h_1,x_2,\dots,h_\rho,x_\rho)$ is a path on $\mathcal{G}_{p\ell}$ satisfying

(8.74) $$t_0 \setminus \{x_0,x_\rho\} \subset S'(\Theta,r_0,s_0) \cap S(b,3.2^{\ell-3}),$$

(8.75) $$x_0 = v_i \text{ and } x_\rho = \text{some vertex of } s_0 \text{ other than } u_0,$$

(8.76) $$x_1,\dots,x_{\rho-1} \text{ are occupied.}$$

Observe first that (8.68) and (8.70) imply

$$(8.77)\qquad S'(\Theta,r_0,s_0) \cap S(b,3.2^{\ell-3}) \subset S'(\Theta,r,s) \cap S(a,3.2^{\ell}).$$

Thus, $t_0 \setminus \{x_0,x_\rho\}$ is also contained in the right hand side of (8.77), and if its final point x_ρ equals a vertex w_j of s with $1 \le j \le \sigma$ (i.e., other than w_0) then v_i is connected in $S'(\Theta,r,s) \cap S(a,2^{\ell})$ to s, i.e., $Y(v_i,a,\ell,r,s) = 1$. The next case to check is when x_ρ is a vertex of s_0 other than $w_1,\dots,w_\sigma$ or u_0. Then x_ρ must be one of the vertices $u_1,\dots,u_\tau$. Moreover, $u_\tau = w_{j_0}$ for some $0 < j_0 < \sigma$ (see (8.64)). Let $x_\rho = u_{k_0}$, $1 \le k_0 \le \tau$ and define $t_1 := (x_0,h_1,\dots,h_\rho,x_\rho = u_{k_0},g_{k_0+1},\dots,g_\tau,u_\tau)$. t_1 consists of t_0 followed by a tail piece of t. All vertices of t_1 other than its initial and final point are occupied, on account of (8.72) and (8.76). Moreover

$$t_1 \setminus \{x_0,u_\tau\} \subset S'(\Theta,r,s) \cap S(a,3.2^{\ell})$$

by virtue of (8.74), (8.77), (8.62), (8.61) and (8.59). Thus, again $Y(v_i,a,\ell,r,s) = 1$. The last case to check for (8.73) is when $v_i = b = v_{i_0} = u_0$. But in this case we already saw, just before (8.73) that $Y(v_{i_0},a,\ell,r,s) = 1$, so that (8.73) has been verified.

The proof of (8.57) is now merely a matter of assembling some of the above results. If there exists a path $t = (u_0,g_1,\dots,g_\tau,u_\tau)$ on $\mathcal{G}_{p\ell}$ which satisfies (8.61)-(8.64) and (8.72) then $b = u_0 \in V$. Then, by definition of V,

$$|a(i)-b(i)| \ge 3(2^{\ell-1}-2^{\ell-3})-1$$

and $S(a,3.2^{\ell-3})$ and $S(b,3.2^{\ell-3})$ are disjoint. Thus, by (8.73)

$$(8.78)\qquad \sum_{\substack{v_i\in r\cap S(a,3.2^{\ell})\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell,r,s)$$

$$\ge \sum_{\substack{v_i\in r\cap S(a,3.2^{\ell-3})\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell,r,s) + \sum_{\substack{v_i\in r\cap S(b,3.2^{\ell-3})\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell,r,s)$$

$$\ge \sum_{\substack{v_i\in r\cap S(a,3.2^{\ell-3})\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell,r,s) + \sum_{\substack{v_i\in r\cap S(b,3.2^{\ell-3})\\ v_i\in\mathcal{M}}} Y(v_i,b,\ell-3,r_0,s_0).$$

Now , as we saw before, the left hand side of (8.78) is independent of the $\omega(v)$ with $v \varepsilon \bar{S}''(\Theta,r,s)$. Therefore, by virtue of (8.58)

$$(8.79) \quad E_{p_H(\mathcal{G})}\{\sum_{\substack{v_i \varepsilon r \cap S(a,3.2^\ell) \\ v_i \varepsilon \mathcal{M}}} Y(v_i,a,\ell,r,s)\,|\,\omega(v) = \varepsilon(v), v \varepsilon \bar{S}''(\Theta,r,s)\}$$

$$= E_{p_H(\mathcal{G})}\{\sum_{\substack{v_i \varepsilon r \cap S(a,3.2^\ell) \\ v_i \varepsilon \mathcal{M}}} Y(v_i,a,\ell,r,s)\}$$

$$\geq Z(\ell-3) + \sum_t P_{p_H(\mathcal{G})}\{T = t\} \quad .$$

$$E_{p_H(\mathcal{G})}\{\sum_{\substack{v_i \varepsilon r \cap S(u_0,3.2^{\ell-3}) \\ v_i \varepsilon \mathcal{M}}} Y(v_i,u_0,\ell-3,r_0,s_0)\,|\,T = t\} \ ,$$

where the sum is over those path t which satisfy (8.61)-(8.64), u_0 is the initial point of t and r_0, s_0 are defined in terms of r, s and t as above. However, by Prop. 2.3 the event $\{T = t\}$ depends only on the occupancies of the vertices in $\bar{J}^-(t) \subset \bar{S}''(\Theta,r_0,s_0)$ (see (8.71)). Therefore

$$E_{p_H(\mathcal{G})}\{\sum_{\substack{v_i \varepsilon r \cap S(u_0,3.2^{\ell-3}) \\ v_i \varepsilon \mathcal{M}}} Y(v_i,u_0,\ell-3,r_0,s_0)\,|\,T = t\}$$

$$\geq \min_{\varepsilon(\cdot)} E_{p_H(\mathcal{G})}\{\sum_{\substack{v_i \varepsilon r \cap S(u_0,3.2^{\ell-3}) \\ v_i \varepsilon \mathcal{M}}} Y(v_i,u_0,\ell-3,r_0,s_0)\,|\,\omega(v) = \varepsilon(v),\ v \varepsilon \bar{S}''(\Theta,r_0,s_0)\}$$

$$\geq Z(\ell-3).$$

Substitution of this estimate into (8.79) and using (8.65) and (8.67) yields

$$E_{p_H(\mathcal{G})}\{\sum_{\substack{v_i \varepsilon r \cap S(a,3.2^\ell) \\ v_i \varepsilon \mathcal{M}}} Y(v_i,a,\ell,r,s)\,|\,\omega(v) = \varepsilon(v), v \varepsilon \bar{S}''(\Theta,r,s)\}$$

$$\geq Z(\ell-3)(1+P_{p_H(\mathcal{G})}\{G(r,s)\}) \geq (1+\delta_{30}^4)Z(\ell-3).$$

(8.57) now follows by minimizing over a,Θ,N,r,s and $\varepsilon(\cdot)$.

