LOGICAL GEOMETRY

> "*It is a very interesting fact that notions originally developed for the purposes of (abstract) algebraic geometry turn out to be intimately related to logic and model theory. Compared to other existing versions of algebraic logic, categorical logic has the distinction of being concerned with objects that appear in mathematical practice.*"
>
> Michael Makkai and Gonzalo Reyes

The theory discussed in this book emerges from an interaction between sheaf theory and logic, and for the most part we have dwelt on the impact of the former on the conceptual framework of the latter. In this chapter we will consider ways in which the application has gone in the opposite direction. Specifically, we study the concept of a *geometric morphism*, a certain kind of functor between topoi that plays a central role in the work of the Grothendieck school (Artin et al. [SGA 4]). In their book *First Order Categorical Logic*, henceforth referred to as [MR], Makkai and Reyes have shown that this notion of morphism can be reformulated in logical terms, and that some important theorems of Pierre Deligne and Michael Barr about the existence of geometric morphisms can be derived by model-theoretic constructions. The essence of their approach is to associate a *theory* (set of axioms) with a given site, and identify functors defined on the site with models of this theory. Conversely, from a certain type of theory a site can be built by a method that adds a new dimension of mathematical significance to the well-known Lindenbaum-algebra construction (cf. §6.5).

These developments will be described below, with our main aim being to account for the fact that Deligne's theorem is actually equivalent to a version of the classical Gödel Completeness Theorem for **Set**-based semantics of first-order logic.

Model theory is both an independent science and an effective technique for studying mathematical structures and explaining their properties. The second of these aspects is perhaps most closely associated with the name of Abraham Robinson, who summarised it in the title of one of his papers – "Model theory as a framework for algebra" (Robinson [73]). Since Robinson was at Yale during the latter part of his career, this attitude has become known as "eastern" model theory, by contrast with the "western" approach, associated with Alfred Tarski at Berkeley, which focuses on the general properties of formal languages and their semantics. The work of Makkai and Reyes is in the eastern style, and constitutes "model theory as a framework for topos theory". One of the goals of this chapter is to exhibit their proof of Deligne's Theorem as a major exercise in applied mathematical logic.

The distinction between western and eastern model theory is given a syntactic expression by H. J. Keisler (cf. page 48 of Barwise [77]): the former is concerned with *all* formulae of first-order languages, while the latter emphasises *universal-existential* formulae – those of the form $\forall v_1 .. \forall v_n \exists w_1 .. \exists w_m \varphi$, with φ quantifier-free – since these suffice to axiomatise the main structures of classical algebra. We will see that the logic of geometric morphisms has an analogous syntactic emphasis, in that it is expressed by formulae, called "geometric" or "coherent", that have the form $\varphi \supset \psi$, where φ and ψ have no occurrence of the symbols $\sim, \supset, \forall$.

16.1. Preservation and reflection

In order to define geometric morphisms we need some general information about how the behaviour of a functor affects the existence of limits and colimits in its domain and codomain. So, let $F : \mathscr{C} \to \mathscr{D}$ be a functor between categories $\mathscr{C}$ and $\mathscr{D}$. F is said to *preserve monics* if, for any $\mathscr{C}$-arrow f, if f is monic in $\mathscr{C}$, then $F(f)$ is monic in $\mathscr{D}$. On the other hand F *reflects* monics if, for any $\mathscr{C}$-arrow f, if $F(f)$ is monic in $\mathscr{D}$ then f is monic in $\mathscr{C}$. Replacing "monic" by "epic" or "iso" here defines what it is for F to preserve or reflect these latter types of arrows.

Similarly, F is said to *preserve equalisers* if whenever e equalises f and g in $\mathscr{C}$, then $F(e)$ equalises $F(f)$ and $F(g)$ in $\mathscr{D}$. If the converse of this last implication always holds, then F is said to *reflect equalisers*. To describe reflection and preservation of categorial constructs in general, it is helpful to invoke the language of diagrams and limits of §3.11. Let D be a

diagram in $\mathscr{C}$, comprising $\mathscr{C}$-objects $d_i, d_j, \ldots$ and $\mathscr{C}$-arrows $g : d_i \to d_j$. The action of F on D produces a diagram $F(D)$ in $\mathscr{D}$ comprising the $\mathscr{D}$-objects $F(d_i)$, $F(d_j)$, ... and $\mathscr{D}$-arrows $F(g) : F(d_i) \to F(d_j)$. F *preserves D-limits* if whenever $\{f_i : c \to d_i\}$ is a collection of arrows forming a limit (universal cone) for D in $\mathscr{C}$, then $\{F(f_i) : F(c) \to F(d_i)\}$ is a limit for $F(D)$ in $\mathscr{D}$. On the other hand, if F always maps a colimit for D in $\mathscr{C}$ to a colimit for $F(D)$ in $\mathscr{D}$, then F *preserves colimits of D*. Reversing the implications in these last two definitions yields the notions of F *reflecting* limits and colimits, respectively, of D.

To be even more general we may simply say that if P is some categorial "property", then F *preserves P* if the image under F of an entity in $\mathscr{C}$ with property P has property P in $\mathscr{D}$, and F *reflects P* if whenever the F-image of an entity from $\mathscr{C}$ has P in $\mathscr{D}$, then that entity itself has P in $\mathscr{C}$.

EXERCISE 1. Show that any functor preserves identities, iso arrows, and commutative diagrams.

EXERCISE 2. If F preserves pullbacks, then F preserves monics. □

A functor $F : \mathscr{C} \to \mathscr{D}$ is *faithful* if it acts injectively on each "hom-set" $\mathscr{C}(a, b)$ (cf. Example 9.1.6). This means that for any pair $f, g : a \to b$ of $\mathscr{C}$-arrows with the same domain and codomain, if $F(f) = F(g)$ then $f = g$.

EXERCISE 3. Show that the forgetful functor **Grp** $\to$ **Set** is faithful but is not bijective on objects or on identity arrows.

EXERCISE 4. Show that a faithful functor reflects monics, epics (and hence iso's if its domain is a topos), and commutative diagrams.

EXERCISE 5. Suppose that $\mathscr{C}$ has an equaliser for any parallel pair of arrows. Show that a parallel pair are equal iff their equaliser is iso. Hence show that if F is a functor on $\mathscr{C}$ that preserves equalisers, then F reflects iso's only if F is faithful. □

It follows by these exercises that a functor, defined on a topos, which preserves equalisers is faithful if, and only if, it reflects iso arrows. There is another important variant of faithfulness, which is the notion of a functor that reflects inclusions of subobjects. To be precise, we need to assume that F preserves monics. Then if f and g are subobjects of a

$\mathscr{C}$-object d, $F(f)$ and $F(g)$ will serve as subobjects of $F(d)$ in $\mathscr{D}$. We say that F is *conservative* if whenever $F(f) \subseteq F(g)$ in Sub($F(d)$), it follows that $f \subseteq g$ in Sub(d).

EXERCISE 6. Suppose that $\mathscr{C}$ has equalisers, and that these are preserved by F. Show that if F is conservative, then F is faithful.

EXERCISE 7. Suppose that $\mathscr{C}$ has pullbacks of all appropriate pairs of arrows, and that these pullbacks are preserved by F. Using the pullback characterisation of intersections (Theorem 7.1.2) show that F reflects iso's only if F is conservative. $\square$

Thus it follows that for a functor which is defined on a topos and preserves equalisers and pullbacks, "faithful", "conservative", and "preserves iso's" are all equivalent.

We will be particularly concerned with functors that preserve all finite limits (i.e. limits of all finite diagrams). Such a functor is called *left exact*, while, dually, a *right exact* functor is one that preserves colimits of all finite diagrams. One that is both left and right exact is simply called *exact*. If a category $\mathscr{C}$ is finitely complete (i.e. has all finite limits, cf. §3.15), then it can be shown that for a functor F defined on $\mathscr{C}$ to be left exact it suffices either that F preserves terminal objects and pullbacks, or that F preserves terminal objects, equalisers, and products of pairs of $\mathscr{C}$-objects (Herrlich and Strecker [73], Theorem 24.2). The dual statement is left to the reader.

Since monics and epics are special cases of limits and colimits respectively (Exercise 3.13.9 and its dual), we see that exact functors preserve epi-monic factorisations. In view of Theorem 5.2.2, we then have the following important fact.

EXERCISE 8. If F is an exact functor between two topoi, then F preserves images of arrows, i.e. $F(\text{im } f)$ is im($F(f)$). $\square$
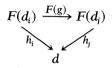
One context in which preservation of certain limits and colimits is guaranteed is that of an adjoint situation (§15.1).

THEOREM 1. *If $\langle F, G, \theta \rangle$ is an adjunction from $\mathscr{C}$ to $\mathscr{D}$, then the left adjoint F preserves all colimits of $\mathscr{C}$, while the right adjoint G preserves all $\mathscr{D}$-limits.*

PROOF. We outline the argument showing that F preserves colimits, giving

enough of the construction to display the role of the adjunction, and leaving the fine detail as a worthy exercise for the reader.

Using the notation of §3.11, let D be a diagram in $\mathscr{C}$ that has a colimit $\{f_i : d_i \to c\}$. Since F preserves commutative diagrams, the collection $\{F(f_i) : F(d_i) \to F(c)\}$ will be a cocone for the diagram $F(D)$ in $\mathscr{D}$. We wish to show that it is co-universal for $F(D)$. So, let $\{h_i : F(d_i) \to d\}$ be another cocone for $F(D)$ in $\mathscr{D}$, meaning that

$$
\begin{array}{ccc}
F(d_i) & \xrightarrow{\ F(g)\ } & F(d_j) \\
& \underset{h_i}{\searrow} \quad \underset{h_j}{\swarrow} & \\
& d &
\end{array}
$$

commutes for each arrow $g : d_i \to d_j$ in D. Applying the components $\theta_{d_i,d}$ of θ we then obtain a family $\{\theta(h_i) : d_i \to G(d)\}$ of $\mathscr{C}$-arrows which proves to be a cocone for D, since the naturalness of θ can be invoked to show that

$$
\begin{array}{ccc}
d_i & \xrightarrow{\ g\ } & d_j \\
\underset{\theta(h_i)}{\searrow} & & \underset{\theta(h_j)}{\swarrow} \\
& G(d) &
\end{array}
$$

always commutes, where g is as above. But then as $\{f_i : d_i \to c\}$ is a colimit for D, there is a unique $\mathscr{C}$-arrow $f : c \to G(d)$ such that

$$
\begin{array}{ccc}
& d_i & \\
f_i \swarrow & & \searrow \theta(h_i) \\
c & \dashrightarrow[f] & G(d)
\end{array}
$$

commutes for all d_i in D.

Applying the inverse of the component θ_{cd} to f, we obtain an arrow $k : F(c) \to d$ such that

$$
\begin{array}{ccc}
& F(d_i) & \\
F(f_i) \swarrow & & \searrow h_i \\
F(c) & \xrightarrow{\ k\ } & d
\end{array}
$$

always commutes. Indeed k is $\varepsilon_d \circ F(f)$, where $\varepsilon : F \circ G \to 1_{\mathscr{D}}$ is the counit of the adjunction. Moreover, the uniqueness of f and the injectivity of θ_{cd} lead us to conclude that k is the only arrow for which this last diagram always commutes (the couniversal property of the unit η of the adjunction expressed in (2) and (3) of §15.1 can be used to prove this). $\qquad\square$

Thus we see that a left exact functor F which has a right adjoint must preserve all finite limits and all colimits (and hence be exact). Functors of this kind lie at the heart of the notion of geometric morphism, which we now proceed to define.

16.2. Geometric morphisms

Let X and Y be topological spaces, with Θ_X and Θ_Y their associated poset categories of open sets. A function $f: X \to Y$ is continuous precisely when each member of Θ_Y pulls back under f to a member of Θ_X, i.e. $V \in \Theta_Y$ only if $f^{-1}(V) \in \Theta_X$, where $f^{-1}(V) = \{x \in X: f(x) \in V\}$ (recall the discussion in Example 3.13.2 of the inverse image $f^{-1}(V)$ as a pullback). In this case, the map f^* taking V to $f^{-1}(V)$ becomes a functor $f^*: \Theta_Y \to \Theta_X$ which is an $\sqcap$-$\sqcup$ map of **CHA**'s (and which is a special case of the pulling-back functor $f^*: \mathscr{C} \downarrow b \to \mathscr{C} \downarrow a$ discussed in §15.3). As a functor, f^* has a right adjoint $f_*: \Theta_X \to \Theta_Y$ defined, for each $U \in \Theta_X$, by

$$f_*(U) = \bigcup \{V: f^{-1}(V) \subseteq U\}.$$

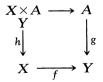EXERCISE 1. Why is f^* left exact?

EXERCISE 2. Show that

$$f^*(V) \subseteq U \quad \text{iff} \quad V \subseteq f_*(U),$$

and hence
$$f^* \dashv f_* \qquad \qquad \square$$

A continuous function $f: X \to Y$ can be lifted to a pair (f^*, f_*) of adjoint functors between the topoi **Top**(Y) and **Top**(X) which generalises the above situation. First we define the functor $f^*: \textbf{Top}(Y) \to \textbf{Top}(X)$, as follows. If $g: A \to Y$ is a **Top**(Y)-object, i.e. a local homeomorphism into Y, we form the pullback h of g along f in **Set**, thus:

$$
\begin{array}{ccc}
X \times_Y A & \longrightarrow & A \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

The domain of h inherits the product topology of X and A, and h proves to be a local homeomorphism, hence a **Top**(X)-object. We put $f^*(g) = h$,

and leave it to the reader to use the universal property of pullbacks to define f^* on $\mathbf{Top}(Y)$-arrows (cf. §15.3) and to show f^* is left exact.

EXERCISE 3. Explain how Θ_Y can be regarded as a subcategory of $\mathbf{Top}(Y)$, and $f^*:\mathbf{Top}(Y) \to \mathbf{Top}(X)$ an extension of the ⊓-⊔ map induced on Θ_Y by f.   □

To define f_* we switch from sheaves of germs to sheaves of sections. We saw in §14.1 how $\mathbf{Top}(X)$ is equivalent to the topos $\mathbf{Sh}(X)$ whose objects are those contravariant functors $F:\Theta_X \to \mathbf{Set}$ which satisfy the axiom COM. But we have just seen that f gives rise to an ⊓-⊔ map $\Theta_Y \to \Theta_X$, and so we can compose this with F to obtain $f_*(F):\Theta_Y \to \mathbf{Set}$. In other words, for $V \in \Theta_Y$, we put

$$f_*(F)(V) = F(f^{-1}(V)).$$

This definition of $f_*(F)$ turns out to produce a sheaf over Y, and gives rise to a functor $f_*:\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. Applying the equivalence of $\mathbf{Sh}$ and $\mathbf{Top}$ then leads to a functor from $\mathbf{Top}(X)$ to $\mathbf{Top}(Y)$ that proves to be right adjoint to f^*.

EXERCISE 4. Explain how this right adjoint can be construed as an extension of the function $f_*:\Theta_X \to \Theta_Y$ defined earlier.

EXERCISE 5. Let $f^*:\Omega \to \Omega'$ be an ⊓-⊔ map between **CHA**'s. If $\mathbf{A}$ is an Ω-set, define an Ω'-set $f^*(\mathbf{A})$, based on the same **Set**-object as $\mathbf{A}$, by putting
$$[\![x \approx y]\!]_{f^*(\mathbf{A})} = f^*([\![x \approx y]\!]_{\mathbf{A}}).$$
Using completions of Ω-sets (§14.7), show that this gives rise to a functor $f^*:\mathbf{Sh}(\Omega) \to \mathbf{Sh}(\Omega')$. Conversely, show that the process of "composing with $f^*:\Omega \to \Omega'$" gives rise to a functor $f_*:\mathbf{Sh}(\Omega') \to \mathbf{Sh}(\Omega)$ that has f^* as a left exact left adjoint (cf. Fourman and Scott [79], §6, for details of this construction).   □

In view of the analysis thus far, we are led to the following definition: a *geometric morphism* $f:\mathscr{E}_1 \to \mathscr{E}_2$ of elementary topoi $\mathscr{E}_1$ and $\mathscr{E}_2$ is a pair (f^*, f_*) of functors of the form

$$\mathscr{E}_1 \underset{f_*}{\overset{f^*}{\longleftrightarrow}} \mathscr{E}_2$$

such that f^* is left exact and left adjoint to f_*. f^* is called the *inverse image part*, and f_* the *direct image part*, of the geometric morphism.

As explained at the end of the last section, the conditions on the inverse image part f^* of a geometric morphism entail that it preserves finite limits and arbitrary colimits. This naturally generalises the notion of an ⊓-⊔ map of **CHA**'s, and hence, ultimately, that of a continuous function between topological spaces.

In any adjoint situation, each functor determines the other up to natural isomorphism, in the sense that any two left adjoints of a given functor are naturally isomorphic to each other and dually (MacLane [71], Chap. IV, or Herrlich and Strecker [73], Cor. 27.4). In this sense each part of a geometric morphism uniquely determines the other.

Further examples of geometric morphisms

EXAMPLE 1. The inclusion functor $\mathbf{Sh}(I) \hookrightarrow \mathbf{St}(I)$ from the topos of sheaves of sections over a topological space I to the topos of presheaves over I (§14.1) is the direct image part of a geometric morphism whose inverse image part is the "sheafification" functor $F \mapsto F_{p_F}$ (Exercise 14.1.9).

EXAMPLE 2. Example 1 extends to any elementary site $(\mathscr{E}, j)$. The inclusion $sh_j(\mathscr{E}) \hookrightarrow \mathscr{E}$ of the j-sheaves into $\mathscr{E}$ has as left adjoint the left exact sheafification functor $\mathscr{S}\!h_j : \mathscr{E} \to sh_j(\mathscr{E})$ mentioned in §14.4. In addition to the references given there, details may also be found in Tierney [73], Johnstone [77] §3.3, and Veit [81]. The latter gives the construction of $\mathscr{S}\!h_j$ and a proof of its left exactness by means of the internal logic of the site.
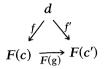
EXAMPLE 3. The fundamental Theorem of Topoi (§15.3) states that if $f : a \to b$ is any arrow in an elementary topos $\mathscr{E}$, then the pulling-back functor $f^* : \mathscr{E} \downarrow b \to \mathscr{E} \downarrow a$ has a right adjoint Π_f. The pair (f^*, Π_f) form a geometric morphism from $\mathscr{E} \downarrow a$ to $\mathscr{E} \downarrow b$.

EXAMPLE 4. If $\mathscr{E}_1$ and $\mathscr{E}_2$ are topoi, the projection functor $\mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_1$ is left exact and left adjoint to the functor taking the $\mathscr{E}_1$-object a to $(a, 1)$.

EXAMPLE 5. **Kan Extensions.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, whose nature will be qualified below. A given functor $F : \mathscr{C} \to \mathscr{D}$ induces a functor $F : \mathbf{St}(\mathscr{D}) \to \mathbf{St}(\mathscr{C})$ between pre-sheaf categories which takes the $\mathbf{St}(\mathscr{D})$-object $G : \mathscr{D} \to \mathbf{Set}$ to $G \circ F : \mathscr{C} \to \mathbf{Set}$, and the arrow $\tau : G \twoheadrightarrow G'$ to $\sigma : G \circ F \twoheadrightarrow G \circ F'$ where the component σ_c is $\tau_{F(c)}$. There is a general

theory, due to Daniel Kan, that produces a left adjoint $F^{\cdot}: \mathbf{St}(\mathscr{C}) \to \mathbf{St}(\mathscr{D})$ to $F_{\cdot}$. Full details are given in [MR], p. 38 (cf. also MacLane [71], Ch. X, and Verdier [SGA4], Exp. I, §5). We will describe the construction of $F^{\cdot}(G)$ for a $\mathbf{St}(\mathscr{C})$-object $G: \mathscr{C} \to \mathbf{Set}$. $F^{\cdot}(G)$ is called the *left Kan extension* of G along F.

If d is a $\mathscr{D}$-object, $F^{\cdot}(G)(d)$ will be an object in **Set**, realised as a colimit of a diagram. First we define a category $d \downarrow F$ whose objects are the pairs (c, f) such that c is a $\mathscr{C}$-object and f a $\mathscr{D}$-arrow of the form $d \to F(c)$. An arrow from (c, f) to (c', f') in $d \downarrow F$ is a $\mathscr{C}$-arrow $g: c \to c'$ such that the diagram

$$
\begin{array}{ccc}
 & d & \\
f \swarrow & & \searrow f' \\
F(c) & \xrightarrow[F(g)]{} & F(c')
\end{array}
$$

commutes. There is a "forgetful" functor $U: d \downarrow F^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$ given by $U(c, f) = c$, $U(g) = g$. The image of $G \circ U$ is then a diagram in **Set**. $F^{\cdot}(G)(d)$ is defined as the colimit of this diagram.

Of course this definition depends on the existence of the colimit in question, and to guarantee this we have to limit the "size" of $\mathscr{C}$ and $\mathscr{D}$. The category **Set** is *bicomplete*, in the sense that it has limits and colimits of all small diagrams (cf. MacLane [71], Ch. V, or Herrlich and Strecker [73], §23). The adjective "small" is applied to a collection which is a set, i.e. a **Set**-object, rather than a proper class (§1.1). Thus a diagram is small if its collection of objects and arrows forms a set, and the same definition of smallness applies to a category. Of course many of the categories we deal with are not small (e.g. **Set**, **Top**(X), **Sh**(X), **St**($\mathscr{C}$), Ω-**Set**, etc.). But they often satisfy the weaker condition of *local* smallness, which means that for any two objects a and b, the collection of all arrows from a to b in the category is small.

Now if $\mathscr{C}$ is a small category, and $\mathscr{D}$ is locally small, then the category $d \downarrow F$ above will be small, and hence the image of $G \circ U$ will be a small diagram in **Set**. Under these conditions then, the functor $F^{\cdot}$ is well-defined, and proves to be left adjoint to $F_{\cdot}$, and left exact if $\mathscr{C}$ has finite limits that are preserved by F ([MR], p. 39).

To sum up: if $\mathscr{C}$ is a finitely complete small category, $\mathscr{D}$ is locally small, and $F: \mathscr{C} \to \mathscr{D}$ is left exact, then the pair $(F^{\cdot}, F_{\cdot})$ form a geometric morphism from $\mathbf{St}(\mathscr{D})$ to $\mathbf{St}(\mathscr{C})$.

We will take up this construction again below in relation to Grothendieck topoi. □

A geometric morphism $f: \mathscr{E}_1 \to \mathscr{E}_2$ is called *surjective* if its inverse image part $f^*: \mathscr{E}_2 \to \mathscr{E}_1$ is a faithful functor. By the work of the previous section, this is equivalent to requiring that f^* be conservative, or that it reflect iso's. The justification for the terminology is contained in the following exercises.

EXERCISE 6. Let $f: X \to Y$ be a continuous function that is surjective, i.e. Im $f = Y$. If

$$A \underset{h}{\overset{g}{\rightrightarrows}} B$$
$$\searrow \swarrow$$
$$Y$$

are two parallel **Top**(Y)-arrows such that $f^*(g) = f^*(h)$ in **Top**(X), show that $g \circ k = h \circ k$, where $k: X \times_Y A \to A$ is the pullback of f along $A \to Y$. Noting that k is onto, conclude that $f^*: \textbf{Top}(Y) \to \textbf{Top}(X)$ is faithful.

EXERCISE 7. Show, with the help of 5.3.1, that the construction of Exercise 6 works for any arrow $f: a \to b$ in any elementary topos $\mathscr{E}$, in the sense that if f is $\mathscr{E}$-epic then the geometric morphism $\mathscr{E} \downarrow a \to \mathscr{E} \downarrow b$ given in Example 3 above is surjective.

EXERCISE 8. If $f: a \to b$ is an $\mathscr{E}$-arrow, show that in $\mathscr{E} \downarrow a$, $f^*(\text{im } f)$ is an iso arrow. Hence show, conversely to the last exercise, that if $f^*: \mathscr{E} \downarrow b \to \mathscr{E} \downarrow a$ reflects iso's, then f is an epic arrow in $\mathscr{E}$.   □

If $\mathscr{E}$ is a topos, then an $\mathscr{E}$-*topos* is a pair $(\mathscr{E}_1, f_1)$ comprising a topos $\mathscr{E}_1$ and a geometric morphism $f_1: \mathscr{E}_1 \to \mathscr{E}$. A morphism $f: \mathscr{E}_1 \to \mathscr{E}_2$ of $\mathscr{E}$-topoi is a geometric morphism which makes the diagram

$$\mathscr{E}_1 \xrightarrow{f} \mathscr{E}_2$$
$$f_1 \searrow \swarrow f_2$$
$$\mathscr{E}$$

commute up to natural isomorphism, i.e. the functors $f_{2*} \circ f_*$ and f_{1*} are naturally isomorphic, as are $f^* \circ f_2^*$ and f_1^*.

An $\mathscr{E}$-topos is said to be *defined over* $\mathscr{E}$, and the arrow f in the above diagram is called a geometric morphism *over* $\mathscr{E}$. A topos defined over **Set** will be called an **S**-topos. The extent to which **Set** determines the structure of an **S**-topos can be seen by examining the reasons behind the

fact that for any topos $\mathscr{E}$ there is, up to natural isomorphism, *at most* one geometric morphism $f : \mathscr{E} \to \mathbf{Set}$. This is because the adjunction of f^* and f_* provides, for each $\mathscr{E}$-object b, a bijection

$$\theta_b : \mathscr{E}(f^*(1), b) \cong \mathbf{Set}(1, f_*(b))$$

which is natural in b. But in $\mathbf{Set}$, arrows of the form $1 \to f_*(b)$ correspond bijectively to elements of the set $f_*(b)$. Also f^*, being left exact, preserves terminal objects, so that $f^*(1)$ is terminal in $\mathscr{E}$. In this way we obtain a bijection

$$\mathscr{E}(1, b) \cong f_*(b)$$

natural in b. Hence if such a geometric morphism exists, its direct image part f_* is determined up to natural isomorphism as the functor $\mathscr{E}(1, -)$ (Example 9.1.7). Since f_* is thus determined, its left adjoint f^* is too.

By pursuing this analysis of f, we can find sufficient conditions for $\mathscr{E}$ to be an $\mathbf{S}$-topos. First, for any two $\mathscr{E}$-objects a and b, $\mathscr{E}$-arrows of the form $a \to b$ correspond bijectively with those of the form $1 \times a \to b$, via the isomorphism $1 \times a \cong a$ (Exercise 3.8.4), and hence bijectively with those of the form $1 \to b^a$, by exponentiation (cf. the discussion of the "name" of an arrow in §4.1). Therefore there is a bijection between $\mathscr{E}(a, b)$ and $\mathscr{E}(1, b^a)$ and so, as above, one between $\mathscr{E}(a, b)$ and the $\mathbf{Set}$-object $f_*(b^a)$. It follows that $\mathscr{E}(a, b)$ is a set, and that $\mathscr{E}$ is a locally small category, in the sense defined previously in our discussion of Kan extensions.

Secondly, the preservation properties of the inverse image part f^* allow us to conclude that $\mathscr{E}$ has arbitrary *set-indexed copowers* of 1. This means that any collection $\{1_s : s \in S\}$ of terminal $\mathscr{E}$-objects, indexed by a *set* S, has a coproduct in $\mathscr{E}$. For, in $\mathbf{Set}$ S is $\varinjlim_{s \in S}\{s\}$, and so as f^* preserves colimits, $f^*(S)$ is $\varinjlim_{s \in S} f^*(\{s\})$. But $\{s\}$ is terminal in $\mathbf{Set}$, and f^* left exact, so $f^*(\{s\}) \cong 1_s$, implying that $f^*(S)$ is a coproduct of $\{1_s : s \in S\}$ as desired.

Thus we see that an $\mathbf{S}$-topos is locally small and has arbitrary set-indexed copowers of 1. But if $\mathscr{E}$ is any topos that has these two properties, we can *define* a geometric morphism $f : \mathscr{E} \to \mathbf{Set}$ by putting $f_*(b) = \mathscr{E}(1, b)$ and $f^*(S) = \varinjlim_{s \in S} 1_s$.

EXERCISE 9. Show that for any topos $\mathscr{E}$ there is at most one geometric morphism $\mathscr{E} \to \mathbf{Finset}$, and that it exists iff $\mathscr{E}(a, b)$ is finite for all $\mathscr{E}$-objects a and b.   □

There is a particularly direct way of showing that $\mathbf{Top}(X)$ is always an $\mathbf{S}$-topos. If $\{*\}$ is a one-point space with the *discrete* topology in which all

subsets are open (this is the only possible topology on $\{*\}$), then the unique function $X \to \{*\}$ is continuous, and so induces a geometric morphism $\mathbf{Top}(X) \to \mathbf{Top}(\{*\})$. But a $\mathbf{Top}(\{*\})$-object is a topological space Y for which $Y \to \{*\}$ is a local homeomorphism. This, however, is only possible when Y itself has the discrete topology, and the latter is determined as soon as we are given the underlying set of Y. Hence $\mathbf{Top}(\{*\})$ is an isomorphic copy of $\mathbf{Set}$.

EXERCISE 10. For any **CHA** Ω, show that there is an $\sqcap$-$\sqcup$ map $2 \to \Omega$. Hence show that $\mathbf{Sh}(\Omega)$ is an **S**-topos. $\qquad\qquad\qquad\square$

It is notable that the existence of set-indexed copowers of 1 in a topos $\mathscr{E}$ implies that the **HA** $\mathrm{Sub}_{\mathscr{E}}(1)$ (or, isomorphically, $\mathscr{E}(1, \Omega)$) is complete (this was mentioned at the end of §14.7). The proof is as follows.

EXERCISE 11. Let $\{a_s \rightarrowtail 1 : s \in S\}$ be a set of subojbects of 1 in $\mathscr{E}$, with characteristic arrow $\chi_s : 1 \to \Omega$ for each $s \in S$. Show that the support of the subobject whose characteristic arrow is the coproduct of the χ_s's is a join of the a_s's in $\mathrm{Sub}(1)$. $\qquad\qquad\qquad\square$

Geometric morphisms of Grothendieck topoi

To discuss these, we are going to modify our earlier notation and terminology a little. Let $\mathbf{C} = (\mathscr{C}, Cov)$ be a site (§14.3), consisting of a pretopology Cov on a category $\mathscr{C}$. The full subcategory of the pre-sheaf category $\mathbf{St}(\mathscr{C})$ generated by the sheaves over $\mathbf{C}$ will now be denoted $\mathbf{Sh}(\mathbf{C})$ instead of $\mathbf{Sh}(Cov)$. $\mathbf{C}$ will be called a *small* site if $\mathscr{C}$ is a small category. The name "Grothendieck topos" will be reserved for categories equivalent to those of the form $\mathbf{Sh}(\mathbf{C})$ for *small* sites $\mathbf{C}$. Moreover we will assume throughout that *all sites are finitely complete*, i.e. have all finite limits.

For small sites $\mathbf{C}, \mathbf{Sh}(\mathbf{C})$ satisfies the two conditions given above that suffice to make it an **S**-topos. The existence of set-indexed copowers of 1 is just a special case of the fact that $\mathbf{Sh}(\mathbf{C})$ is *bicomplete* in the sense that every *small* diagram has a limit and a colimit. This fact derives ultimately from the bicompleteness of **Set** itself, which allows all set-indexed limits and colimits to be constructed "component-wise" in the pre-sheaf categ-ory $\mathbf{St}(\mathscr{C})$ (cf. §9.3, or MacLane [71], V.3). Then if D is a small diagram in $\mathbf{Sh}(\mathbf{C})$, the limit of D in $\mathbf{St}(\mathscr{C})$ proves to be a sheaf, and hence a D-limit in $\mathbf{Sh}(\mathbf{C})$. On the other hand the colimit for D in $\mathbf{St}(\mathscr{C})$ is transferred by

the colimit preserving sheafification functor $\mathbf{St}(\mathscr{C}) \to \mathbf{Sh}(\mathbf{C})$ to a colimit for D in $\mathbf{Sh}(\mathbf{C})$.

For local smallness of $\mathbf{Sh}(\mathbf{C})$ we note first that the axioms of ZF set theory allow us to form the product $\varprojlim_{i \in I} A_i$ of a collection of sets A_i, indexed by a set I, as the $\mathbf{Set}$-object

$$\{f: f \text{ is a function \& } \operatorname{dom} f = I \& f(i) \in A_i \text{ for all } i \in I\}.$$

Now an arrow $\tau : F \to G$ in $\mathbf{Sh}(\mathbf{C})$ is a natural transformation, and hence is a function assigning to each $\mathscr{C}$-object c a set-function $\tau_c : F(c) \to G(c)$, i.e. a member of the set $\mathbf{Set}(F(c), G(c))$. But if $\mathscr{C}$ is small, then the collection $|\mathscr{C}|$ of $\mathscr{C}$-objects is small, so the collection $\mathbf{Sh}(\mathbf{C})(F, G)$ of $\mathbf{Sh}(\mathbf{C})$-arrows from F to G is included in the set

$$\varprojlim_{c \in |\mathscr{C}|} \mathbf{Set}(F(c), G(c))$$

and thus is itself small.

Assuming only that $\mathscr{C}$ is locally small, a functor $E_{\mathbf{C}} : \mathscr{C} \to \mathbf{Sh}(\mathbf{C})$, known as the *canonical* functor ([SGA4], II 4.4), can be defined as the composite of two other functors $\mathscr{Y} : \mathscr{C} \to \mathbf{St}(\mathscr{C})$ and $Sh : \mathbf{St}(\mathscr{C}) \to \mathbf{Sh}(\mathbf{C})$. The second of these is the sheafification or "associated-sheaf" functor that forms the inverse image part of the geometric morphism whose direct image part is the inclusion $\mathbf{Sh}(\mathbf{C}) \hookrightarrow \mathbf{St}(\mathscr{C})$. For a detailed account of Sh the reader is referred to the work of Verdier [SGA4] II.2, [MR]1.2, or Schubert [72], §20.3.

The functor $\mathscr{Y}$ is the dual form of the fundamental *Yoneda functor*. It takes the $\mathscr{C}$-object c to the contravariant hom-functor $\mathscr{C}(-, c) : \mathscr{C} \to \mathbf{Set}$ of Example 9.1.10, and the $\mathscr{C}$-arrow $f : c \to d$ to the natural transformation $\mathscr{C}(-, f) : \mathscr{C}(-, c) \to \mathscr{C}(-, d)$ where, for any $\mathscr{C}$-object a, the component assigned to a by $\mathscr{C}(-, f)$ is the "composing with f" function $\mathscr{C}(a, f) : \mathscr{C}(a, c) \to \mathscr{C}(a, d)$. Note that the local smallness of $\mathscr{C}$ is essential here in order for the functor $\mathscr{Y}(c)$, i.e. $\mathscr{C}(-, c)$, to have its values in $\mathbf{Set}$.

Underlying the definition of $\mathscr{Y}$ is a very important piece of category theory known as the *Yoneda Lemma* (MacLane [71] III §2, Herrlich and Strecker [73] §30). In its dual form it states that for any $\mathscr{C}$-object c and presheaf $F : \mathscr{C}^{op} \to \mathbf{Set}$, there is a bijection

$$\mathbf{St}(\mathscr{Y}(c), F) \cong F(c)$$

between $\mathbf{St}(\mathscr{C})$-arrows (i.e. natural transformations) from $\mathscr{C}(-, c)$ to F and elements of the set $F(c)$.