To obtain (8.55) from (8.57) we merely have to show that $Z(\ell) > 0$ for each $\ell \geq 0$. This is easy to see, though, since by Def. 8.1 always $Y(a,a,\ell,r,s) = 1$. If a is a central vertex of $\mathcal{G}$ - and hence lies inside a face F of $\mathcal{M}$, but is not a vertex of $\mathcal{M}$ (see Sect. 2.3) - then a is not to be counted as one of the v_i in the sum in (8.51). However, in this case r and s both have vertices on the perimeter of F, and these vertices belong to $\mathcal{G}$ (and hence $\mathcal{M}$). In particular there will be a vertex v of $\mathcal{M}$ on r and a vertex w of $\mathcal{M}$ on s on the perimeter of F, such that an open arc of the perimeter of F from v to w lies inside $S'(\Theta,r,s) \cap S(a,3.2^{\ell})$. (See Fig. 8.5 for an illustration which applies when $\mathcal{G} = \mathcal{G}_0^*, \mathcal{G}_1$ or $\mathcal{G}_1^*$; a cannot be a central vertex when $\mathcal{G} = \mathcal{G}_0$.)

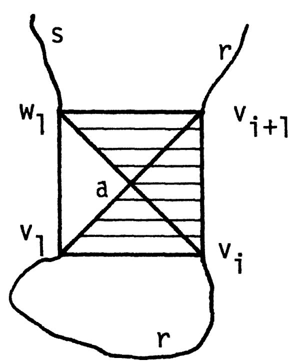


Figure 8.5 The center is the vertex a; it is a central vertex in the square, which is a face F of $\mathcal{M}$. The hatched region belongs to $S'(\theta,N)$. The edges from a to v_1 and from v_i to v_{i+1} belong to r. The edge from a to w_1 is the first edge of s. In this illustration the open edge between v_{i+1} and w_1 belongs to $S'(\Theta,r,s) \cap S(a,3.2^{\ell})$ and $Y(v_{i+1},a,\ell,r,s) = 1$.

This open arc contains at most two vertices of $\mathcal{G}$ and hence the vertices on this open arc are all occupied with a probability at least $(p_H(\mathcal{G}))^2$. If this happens, then $Y(v,a,\ell,r,s) = 1$. Consequently

$$Z(\ell) \geq (p_H(\mathcal{G}))^2 > 0.$$

This completes the proof. $\square$

We remind the reader that $N^*(r^*) = N^*(r^*,N)$ was defined in (8.30).

Lemma 8.3. There exist a constant $0 < C_{13} < \infty$ such that for $\mathcal{G}$ equal to $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_0^*$ or $\mathcal{G}_1^*$ and any path $r^* = (v_0^*,e_1^*,\dots,e_\nu^*,v_\nu^*)$ on $\mathcal{G}_{p\ell}^*$ which satisfies (8.23)-(8.25) one has

$$E_{p_H(\mathcal{G})}N^*(r^*,N) \geq C_{13}N^{\alpha_1}, \tag{8.80}$$

$$\tau^*(N,N);i,p,\mathcal{G}) \leq \exp -C_{13}(p-p_H(\mathcal{G}))N^{\alpha_1}, \; p > p_H, \; i = 1,2 \tag{8.81}$$

and

$$\tau((N,N);i,p,\mathcal{G}) \leq \exp -C_{13}(p_H(\mathcal{G})-p)N^{\alpha_1}, \; p < p_H, \; i = 1,2. \tag{8.82}$$

Proof: Again fix $\mathcal{G}$. Let J be the perimeter of $[0,N]\times[0,3N]$ viewed as a Jordan curve, and set

$$A = [0,N]\times\{0\} = \text{bottom edge of } J,$$

$$C = [0,N]\times\{3N\} = \text{top edge of } J.$$

Also fix a path $r^* = (v_0^*,e_1^*,\dots,e_\nu^*,v_\nu^*)$ on $\mathcal{G}_{p\ell}^*$ which satisfies (8.23)-(8.25). It will turn out to be convenient to estimate the left hand side of (8.80) somewhat indirectly, by means of the expected number of occupied connections above r^* to the interior of

$$C_1 := [0,N]\times\{4N\}$$

(rather than to C itself). To be more specific, let J_1 be the perimeter of $[0,N]\times[0,4N]$. Then A is also the bottom edge of J_1 and C_1 is the top edge of J_1. The path r^* is also a horizontal crossing of J_1, and we define $J_1^-(r^*)$ and $J_1^+(r^*)$ as the components of $\text{int}(J_1)\setminus r^*$ with A and C_1 in their boundary, respectively. We say that a vertex v^* on $r^*\cap\text{int}(J) = r^*\cap\text{int}(J_1)$ has an occupied connection to $\mathring{C}_1$ above r^* if there exists a path $s = (v_0,e_1,\dots,e_\rho,v_\rho)$ on $\mathcal{G}_{p\ell}$ which satisfies (8.26)-(8.29) with J and C replaced by J_1 and C_1. Analogously to (8.30) we write $N_1^*(r^*,N)$ for the number of vertices v^* of $\mathcal{M}$ on $r^*\cap\text{int}(J_1)$ which have an occupied connection above v^* to $\mathring{C}_1$. If v^* has an occupied connection $s = (v_0,e_1,\dots,e_\rho,v_\rho)$ above r^* to $\mathring{C}_1$, then s must intersect $\mathring{C}$, necessarily in one of the v_i (see Fig. 8.6). If i_0 is the smallest index i with $v_{i_0}\in C$, then $(v_0,e_1,\dots,e_{i_0},v_{i_0})$

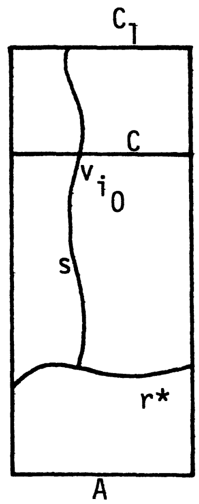


Figure 8.6

(an initial piece of s) is an occupied connection above r^* from v^* to $\mathring{C}$. Thus any vertex counted in $N_1^*(r^*,N)$ must also be counted in $N^*(r^*,N)$ so that

(8.83) $$N^*(r^*,N) \geq N_1^*(r^*,N).$$

The first step in estimating the expectation of N_1 is again an imitation of Lemma 7.4. Let ℓ be the unique integer for which

(8.84) $$3.2^{\ell} < \frac{N}{2} \leq 3.2^{\ell+1}$$

and let

$$\chi = [1,3.2^{\ell}+1] \times \mathbb{R} .$$

We denote by $F^*(r^*)$ the event that there exists an occupied path $\tilde{s} = (w_0,f_1,\ldots,f_\sigma,w_\sigma)$ on $\mathcal{G}_{p\ell}$ with the following properties:

(8.85) w_0 is a vertex of $\mathcal{M}$ on $r^* \cap \mathrm{int}(J) \cap \chi$

(8.86) $$w_\sigma \in C_1 \cap \chi \subset \mathring{C}_1$$

(8.87) $$(f_1 \setminus \{w_0\},w_1,f_2,\ldots,w_{\sigma-1},f_\sigma \setminus \{w_\sigma\}) = \tilde{s} \setminus \{w_0,w_\sigma\} \subset J_1^+(r^*) \cap \chi \quad .$$

Of course w_0 has an occupied connecton to $\mathring{C}_1$ above r^*, whenever such an $\tilde{s}$ exists. Furthermore, if we denote the perimeter of the Jordan domain $J_1^+(r^*)$ by J_2, then such an $\tilde{s}$ is a crosscut of $J_1^+(r^*) = \mathrm{int}(J_2)$ and divides this domain into two components, $J_2^L(\tilde{s})$ and $J_2^R(\tilde{s})$ say. $J_2^L(\tilde{s})$ $(J_2^R(\tilde{s}))$ is the component with a piece of the left edge of J_1, $\{0\} \times [0,4N]$, (the right edge of J_1, $\{N\} \times [0,4N]$) in its boundary. By Prop. 2.3, if $F^*(r^*)$ occurs, then there is a unique occupied connection $\tilde{s}$ with the properties (8.85)-(8.87) with minimal $J_2^L(\tilde{s})$. We shall call this the "left-most occupied connection" and denote it by $\tilde{S}$ whenever it exists. As in (7.60) any occupied vertical crossing t on $\mathcal{G}_{p\ell}$ of $[1,3.2^{\ell}+1] \times [0,4N]$ contains an occupied connection from some point of r^* to $\mathring{C}_1$ inside X. Thus

$$
\begin{aligned}
(8.88) \quad P_{p_H}\{\tilde{S} \text{ exists}\} &= P_{p_H}\{F^*(r^*) \text{ occurs}\} \\
&\geq P_{p_H}\{\exists \text{ occupied vertical crossing on } \mathcal{G}_{p\ell} \text{ of} \\
&\qquad [1,3.2^{\ell}+1] \times [0,4N]\} \\
&\geq \sigma((3.2^{\ell}, 3.2^{\ell+4}); 2, p_H, \mathcal{G}_{p\ell}) \geq \delta_{96} \,.
\end{aligned}
$$

For the one but last inequality we used (8.84) and Comment 3.3 (v), while the last inequality comes from (8.36).

Now let $\tilde{s}$ be a fixed path satisfying (8.85)-(8.87). This brings us to the setup for Lemma 8.2. Take $a = w_0$, $\Theta = (-3N,0)$ and r_1^* = the piece of r^* from w_0 to the right edge $\{N\} \times [0,4N]$ of J_1 (i.e., if $w_0 = v_i^*$, then $r_1 = (w_0 = v_i^*, e_{i+1}^*, \dots, e_\nu^*, v_\nu^*)$). Then (8.42)-(8.46) with r replaced by r_1, s replaced by $\tilde{s}$ and N replaced by $4N$ are clearly satisfied, since

$$S(\Theta,4N) = [-7N,N] \times [-4N,4N] \supset [0,N] \times [0,4N]$$

and the top right corners of the two rectangles coincide. (8.41) is replaced by

$$S(a,3.2^{\ell}) \subset S(\Theta,4N),$$

which holds by virtue of (8.84) and the fact that $a = w_0$ lies in

$$r^* \cap X \subset [0,3.2^{\ell}+1] \times [0,3N].$$

We now take for $S'(\Theta, r_1, \tilde{s})$ the component of $S(\Theta,4N) \setminus r_1 \cup \tilde{s})$ $= [-7N,N] \times [-4N,4N] \setminus r_1 \cup \tilde{s}$ which is in the "upper right corner" of

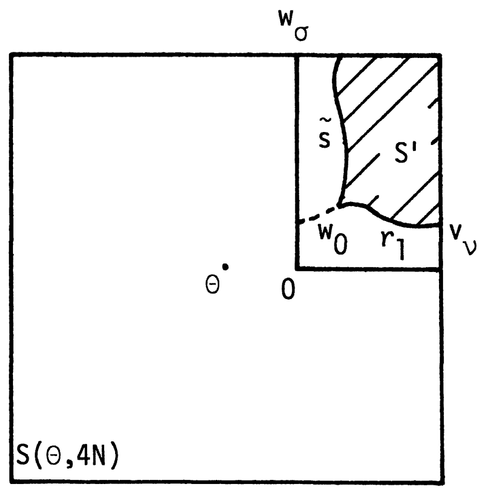


Figure 8.7. r consists of the dashed curve followed by r_1. $S'(\Theta,r_1,\tilde{s}) = J_2^R(\tilde{s})$ is the hatched region.