EXERCISE 12. If $x \in F(c)$ and d is any $\mathscr{C}$-object, show that the equation

$$x_d(f) = F(f)(x)$$

defines a function $x_d : \mathscr{C}(d, c) \to F(d)$. Show that the x_d's form the components of a natural transformation $\mathscr{Y}(c) \to F$, and that this construction gives the bijection asserted above.

Formulate precisely, and prove, the condition that this bijection be "natural" in c and F.   □

In particular, the Yoneda Lemma implies, for any $\mathscr{C}$-objects c and d, that
$$\mathbf{St}(\mathscr{Y}(c), \mathscr{Y}(d)) \cong \mathscr{C}(c, d),$$

so that $\mathscr{Y}$ acts bijectively on hom-sets. It is also injective on objects, and so embeds $\mathscr{C}$ isomorphically into $\mathbf{St}(\mathscr{C})$, making it possible to identify c and $\mathscr{Y}(c)$, and regard $\mathscr{C}$ as a full subcategory of $\mathbf{St}(\mathscr{C})$.

Now in a cocomplete topos, the existence of set-indexed coproducts allows us to form the *union* of any set $\{G_x \rightarrowtail F : x \in X\}$ of subobjects of an object F, by defining $\bigcup_X G_x \rightarrowtail F$ to be the image arrow of the coproduct arrow $(\varinjlim G_x) \to F$ (thereby extending the formation of unions given by Theorem 3 of §7.1). This construction enables us to make the topos itself into a site! A set $\{F_x \xrightarrow{f_x} F : x \in X\}$ of arrows is defined to be a cover of F if, in $\mathrm{Sub}(F)$, $\bigcup_X \mathrm{im} f_x$ is $\mathbf{1}_F$ (and so $\bigcup_X f_x(F_x) \cong F$). Equivalently, the definition requires that the coproduct arrow $[\mathrm{im} f_x]$ of the arrows $\mathrm{im} f_x : f_x(F_x) \rightarrowtail F$ be epic.

This notion of cover defines the *canonical* pre-topology, which in the case of a Grothendieck topos $\mathbf{Sh}(\mathbf{C})$ proves to have the property that all the hom-functors $\mathscr{Y}(c)$ are sheaves, so that the Yoneda functor maps $\mathscr{C}$ into $\mathbf{Sh}(\mathbf{C})$. There is another way of defining canonical covers in $\mathbf{Sh}(\mathbf{C})$ which is formally simpler to express and avoids reference to colimits. We say that $C = \{F_x \xrightarrow{f_x} F : x \in X\}$ is an *epimorphic family* if, for any pair $f, g : F \to G$ of parallel arrows with domain F, if $f \circ f_x = g \circ f_x$ for all $x \in X$, then $f = g$.
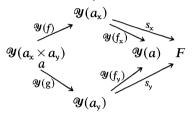
EXERCISE 13. Show that C as above is an epimorphic family iff the coproduct $[f_x] : \varinjlim F_x \to F$ of the f_x's is epic.

EXERCISE 14. Show that the epic parts $F_x \twoheadrightarrow f_x(F_x)$ of the arrows f_x give rise to an epic arrow $\varinjlim F_x \twoheadrightarrow \varinjlim f_x(F_x)$ which factors $[f_x]$ through $[\mathrm{im} f_x] : \varinjlim f_x(F_x) \to F$. Hence show that $[\mathrm{im} f_x]$ is epic iff $[f_x]$ is epic, and so the canonical covers are precisely the epimorphic families.   □
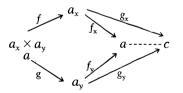
To place the canonical pretopology in broader perspective, we need to examine the general conditions under which $\mathscr{Y}(c)$ is a sheaf over **C**. To do this, we reformulate the sheaf axiom COM of §14.3 in the terms given by the Yoneda Lemma. Let $F: \mathscr{C} \to \mathbf{Set}$ be a presheaf, and $\{a_x \xrightarrow{f_x} a : x \in X\}$ a cover of the site **C**. Instead of dealing with elements $s_x \in F(a_x)$ we deal, via Exercise 12, with arrows $s_x : \mathscr{Y}(a_x) \to F$. Compatibility of a selection of such "elements" s_x for each $x \in X$ requires that for all $x, y \in X$ we have that $s_x \circ \mathscr{Y}(f) = s_y \circ \mathscr{Y}(g)$, where f and g are the pullback in $\mathscr{C}$ of f_x and f_y:

Fulfillment of COM for this situation requires a unique arrow $\mathscr{Y}(a) \to F$ that for all $x, y \in X$ makes this diagram commute.

Now if F is of the form $\mathscr{Y}(c)$, the fact that $\mathscr{Y}$ is injective on objects and bijective on hom-sets allows us to pull the above diagram back into $\mathscr{C}$ itself. This leads to the following notion.

A collection $C = \{a_x \xrightarrow{f_x} a : x \in X\}$ of $\mathscr{C}$-arrows is called an *effectively epimorphic family* if for any $\mathscr{C}$-object c, and for any collection $D = \{a_x \xrightarrow{g_x} c : x \in X\}$ of $\mathscr{C}$-arrows such that for all $x, y \in X$ we have

$g_x \circ f = g_y \circ g$, where f and g are the pullback of f_x and f_y, there is a *unique* $\mathscr{C}$-arrow $g : a \to c$ such that $g \circ f_x = g_x$ for all $x \in X$.

A collection D satisfying the hypothesis of this definition will be called *compatible* with C. Thus the definition requires that any collection compatible with C is factored through C by a unique arrow.

EXERCISE 15. Show that an effectively epimorphic family is epimorphic.

EXERCISE 16. If C is the empty set of arrows with condomain a, show that C is effectively epimorphic iff a is an initial object.   □

It is apparent from our discussion that for a site $\mathbf{C}$ in which every cover is effectively epimorphic, the hom-functors are all sheaves, and so $\mathscr{Y}$ embeds $\mathbf{C}$ in $\mathbf{Sh(C)}$. Such a pretopology is called *precanonical*. In the case of a general finitely complete category $\mathscr{C}$, an effectively epimorphic family is called *stable* (or *universal*) if its pullback along any arrow is also effectively epimorphic. The stable effectively epimorphic families form a precanonical pretopology on $\mathscr{C}$ that includes any other precanonical one ([MR], Proposition 1.1.9). Hence it is known as the *canonical* pretopology on $\mathscr{C}$.

In a Grothendieck topos $\mathbf{Sh(C)}$, the stable effectively epimorphic families prove to be precisely the epimorphic families as defined prior to Exercise 13 ([MR], Proposition 3.4.11). Whenever we refer to $\mathbf{Sh(C)}$ as a site, we will thus be referring to epimorphic families as covers. The canonical functor $E : \mathbf{Sh(C)} \to \mathbf{Sh(Sh(C))}$ from $\mathbf{Sh(C)}$ to the category of sheaves on the site $\mathbf{Sh(C)}$ will then just be the Yoneda embedding. It turns out that E is an equivalence, so that $\mathbf{Sh(C)}$ and $\mathbf{Sh(Sh(C))}$ are equivalent categories in the sense of §9.2, allowing us to think of any Grothendieck topos as being the topos of sheaves on a canonical site. The proof of this fact is part of a number of fundamental characterisations of Grothendieck topoi that may be found in [SGA4], IV.1, or [MR], Theorem 1.4.5. The fact itself is needed to show that geometric morphisms between Grothendieck topoi are determined by certain "continuous morphisms" between sites, as we shall now see.

If $\mathbf{C} = (\mathscr{C}, Cov)$ and $\mathbf{D} = (\mathscr{D}, Cov')$ are sites, a *continuous morphism* $F : \mathbf{C} \to \mathbf{D}$ is a functor $F : \mathscr{C} \to \mathscr{D}$ that is left exact (remember sites are presumed to be finitely complete) and preserves covers, i.e. has $\{f_x : x \in X\} \in Cov(c)$ only if $\{F(f_x) : x \in X\} \in Cov'(F(c))$. For example, if $f : V \to W$ is a continuous function of topological spaces, then $f^* : \Theta_W \to \Theta_v$ preserves open covers in the usual topological sense. Similarly, an $\sqcap$-$\sqcup$ map $f^* : \Omega \to \Omega'$ between $\mathbf{CHA}$'s is continuous with respect to the definition of Cov_Ω introduced just prior to Exercise 14.7.11 – indeed left exactness amounts to preservation of $\sqcap$, and preservation of members of Cov_Ω means preservation of $\sqcup$.

The examples indicate that the concept of continuous morphism of sites generalises that of continuous function of topological spaces, and hence is linked to the notion of geometric morphism. Indeed, if $f^* : \mathbf{Sh(C)} \to \mathscr{E}$ is the inverse image part of a geometric morphism of Grothendieck topoi, then f^* is continuous with respect to the associated canonical sites. This is because in that context the notion of epimorphic family is characterised by colimits (viz. coproducts and epic arrows), and colimits are preserved

by f^*. Moreover, the canonical functor $E_{\mathbf{C}} : \mathscr{C} \to \mathbf{Sh}(\mathbf{C})$ proves to be continuous. In fact, $E_{\mathbf{C}}$ both preserves and *reflects* covers in the sense that $\{f_x : x \in X\} \in Cov(c)$ in $\mathbf{C}$ if and only if $\{E_{\mathbf{C}}(f_x) : x \in X\}$ is an epimorphic family in $\mathbf{Sh}(\mathbf{C})$ ([SGA4], II.4.4, and [MR], Proposition 1.3.3). Thus we can compose $E_{\mathbf{C}}$ and f^* to get a continuous morphism $\mathbf{C} \to \mathscr{E}$. Conversely, and more importantly, every geometric morphism $\mathscr{E} \to \mathbf{Sh}(\mathbf{C})$ can be obtained uniquely as an extension of a continuous morphism of this type. To show this we need the following result.

THEOREM 1. *Let $F : \mathbf{C} \to \mathbf{D}$ be a continuous morphism of sites, with $\mathbf{C}$ small and $\mathbf{D}$ locally small. Then there is a geometric morphism $f : \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$ such that the diagram*

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\ E_{\mathbf{C}}\ } & \mathbf{Sh}(\mathbf{C}) \\
\scriptstyle F \downarrow & & \Big\downarrow \scriptstyle f^* \\
\mathbf{D} & \xrightarrow[\ E_{\mathbf{D}}\ ]{} & \mathbf{Sh}(\mathbf{D})
\end{array}
$$

commutes. Moreover there is, up to natural isomorphism, at most one continuous $\mathbf{Sh}(\mathbf{C}) \to \mathbf{Sh}(\mathbf{D})$ that makes this diagram commute, so that f is unique up to natural isomorphism.

This theorem is proven in Proposition 1.2 of Expose III of [SGA4]. In [MR], the reference is Theorem 1.3.10, with the uniqueness clause coming from 1.3.12. We will do no more here than outline the definition of f.

Recall, from the discussion of Kan extensions in Example 5 of our list of geometric morphisms, that F induces a functor $F_. : \mathbf{St}(\mathscr{D}) \to \mathbf{St}(\mathscr{C})$ that has a left exact left adjoint $F^.$. Now consider the diagram

$$
\begin{array}{ccccc}
\mathscr{C} & \xrightarrow{\ \mathscr{Y}_{\mathscr{C}}\ } & \mathbf{St}(\mathscr{C}) & \underset{\mathscr{I}_{\mathbf{C}}}{\overset{Sh_{\mathbf{C}}}{\rightleftarrows}} & \mathbf{Sh}(\mathbf{C}) \\
\scriptstyle F \downarrow & & \scriptstyle F^. \big\uparrow\big\downarrow \scriptstyle F_. & & \\
\mathscr{D} & \xrightarrow[\ \mathscr{Y}_{\mathscr{D}}\ ]{} & \mathbf{St}(\mathscr{D}) & \underset{\mathscr{I}_{\mathbf{D}}}{\overset{Sh_{\mathbf{D}}}{\rightleftarrows}} & \mathbf{Sh}(\mathbf{D})
\end{array}
$$

Here, $\mathscr{Y}$ denotes a Yoneda functor, Sh a sheafification functor, and $\mathscr{I}$ an inclusion. f^* is defined to be $Sh_{\mathbf{D}} \circ F^. \circ \mathscr{I}_{\mathbf{C}}$, and f_* is $Sh_{\mathbf{C}} \circ F_. \circ \mathscr{I}_{\mathbf{D}}$. (Since, in any adjoint situation, each adjoint determines the other up to natural isomorphism, the uniqueness of f^* implies that of f_*, and hence of f.)

If we now apply Theorem 1 in the case that $\mathbf{D}$ is itself a Grothendieck topos $\mathscr{E}$, with the canonical pretopology, then $E_{\mathbf{D}}$ is an equivalence whose

"inverse" $\mathbf{Sh}(\mathscr{E}) \to \mathscr{E}$ may be composed with f^* to yield a continuous morphism $\mathbf{Sh}(\mathbf{C}) \to \mathscr{E}$. This leads to the following central result.

THEOREM 2. (Reduction Theorem). *If* $\mathbf{C}$ *is a small size, and* $\mathscr{E}$ *a Grothendieck topos, then for any continuous morphism* $F : \mathbf{C} \to \mathscr{E}$ *there exists a continuous* $f^* : \mathbf{Sh}(\mathbf{C}) \to \mathscr{E}$, *unique up to natural isomorphism, such that*
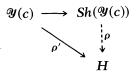
$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\ E_{\mathbf{C}}\ } & \mathbf{Sh}(\mathbf{C}) \\
& {\scriptstyle F}\searrow & \downarrow{\scriptstyle f^*} \\
& & \mathscr{E}
\end{array}
$$

commutes. Moreover f^* *is the inverse image part of a* (*thereby unique up to natural isomorphism*) *geometric morphism* $f : \mathscr{E} \to \mathbf{Sh}(\mathbf{C})$. $\qquad\square$

Thus we see that any geometric morphism $f : \mathscr{E} \to \mathbf{Sh}(\mathbf{C})$ is determined uniquely up to natural isomorphism by the continuous functor $f^* \circ E_{\mathbf{C}} : \mathbf{C} \to \mathscr{E}$, and by this result that the construction of geometric morphisms between Grothendieck topoi reduces to the construction of continuous morphisms defined on small sites. In the next section, the later notion will be reformulated in terms of models of logical theories.

As a final topic on this theme we consider the question as to when the functor f^* in Theorem 2 is faithful, so that the associated geometric morphism is surjective. To discuss this we need to know the fact that the $E_{\mathbf{C}}$-image of $\mathbf{C}$ in $\mathbf{Sh}(\mathbf{C})$ forms a set of *generators* for $\mathbf{Sh}(\mathbf{C})$. This means that for any $\mathbf{Sh}(\mathbf{C})$-object H, the family of arrows from objects of the form $E_{\mathbf{C}}(c)$ to H is epimorphic. In other words, if $\sigma, \tau : H \to G$ are *distinct* arrows in $\mathbf{Sh}(\mathbf{C})$, then there is a $\mathscr{C}$-object c and an arrow $\rho : E_{\mathbf{C}}(c) \to H$ such that $\sigma \circ \rho \neq \tau \circ \rho$.

To prove this, observe that if $\sigma \neq \tau$, then for some c, and some $x \in H(c)$, $\sigma_c(x) \neq \tau_c(x)$. But by the Yoneda Lemma (Exercise 12), x determines an arrow $\rho' : \mathscr{Y}(c) \to H$ such that $\rho'_c(\mathbf{1}_c) = x$, and so $\sigma \circ \rho' \neq \tau \circ \rho'$. Then by the co-universal property associated with the left adjoint sheafification functor $Sh : \mathbf{St}(\mathscr{C}) \to \mathbf{Sh}(\mathbf{C})$ (cf. (2) of §15.1), ρ' factors uniquely

$$
\begin{array}{ccc}
\mathscr{Y}(c) & \longrightarrow & Sh(\mathscr{Y}(c)) \\
& {\scriptstyle \rho'}\searrow & \downarrow{\scriptstyle \rho} \\
& & H
\end{array}
$$

through an arrow $\rho: E_{\mathbf{C}}(c) \to H$ (using the fact that the right adjoint of Sh is the inclusion) which must then have $\sigma \circ \rho \neq \tau \circ \rho$.

We see then that in $\mathbf{Sh}(\mathbf{C})$, every object is "covered" by a family of objects of the form $E_{\mathbf{C}}(c)$. This generating role of these objects gives rise to the following result, whose proof may be found in [MR], Lemma 1.3.8.

LEMMA. *If* $e: K \rightarrowtail E_{\mathbf{C}}(c)$ *is monic in* $Sh(\mathbf{C})$, *then there is an epimorphic family* $\{E_{\mathbf{C}}(c_x) \xrightarrow{h_x} K: x \in X\}$ *such that each composite* $e \circ h_x$ *is* $E_{\mathbf{C}}(g_x)$ *for some* $\mathscr{C}$-*arrow* $g_x: c_x \to c$. $\qquad \square$

THEOREM 3. *Let* $F: \mathbf{C} \to \mathscr{E}$ *be a continuous morphism as in Theorem 2. Then the extension* $f^*: \mathbf{Sh}(\mathbf{C}) \to \mathscr{E}$ *of* F *along* $E_{\mathbf{C}}$ *is faithful if, for any set* $\{g_x: x \in X\}$ *of* $\mathscr{C}$-*arrows with a common codomain,* $\{F(g_x): x \in X\}$ *is epimorphic in* $\mathscr{E}$ *only if* $\{E_{\mathbf{C}}(g_x): x \in X\}$ *is epimorphic in* $\mathbf{Sh}(\mathbf{C})$.