$S(\Theta,4N)$, i.e., the component which is bounded by $r_1 \cup \tilde{s}$ and the arc of $\Delta S(\Theta,4N)$ from w_σ to v_ν which goes through the upper right corner $(N,4N)$ of $S(\Theta,4N)$ (see Fig. 8.7). The latter arc is also an arc of J_2 and one easily sees that $S'(\Theta,r_1,\tilde{s})$ is precisely $J_2^R(\tilde{s})$. $S''(\Theta,r_1,\tilde{s})$ will be the other component of $S(\Theta,4N)\setminus r_1 \cup \tilde{s}$. Then (8.56) implies

$$\begin{aligned}(8.89)\quad & E_{p_H}\{\# \text{ of vertices of } \mathcal{M} \text{ on } r_1 \cap S(a,3.2^\ell) \text{ connected to} \\ & \tilde{s} \text{ in } S'(\Theta,r_1,\tilde{s}) \cap S(a,3.2^\ell) \,|\, \omega(v) = \varepsilon(v), \\ & v \in \bar{S}''(\Theta,r_1,\tilde{s})\} \geq C_{12} 2^{\alpha_1 \ell}\end{aligned}$$

for any choice of $\varepsilon(v) = \pm 1$, $v \in \bar{S}''$.

We can derive the required estimate (8.80) easily from (8.89) by an argument already used in Lemma 7.4. Firstly

$$\begin{aligned}(8.90)\quad E_{p_H}\{N^*(r^*,N)\} &\geq E_{p_H}\{N_1^*(r^*,N)\} \\ &= \sum_{\tilde{s}} P_{p_H}\{\tilde{S} = \tilde{s}\} E_{p_H}\{N_1^*(r^*,N) \,|\, \tilde{S} = \tilde{s}\} \\ &\geq P_{p_H}\{\tilde{S} \text{ exists}\} \min_{\tilde{s}} E_{p_H}\{N_1^*(r^*,N) \,|\, \tilde{S} = \tilde{s}\} \\ &\geq \delta_{96} \min_{\tilde{s}} E_{p_H}\{N_1^*(r^*,N) \,|\, \tilde{S} = \tilde{s}\} \qquad \text{(by (8.88)).}\end{aligned}$$

The sum and minimum in (8.90) are over all $\tilde{s}$ satisfying (8.85)-(8.87). Secondly, any vertex v on $r_1 \cap S(a,3.2^\ell)$ which is connected to $\tilde{s}$ in $S'(\Theta,r_1,\tilde{s}) \cap \mathring{S}(a,3.2^\ell)$ (in the sense of Def. 8.1) has an occupied connection above r^* to C_1. The argument for this is practically identical to the argument following (7.66)-(7.69) in Lemma 7.4. Consequently

$N_1^*(r^*,N) \geq$ # of vertices of $\mathcal{M}$ on $r_1 \cap S(a,3.2^\ell)$ which is connected to $\tilde{s}$ in $S'(\Theta,r_1,\tilde{s}) \cap S(a,3.2^\ell)$.

Lastly, by Prop. 2.3 the event $\tilde{S} = \tilde{s}$ depends only on the occupancies of the vertices in

$$\bar{J}_2^L(\tilde{s}) \subset \bar{J}_2 \setminus J_2^R(\tilde{s}) \subset S(\Theta,4N) \setminus S'(\Theta,r_1,\tilde{s}) = \bar{S}''(\Theta,r_1,\tilde{s}).$$

Consequently

$$E_{p_H}\{N_1^*(r^*,N) \mid \tilde{S} = s\} \geq \min_{\varepsilon} E_{p_H}\{\text{\# of vertices of } \mathcal{M} \text{ on}$$

$$r_1 \cap S(a,3.2^\ell) \text{ connected to } \tilde{s}' \text{ in } S'(\Theta,r_1,\tilde{s}) \cap S(a,3.2^\ell) \mid$$

$$\omega(v) = \varepsilon(v),\ v \in \bar{S}''(\Theta,r_1,\tilde{s})\} \quad .$$

This, together with (8.90), (8.89) and (8.84), gives

$$E_{p_H}\{N^*(r^*,N)\} \geq \delta_{96} C_{12} \left(\frac{N}{12}\right)^{\alpha_1} ,$$

whence (8.80).

(8.81) is immediate from (8.80) and (8.31) (and the symmetry between horizontal and vertical for the graphs under consideration). Finally (8.82) is nothing but (8.81) with $\mathcal{G}^*$ and "vacant" replaced by $\mathcal{G}$ and "occupied". (Recall that

$$P_p\ \{v \text{ vacant}\} = 1-p = 1-P_p\ \{v \text{ occupied}\}$$

and

$$p_H(\mathcal{G}) = 1-p_H(\mathcal{G}^*) \tag{8.91}$$

for the graphs of this chapter, by virtue of Applications ii) and iv) in Sect. 3.4.) □

Another application of Lemma 8.2 will be needed for Theorems 8.1 and 8.2. It provides us with a lower bound for $P_{p_H}(B(N))$, where

$$B(N) := \{ \exists \text{ occupied path on } \mathcal{G} \text{ in } S(z_0,N) \text{ which connects } z_0 \text{ with a point on } \Delta S(z_0,N)\} . \tag{8.92}$$

Here z_0 is as before, i.e., z_0 = the origin if $\mathcal{G}$ is $\mathcal{G}_0$ or $\mathcal{G}_0^*$, and $z_0 = (\frac{1}{2},0)$ if $\mathcal{G}$ is $\mathcal{G}_1$ or $\mathcal{G}_1^*$.

<u>Lemma 8.4</u>. <u>There exists a constant</u> $0 < C_{14} < \infty$ <u>such that</u>

$$P_{p_H}\{B(N)\} \geq C_{14}N^{\alpha_1-1} . \tag{8.93}$$

<u>Remark</u>.

It is easy to use the argument at the end of the proof below and (8.36) to obtain

$$P_{p_H}\{B(N)\} \geq C_{14}N^{-1} .$$

Such an estimate already appears in Smythe and Wierman (1978), formula (3.34). However, to obtain the lower bounds in (8.5) and (8.6) it is crucial to have an estimate like (8.93) which decreases only as a power of N which is strictly larger than the minus first power. Lemma 8.5 below will give an upper bound for $P_{p_H}\{B(N)\}$ which decreases like a negative power of N. It is not known whether there exists an α for which $N^{\alpha}P_{p_H}\{B(N)\}$ has a nonzero (but finite) limit as $N \to \infty$. If such an α exists it must lie strictly between zero and one by (8.93) and (8.101). This is closely related to questions about the behavior of $P_{p_H}\{\#W \geq N\}$ for large N, or the cluster exponent τ of Stauffer (1979).