PROOF. Let $\sigma, \tau: H \to G$ be a pair of $\mathbf{Sh}(\mathbf{C})$-arrows such that $f^*(\sigma) = f^*(\tau)$. If $\sigma \neq \tau$, then by what we have just seen, there is a $\mathscr{C}$-object c and an arrow $\rho: E_{\mathbf{C}}(c) \to H$ such that $\sigma \circ \rho \neq \tau \circ \rho$. Let $e: K \rightarrowtail E_{\mathbf{C}}(c)$ be the equaliser in $\mathbf{Sh}(\mathbf{C})$ of $\sigma \circ \rho$ and $\tau \circ \rho$. By the Lemma there is an epimorphic family of arrows $h_x: E_{\mathbf{C}}(c_x) \to K$, for all x in some set X, such that each $e \circ h_x$ is $E_{\mathbf{C}}(g_x)$ for some $g_x: c_x \to c$. Since f^* is continuous, $\{f^*(h_x): x \in X\}$ is epimorphic in $\mathscr{E}$. But since f^* is left exact, $f^*(e)$ equalises $f^*(\sigma \circ \rho)$ and $f^*(\tau \circ \rho)$ in $\mathscr{E}$, and these last two arrows are equal, since $f^*(\sigma) = f^*(\tau)$ and f^* preserves composites. Therefore $f^*(e)$ is iso, from which it follows readily that $\{f^*(e) \circ f^*(h_x): x \in X\}$ is an epimorphic family. But $f^*(e) \circ f^*(h_x) = f^*(e \circ h_x) = f^*(E_{\mathbf{C}}(g_x)) = F(g_x)$, so the hypothesis of the Theorem implies that $\{E_{\mathbf{C}}(g_x): x \in X\}$ is epimorphic. However $(\sigma \circ \rho) \circ E_{\mathbf{C}}(g_x) = (\sigma \circ \rho) \circ e \circ h_x = (\tau \circ \rho) \circ e \circ h_x = (\tau \circ \rho) \circ E_{\mathbf{C}}(g_x)$ (by definition of e), so this entails that $\sigma \circ \rho = \tau \circ \rho$ – contrary to hypothesis. Thus our assumption that $\sigma \neq \tau$ must be false. $\qquad \square$

COROLLARY 4. *If* F *reflects covers, then* f^* *is faithful.*

PROOF. This follows immediately from the fact that $E_{\mathbf{C}}$ preserves covers, i.e. if $\{g_x: x \in X\}$ is a cover in $\mathbf{C}$ then $\{E_{\mathbf{C}}(g_x): x \in X\}$ is a cover in $\mathbf{Sh}(\mathbf{C})$. $\square$

Points

If Y is a topological space, then a point $y \in Y$ determines a continuous function $\{*\} \to Y$, where $\{*\}$ is the one-point space. Since $\mathbf{Top}(\{*\})$ is isomorphic to $\mathbf{Set}$, this in turn gives rise to a geometric morphism $p_y: \mathbf{Set} \to \mathbf{Top}(Y)$.

EXERCISE 17. Show that the inverse image functor p_y^* takes each **Top**(Y)-object to its stalk over y, and each arrow to its restriction to this stalk.

EXERCISE 18. Show that for any **CHA** Ω, an $\sqcap$-$\sqcup$ map $\Omega \to 2$ (i.e. a point of Ω in the sense of §14.8) gives rise to a geometric morphism from **Set** to **Sh**(Ω). $\square$

In view of these examples we define a *point* of an **S**-topos $\mathcal{E}$ to be a geometric morphism $p: \mathbf{Set} \to \mathcal{E}$. By left exactness, a subobject $a \rightarrowtail 1$ of 1 in $\mathcal{E}$ will be mapped by p^* to a subobject of 1 in **Set**, so $p^*(a) \in \{0, 1\}$. As p^* also preserves colimits, we obtain thereby an $\sqcap$-$\sqcup$ map $\Omega_{\mathcal{E}} \to 2$, where, in the notation of §14.7, $\Omega_{\mathcal{E}}$ is the **CHA** $\mathrm{Sub}_{\mathcal{E}}(1)$ of subobjects of 1 in $\mathcal{E}$. Thus a point of $\mathcal{E}$ gives rise to a point of $\Omega_{\mathcal{E}}$ (recall from Exercise 11 that constraining $\mathcal{E}$ to be an **S**-topos ensures that $\Omega_{\mathcal{E}}$ is a *complete* **HA**).

In the topological case, subobjects of 1 in **Top**(Y) correspond to open subsets of Y, and Sub(1) can be identified with Θ_Y (cf. §4.5). If Y is sober, in the sense (defined in §14.8) that every **CHA**-point $f: \Theta_Y \to 2$ is of the form

$$f(V) = \begin{cases} 1 & \text{if } y \in V, \\ 0 & \text{if } y \notin V \end{cases}$$

for some $y \in Y$, then the geometric points **Set** $\to$ **Top**(Y) are precisely those that arise from elements of Y in the above manner.

More generally, we can define a topology on the class of points of an **S**-topos $\mathcal{E}$ by taking as opens the collections

$$V_a = \{p : p^*(a) = 1\}$$

for each $a \rightarrowtail 1$ in $\mathrm{Sub}_{\mathcal{E}}(1)$. In the case of **Top**(Y), this produces a space topologically isomorphic to the sober space $\beta(\Theta_Y)$ of all points of Θ_Y (called the "soberification" of Y – cf. Wraith [75], §4, and Johnstone [77], §7.2).

Now if P is a class of points of $\mathcal{E}$, we call P *sufficient* if any $\mathcal{E}$-arrow f with the property that $p^*(f)$ is iso in **Set** for all $p \in P$ must itself be iso in $\mathcal{E}$. In other words, whenever f is not iso in $\mathcal{E}$, then there is at least one $p \in P$ such that $p^*(f)$ is not iso in **Set**. By the work of §16.1, the reader should recognise that this concept is linked to those of conservative and faithful functors.

EXERCISE 19. P is sufficient iff for any parallel pair $f, g : a \to b$ of $\mathcal{E}$-arrows, if $p^*(f) = p^*(g)$ for all $p \in P$, then $f = g$.

EXERCISE 20. P is sufficient iff for any two subobjects f, g of any $\mathscr{E}$-object, if $p^*(f) \subseteq p^*(g)$ for all $p \in P$, then $f \subseteq g$.

EXERCISE 21. There exists a sufficient class of $\mathscr{E}$-points if and only if the class of all $\mathscr{E}$-points is sufficient.   □

We say that $\mathscr{E}$ *has enough points* if the class of all points of $\mathscr{E}$ is sufficient. In the case of **Top**(Y), a pair f, g of parallel arrows

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$
$$\searrow \quad \swarrow$$
$$Y$$

are equal if and only if they agree on the stalk of A over each point $y \in Y$. By Exercises 17 and 19 then, it is clear that the topos **Top**(Y) has enough points, and indeed that the set $\{p_y : y \in Y\}$ of points is sufficient.

The question as to when a topos $\mathscr{E}$ has enough points has some interesting answers in the case that $\mathscr{E}$ is the Grothendieck topos **Sh**(**C**) of sheaves over a small site **C**. First there is the fact that if **Sh**(**C**) does have enough points, then it has a sufficient *set* of points. The proof of this ([SGA4], IV 6.5(b), Johnstone [77], 7.17) is too involved to give here, but an inkling of why such a size reduction is plausible comes from the knowledge that, with the aid of the Yoneda Lemma, it can be shown that any functor from **C** to **Set** is constructible as the colimit of a diagram in **Set**$^{\mathscr{C}}$ whose objects are hom-functors on $\mathscr{C}$. Since $\mathscr{C}$ is small, the class of all such hom-functors is small. But any geometric morphism **Set** → **Sh**(**C**) is determined by a continuous functor from **C** to **Set** (**Set** is of course a Grothendieck topos, being equivalent to **Sh**($\{*\}$)).

Now a set P of points of **Sh**(**C**) can be combined into a single geometric morphism $\pi : $ **Set**$^P \to$ **Sh**(**C**). Here **Set**^{P} is the Boolean topos of set-valued functions $f : P \to$ **Set** on the discrete category P, and is equivalent to **Bn**(P) (§9.3). Alternatively, by §14.1.II, viewing P as a discrete poset makes **Set**^{P} equivalent to **Sh**(P), where P becomes a space under the discrete topology $\Theta_P = \mathscr{P}(P)$. Yet another way of looking at this category is to identify it as the Grothendieck topos **Sh**(Ω), defined in §14.7, where we take the **CHA** Ω to be the Boolean power-set algebra $\mathscr{P}(P)$.
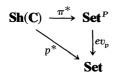
To define π, it suffices by the Reduction Theorem (Theorem 2) to specify its inverse image part $\pi^* : $ **Sh**(**C**) → **Set**^{P} as a continuous morphism, and indeed it would be enough to specify the continuous morphism

$\pi^* \circ E_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set}^P$. In a similar vein, we can regard each geometric morphism $p \in P$ as being a continuous morphism $p : \mathbf{C} \to \mathbf{Set}$ that extends, uniquely up to isomorphism, to a continuous $p^* : \mathbf{Sh}(\mathbf{C}) \to \mathbf{Set}$ making

$$\mathbf{C} \xrightarrow{E_{\mathbf{C}}} \mathbf{Sh}(\mathbf{C})$$
$$p \searrow \quad \downarrow p^*$$
$$\mathbf{Set}$$

commute.

$\mathbf{Set}^P$ is the *P-indexed power* of $\mathbf{Set}$, i.e. the "*P*-fold product of $\mathbf{Set}$ with itself", having projection (evaluation) functors $ev_p : \mathbf{Set}^P \to \mathbf{Set}$, for each $p \in P$, where $ev_p(f) = f(p)$, and $ev_p(\sigma) = \sigma_p$ for each $\mathbf{Set}^P$-arrow $\sigma : f \twoheadrightarrow g$. π^* is then the product arrow of $\{p^* : p \in P\}$, i.e. the unique functor making

$$\mathbf{Sh}(\mathbf{C}) \xrightarrow{\pi^*} \mathbf{Set}^P$$
$$p^* \searrow \quad \downarrow ev_p$$
$$\mathbf{Set}$$

commute for all $p \in P$. Thus $\pi^*(F) : P \to \mathbf{Set}$ is the function that takes p to $p^*(F)$, while $\pi^*(\tau) : \pi^*(F) \to \pi^*(G)$ is the natural transformation with pth component $p^*(\tau) : p^*(F) \to p^*(G)$.

Our earlier remark about the link between sufficiency and faithfulness can now be made precise:

EXERCISE 22. P is sufficient iff π^* is faithful.   □

In order for π^* to determine a geometric morphism, it must be continuous, and in particular preserve canonical covers, i.e. epimorphic families.

LEMMA. *A set* $A = \{f^x \xrightarrow{\sigma^x} f : x \in X\}$ *of* $\mathbf{Set}^P$-*arrows is epimorphic in* $\mathbf{Set}^P$ *iff for each* $p \in P$ *the set* $ev_p(A) = \{ev_p(\sigma^x) : x \in X\}$ *is epimorphic in* $\mathbf{Set}$.

PROOF. We prove necessity, the converse being more straightforward. Note that to define an arrow $\sigma : f \twoheadrightarrow g$ in $\mathbf{Set}^P$ requires us just to specify a function $\sigma_p : f(p) \to g(p)$ for each $p \in P$. As P is a discrete category (i.e. has only identity arrows), σ is then automatically natural in p, so *any* P-indexed collection of functions $f(p) \to g(p)$ defines an arrow.

Suppose that A is epimorphic, and take $p \in P$. Let $k, l : f(p) \to B$ be

arrows in **Set** such that $k \circ ev_p(\sigma^x) = l \circ ev_p(\sigma^x)$ for all $x \in X$. We need to show that $k = l$.

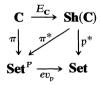Define a $\mathbf{Set}^P$-object $g : P \to \mathbf{Set}$ by putting

$$g(r) = \begin{cases} B & \text{if } r = p, \\ f(r) & \text{if } r \neq p, \end{cases}$$

and define arrows $\tau, \rho : f \rightarrowtail g$ by putting $\tau_p = k$, $\rho_p = l$, and $\tau_r = \rho_r = \text{id}_{f(r)}$ for $r \neq p$. Then $\tau \circ \sigma^x = \rho \circ \sigma^x$ for all $x \in X$. Since A is epimorphic, it follows that $\tau = \rho$, and so $\tau_p = \rho_p$ as desired.   □

Now if C is a cover in $\mathbf{Sh(C)}$, then for each p, continuity of p^* implies that $p^*(C)$, i.e. $ev_p(\pi^*(C))$ is epimorphic in **Set**. Hence, by the Lemma, $\pi^*(C)$ is epimorphic in $\mathbf{Set}^P$. This shows that π^* preserves covers. Left exactness of π^* is established in a similar way, using the left-exactness of each p^*, and the fact that limits are constructed in $\mathbf{Set}^P$ by pointwise evaluation, i.e. a cone U for a diagram D in $\mathbf{Set}^P$ is a D-limit if $ev_p(U)$ is an $ev_p(D)$-limit in **Set** for all $p \in P$.

EXERCISE 23. $\mathbf{Sh(C)}$ has enough points iff there exists a set P and a surjective geometric morphism $\mathbf{Set}^P \to \mathbf{Sh(C)}$.   □

The question of faithfulness of π^* can also be approached in terms of the criterion given in Corollary 4. If $\pi : \mathbf{C} \to \mathbf{Set}^P$ is the continuous morphism $\pi^* \circ E_\mathbf{C}$, then the criterion is that π reflects covers, i.e. if C is a set of $\mathscr{C}$-arrows with a common codomain, and $\pi(C)$ is an epimorphic family in $\mathbf{Set}^P$, then C is a cover in **C**. But $\pi(C)$ will be epimorphic iff $ev_p(\pi(C))$ is epimorphic in **Set** for all $p \in P$. Since we have

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\ E_\mathbf{C}\ } & \mathbf{Sh(C)} \\ {\scriptstyle \pi}\downarrow & {\scriptstyle \pi^*}\swarrow & \downarrow{\scriptstyle p^*} \\ \mathbf{Set}^P & \xrightarrow[\ ev_p\ ]{} & \mathbf{Set} \end{array}$$

$ev_p \circ \pi = E_\mathbf{C} \circ p^* = p$, this leads to the following result.

THEOREM 5. ([SGA4], IV.6.5(a)). *A set P of points of $\mathbf{Sh(C)}$ is sufficient if, and only if, for any set C of $\mathscr{C}$-arrows that is not a cover in **C** there exists some $p \in P$ such that $p(C)$ is not epimorphic in **Set**.*   □

This brings the theory of geometric morphisms to a point from which logical methods can be applied to give a proof of a theorem, due to Pierre Deligne ([SGA4], VI.9) about sufficiency of points for topoi that are called *coherent*. The definition of these categories can be motivated in part by the fundamental topological concept of *compactness*.

In a topological space I, a subset $A \subseteq I$ is *compact* if every open cover of A, i.e. every $C \subseteq \Theta$ such that $A \subseteq \bigcup C$, has a finite subcover, i.e. there is a finite subset C_0 of C such that $A \subseteq \bigcup C_0$. If a member V of Θ is compact, then the topological site (Θ, Cov_Θ) (Exercise 14.3.1) can be modified by changing $Cov_\Theta(V)$ to the set of *finite* open covers $C_0 \subseteq \Theta_V$, without altering the associated class of sheaves. This is seen as follows.

EXERCISE 24. Let F be a presheaf on I that fulfills the sheaf condition COM with respect to all finite open covers of an open set V. Show that if V is compact, then F fulfills COM with respect to all open covers of V. $\square$

A site $(\mathscr{C}, Cov)$ will be called *finitary* if $\mathscr{C}$ is a small finitely complete category and every member of $Cov(c)$ is finite, for all $\mathscr{C}$-objects c. A *coherent* topos is a category that is equivalent to **Sh(C)** for some finitary site **C**. The significance of this class of categories cannot really be conveyed here, except to say that it includes many of the sheaf categories of algebraic geometry to which the theory of Grothendieck topoi is addressed.

DELIGNE'S THEOREM. *Every coherent topos has enough points.* $\square$

This theorem does not hold for all Grothendieck topoi. Several examples have been given of such categories that do not have enough points. One due to Deligne, constructed out of measure spaces, appears in [SGA4], IV.7.4. Wraith [75], Corollary 7.6, shows that for a "Hausdorff" topological space I in which no singletons are open (e.g. the real line $\mathbb{R}$ is such a space), the Boolean topos $sh_{\neg\neg}(\textbf{Top}(I))$ of double negation sheaves on I has no points at all! (cf. also Johnstone [77], 7.12(iii)). A particularly apposite example is given by Barr [74], using atomless Boolean algebras, which we will now study.

Now an *atom* in a poset with a zero (minimum) element 0 is an element $a \neq 0$ such that there is no non-zero element strictly less than a (i.e. if $y \sqsubseteq a$, then $y = 0$ or $y = a$). A poset is *atomic* if for every non-zero element x there is an atom a such that $a \sqsubseteq x$. For any set P, the complete **BA** $\mathscr{P}(P)$ of all subsets of P is atomic, the atoms being the singletons $\{p\}$

corresponding to the points $p \in P$. Conversely any atomic complete **BA B** is isomorphic to $\mathcal{P}(P_{\mathbf{B}})$, where $P_{\mathbf{B}}$ is the set of all atoms in **B**. The isomorphism assigns to each **B**-element b the set $\{p \in P_{\mathbf{B}} : p \sqsubseteq b\}$.

EXERCISE 25. Show that in any **BA**, an element $a \neq 0$ is an atom iff for any y, $a \sqsubseteq y$ or $a \sqsubseteq y'$.

EXERCISE 26. Let $\mathcal{E}$ be an **S**-topos in which $\mathrm{Sub}_{\mathcal{E}}(1)$ is a Boolean algebra. If $p : \mathbf{Set} \to \mathcal{E}$ is a geometric morphism, show that the $\sqcap$-$\sqcup$ map $p^* : \mathrm{Sub}(1) \to 2$ induced by the inverse image part of p preserves Boolean complements, and thus preserves meets $\sqcap$. Hence show that $\sqcap \{f : p^*(f) = 1\}$ is an atom in $\mathrm{Sub}(1)$.   □

Now let **B** be a complete Boolean algebra that has no atoms at all (e.g. the algebra of "regular" open subsets of the real line – Mendelson [70], 5.48). As Barr suggests, **B** may be thought of as a "set without points". But in the Grothendieck topos **Sh(B)**, or equivalently **CB-Set**, $\mathrm{Sub}(1)$ is in fact isomorphic to **B** itself. This can be seen from the fact that in **CB-Set**, elements of **B** correspond to global elements of the subobject classifier, and the latter correspond to subobjects of 1 (cf. Exercise 14.7.46). (Alternatively, note that in **CB-Set**, the terminal object is **B** itself, and associate each subobject of **B** with its join in **B**.) Thus it follows by Exercise 26 that the topos **Sh(B)** does not have any points.