<u>Proof of Lemma 8.4</u>. For simplicity take $\mathcal{G}$ equal to $\mathcal{G}_0$, $\mathcal{G}_0^*$ or $\mathcal{G}_1$ so that $\mathcal{G}_{p\ell}$ has edges along the lines $x(i) = k$, $i = 1,2$, $k \in \mathbb{Z}$. Since the left hand side of (8.93) has the same value on $\mathcal{G}_1$ as on $\mathcal{G}_1^*$ these choices for $\mathcal{G}$ suffice.

Fix ℓ as the unique integer with

$$2^{\ell+2} \leq N < 2^{\ell+3} . \tag{8.94}$$

Consider the collection of occupied vertical crossings on $\mathcal{G}_{p\ell}$ of $[-2^{\ell},2^{\ell}] \times [0,N]$, i.e., the collection of occupied paths

$s = (w_0, f_1, \ldots, f_\sigma, w_\sigma)$ on $\mathcal{G}_{p\ell}$ which satisfy

(8.95) $\quad (f_1 \setminus \{w_0\}, w_1, \ldots, w_{\sigma-1}, f_\sigma \setminus \{w_\sigma\}) = s \setminus \{w_0, w_\sigma\} \subset (-2^\ell, 2^\ell) \times (0,N),$

(8.96)
$$w_0 \in [-2^\ell, 2^\ell] \times \{0\} \quad \text{and}$$
$$w_\sigma \in [-2^\ell, 2^\ell] \times \{N\} \quad .$$

Denote by F the event that there exists at least one such occupied crossing. Then, by (8.36) and Comment 3.3 (v)

(8.97)
$$P_{p_H}\{F\} = \sigma((2^{\ell+1}, N); 2, p_H, \mathcal{G}_{p\ell}) \geq \delta_{32} \quad .$$

Let J be the perimeter of $[-2^\ell, 2^\ell] \times [0,N]$ and $A = \{-2^\ell\} \times [0,N]$ its left edge, $C = \{2^\ell\} \times [0,N]$ its right edge. For any crossing s satisfying (8.95) and (8.96) $J^\pm(s)$ are defined as before (see Def. 2.11). Prop. 2.3 tells us that whenever F occurs there is a unique left-most occupied crossing s of J, i.e., an occupied path s with minimal $J^-(s)$ among all occupied paths satisfying (8.95) and (8.96). We denote this left-most crossing by S whenever it exists.

Now let $s = (w_0, f_1, \ldots, f_\sigma, w_\sigma)$ be a fixed path on $\mathcal{G}_{p\ell}$ satisfying (8.95) and (8.96). We shall apply Lemma 8.2 with the following choices: Θ = the origin, $a = w_0$, r = the path along the first coordinate axis, $x(2) = 0$, from w_0 to the point $(N,0)$ on the right edge of $S(\Theta,N)$. N and ℓ satisfy (8.94), so that (8.41) holds since $a = w_0 = (w_0(1),0)$ with $-2^\ell \leq w_0(1) \leq 2^\ell$ (by (8.96)). We view r as a path on $\mathcal{G}_{p\ell}$. (8.42)-(8.46) are trivially fulfilled for r and s. For $S'(\Theta,r,s)$ we take the "upper right corner" of $S(\Theta,N) \setminus r \cup s$, i.e., the component of $S(\Theta,N) \setminus r \cup s$ which contains the corner vertex (N,N) in its boundary (see Fig. 8.8). $S''(\Theta,N)$ is the other component of $S(\Theta,N) \setminus r \cup s$. It is clear that $Fr(J^-(s))$ intersects $Fr(S'(\Theta,N))$ only in the path s, which is common to both these boundaries. Moreover, the point (N,N) of $Fr(S')$ lies in $ext(J^-(s))$. Consequently $Fr(S') \subset$ closure of $ext(J^-(s))$. Therefore $J^-(s)$ either lies entirely in S' or entirely in S''. Since $A \subset Fr(J^-(s))$ can be connected by a horizontal line segment to the left edge $\{-N\} \times [-N,+N]$ of $S(\Theta,N)$ without entering S' it follows that

(8.98)
$$J^-(S) \subset S''(\Theta,r,s).$$

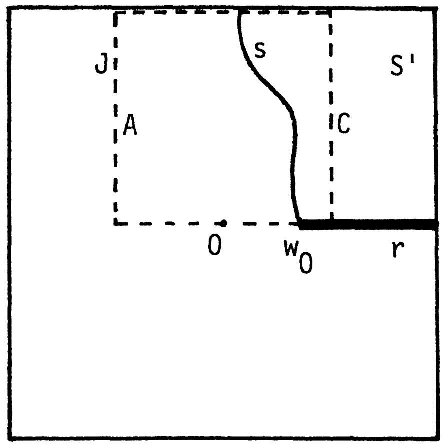


Figure 8.8 r is the boldly drawn path. J is the dashed rectangle. The large square is $S(\Theta,N)$.

We shall write B for the upper edge, $[-N,N]\times\{N\}$, of $S(\Theta,N)$. We shall say that a vertex v of $\mathcal{M}$ in $S(\Theta,N)$ has an occupied connection to B if there exists an occupied path $t = (u_0,g_1,\dots,g_\rho,u_\rho)$ on $\mathcal{G}_{p\ell}$ which satisfies

$$(g_1\setminus\{u_0\},u_1,\dots,u_{\rho-1},g_\rho\setminus\{u_\rho\}) = t\setminus\{u_0,u_\rho\} \subset \overset{\circ}{S}(\Theta,N),$$

$$u_0 = v \quad\text{and}\quad u_\rho \in B.$$

Assume now that $\{S = s\}$ occurs so that s is occupied. Exactly as in the argument following (7.66)-(7.69) one now sees that any vertex v of $\mathcal{M}$ on r which is connected to s in $S'(\Theta,r,s)\cap S(a,3.2^\ell)$ (in the sense of Def. 8.1) automatically has an occupied connection to B. Therefore

$$(8.99)\quad E_{p_H}\{\#\text{ of vertices of } \mathcal{M} \text{ on } r \text{ which have an occupied connection to } B \mid S = s\}$$

$$\geq E_{p_H}\{\sum_{\substack{v_i\in r\cap S(a,3.2^\ell)\\ v_i\in\mathcal{M}}} Y(v_i,a,\ell,r,s)\mid S = s\}\ .$$