Returning to Deligne's Theorem, it follows from all that we have said that if $\mathcal{E}$ is a coherent topos, then there is a set P and a surjective geometric morphism $\pi : \mathbf{Sh}(\mathcal{P}(P)) \to \mathcal{E}$ (since $\mathbf{Sh}(\mathcal{P}(P))$ is a Boolean topos, π is sometimes called a "Boolean-valued point"). In this form the theorem has an appropriate generalisation to Grothendieck topoi (first conjectured by Lawvere, and proven in Barr [74]), obtained by abandoning the atomicity requirement on complete **BA**'s.

BARR'S THEOREM. *If $\mathcal{E}$ is a Grothendieck topos, then there is a complete Boolean algebra* **B** *and a surjective geometric morphism* **Sh(B)** $\to \mathcal{E}$.   □

This section has been a descriptive sketch of what is an extensive mathematical theory, and has only attempted to reproduce enough of it to allow a statement of the theorems of Deligne and Barr and an explanation of their model-theoretic content (to follow). A deeper understanding of this theory may be gained from Chapter 1 of [MR]. Its ultimate source is, of course, the monumental treatise [SGA4].

16.3. Internal logic

In this section we introduce the ideas of *many-sorted* languages and structures and show how to use them to express the internal structure of a category.

A model $\mathfrak{A} = \langle A, \ldots \rangle$ for an elementary language, as described in §11.2, consists of a single set A that carries certain operations $g : A^n \to A$, relations $R \subseteq A^n$, and distinguished elements $c \in A$. The corresponding first-order language has a single set $\{v_1, v_2, \ldots\}$ of individual variables that "range over A". But it is common in mathematics to deal with operations whose various arguments are of different *sorts*, i.e. come from different specified sets. A classic (two-sorted) example is the notion of a *vector space*, which involves a set V of "vectors", a set S of "scalars", and an operation of the form $S \times V \to V$ of "scalar multiplication of vectors". We formalise this sort of situation as follows.

Let $\mathscr{S}$ be a class, whose members will be called *sorts*. The basic alphabet for elementary languages of §11.2 is now adapted to an alphabet for $\mathscr{S}$-sorted languages by retaining the symbols $\wedge$, $\vee$, $\sim$, $\supset$, $\forall$, $\exists$, $\approx$,), (, and replacing the single list of individual variables by a denumerable set V_a of such variables for each $a \in \mathscr{S}$, with V_a disjoint from V_b whenever $a \neq b$. We often write $v \colon a$, and say "v is of sort a", when $v \in V_a$.

An $\mathscr{S}$-sorted language $\mathscr{L}$ is a collection of operation and relation symbols, and individual constants, such that:

(1) each relation symbol **R** has assigned to it a natural number n, called its *number of places*, and a sequence $\langle a_1, \ldots, a_n \rangle$ of sorts. We write **R**$\colon \langle a_1, \ldots, a_n \rangle$ to indicate this;

(2) each operation symbol **g** has an assigned *number of places* n, and a sequence $\langle a_1, \ldots, a_{n+1} \rangle$ of sorts. We indicate this by **g**$\colon \langle a_1, \ldots, a_n \rangle \to a_{n+1}$;

(3) each individual constant **c** is assigned a sort $a \in \mathscr{S}$, indicated by **c**$\colon a$ (this could be seen as a special case of (2) – an individual constant is a 0-placed operation symbol).

Terms and formulae of $\mathscr{L}$ are defined inductively as usual, with additional qualifications relating to the sort of each term. Thus variables and constants of sort a are terms of sort a, and if **g**$\colon \langle a_1, \ldots, a_n \rangle \to a_{n+1}$, and $t_1, \ldots, t_n$ are terms of respective sorts $a_1, \ldots, a_n$, then **g**$(t_1, \ldots, t_n)$ is a term of sort a_{n+1}. Atomic formulae are those of the form $(t \approx u)$, where t and u are terms of the same sort, and of the form **R**$(t_1, \ldots, t_n)$, where if **R**$\colon \langle a_1, \ldots, a_n \rangle$ then $t_1 \colon a_1, \ldots, t_n \colon a_n$. Other $\mathscr{L}$-formulae are built up from the atomic ones in the standard manner. We also include two *atomic sentences*, denoted $\top$ and $\bot$, in any language $\mathscr{L}$.

If $\mathscr{E}$ is an elementary topos, then an $\mathscr{E}$-model for an $\mathscr{S}$-sorted language $\mathscr{L}$ is a function $\mathfrak{A}$ with domain $\mathscr{S} \cup \mathscr{L}$ such that

(1) for each sort $a \in \mathscr{S}$, $\mathfrak{A}(a)$ is an $\mathscr{E}$-object;

(2) for each operation symbol $\mathbf{g}:\langle a_1,\ldots,a_n\rangle \to a_{n+1}$ in $\mathscr{L}$, $\mathfrak{A}(\mathbf{g})$ is an $\mathscr{E}$-arrow from $\mathfrak{A}(a_1) \times \cdots \times \mathfrak{A}(a_n)$ to $\mathfrak{A}(a_{n+1})$;

(3) for each relation symbol $\mathbf{R}:\langle a_1,\ldots,a_n\rangle$ in $\mathscr{L}$, $\mathfrak{A}(\mathbf{R})$ is a subobject of $\mathfrak{A}(a_1) \times \cdots \times \mathfrak{A}(a_n)$;

(4) for each individual constant $\mathbf{c}: a$, $\mathfrak{A}(\mathbf{c})$ is an arrow $1 \to \mathfrak{A}(a)$, i.e. a "global element" of $\mathfrak{A}(a)$.

We will use the notation $\mathfrak{A}:\mathscr{L} \to \mathscr{E}$ to indicate that $\mathfrak{A}$ is an $\mathscr{E}$-model for $\mathscr{L}$.

It is important to realise that this definition of model departs from that of §11.4 in that we now allow $\mathfrak{A}(a)$ to be any $\mathscr{E}$-object, including the initial object 0, or any other $\mathscr{E}$-object d that may have no global elements $1 \to d$ at all. This takes us into the domain of "free" logic (§11.8), but instead of using objects of partial elements, and existence predicates, we are following the approach of the Montreal school ([MR], Chapter 2), in which the notion of "model" directly abstracts the classical Tarskian one, while the standard rules of inference undergo restriction.

If $\mathbf{v} = \langle v_1,\ldots,v_m\rangle$ is a sequence of distinct variables, with $v_i: a_i$, we let $\mathfrak{A}(\mathbf{v})$ be $\mathfrak{A}(a_1) \times \cdots \times \mathfrak{A}(a_m)$. We also adopt the convention of declaring that if $\mathbf{v}$ is the *empty* sequence of variables then $\mathfrak{A}(\mathbf{v}) = 1$ (n.b., 1 is the product of the empty diagram). This is relevant to the interpretation of sentences (see below).

If t is a term of sort a, and $\mathbf{v} = \langle v_1,\ldots,v_m\rangle$ is *appropriate to t* in the sense that all of the variables of t occur in the list $\mathbf{v}$, then an $\mathscr{E}$-arrow $\mathfrak{A}^{\mathbf{v}}(t):\mathfrak{A}(\mathbf{v}) \to \mathfrak{A}(a)$ is defined inductively as follows.

(1) If t is the variable v_i, $\mathfrak{A}^{\mathbf{v}}(t)$ is the projection arrow $\mathfrak{A}(\mathbf{v}) \to \mathfrak{A}(a_i)$.

(2) If t is the constant $\mathbf{c}$, $\mathfrak{A}^{\mathbf{v}}(t)$ is the composite of

$$\mathfrak{A}(\mathbf{v}) \to 1 \xrightarrow{\mathfrak{A}(\mathbf{c})} \mathfrak{A}(a).$$

(3) If t is $\mathbf{g}(t_{i_1},\ldots,t_{i_n})$, where $\mathbf{g}:\langle a_{i_1},\ldots,a_{i_n}\rangle \to a$, then we inductively define $\mathfrak{A}^{\mathbf{v}}(t)$ to be the composite of

$$\mathfrak{A}(\mathbf{v}) \xrightarrow{f} \mathfrak{A}(a_{i_1}) \times \cdots \times \mathfrak{A}(a_{i_n}) \xrightarrow{\mathfrak{A}(\mathbf{g})} \mathfrak{A}(a),$$

where f is the product arrow $\langle \mathfrak{A}^{\mathbf{v}}(t_{i_1}),\ldots,\mathfrak{A}^{\mathbf{v}}(t_{i_n})\rangle$.

If φ is an $\mathscr{L}$-formula, and the list $\mathbf{v}$ is *appropriate to φ* in that all *free* variables of φ appear in $\mathbf{v}$, then φ is interpreted by the model $\mathfrak{A}$ as a

subobject $\mathfrak{A}^{\mathbf{v}}(\varphi)$ of $\mathfrak{A}(\mathbf{v})$. We often present this subobject as $\mathfrak{A}^{\mathbf{v}}(\varphi) \rightarrowtail \mathfrak{A}(\mathbf{v})$, so that the symbol "$\mathfrak{A}^{\mathbf{v}}(\varphi)$" tends to be associated with an object of $\mathscr{E}$, even though strictly speaking it denotes a subobject, whose domain is only determined up to isomorphism. The inductive definition of $\mathfrak{A}^{\mathbf{v}}(\varphi)$ is as follows.
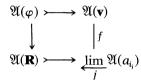
(1) $\mathfrak{A}^{\mathbf{v}}(\top)$ is the maximum subobject $1: \mathfrak{A}(\mathbf{v}) \to \mathfrak{A}(\mathbf{v})$, i.e. the subobject whose character is $true!: \mathfrak{A}(\mathbf{v}) \to \Omega$.

$\mathfrak{A}^{\mathbf{v}}(\bot)$ is the minimum subobject $0 \to \mathfrak{A}(\mathbf{v})$, with character $false!: \mathfrak{A}(\mathbf{v}) \to \Omega$.

(2) If t and u are terms of sort a, $\mathfrak{A}^{\mathbf{v}}(t \approx u)$ is the equaliser of

$$\mathfrak{A}(\mathbf{v}) \underset{\mathfrak{A}^{\mathbf{v}}(u)}{\overset{\mathfrak{A}^{\mathbf{v}}(t)}{\rightrightarrows}} \mathfrak{A}(a).$$

(3) If φ is $\mathbf{R}(t_{i_1}, \ldots, t_{i_n})$, then $\mathfrak{A}^{\mathbf{v}}(\varphi)$ is the pullback

$$
\begin{array}{ccc}
\mathfrak{A}(\varphi) & \rightarrowtail & \mathfrak{A}(\mathbf{v}) \\
\downarrow & & \downarrow{\scriptstyle f} \\
\mathfrak{A}(\mathbf{R}) & \rightarrowtail & \varprojlim_{j} \mathfrak{A}(a_{i_j})
\end{array}
$$

where $\mathbf{R}: \langle a_{i_1}, \ldots, a_{i_n} \rangle$ and f is $\langle \mathfrak{A}^{\mathbf{v}}(t_{i_1}), \ldots, \mathfrak{A}^{\mathbf{v}}(t_{i_n}) \rangle$.

(4) The connectives $\wedge, \vee, \sim, \supset$ are interpreted as the operations $\cap, \cup, -, \Rightarrow$ in the Heyting algebra $\mathrm{Sub}_{\mathscr{E}}(\mathfrak{A}(\mathbf{v}))$ (cf. §§7.1, 7.5).

(5) The quantifiers $\forall, \exists$, are interpreted by the functors $\forall_f$, $\exists_f: \mathrm{Sub}(\mathrm{dom}\, f) \to \mathrm{Sub}(\mathrm{cod}\, f)$ associated with an $\mathscr{E}$-arrow f, as defined in §15.4. If φ is $\exists w\psi$, or $\forall w\psi$, then all free variables of ψ appear in the list $\mathbf{v}, w = \langle v_1, \ldots, v_m, w \rangle$. Then if $pr: \mathfrak{A}(\mathbf{v}, w) \to \mathfrak{A}(\mathbf{v})$ is the evident projection, we put

$$\mathfrak{A}^{\mathbf{v}}(\exists w\psi) = \exists_{pr}(\mathfrak{A}^{\mathbf{v}, w}(\psi)),$$
$$\mathfrak{A}^{\mathbf{v}}(\forall w\psi) = \forall_{pr}(\mathfrak{A}^{\mathbf{v}, w}(\psi))$$

(cf. the beginning of §15.4 for motivation).

Note that if w is the only free variable of ψ, we need to allow that $\mathbf{v}$ be the empty sequence here. But in that case, pr can be identified with the unique arrow $\mathfrak{A}(w) \to 1$, so that the sentences $\exists w\psi$ and $\forall w\psi$ are interpreted as subobjects of 1.

Now if φ is any $\mathscr{L}$-formula, and $\mathbf{v}$ is the (possibly empty) sequence consisting of all and only the free variables of φ, we say that φ is *true in* $\mathfrak{A}$, or that $\mathfrak{A}$ is an $\mathscr{E}$-model of φ, denoted $\mathfrak{A} \vDash \varphi$, if $\mathfrak{A}^{\mathbf{v}}(\varphi)$ is the maximum

subobject of $\mathfrak{A}(\mathbf{v})$ (i.e. if $\mathfrak{A}^{\mathbf{v}}(\varphi)$ is $\mathfrak{A}^{\mathbf{v}}(\top)$). If $\mathbb{T}$ is a class of formulae, then $\mathfrak{A}$ is a $\mathbb{T}$-model, $\mathfrak{A} \vDash \mathbb{T}$, if every member of $\mathbb{T}$ is true in $\mathfrak{A}$.

We may tend to drop the symbol "$\mathbf{v}$" from "$\mathfrak{A}^{\mathbf{v}}(\varphi)$" if the intention is clear, and especially if $\mathbf{v}$ is the list of all free variables of φ.

EXERCISE 1. Develop the notion of a many-sorted model in **Set** along the Tarskian set-theoretic lines of §11.2, allowing for the presence of empty sorts, and defining a satisfaction relation

$$\mathfrak{A} \vDash \varphi[x_1, \ldots, x_m].$$

Show that in these terms the categorial notion $\mathfrak{A}^{\mathbf{v}}(\varphi)$ corresponds to the set

$$\{\langle x_1, \ldots, x_m \rangle : \mathfrak{A} \vDash \varphi[x_1, \ldots, x_m]\}.$$

EXERCISE 2 (Substitution). Let $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$ be appropriate to a term t of sort a. Let u be a term of the same sort as v_i, and let $\mathbf{u}$ be a sequence appropriate to the term $t(v_i/u)$. Define $\mathfrak{A}|v_i/u| : \mathfrak{A}(\mathbf{u}) \to \mathfrak{A}(\mathbf{v})$ to be the product arrow

$$\langle \mathfrak{A}^{\mathbf{u}}(v_1), \ldots, \mathfrak{A}^{\mathbf{u}}(v_{i-1}), \mathfrak{A}^{\mathbf{u}}(u), \mathfrak{A}^{\mathbf{u}}(v_{i+1}), \ldots, \mathfrak{A}^{\mathbf{u}}(v_m) \rangle.$$

(i) Show that

$$
\begin{array}{ccc}
 & \mathfrak{A}(\mathbf{u}) & \\
\mathfrak{A}|v_i/u| \swarrow & & \searrow \mathfrak{A}^{\mathbf{u}}(t(v_i/u)) \\
\mathfrak{A}(\mathbf{v}) & \xrightarrow[\mathfrak{A}^{\mathbf{v}}(t)]{} & \mathfrak{A}(a)
\end{array}
$$

commutes.

(ii) If $\mathbf{v}$ is appropriate to φ, v_i free for u in φ, and $\mathbf{u}$ is appropriate to $\varphi(v_i/u)$, show that $\mathfrak{A}^{\mathbf{u}}(\varphi(v_i/u))$ is the pullback

$$
\begin{array}{ccc}
\mathfrak{A}^{\mathbf{u}}(\varphi(v_i/u)) & \rightarrowtail & \mathfrak{A}(\mathbf{u}) \\
\downarrow & & \downarrow \mathfrak{A}|v_i/u| \\
\mathfrak{A}^{\mathbf{v}}(\varphi) & \rightarrowtail & \mathfrak{A}(\mathbf{v})
\end{array}
$$

of $\mathfrak{A}^{\mathbf{v}}(\varphi)$ along $\mathfrak{A}|v_i/u|$.

EXERCISE 3. $\mathfrak{A} \vDash \varphi \wedge \psi$ iff $\mathfrak{A} \vDash \varphi$ and $\mathfrak{A} \vDash \psi$.

EXERCISE 4. $\mathfrak{A} \vDash \varphi \supset \psi$ iff $\mathfrak{A}(\varphi) \subseteq \mathfrak{A}(\psi)$, and hence $\mathfrak{A} \vDash \varphi \equiv \psi$ iff $\mathfrak{A}(\varphi) = \mathfrak{A}(\psi)$.

EXERCISE 5. $\mathfrak{A}(\varphi) = \mathfrak{A}(\top \supset \varphi)$.