Proposition 2.3 shows that the event $\{S = s\}$ depends only on the occupancies of vertices in $\overline{J}^-(s) \subset \overline{S}''(\Theta,r,s)$ (see (8.98)). Consequently the right hand side of (8.99) is at least

$$\min_{\varepsilon} E_{p_H}\{\sum_{\substack{v_i \in r \cap S(a,3.2^\ell) \\ v_i \in \mathcal{M}}} Y(v_i,a,\ell,r,s) | \omega(v) = \varepsilon(v), v \in \bar{S}''(\Theta,r,s)\}$$

$$\geq Z(\ell) \geq C_{12} 2^{\alpha_1 \ell} \geq C_{12} 2^{-3\alpha_1} N^{\alpha_1}$$

(see (8.55) and (8.94)). Finally, since $r \subset [-N,+N] \times \{0\}$,

(8.100) $E_{p_H}\{$# of vertices of $\mathcal{M}$ on $[-N,N] \times \{0\}$ which have an occupied connection to $B\}$

$\geq \sum_s P_{p_H}\{S = s\} E_{p_H}\{$# of vertices of $\mathcal{M}$ on r which are connected to s in $S'(\Theta,r,s) \cap S(w_0,3.2^\ell) | S = s\}$

$$\geq C_{12} 2^{-3\alpha_1} N^{\alpha_1} \sum_s P_{p_H}\{S = s\}$$

$$\geq C_{12} 2^{-3\alpha_1} N^{\alpha_1} P_{p_H}\{F\} \geq \delta_{32} C_{12} 2^{-3\alpha_1} N^{\alpha_1} .$$

In (8.100) the sum is over all s which satisfy (8.95) and (8.96), and the last inequality relies on (8.97). (8.93) follows from (8.100) since any vertex v of $\mathcal{M}$ on $[-N,N] \times \{0\}$ which has an occupied connection to B also is connected by an occupied path on $\mathcal{G}_{p\ell}$ to a vertex on $\Delta S(v,N)$, because B lies in the complement of $\mathring{S}(v,N)$. From Lemma 2.1a we see that any such v is then also connected by an occupied path on $\mathcal{G}$ to a point on $\Delta S(v,N-1)$. The probability of this event is $P_{p_H}\{B(N-1)\}$, the same for all v of $\mathcal{M}$ on $[-N,N] \times \{0\}$. There are at most $(2N+1)$ such vertices v on $[-N,N] \times \{0\}$, so that the left hand side of (8.100) is at most equal to $(2N+1) P_{p_H}\{B(N-1)\}$. (8.93) follows. $\square$

We turn to the upper bound for $P_{p_H}\{B(N)\}$. The method for this estimation is due to Russo (1978) and Seymour and Welsh (1978).

<u>Lemma 8.5</u>. <u>There exist constants</u> $0 < C_{15}, \alpha_2 < \infty$ <u>such that</u>

(8.101) $$P_{p_H}\{B(N)\} \leq C_{15} N^{-\alpha_2} .$$

<u>Proof</u>: Consider the disjoint annuli

$$(8.102)\qquad V_\ell := S(0,3.2^{2\ell})\setminus \mathring{S}(0,3.2^{2\ell-1}) \quad \text{for}$$
$$\ell = 1,2,\dots,\ell_0 := \left\lfloor \frac{\log \frac{1}{3}(N-2)}{2\log 2} \right\rfloor .$$

These are all contained in $S(z_0,N-1)$. If any one of them contains a vacant circuit c^* on $\mathcal{G}^*_{p\ell}$ surrounding the origin, then there cannot exist an occupied path on $\mathcal{G}$ from z_0 to a point on $\Delta S(z_0,N)$. Indeed such an occupied path would start at z_0 in the interior of c^* and end in the exterior of c^*, and hence would have to intersect c^*. But if a path on $\mathcal{G}$ intersects a path on $\mathcal{G}^*_{p\ell}$, then the two paths must have a vertex of $\mathcal{G}$ in common (cf. Comment 2.3 (v)). In our case there would have to be a vertex on c^* (hence vacant) which would also be a vertex on an occupied path from z_0 to $\Delta S(z_0,N)$, which is impossible.

It follows from the above that

$$(8.103)\qquad P_p\{B(N)\} \le P_p\{\text{there is no vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding the origin in any } V_\ell,\ 1 \le \ell \le \ell_0\}$$
$$= \prod_{\ell=1}^{\ell_0} P_p\{\text{there is no vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding the origin in } V_\ell\} .$$

But (8.37) applied to $\mathcal{G}^*$ (and with ℓ replaced by $2^{2\ell-2}$) states

$$P_{p_H(\mathcal{G}^*)}\{\exists \text{ occupied circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding the origin in } V_\ell\} \ge \delta_4^4 ,\quad \ell \ge 1.$$

It we interchange "occupied" and "vacant" and take (8.91) into account, this means

$$(8.104)\qquad P_{p_H(\mathcal{G})}\{\text{there is no vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding the origin in } V_\ell\} \le 1-\delta_4^4 ,\quad \ell \ge 1.$$

Substituting this estimate into the right hand side of (8.103) yields

$$P_{p_H(\mathcal{G})}\{B(N)\} \le (1-\delta_4^4)^{\ell_0} ,$$

from which (8.101) is immediate. $\square$

Proof of (8.4). The left hand inequality will be seen to follow quickly from (8.13), (8.93), (8.19) and (8.81). Indeed, by virtue of (8.81), (8.17) holds for

$$N = C_{16}(p-p_H(\mathcal{G}))^{-1/\alpha_1} .$$

(We continue to denote by C_k various finite but strictly positive constants which depends on $\mathcal{G}$ only.) (8.19) now shows that

$$P_p\{\#W^*(z_0) \geq \ell\} \leq C_1 \exp - C_{17}(p-p_H(\mathcal{G}))^{2/\alpha_1}\ell \quad ,p > p_H(\mathcal{G}). \tag{8.105}$$

Since

$$\sum_{\ell=M}^{\infty} \ell e^{-x\ell} = -\frac{d}{dx}\Big(\sum_{\ell=M}^{\infty} e^{-x\ell}\Big) = \frac{e^{-xM}}{1-e^{-x}}\Big(M+\frac{1}{1-e^{-x}}\Big) \tag{8.106}$$

one easily sees from this that

$$\sum_{\ell=M}^{\infty} \ell P_p\{\#W^*(z_0) \geq \ell\} \leq \frac{1}{2}$$

as soon as

$$M \geq C_{18}(p-p_H(\mathcal{G}))^{-3/\alpha_1} . \tag{8.107}$$

Thus, by (8.13), and the definition (8.92) of $B(M)$

$$\theta(p) \geq \frac{1}{2} P_p\{B(M)\}, \quad p > p_H$$

for any M which satisfies (8.107). Finally, since $B(M)$ is an increasing event we obtain from Lemma 4.1 and (8.93)

$$\theta(p) \geq \frac{1}{2} P_p\{B(M)\} \geq \frac{1}{2} P_{p_H(\mathcal{G})}\{B(M)\}$$

$$\geq \frac{1}{2} C_{14}\{C_{18}(p-p_H(\mathcal{G}))^{-3/\alpha_1}\}^{\alpha_1 - 1}, \quad p > p_H(\mathcal{G}) .$$

This gives the left hand inequality in (8.4).

The right hand inequality in (8.4) is much easier to prove. Indeed, if $\#W(z_0) = \infty$ then z_0 is connected by occupied paths to $\Delta S(z_0,N)$ for all N. Consequently, for each N

$$\theta(p) \leq P_p\{B(N)\} .$$

However, $B(N)$ is an increasing event which depends only on the occupancies of vertices in $S(z_0,N)$, and for our graphs there are at most $2(2N+1)^2$ vertices of $\mathcal{G}$ in $S(z_0,N)$. Thus for $p_1 \le p_2$ (4.2) applied to $f = I_{B(N)}$ gives

$$(8.108) \qquad P_{p_2}\{B(N)\} \le \left(\frac{p_2}{p_1}\right)^{2(2N+1)^2} P_{p_1}\{B(N)\} \quad .$$

If we use (8.108) with $p_1 = p_H < p_2 = p$ then we obtain from (8.101)

$$\theta(p) \le \left(\frac{p}{p_H}\right)^{2(2N+1)^2} P_{p_H}\{B(N)\} \le \left(\frac{p}{p_H}\right)^{2(2N+1)^2} C_{15}N^{-\alpha_2}, \; p \ge p_H \; .$$

This holds for all N, and the right hand inequality of (8.4) now follows by choosing

$$N = \left\lfloor \left(\log \frac{p}{p_H}\right)^{-1/2} \right\rfloor \sim (p_H)^{1/2}(p-p_H)^{-1/2}, \; p > p_H \; . \qquad \square$$

Proof of the left hand inequalities in (8.5) and (8.6). Whenever $B(N)$ occurs, then $W(z_0)$ contains an occupied path from z_0 to $\Delta S(z_0,N)$, and any such path contains at least N vertices of $\mathcal{G}$. Therefore

$$E_p\{\#W;\#W < \infty\} \ge NP_p\{B(N) \text{ occurs and } \exists \text{ vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding } 0 \text{ in } V_{\ell_0+3}\} \, .$$

Here V_ℓ and ℓ_0 are as in (8.102) and we again use the fact that any vacant circuit on $\mathcal{G}^*_{p\ell}$ which surrounds z_0 must contain all of $W(z_0)$ in its interior (cf. proof of Lemma 8.5). But $B(M)$ depends only on the vertices in $S(z_0,N) \subset S(0,N+1)$ in the graphs $\mathcal{G}$ under consideration. Moreover $S(0,N+1)$ is disjoint from V_{ℓ_0+3} for $N > 5$. It follows that for $N > 5$

$$E_p\{\#W;\#W < \infty\} \ge NP_p\{B(N)\}$$
$$. P_p\{\exists \text{ vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding } 0 \text{ in } V_{\ell_0+3}\}.$$

Now we first take $p \ge p_H$. Then we obtain from the fact that $B(N)$ is an increasing event and (4.1), (8.93)

$$(8.109) \quad E_p\{\#W;\#W < \infty\} \ge N\, C_{14}N^{\alpha_1-1}$$
$$. P_p\{\exists \text{ vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding } 0 \text{ in } V_{\ell_0+3}\} \, .$$

Next we use Lemma 4.1 with f the indicator function of the event that there exists a vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding 0 in V_{ℓ_0+3}. This event depends only on the occupancies of the vertices of $\mathcal{G}^*_{p\ell}$ in V_{ℓ_0+3}. Actually, it depends only on the vertices of $\mathcal{M}$ (or $\mathcal{G}^*$) in V_{ℓ_0+3}, since the other vertices of $\mathcal{G}^*_{p\ell}$ are all vacant by our convention (7.3). There are at most $C_{19}N^2$ vertices of $\mathcal{G}^*$ in V_{ℓ_0+3}. Therefore by the version of (4.2) for decreasing f, and (8.104)

$$\begin{aligned}
&P_p\{\exists \text{ vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding } 0 \text{ in } V_{\ell_0+3}\} \\
&\geq \left(\frac{1-p}{1-p_H}\right)^{C_{19}N^2} P_{p_H}\{\exists \text{ vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding } 0 \\
&\qquad\qquad \text{in } V_{\ell_0+3}\} \\
&\geq \left(\frac{1-p}{1-p_H}\right)^{C_{19}N^2} \delta_4^4, \quad p \geq p_H .
\end{aligned}$$

From this and (8.109) we obtain

$$E_p\{\#W;\#W < \infty\} \geq NC_{14}N^{\alpha_1-1}\left(\frac{1-p}{1-p_H}\right)^{C_{19}N^2} \delta_4^4,\ p \geq p_H .$$

The left hand inequality in (8.6) follows by taking

$$N = \left\lfloor \left(\log \frac{1-p_H}{1-p}\right)^{-1/2} \right\rfloor \sim (1-p_H)^{1/2}(p-p_H)^{-1/2},\ p > p_H .$$

To obtain the left hand inequality of (8.5) we take $p \leq p_H$. Then

$$P_p\{\#W(z_0) < \infty\} = 1-\theta(p) = 1.$$

(This is true even at $p = p_H$ by Sect. 3.3 or by (8.4).) We therefore have the simple bound

$$\begin{aligned}
E_p\{\#W\} &\geq NP_p\{B(N)\} \\
&\geq N\left(\frac{p}{p_H}\right)^{2(2N+1)^2} P_{p_H}\{B(N)\} \qquad \text{(see (8.108))} \\
&\geq N\left(\frac{p}{p_H}\right)^{2(2N+1)^2} C_{14}N^{\alpha_1-1} .
\end{aligned}$$