EXERCISE 6. $\mathfrak{A}(\sim\varphi) = \mathfrak{A}(\varphi \supset \bot)$.   □

The general existence in $\mathscr{E}$ of the interpretation $\mathfrak{A}(\varphi)$ of the formula φ depends on the possibility of performing certain categorial constructions in $\mathfrak{A}$. For instance, the interpretation of universal quantifiers requires the functors $\forall_f$, whose definition in §15.3 used properties that are very special to topoi. On the other hand, the definition of "$\mathscr{L}$-model" itself refers only to products and their subobjects. Indeed if $\mathscr{L}$ has only *one-placed* operation symbols, and no relation symbols or constants, we can construct $\mathscr{L}$-models in any category $\mathscr{C}$. $\mathscr{C}$ would have to have finite products for $\mathfrak{A}(\mathbf{v})$ to exist for all sequences $\mathbf{v}$ of variables, including a terminal object (empty product) for the case that $\mathbf{v}$ is the empty sequence. If $\mathscr{C}$ also had equalisers, then all *equations*, i.e. atomic identities $(t \approx u)$, would have interpretations in a $\mathscr{C}$-model $\mathfrak{A}$. Since Sub(d) is always a poset with a maximum element $\mathbf{1}_d$ (§4.1), we could then talk about the truth in $\mathfrak{A}$ of such equations. But if a category has a terminal object, equalisers, and a product for any pair of objects, then it has all finite limits (cf. §3.15). In sum then, provided that we assume that $\mathscr{C}$ is finitely complete, we can at least construct $\mathscr{C}$-models of equational logic.

The general question as to what categorial structure needs to be present for various types of $\mathscr{L}$-formulae to be interpretable is discussed in Reyes [74], [MR], and Kock and Reyes [77], and leads to notions of "Heyting" and "logical" categories. Similar work is carried out by Volger [75].

The language of a category

Let $\mathscr{C}$ be a finitely complete category. We associate with $\mathscr{C}$ a many-sorted language $\mathscr{L}_\mathscr{C}$ and a *canonical* $\mathscr{C}$-model $\mathfrak{A}_\mathscr{C} : \mathscr{L}_\mathscr{C} \to \mathscr{C}$ of $\mathscr{L}_\mathscr{C}$:

   (1) the collection of sorts of $\mathscr{L}_\mathscr{C}$ is the class $|\mathscr{C}|$ of $\mathscr{C}$-objects, i.e. each $\mathscr{C}$-object *is* a sort;

   (2) each $\mathscr{C}$-arrow $f : a \to b$ is declared to be a one-placed operation symbol, with associated sequence $\langle a, b \rangle$ of sorts. These are the only operation symbols of $\mathscr{L}_\mathscr{C}$, and there are no constants or relation symbols;

   (3) the model $\mathfrak{A}_\mathscr{C}$ is simply the identity function on $|\mathscr{C}| \cup \mathscr{L}_\mathscr{C}$. Thus if a is a sort, $\mathfrak{A}_\mathscr{C}(a)$ is a as a $\mathscr{C}$-object, and if f is an operation symbol, $\mathfrak{A}_\mathscr{C}(f)$ is f as a $\mathscr{C}$-arrow $\mathfrak{A}_\mathscr{C}(a) \to \mathfrak{A}_\mathscr{C}(b)$.

Now if $\mathscr{C}$ is a finitely complete category, the truth of certain equations $(t \approx u)$ in $\mathfrak{A}_\mathscr{C}$ can be used to characterise the structure of $\mathscr{C}$ as a category.

To see this, consider the question as to whether a triangle

$$a \xrightarrow{\ g\ } b$$

with h and f mapping to c

of $\mathscr{C}$-arrows commutes. If v is a variable of sort a. Then $\mathfrak{A}^v_{\mathscr{C}}(g(v))$ is just g, and correspondingly for the term $h(v)$, while $\mathfrak{A}^v_{\mathscr{C}}(f(g(v)))$ is $f \circ g$. Thus the equation $(f(g(v)) \approx h(v))$ is interpreted by $\mathfrak{A}^v_{\mathscr{C}}$ as the equaliser of $f \circ g$ and h. Since parallel arrows are equal if, and only if, their equaliser is iso, we get

$$\mathfrak{A}_{\mathscr{C}} \vDash (f(g(v)) \approx h(v)) \quad \text{iff} \quad f \circ g = h,$$

and, in particular,

$$\mathfrak{A}_{\mathscr{C}} \vDash (f(g(v)) \approx f \circ g(v)).$$

EXERCISE 7. Let $f : a \to a$ be an endo $\mathscr{C}$-arrow. Show that if $v : a$, then

$$\mathfrak{A}_{\mathscr{C}} \vDash (f(v) \approx v) \quad \text{iff} \quad f = \mathbf{1}_a,$$

and so

$$\mathfrak{A}_{\mathscr{C}} \vDash (\mathbf{1}_a(v) \approx v). \qquad \qquad \Box$$

Now if $\mathfrak{A} : \mathscr{L}_{\mathscr{C}} \to \mathscr{D}$ is an $\mathscr{L}_{\mathscr{C}}$-model in a category $\mathscr{D}$, then $\mathfrak{A}$ assigns a $\mathscr{D}$-object $\mathfrak{A}(a)$ to each $\mathscr{C}$-object (i.e. $\mathscr{L}_{\mathscr{C}}$-sort) a, and a $\mathscr{D}$-arrow $\mathfrak{A}(f) : \mathfrak{A}(a) \to \mathfrak{A}(b)$ to each $\mathscr{C}$-arrow ($\mathscr{L}_{\mathscr{C}}$ operation symbol) $f : a \to b$. Thus $\mathfrak{A}$ is *exactly* the same type of function as is a functor $\mathfrak{A} : \mathscr{C} \to \mathscr{D}$. To actually qualify as a functor, $\mathfrak{A}$ is required to preserve identity arrows and commutative triangles. Since these two notions have been expressed as $\mathscr{L}_{\mathscr{C}}$-equations, we can repeat the above arguments and exercises in $\mathscr{D}$ to show that the truth in $\mathfrak{A}$ of these equations exactly captures the required preservation property. Given a triangle f, g, h of $\mathscr{C}$-arrows, let

$$\mathbf{id}(f) \quad \text{be} \quad (f(v) \approx v),$$

and

$$\mathbf{com}(f, g, h) \quad \text{be} \quad (f(g(v)) \approx h(v)).$$

(In each case v is a variable of the required sort to make the formula well-formed, so to be precise $\mathbf{id}(f)$ is a formula schema, representing a different formula for each choice of v. We will in future gloss over this point).

THEOREM 1. *If $\mathcal{D}$ is finitely complete, then a $\mathcal{D}$-model $\mathfrak{A}:\mathscr{L}_{\mathscr{C}}\to\mathcal{D}$ for $\mathscr{L}_{\mathscr{C}}$ is a functor $\mathfrak{A}:\mathscr{C}\to\mathcal{D}$ if, and only if, for all $\mathscr{C}$-objects a, and all composable pairs f, g of $\mathscr{C}$-arrows, the equations* **id**(1_a) *and* **com**$(f, g, f \circ g)$ *are true in $\mathfrak{A}$.* $\qquad\square$

This result displays the essential idea of the logical characterisation of categorial properties (the reader familiar with model theory will recognise it as a variant of the "method of diagrams"). Note that the result does not depend on the existence of any limits in $\mathscr{C}$.

Continuing in this vein, we develop logical axioms for products, equalisers etc. This will involve us in the use of existential quantifiers, and hence the subobject functors $\exists_f$. So, from now on we will assume that the category in which our model exists is a topos (although this is stronger than is needed for $\exists_f$ to exist).

Recall that $\exists_f:\mathrm{Sub}(a)\to\mathrm{Sub}(b)$ takes $g:c\rightarrowtail a$ to the image arrow of $f\circ g$, so

$$
\begin{array}{ccc}
c & \xrightarrowtail{\;g\;} & a \\
\downarrow & & \downarrow f \\
f\circ g(c) & \xrightarrowtail[\exists_f(g)]{} & b
\end{array}
$$

$\exists_f(g)$ is the smallest subobject of b through which $f\circ g$ factors (Theorem 5.2.1). The interplay between $\exists$ and im is very much to the fore in the next series of exercises, for which we assume that $\mathfrak{A}$ is the canonical $\mathscr{L}_{\mathscr{C}}$-model $\mathscr{L}_{\mathscr{C}}\to\mathscr{C}$ in a topos $\mathscr{C}$.

EXERCISE 8. Let a be an $\mathscr{C}$-object, and v, w variables of sort a.

(1) Show that $\mathfrak{A}(\exists v(v\approx v))$ is the support $\sup(a)\rightarrowtail 1$ of a (§12.1), and hence that $\mathfrak{A}\vDash\exists v(v\approx v)$ iff the unique arrow $a\to 1$ is epic.

(2) Show that the two projection arrows $a\times a\to a$ are equal iff $a\to 1$ is monic. Hence show that this last arrow is monic iff $\mathfrak{A}\vDash(v\approx w)$.

(3) Let **term**(a) be the conjunction of the formulae $\exists v(v\approx v)$ and $(v\approx w)$. Show that

$$\mathfrak{A}\vDash\textbf{term}(a)\quad\text{iff}\quad a\text{ is a terminal object}\qquad\square$$

The formula **term**(a) may be regarded as expressing "there exist a unique v of sort a".

EXERCISE 9. If $f:a\to b$ is an arrow, let **mon**(f) be the formula $(f(v)\approx f(w)\supset v\approx w)$. Show that

$$\mathfrak{A}\vDash\textbf{mon}(f)\quad\text{iff}\quad f\text{ is monic.}$$

EXERCISE 10. Given $f: a \to b$, $v: a$, and $w: b$, show that the "graph" $\langle 1_a, f \rangle: a \rightarrowtail a \times b$ of f is $\mathfrak{A}(f(v) \approx w)$. Hence show that $\operatorname{im} f: f(a) \rightarrowtail b$ is $\mathfrak{A}(\mathbf{ep}(f))$, where $\mathbf{ep}(f)$ is the formula $\exists v(f(v) \approx w)$. Thus

$$\mathfrak{A} \vDash \mathbf{ep}(f) \quad \text{iff} \quad f \text{ is epic.} \qquad \qquad \square$$

Thus the condition "f is iso" is characterised by the truth of $\mathbf{mon}(f) \wedge \mathbf{ep}(f)$, a formula that expresses "there is a unique v such that $f(v) = w$".

Next we consider equalisers. If $i: e \rightarrowtail a$ equalises $f, g: a \to b$, then i, as a subobject, is precisely $\mathfrak{A}(f(v) \approx g(v))$. On the other hand, since i is monic it can be identified with $\operatorname{im} i$, and hence (Exercise 10) with $\mathfrak{A}(\mathbf{ep}(i))$, so that

$$\mathfrak{A} \vDash (f(v) \approx g(v)) \equiv \exists w(i(w) \approx v).$$

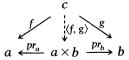Now if the arrow h in

$$c \xrightarrow{h} a \underset{g}{\overset{f}{\rightrightarrows}} b$$

is monic and has $f \circ h = g \circ h$, then h, or equivalently $\operatorname{im} h$, is a subobject of the equaliser of f and g, which means that $\mathfrak{A}(\mathbf{ep}(h)) \subseteq \mathfrak{A}(f(v) \approx g(v))$. Therefore, if the converse of this last inclusion holds, h itself is an equaliser of f and g. These observations lead to the following result.

EXERCISE 11. Given $f, g: a \to b$, and $h: c \to a$, let $\mathbf{equ}(h, f, g)$ be the conjunction of the three formulae

$\mathbf{mon}(h)$,

$f(h(w)) \approx g(h(w))$,

$f(v) \approx g(v) \supset \exists w(h(w) \approx v)$.

Then

$$\mathfrak{A} \vDash \mathbf{equ}(h, f, g) \quad \text{iff} \quad h \text{ is an equaliser of } f \text{ and } g. \qquad \square$$

For the case of products, given $f: c \to a$ and $g: c \to b$, then c will be a product of a and b, with f and g as projection arrows, precisely when the product arrow $\langle f, g \rangle$

is iso (cf. §3.8), i.e. monic and epic. Ostensibly then we could express this by the formulae $\mathbf{mon}$ and $\mathbf{ep}$ applied to the arrow $\langle f, g \rangle$. But it is

desirable that we have a formula that explicitly refers only to f and g. After all, the notation "$\langle f, g \rangle$" does not refer to a uniquely determined arrow (unlike "$f \circ g$"), but is only unique up to isomorphism and depends on the choice of the product $a \times b$ and projections pr_a and pr_b. Thus we reduce the two desired properties of $\langle f, g \rangle$ to properties of f and g.

EXERCISE 12. Let f, g, pr_a, pr_b be as above.
  (1) Show that

$$\mathfrak{A}(\langle f, g \rangle(v) \approx \langle f, g \rangle(w)) = \mathfrak{A}(f(v) \approx f(w) \wedge g(v) \approx g(w)).$$

  (2) Show that the graph $\langle 1_c, \langle f, g \rangle \rangle : c \to c \times (a \times b)$ of $\langle f, g \rangle$ is $\mathfrak{A}(f(v) \approx w \wedge g(v) \approx z)$, where $v : c$, $w : a$, $z : b$.
  (3) Let **prod**(f, g) be the conjunction of the formulae

$$(f(v) \approx f(w) \wedge g(v) \approx g(w)) \supset v \approx w$$
$$\exists v (f(v) \approx w \wedge g(v) \approx z).$$

Show that

$$\mathfrak{A} \vDash \mathbf{prod}(f, g) \quad \text{iff} \quad c \text{ is a product of } a \text{ and } b \text{ with projections}$$
$$f \text{ and } g$$

(cf. Exercises 9 and 10 above).   □

By adapting these exercises to a model of the form $\mathfrak{A} : \mathscr{L}_\mathscr{C} \to \mathscr{E}$, we can extend Theorem 1 above to characterise left exactness of $\mathfrak{A}$, as a functor $\mathscr{C} \to \mathscr{E}$, in terms of the truth in $\mathfrak{A}$ of the formulae of the type **term, equ**, and **prod** determined by the terminal objects, equalisers, and products of pairs of objects in $\mathscr{C}$ (left exactness being equivalent to preservation of these particular limits). Our use of this logical characterisation will be in the context of continuous morphisms from a site on $\mathscr{C}$ to a topos $\mathscr{E}$ with its canonical pretopology. In the latter case, a set $C = \{f_x : x \in X\}$ of $\mathscr{E}$-arrows with the same codomain c is a cover of c iff $\bigcup_X \operatorname{im} f_X$ is the maximum element $\mathbf{1}_c$ of $\operatorname{Sub}_\mathscr{E}(c)$. But in the canonical model $\mathfrak{A} : \mathscr{L}_\mathscr{C} \to \mathscr{E}$, $\operatorname{im} f_x$ is $\mathfrak{A}(\mathbf{ep}(f_x))$, where $\mathbf{ep}(f_x)$ is the formula $\exists v_x (f_x(v_x) \approx v)$ for $v_x : \operatorname{dom} f_x$ and $v : c$. Moreover, *if X is finite*, we can form the disjunction

$$\bigvee_{x \in X} \mathbf{ep}(f_x)$$

as an $\mathscr{L}_\mathscr{C}$-formula. Since $\mathfrak{A}$ interprets disjunction as union in $\operatorname{Sub}(c)$, we have that $\mathfrak{A}(\bigvee \mathbf{ep}(f_x)) = \bigcup \operatorname{im} f_x$. Thus it follows in this case that C is a cover for the canonical site on $\mathscr{E}$ if, and only if, $\mathfrak{A} \vDash \mathbf{cov}(C)$, where $\mathbf{cov}(C)$

is the formula

$$\bigvee_{x \in X} \exists v_x (f_x(v_x) \approx v).$$

The restriction to finite X is of course because formulae in the first-order languages we are currently using are finite sequences of symbols, and we are disbarred from disjoining infinitely many formulae at once (the possibility of allowing this will be taken up later). So our present theory is appropriate to the case of *finitary* sites, in which all covers are finite.

If $\mathbf{C} = (\mathscr{C}, Cov)$ is a finitary site, we define a collection $\mathbb{T}_\mathbf{C}$ of formulae of the canonical language $\mathscr{L}_\mathscr{C}$ of the small category $\mathscr{C}$. $\mathbb{T}_\mathbf{C}$ is called the *theory* of the site $\mathbf{C}$, and consists of

(1) $\mathbf{id}(1_a)$, for each $\mathscr{C}$-object a;

(2) $\mathbf{com}(f, g, f \circ g)$ for each composable pair f, g of $\mathscr{C}$-arrows;

(3) $\mathbf{term}(a)$, for each terminal object a in $\mathscr{C}$;

(4) $\mathbf{equ}(h, f, g)$, for each equaliser h of a parallel pair f, g of $\mathscr{C}$-arrows;

(5) $\mathbf{prod}(f, g)$, for each pair of $\mathscr{C}$-arrows with dom $f =$ dom g that forms a product diagram in $\mathscr{C}$;

(6) $\mathbf{cov}(C)$, for each cover C in $\mathbf{C}$.

Notice that since $\mathbf{C}$ is small, so too is $\mathbb{T}_\mathbf{C}$.

THEOREM 2. *If $\mathscr{E}$ is a Grothendieck topos, and $\mathbf{C}$ a finitary site, then an $\mathscr{E}$-model $\mathfrak{A} : \mathscr{L}_\mathscr{C} \to \mathscr{E}$ for the canonical language of $\mathscr{C}$ is a continuous morphism $\mathfrak{A} : \mathbf{C} \to \mathscr{E}$ if, and only if, $\mathfrak{A} \vDash \mathbb{T}_\mathbf{C}$.*   □

In view of the Reduction Theorem 16.2.2, we now see from Theorem 2 that the existence of geometric morphisms $\mathscr{E} \to \mathbf{Sh}(\mathbf{C})$ reduces to the existence of $\mathscr{E}$-models of $\mathbb{T}_\mathbf{C}$. In particular, points of the form $\mathbf{Set} \to \mathbf{Sh}(\mathbf{C})$ correspond to classical $\mathbf{Set}$-based models of $\mathbb{T}_\mathbf{C}$. Since Deligne's Theorem is about the existence of sufficiently many points, while the classical Completeness Theorem is about the existence of sufficiently many $\mathbf{Set}$-models (a falsifying one for each non-theorem), we begin to see why, and how, these two basic results are related.

The exacting reader will be dissatisfied with the gap between Theorems 1 and 2 of this section and the given arguments and exercises for $\mathbf{id}$, $\mathbf{com}$, $\mathbf{term}$, $\mathbf{prod}$, $\mathbf{equ}$, and $\mathbf{cov}$ that lie behind them. The latter were stated in terms of canonical models $\mathscr{L}_\mathscr{C} \to \mathscr{C}$, whereas the Theorems refer to models $\mathscr{L}_\mathscr{C} \to \mathscr{E}$ of $\mathscr{L}_\mathscr{C}$ in other categories than $\mathscr{C}$. The only comment made about the connection was that the arguments and exercises given

for $\mathscr{C}$ could be "repeated" in $\mathscr{E}$. This can be made precise by observing that a model $\mathfrak{A}:\mathscr{L}_\mathscr{C} \to \mathscr{E}$ can be regarded as a function $\mathfrak{A}:\mathscr{L}_\mathscr{C} \to \mathscr{L}_\mathscr{E}$ between canonical languages, takings sorts and operation symbols of $\mathscr{L}_\mathscr{C}$ to the corresponding entities in $\mathscr{L}_\mathscr{E}$. This induces a translation $(-)^\mathfrak{A}$ of $\mathscr{L}_\mathscr{C}$-formulae φ to $\mathscr{L}_\mathscr{E}$-formulae $\varphi^\mathfrak{A}$, obtained by replacing each operation symbol $\mathbf{g}$ in φ by $\mathfrak{A}(\mathbf{g})$, and regarding variables of sort a in φ as being of sort $\mathfrak{A}(a)$ in $\varphi^\mathfrak{A}$. It is then readily seen that each "axiom" associated with a diagram D in $\mathscr{C}$ translates under $(-)^\mathfrak{A}$ to the axiom associated with the image diagram $\mathfrak{A}(D)$ in $\mathscr{E}$. In other words, $(\mathbf{term}(a))^\mathfrak{A} = \mathbf{term}(\mathfrak{A}(a))$, $(\mathbf{cov}(C))^\mathfrak{A} = \mathbf{cov}(\mathfrak{A}(C))$, and so on. It is also straightforward to show that the interpretation of any $\mathscr{L}_\mathscr{C}$-formula φ in $\mathfrak{A}$ is the same as the interpretation of its translate $\varphi^\mathfrak{A}$ in the canonical $\mathscr{E}$-model $\mathfrak{A}_\mathscr{E} : \mathscr{L}_\mathscr{E} \to \mathscr{E}$. That is, we have ([MR] Proposition 3.5.1)

$$\mathfrak{A}^\mathbf{v}(\varphi) = \mathfrak{A}^\mathbf{v}_\mathscr{E}(\varphi^\mathfrak{A}),$$

and so

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathfrak{A}_\mathscr{E} \vDash \varphi^\mathfrak{A}.$$

Now suppose that D is one of the types of diagram in $\mathscr{C}$ that we have been considering (finite limit, cover etc.), with its categorial property P characterised by the $\mathfrak{A}$-truth of some $\mathscr{L}_\mathscr{C}$-formula φ_P (where φ_P has one of the forms **term**, **prod**, **cov** etc.). Then it follows that the same property for $\mathfrak{A}(D)$ in $\mathscr{E}$ is characterised by the truth in $\mathfrak{A}_\mathscr{E}$ of $(\varphi_P)^\mathfrak{A}$. In view of the last equation, this establishes the principle ([MR], Metatheorem 3.5.2) that

$$\mathfrak{A} \text{ preserves the property } P \text{ of } D \quad \text{iff} \quad \mathfrak{A} \vDash \varphi_P.$$

16.4. Geometric logic

A formula will be called *positive-existential* if, in addition to atomic formulae, it contains no logical symbols other than $\top, \bot, \wedge, \vee, \exists$. The class of all positive-existential $\mathscr{L}$-formulae will be denoted $\mathscr{L}^\mathrm{g}$. A *geometric*, or *coherent* L-formula is one of the form $\varphi \supset \psi$, where φ and ψ are in $\mathscr{L}^\mathrm{g}$. Since any φ can be identified with $(\top \supset \varphi)$, in the sense that $\mathfrak{A}(\varphi) = \mathfrak{A}(\top \supset \varphi)$ (Ex. 16.3.5), each positive existential formula can be regarded as being geometric. Also in this sense, the negation of an $\mathscr{L}^\mathrm{g}$-formula is geometric, as in general $\sim\varphi$ is equivalent to $(\varphi \supset \bot)$. A *set* $\mathbb{T}$ of geometric formulae will be called a *geometric theory*.

The concept of a geometric theory is central to our present context, as

all members of the theory $\mathbb{T}_{\mathbf{C}}$ of a finitary site are geometric. Moreover, all formulae of this type have their "truth-value" preserved by the inverse image parts of geometric morphisms. To see this, let $f : \mathscr{F} \to \mathscr{E}$ be a geometric morphism of topoi, and $\mathfrak{A} : \mathscr{L} \to \mathscr{E}$ an $\mathscr{E}$-model for some language $\mathscr{L}$. We define an $\mathscr{F}$-model $f^*\mathfrak{A}$ of $\mathscr{L}$, which, as a function on the collection of sorts, operation symbols, and relation symbols of $\mathscr{L}$, is just the composite of f^* and $\mathfrak{A}$. Thus for each $\mathscr{L}$-sort a, the $\mathscr{E}$-object $f^*\mathfrak{A}(a)$ is $f^*(\mathfrak{A}(a))$, the result of applying the functor f^* to the $\mathscr{E}$-object $\mathfrak{A}(a)$, and so on. The fact that f^* preserves products and subobjects and is functorial ensures that the definition of "model" is thereby satisfied.

THEOREM 1. (1) *For any positive-existential formula* φ,

$$(f^*\mathfrak{A})^{\mathbf{v}}(\varphi) = f^*(\mathfrak{A}^{\mathbf{v}}(\varphi)).$$

(2) *If* θ *is geometric, then*

$$\mathfrak{A} \vDash \theta \quad implies \quad f^*\mathfrak{A} \vDash \theta.$$

(3) *If* f^* *is faithful, then for geometric* θ,

$$\mathfrak{A} \vDash \theta \quad iff \quad f^*\mathfrak{A} \vDash \theta.$$

PROOF. (1) (Outline). This is proven by induction over the formation of φ. The essential point is that, being an exact functor, f^* preserves all finite limits and colimits, and hence preserves all the categorial structure involved in interpreting a positive-existential formula.

First of all, preservation of monics ensures that $f^*\mathfrak{A}^{\mathbf{v}}(\varphi)$ is a subobject of $f^*\mathfrak{A}(\mathbf{v})$. Functoriality of f^* and preservation of products makes $f^*\mathfrak{A}(\mathbf{v}) = f^*(\mathfrak{A}(\mathbf{v}))$ for sequences of variables, and $f^*\mathfrak{A}^{\mathbf{v}}(t) = f^*(\mathfrak{A}^{\mathbf{v}}(t))$ for terms t to which $\mathbf{v}$ is appropriate.

Preservation of terminal and initial objects ensures that the desired result holds when φ is $\top$ or $\bot$, while the cases of the other atomic formulae use equalisers, products, and pullbacks. Pullbacks are used in the inductive case for $\wedge$, and coproducts and images (Ex. 16.1.8) are needed for $\vee$. Finally, preservation of images (and projection arrows) is needed for the inductive case of $\exists$.

(2) If θ is $\varphi \supset \psi$, where φ and ψ are in $\mathscr{L}^{\mathsf{g}}$, and $\mathfrak{A} \vDash \theta$, then $\mathfrak{A}(\varphi) \subseteq \mathfrak{A}(\psi)$, so that there is an arrow h factoring $\mathfrak{A}(\varphi)$ through $\mathfrak{A}(\psi)$ in $\mathscr{E}$, i.e. $\mathfrak{A}(\varphi) = \mathfrak{A}(\psi) \circ h$. But then as f^* is a functor, $f^*(h)$ factors $f^*(\mathfrak{A}(\varphi))$ through $f^*(\mathfrak{A}(\psi))$. Hence by (1), we have $f^*\mathfrak{A}(\varphi) \subseteq f^*\mathfrak{A}(\psi)$, so that $f^*\mathfrak{A} \vDash \varphi \supset \psi$.

(3) If f^* is faithful, or equivalently conservative (§16.1), then f^* reflects

subobjects, so that, by (1), $f^*\mathfrak{A}(\varphi) \subseteq f^*\mathfrak{A}(\psi)$ only if $\mathfrak{A}(\varphi) \subseteq \mathfrak{A}(\psi)$. In combination with (2), the result then follows.   □

Now if $\mathscr{E}$ is a *coherent* topos, and $\mathfrak{A}$ an $\mathscr{E}$-model of a geometric theory $\mathbb{T}$, then for any point $p : \mathbf{Set} \to \mathscr{E}$ it follows that $p^*\mathfrak{A}$ is a **Set**-model of $\mathbb{T}$. On the other hand, if $\mathfrak{A} \nvDash \mathbb{T}$ then there is some (geometric) formula $(\varphi \supset \psi)$ in $\mathbb{T}$ such that $\mathfrak{A}(\varphi) \nsubseteq \mathfrak{A}(\psi)$. But then, by Deligne's Theorem there exists a point p of $\mathscr{E}$ such that $p^*(\mathfrak{A}(\varphi)) \nsubseteq p^*(\mathfrak{A}(\psi))$, so that $p^*\mathfrak{A} \nvDash (\varphi \supset \psi)$. In this way, truth of geometric formulae in $\mathscr{E}$ reduces to the question of their truth in standard set-theoretic models. We have

THEOREM 2. *If $\mathfrak{A}$ is an $\mathscr{L}$-model in a coherent topos $\mathscr{E}$, and $\mathbb{T}$ a geometric theory, then*

$$\mathfrak{A} \vDash \mathbb{T} \ \ in \ \mathscr{E} \quad iff \quad for \ all \ points \ p \ of \ \mathscr{E}, p^*\mathfrak{A} \vDash \mathbb{T}$$

$$in \ \mathbf{Set}.$$

EXERCISE 1. By appropriate choice of $\mathscr{L}, \mathfrak{A}$, and $\mathbb{T}$, deduce Deligne's Theorem from Theorem 2.   □

The Theorem of Barr on the existence of Boolean-valued points for Grothendieck topoi also leads to an important metatheorem about models of geometric theories. Let us write $\mathbb{T} \vdash_c \theta$ to mean that θ is derivable from $\mathbb{T}$ by classical logic. This notion is defined by admitting proof sequences for θ that may contain as "axioms" members of $\mathbb{T}$ and classically valid axioms like $\varphi \vee \sim \varphi$. There is a standard completeness Theorem to the effect that $\mathbb{T} \vdash_c \varphi$ iff φ is true in every **Set**-model of $\mathbb{T}$ (Henkin [49]). But then from Barr's Theorem, we get

THEOREM 3. *If $\mathbb{T}$ and θ are geometric, and $\mathbb{T} \vdash_c \theta$, then θ is true in every $\mathbb{T}$-model in every Grothendieck topos.*

PROOF. Let $\mathfrak{A} : \mathscr{L} \to \mathscr{E}$ be a model of $\mathbb{T}$, with $\mathscr{E}$ a Grothendieck topos. By Barr's Theorem there exists a surjective geometric morphism of the form $f : \mathbf{Sh}(\mathbf{B}) \to \mathscr{E}$ for some complete **BA B**. Then as $\mathbb{T}$ is geometric, Theorem 1(2) implies that $f^*\mathfrak{A} \vDash \mathbb{T}$. But the laws of classical logic hold in the *Boolean* topos $\mathbf{Sh}(\mathbf{B})$, and so as $\mathbb{T} \vdash_c \theta$, $f^*\mathfrak{A} \vDash \theta$. Since f^* is faithful, Theorem 1(3) then gives $\mathfrak{A} \vDash \theta$.   □

EXERCISE 2. Show that the restriction of Theorem 3 to coherent topoi follows from Deligne's Theorem.   □

One interpretation of Theorem 3 is that for geometric formulae, anything inferrable by classical logic is inferrable by the weaker intuitionistic logic, and so we gain no new geometric theorems by using principles that are not intuitionistically valid (note of course that $\varphi \lor \sim \varphi$ is not geometric). But the importance of the result resides in its use in lifting mathematical constructions from **Set** to non-Boolean topoi. For example, suppose $\mathbb{T}$ consists of the axioms for the notion of a group. Then to show that $\mathbb{T}$-models in Grothendieck topoi have a certain property, then provided that the property can be expressed by a geometric formula θ, it suffices to show that all standard groups, i.e. all $\mathbb{T}$-models in **Set**, satisfy θ. But for the latter we have at our disposal the power of classical logic, and the techniques of standard group theory.

An application of this method to the "Galois theory of local rings" is given by Wraith [79].

Proof theory

By a *sequent* we mean an expression $\Gamma \supset \psi$, where Γ is a finite set of formulae, and ψ a single formula. A sequent is *geometric* if all members of $\Gamma \cup \{\psi\}$ are positive-existential.

A sequent is not a formula, since Γ is not, but if $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, then $\Gamma \supset \psi$ is "virtually the same thing as" the formula $(\varphi_1 \land \ldots \land \varphi_n) \supset \psi$. Thus if this last formula is true in a model $\mathfrak{A}$, we will say that the sequent $\Gamma \supset \psi$ is *true in* $\mathfrak{A}$. A set $\mathbb{T}$ of sequents will be called a *theory*, just as for a set of formulae.

It is clear that the notions of geometric sequent and geometric formula can be interchanged, and we will tend to do this at times. The point of introducing sequents at all is to provide a convenient notation for expressing axioms and rules of inference that enable us to derive geometric formulae. Given a theory $\mathbb{T}$, and sequent θ, we are going to define the relation "θ is provable from $\mathbb{T}$", denoted $\mathbb{T} \vdash \theta$. The aim of this proof-theoretic approach will be to obtain θ from $\mathbb{T}$ by operations on sequents that depend only on their syntactic form (i.e. on the nature of the symbols that occur in them), and not on any semantic notions of "truth", "implication", etc.

In the rules to follow, the union $\Gamma \cup \Delta$ of sets Γ and Δ will be written Γ, Δ, or Γ, φ if $\Delta = \{\varphi\}$. If $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, then $\bigwedge \Gamma$ denotes the conjunction $\varphi_1 \land \ldots \land \varphi_n$, while $\bigvee \Gamma$ is the disjunction $\varphi_1 \lor \ldots \lor \varphi_n$. If Γ is the empty set, $\bigvee \Gamma$ is $\bot$, while $\bigwedge \Gamma$ is $\top$, so that $\Gamma \supset \psi$ is identified with $\top \supset \psi$, or simply ψ in conformity with our conventions stated earlier.

We write $\Gamma(v_1, \ldots, v_n)$ to indicate that any free variable occurring in any member of Γ is amongst $v_1, \ldots, v_n$. $\Gamma(t_1, \ldots, t_n)$ denotes the set of formulae obtained by uniformly substituting the term t_i for v_i through out Γ.

Given a set $\mathbb{T}$ of geometric sequents, $\mathbb{T}^=$ will denote the union of $\mathbb{T}$ and all the following

AXIOMS OF IDENTITY

$$v \approx v,$$

$$v \approx w \supset w \approx v,$$

$$v \approx w, \varphi \supset \varphi(v/w),$$

where v and w are variables of the same sort, and φ is atomic.

We can now set out the axiom system for geometric sequents developed by Makkai and Reyes ([MR], §5.2), which we will call GL.

AXIOM

$$\Gamma \supset \psi, \quad \text{if} \quad \psi \in \Gamma.$$

Rules of inference

The rules all have the form

$$\frac{\{\theta_i : i \in I\}}{\theta},$$

the intended meaning being that the sequent θ is derivable if all of the sequents θ_i have been derived, i.e. the *conclusion* θ is a consequence of the *premisses* θ_i.

$(R \bigwedge_1)$ $\quad \dfrac{\Delta, \bigwedge \Gamma, \varphi \supset \psi}{\Delta, \bigwedge \Gamma \supset \psi}, \quad \text{if } \varphi \in \Gamma,$

$(R \bigwedge_2)$ $\quad \dfrac{\Delta, \bigwedge \Gamma \supset \psi}{\Delta \supset \psi}, \quad \text{if } \Gamma \subseteq \Delta,$

$(R \bigvee_1)$ $\quad \dfrac{\Delta, \varphi, \bigvee \Gamma \supset \psi}{\Delta, \varphi \supset \psi}, \quad \text{if } \varphi \in \Gamma,$

and all free variables occurring in Γ also occur free in the conclusion.

$$(R \bigvee_2) \qquad \frac{\{\Delta, \bigvee \Gamma, \varphi \supset \psi : \varphi \in \Gamma\}}{\Delta, \bigvee \Gamma \supset \psi},$$

$$(R\exists_1) \qquad \frac{\Delta, \varphi(v/t), \exists w \varphi(v/w) \supset \psi}{\Delta, \varphi(v/t) \supset \psi},$$

$$(R\exists_2) \qquad \frac{\Delta, \exists w \varphi(v/w), \varphi \supset \psi}{\Delta, \exists w \varphi(v/w) \supset \psi},$$

if v does not occur free in the conclusion.

$$(R\mathbb{T}) \qquad \frac{\Delta, \Gamma(t_1, \ldots, t_n), \varphi(t_1, \ldots, t_n) \supset \psi}{\Delta, \Gamma(t_1, \ldots, t_n) \supset \psi},$$

provided that all free variables in the premiss occur free in the conclusion, and for some $v_1, \ldots, v_n$, $\Gamma(v_1, \ldots, v_n) \supset \varphi(v_1, \ldots, v_n)$ belongs to $\mathbb{T}^=$.