This time we take

$$N = \lfloor (\log \frac{p_H}{p})^{-1/2} \rfloor \sim p_H^{1/2}(p_H-p)^{-1/2}, \ p < p_H \ . \qquad \square$$

Proof of right hand inequalities in (8.5) and (8.6). The right hand inequality in (8.5) comes from (8.105) with $\mathcal{G}$ and $\mathcal{G}^*$ interchanged and "occupied" and "vacant" interchanged. With these changes (8.105) turns into

$$(8.110) \qquad P_p\{\#W(z_0) \geq \ell\} \leq C_1 \exp - C_{17}(p_H(\mathcal{G})-p)^{2/\alpha_1} \ell \ , \ p < p_H(\mathcal{G})$$

(recall (8.91)). The right hand inequality in (8.5) is now obtained by summing over ℓ.

For the right hand inequality in (8.6) we need one more observation. Since $S(z_0,N)$ contains no more than $2(2N+1)^2$ vertices of $\mathcal{G}$, $\#W(z_0) > 2(2N+1)^2$ implies that $W(z_0)$ must contain vertices outside $S(z_0,N)$. This can only happen if z_0 is connected by an occupied path to the exterior of $S(z_0,N)$, and hence $B(N)$ occurs. If in addition $\#W(z_0) < \infty$, then - as we saw in the derivation of (8.12) - there must exist a vacant circuit on $\mathcal{G}^*$ surrounding z_0 and containing at least N vertices of $\mathcal{G}^*$. Therefore

$$(8.111) \qquad P_p\{2(2N+1)^2 < \#W(z_0) < \infty\} \leq P_p\{B(N) \text{ and there exists a vacant circuit on } \mathcal{G}^* \text{ surrounding } z_0 \text{ and containing at least } N \text{ vertices}\} \ .$$

By the estimate used for the second factor in the right hand side of (8.12) we obtain by means of (8.105), (8.106)

$$\begin{aligned}(8.112) \qquad & P_p\{2(2N+1)^2 < \#W(z_0) < \infty\} \\ & \leq \sum_{\ell=N}^{\infty} \ell P_p\{\#W^*(z_0) \geq \ell\} \\ & \leq \sum_{\ell=N}^{\infty} C_1 \ell \exp - C_{17}(p-p_H(\mathcal{G}))^{2/\alpha_1} \ell \ , \ p > p_H(\mathcal{G}) \\ & \leq C_{20}(p-p_H(\mathcal{G}))^{-2/\alpha_1}\{N+(p-p_H(\mathcal{G}))^{-2/\alpha_1}\} \\ & \qquad \cdot \exp - C_{17}(p-p_H(\mathcal{G}))^{2/\alpha_1} N \ .\end{aligned}$$

Since

$$E_p\{\#W(z_0);\#W(z_0) < \infty\} \le 2+16 \sum_{N=0}^{\infty}(N+1)P_p\{2(2N+1)^2 < \#W(z_0) < \infty\}$$

the right hand inequality in (8.6) follows easily. □

The above proofs of (8.4)-(8.6) constitute the proof of Theorem 8.1.

Proof of Theorem 8.2. We begin with the proof of (8.7). For $p \le p_H$ we have the simple estimate

$$P_p\{n \le \#W(z_0) < \infty\} \le P_p\{B(\tfrac{1}{2}(\tfrac{n-1}{2})^{\frac{1}{2}} - \tfrac{1}{2})\} \text{ (by (8.111))}$$

$$\le P_{p_H}\{B(\tfrac{1}{2}(\tfrac{n-1}{2})^{1/2} - \tfrac{1}{2})\} \text{ (since } B \text{ is an increasing event)}$$

$$\le C_{19}n^{-\alpha_2} \quad \text{(by (8.101))} \quad .$$

For $p > p_H$ we estimate (8.7) more or less in the same way, as long as p is close to p_H, and by means of (8.112) for $p-p_H$ large. To be specific, take

$$m = \frac{1}{3}\min(n^{1/2}, n^{\alpha_1/16}).$$

Then, for large n $2(2m+1)^2 < n$ so that by (8.111), (8.108) and (8.101)

$$(8.113) \quad P_p\{n \le \#W(z_0) < \infty\} \le P_p\{2(2m+1)^2 < \#W(z_0) < \infty\}$$

$$\le P_p\{B(m)\} \le (\frac{p}{p_H})^{2(2m+1)^2} P_{p_H}\{B(m)\}$$

$$\le C_{15}(\frac{p}{p_H})^{2(2m+1)^2} m^{-\alpha_2} \quad , \quad p > p_H \, .$$

For

$$0 < p-p_H \le n^{-\alpha_1/8}$$

the factor

$$(\frac{p}{p_H})^{2(2m+1)^2} \le \exp\{2(2m+1)^2\log(1+p_H^{-1}n^{-\alpha_1/8})\}$$

is bounded, so that (8.7) holds with $\gamma_5 = \min(\alpha_2/2, \alpha_1\alpha_2/16)$ for such p. On the remaining interval

$$p-p_H \ge n^{-\alpha_1/8} \quad ,$$

we use (8.112) with

$$N = \frac{1}{3} n^{1/2} \geq \frac{1}{3}(p-p_H)^{-4/\alpha_1} \quad .$$

This gives

$$P_p\{n \leq \#W(z_0) < \infty\} \leq P_p\{2(2N+1)^2 \leq \#W(z_0) < \infty\}$$

$$\leq C_{21} N^{3/2} \exp - C_{17} N^{1/2} = O(n^{-\gamma_5}),$$

for any choice of $\gamma_5 > 0$.

The above proves (8.7) in all cases. (8.8) and the last inequality in (8.9) are immediate from (8.7). Finally, the left hand inequality in (8.9) follows from the observation - made already in the proof of the left hand inequalities in (8.5) and (8.6) - that $\#W(z_0) \geq n$ on the event $B(n)$. Thus, by Lemma 8.4

$$P_{p_H}\{\#W(z_0) \geq n\} \geq P_{p_H}\{B(n)\} \geq C_{14} n^{\alpha_1 - 1} \quad . \qquad \square$$