Note that the last rule $R\mathbb{T}$ depends on the particular theory $\mathbb{T}$. The restriction on free variables in $R \bigvee_1$, and $R\mathbb{T}$ are necessary for these rules to be truth-preserving, as our models now may involve "empty" objects (cf. the discussion of Detachment in §11.9).

We say that geometric sequent θ is *derivable from* $\mathbb{T}$ in the system GL, which we denote simply by $\mathbb{T} \vdash \theta$, if there is a proof sequence for θ from $\mathbb{T}$, i.e. a finite list of sequents ending in θ and such that each member of the list is either an axiom or a consequence of earlier members of the list by one of the above rules. It can be shown (cf. [MR] Theorem 3.2.8), that all of the axioms, including the Axioms of Identity, are true in any model in any topos, and that the rules of GL preserve this property. Hence the

SOUNDNESS THEOREM. *If* $\mathbb{T} \vdash \theta$, *and if* $\mathfrak{A}$ *is a model of* $\mathbb{T}$ *in a topos* $\mathscr{E}$, *then* $\mathfrak{A} \models \theta$.   □

The converse of Soundness asserts that if $\mathbb{T} \nvdash \theta$ (i.e. if θ is not GL-derivable from $\mathbb{T}$), then there is a $\mathbb{T}$-model in some topos that falsifies θ. If such a model can be found in a Grothendieck topos, then, in view of the discussion of the logical significance of Barr's Theorem (Theorem 3 above), one must exist in **Set**. Indeed we have the

CLASSICAL COMPLETENESS THEOREM (cf. [MR], 5.2.3(b)). *If* $\mathbb{T} \nvdash \theta$, *then there is a* **Set**-*model* $\mathfrak{A}$ *such that* $\mathfrak{A} \models \mathbb{T}$ *and* $\mathfrak{A} \nVdash \theta$.

There is a systematic technique for proving theorems of this kind. It is known as the *Henkin method*, after the work of Leon Henkin [49], who introduced it as a way of proving completeness for systems of the type described in §11.3. The basic idea is as follows.

If θ is $\Gamma \supset \varphi$, then since $\mathbb{T} \nvdash \theta$, the GL Axiom implies that $\varphi \notin \Gamma$. We then attempt to expand Γ to a set Σ of formulae that still does not include φ and which will be the full theory of the desired $\mathbb{T}$-model $\mathfrak{A}$, i.e. will have

$$\psi \in \Sigma \quad \text{iff} \quad \mathfrak{A} \vDash \psi.$$

Given Σ, the model is defined through specification of an equivalence relation on the terms of a given sort by

$$t \sim u \quad \text{iff} \quad (t \approx u) \in \Sigma.$$

The resulting equivalence classes $\hat{t}$ then become the individuals of the given sort, and relation and operation symbols are interpreted by putting

$$\mathfrak{A}(\mathbf{R})(\hat{t}_1, \ldots, \hat{t}_n) \quad \text{iff} \quad \mathbf{R}(t_1, \ldots, t_n) \in \Sigma,$$
$$\mathfrak{A}(\mathbf{g})(\hat{t}_1, \ldots, \hat{t}_n) = \hat{t} \quad \text{iff} \quad (\mathbf{g}(t_1, \ldots, t_n) \approx t) \in \Sigma.$$

Note that since $\mathbb{T}$ is a set, by ignoring any symbols extraneous to $\mathbb{T} \cup \{\theta\}$ we can assume we are dealing with a *small* language. This guarantees that the $\mathfrak{A}$-individuals of a given sort form a set, so that $\mathfrak{A}$ is indeed **Set**-based.

Now if Σ is to correspond to a model in this way, then it must satisfy certain closure properties, e.g. $\bigwedge \Delta \in \Sigma$ iff $\Delta \subseteq \Sigma$; if $(\Delta \supset \psi) \in \mathbb{T}^=$ and $\Delta \subseteq \Sigma$ then $\psi \in \Sigma$; if $\Sigma \vdash \theta$ then $\theta \in \Sigma$, and so on. Reflection on the desired properties of $\mathfrak{A}$ tells us exactly what properties Σ must have. The procedure then is to work through an enumeration of the formulae of the language, deciding of each formula in turn whether or not to add it into Σ, in such a way that the end result is as desired. In trying to do this we discover what rules of inference our axiom system needs to admit. If these rules are in turn truth-preserving, so that the Soundness Theorem is fulfilled, then the whole procedure becomes viable, and actually gives a systematic technique for constructing an axiomatisation of the class of "true" or "valid" formulae determined by a given notion of "model".

The reader will find a construction and proof of this Henkin type in almost any standard text on mathematical logic. A significant point for us to note about the method here is that it is entirely independent of category theory.

It follows from Classical Completeness that in order for $\mathbb{T}$ to have a set-theoretic model at all, it is sufficient that $\mathbb{T}$ be *consistent*, which means

that $\mathbb{T} \not\vdash \bot$. But since our proof theory is finitary, i.e. proof sequences are of finite length, it is the case in general that if $\mathbb{T} \vdash \theta$, then $\mathbb{T}_0 \vdash \theta$ for some finite subset $\mathbb{T}_0$ of $\mathbb{T}$. Consequently, in order for $\mathbb{T}$ to be consistent, and therefore have a **Set**-model, it suffices that each finite subset of $\mathbb{T}$ be consistent. Since Soundness implies that any theory having a model must be consistent, we have the following fundamental feature of finitary logic.

COMPACTNESS THEOREM. *If every finite subset of* $\mathbb{T}$ *has a* **Set**-*model, so too does* $\mathbb{T}$. ☐

Proof of Deligne's Theorem

We now apply Classical Completeness to the theory of a finitary site **C** to show that the coherent topos **Sh(C)** has enough points. For this we use the criterion stated as Theorem 16.2.5.

Let $C = \{f_x : x \in X\}$ be a set of **C**-arrows with the same codomain c, such that $C \notin Cov(c)$. We need to construct a continuous morphism $p : \mathbf{C} \to \mathbf{Set}$, i.e. by Theorem 16.3.2 a **Set**-model of the theory $\mathbb{T}_{\mathbf{C}}$, such that $p(C)$ is not epimorphic in **Set**.

Now let $\mathscr{L} = \mathscr{L}_{\mathbf{C}} \cup \{\mathbf{c}\}$, where $\mathscr{L}_{\mathbf{C}}$ is the language of the category underlying **C**, and $\mathbf{c}$ is a new individual constant of sort c. Let v be a variable of sort c, and for each $x \in X$ let v_x be a variable of sort dom f_x. Let $\varphi_x(v)$ be the $\mathscr{L}$-formula $\exists v_x(f_x(v_x) \approx v)$, and let $\varphi_x(\mathbf{c})$ be $\exists v_x(f_x(v_x) \approx \mathbf{c})$. Put

$$\mathbb{T} = \mathbb{T}_{\mathbf{C}} \cup \{\sim\varphi_x(\mathbf{c}): x \in X\}.$$

Then $\mathbb{T}$ is geometric, and is a set in view of the smallness of the site **C**. We will show below that $\mathbb{T}$ is consistent, and therefore by Classical Completeness that there is a set-theoretic model $\mathfrak{A}_C : \mathscr{L} \to \mathbf{Set}$ such that $\mathfrak{A}_C \vDash \mathbb{T}$. But then $\mathfrak{A}_C$ is a model of $\mathbb{T}_{\mathbf{C}}$, and so determines a continuous morphism $\mathbf{C} \to \mathbf{Set}$, and hence a point of **Sh(C)**. If A is the set $\mathfrak{A}_C(c)$, then $\mathfrak{A}_C$ interprets the constant $\mathbf{c}$ as an element $a \in A$, and interprets each $\mathscr{L}_{\mathbf{C}}$-operation-symbol f_x as a function $g_x : \mathfrak{A}_C(\text{dom } f_x) \to A$. But for each $x \in X$,

$$\mathfrak{A}_C \vDash \sim\exists v_x(f_x(v_x) \approx \mathbf{c}),$$

which means that $a \notin \text{Im } g_x$, Thus $A \neq \bigcup_x \text{Im } g_x$, showing that the family $\mathfrak{A}_C(C) = \{g_x : x \in X\}$ is not epimorphic in **Set**, as desired.

To prove that $\mathbb{T}$ is consistent, it is enough to show that each finite subset of $\mathbb{T}$ is consistent. Hence it is enough to show $\mathbb{T}_0 = \mathbb{T}_{\mathbf{C}} \cup \{\sim\varphi_x(\mathbf{c}): x \in X_0\}$ is consistent for any finite $X_0 \subseteq X$. To this end, let

$E_\mathbf{C}$ be the canonical functor $\mathbf{C} \to \mathbf{Sh}(\mathbf{C})$. Then $E_\mathbf{C}$ is a continuous morphism, and so (16.3.2) serves as an $\mathscr{L}_\mathbf{C}$-model in $\mathbf{Sh}(\mathbf{C})$ of $\mathbb{T}_\mathbf{C}$. Moreover $E_\mathbf{C}$ reflects covers, so that $E_\mathbf{C}(C)$ is not epimorphic in $\mathbf{Sh}(\mathbf{C})$. Thus $\bigcup \{\mathrm{im}\, E_\mathbf{C}(f_x) : x \in X\}$ is not the maximum subobject of $E_\mathbf{C}(c)$, and hence $\bigcup \{\mathrm{im}\, E_\mathbf{C}(f_x) : x \in X_0\}$ is not the maximum either. But $E_\mathbf{C}$, as a model, interprets f_x, as a symbol, as the arrow $E_\mathbf{C}(f_x)$, and so

$$E_\mathbf{C} \not\models \bigvee \{\varphi_x(v) : x \in X_0\}.$$

As $E_\mathbf{C} \models \mathbb{T}_\mathbf{C}$, it follows by Soundness that $\mathbb{T}_\mathbf{C} \not\vdash \bigvee \{\varphi_x(v) : x \in X_0\}$. Hence by Classical Completeness it follows that there is a model $\mathfrak{A} : \mathscr{L}_\mathbf{C} \to \mathbf{Set}$ such that $\mathfrak{A} \models \mathbb{T}_\mathbf{C}$ and

$$\mathfrak{A} \not\models \bigvee \{\varphi_x(v) : x \in X_0\}.$$

But this means that $\bigcup \{\mathrm{Im}\, \mathfrak{A}(f_x) : x \in X_0\} \neq \mathfrak{A}(c)$, so that there is an element $a \in \mathfrak{A}(c)$ such that $a \notin \mathrm{Im}\, \mathfrak{A}(f_x)$, for all $x \in X_0$. Then defining $\mathfrak{A}(\mathbf{c}) = a$ allows us to extend $\mathfrak{A}$ to become an $\mathscr{L}$-model in which all members of $\mathbb{T}_0$ are true. But if $\mathbb{T}_0$ has a model it must, by Soundness, be consistent as desired.

This finishes the proof that $\mathfrak{A}_C$ exists and has the required property that $\mathfrak{A}_C(C)$ is not epimorphic in $\mathbf{Set}$. If we define P to be the set of such models $\mathfrak{A}_C$ for all sets C of $\mathbf{C}$-arrows that are not covers, then by Theorem 16.2.5, P is a sufficient set of points for $\mathbf{Sh}(\mathbf{C})$, and Deligne's Theorem is proved. $\qquad\square$

Infinitary generalisation

In defining the theory $\mathbb{T}_\mathbf{C}$ we noted that the finiteness of first-order formulae restricted us to finitary sites, and that if we wanted to treat the general case we would have to be able to form disjunctions of infinite sets of formulae. There is no technical obstacle to doing this. We add to the inductive rules for generating formulae the condition that for any set Γ of formulae that has altogether finitely many free variables occurring in its members, there is a formula $\bigvee \Gamma$ with the set-theoretic semantics (§§11.3, 11.4).

$$\mathfrak{A} \models \bigvee \Gamma[x_1, \ldots, x_m] \quad \text{iff} \quad \text{for some } \varphi \in \Gamma, \mathfrak{A} \models \varphi[x_1, \ldots, x_m].$$

In any topos in which $\mathrm{Sub}(d)$ is always a complete lattice (which includes any Grothendieck topos), we can interpret infinitary disjunction by

$$\mathfrak{A}^v(\bigvee \Gamma) = \bigcup \{\mathfrak{A}^v(\varphi) : \varphi \in \Gamma\}.$$

We denote by $\mathscr{L}^\infty$ the class (in fact proper class) of infinitary formulae generated from $\mathscr{L}$ by allowing formation of $\bigvee \Gamma$ for sets Γ as above (cf. Barwise [75], §III.1 for a careful presentation of the syntax of infinitary formulae). $\mathscr{L}^{g\infty}$ denotes the positive-existential members of $\mathscr{L}^\infty$, i.e. those with no occurrence of the symbols $\forall$, $\supset$, $\sim$. The definition of derivability for $\mathscr{L}^\infty$ can no longer be given in terms of proof sequences of finite length, since even though a sequent $\Gamma \supset \varphi$ will continue to have Γ as a finite set of formulae, members of Γ can themselves have infinitely many subformulae, and so an instance of the rule $R \bigvee_2$ may well involve infinitely many premisses. Thus we will now stipulate that the relation $\mathbb{T} \vdash \theta$ holds when θ belongs to the smallest collection of formulae that contains $\mathbb{T}$ as well as all axioms and is closed under the rules of inference of the system GL. In other words, $\{\theta : \mathbb{T} \vdash \theta\}$ is the intersection of all collections that have these properties. (For finitary logic, this definition is equivalent to the one given in terms of finite proof sequences). For any $\mathbb{T}$-model $\mathfrak{A}$ in a topos, $\{\theta : \mathfrak{A} \vDash \theta\}$ is such a collection, and so contains all θ such that $\mathbb{T} \vdash \theta$. Hence we retain the Soundness Theorem.

If $\mathbf{C}$ is any small site, then by extending the definition of $\mathbb{T}_\mathbf{C}$ to include $\bigvee \{\exists v_x (f_x(v_x) \approx v) : x \in X\}$ for any cover $\{f_x : x \in X\}$ in $\mathbf{C}$, we obtain the theory of $\mathbf{C}$ as a set of $\mathscr{L}_\mathbf{C}^\infty$-formulae. However we cannot now use $\mathbb{T}_\mathbf{C}$ in the way we did for Deligne's Theorem. Infinitary formulae do not enjoy the properties in **Set** that finitary ones do. To see this, let φ be the formula $\bigvee \{v \approx \mathbf{c}_n : n \in \omega\}$, where $\mathbf{c}_0, \mathbf{c}_1, \ldots$ is an infinite list of distinct individual constants. If $\mathbf{d}$ is a constant distinct from all the $\mathbf{c}_n$'s, then $\{\varphi\} \cup \{\sim(\mathbf{d} \approx \mathbf{c}_n) : n \in \omega\}$ cannot have a **Set**-model, even though each of its finite subsets does. Thus the Compactness Theorem fails for infinitary logic. Moreover, the Completeness Theorem no longer holds. It can be shown ([MR], p. 162) that if we admit disjunctions of *countable* sets Γ only, then a countable set of geometric sequents has a **Set**-model if it is consistent with respect to GL. On the other hand there exist uncountable sets of infinitary formulae that are consistent but have no **Set**-model at all (cf. Scott [65] for an example).

It was shown by Mansfield [72] that an Infinitary Completeness Theorem can be obtained if we replace standard set-theoretic models by **B**-valued models, for complete Boolean algebras **B**. Such a model interprets a formula $\varphi(v_1, \ldots, v_n)$ as a function of the form $A^n \to \mathbf{B}$, where A is the set of individuals of the model. This is very similar to the notion of model in the topos Ω-**Set** (for $\Omega = \mathbf{B}$) outlined at the end of §11.9.

Makkai and Reyes have adapted Mansfield's approach to their axioms for many-sorted geometric logic without existence assumptions. They

show that for any set $\mathbb{T}$ of geometric sequents there is a complete **BA B**$_{\mathbb{T}}$ and a model $\mathfrak{A}_{\mathbb{T}}$ of $\mathbb{T}$ in the Grothedieck topos **Sh(B**$_{\mathbb{T}}$**)** of sheaves over **B**$_{\mathbb{T}}$ such that

$$\mathbb{T} \vdash \theta \quad \text{iff} \quad \mathfrak{A}_{\mathbb{T}} \vDash \theta$$

for "suitable" θ (see below).

It could be held that the **B**-valued approach recovers Completeness by generalising the notion of "model". But from the point of view of categorial logic one could say that the notion of model is invariant, in that the definition of $\mathscr{E}$-model is the same for all topoi $\mathscr{E}$, including $\mathscr{E} = $ **Set**, but that in order to obtain Completeness we have to allow the category in which the model lives to change as we change the theory $\mathbb{T}$.

Let us now sketch the definition of $\mathfrak{A}_{\mathbb{T}}$, and see how it can be used to prove Barr's Theorem. Given a geometric theory $\mathbb{T}$ in $\mathscr{L}^{\infty}$, i.e. a set $\mathbb{T} \subseteq \mathscr{L}^{g\infty}$, let L be any sub*set* of $\mathscr{L}^{g\infty}$ that contains $\mathbb{T}$ and is closed under (i) subformulae, and (ii) substitution for free variables of terms whose variables all occur in L. A subclass of $\mathscr{L}^{g\infty}$ satisfying (i) and (ii) is called a *fragment*. Since $\mathbb{T}$ is small, there do in fact exist small fragments containing $\mathbb{T}$. Let P be the collection of all sequents $\Gamma \supset \varphi$ of formulae from L such that $\mathbb{T} \nvdash \Gamma \supset \varphi$. If $p = (\Gamma \supset \varphi)$ is in P, we write Γ_p for Γ and φ_p for φ. Then a partial ordering on P is given by

$$p \sqsubseteq q \quad \text{iff} \quad \Gamma_p \subseteq \Gamma_q \quad \text{and} \quad \varphi_p = \varphi_q.$$

The Boolean algebra we want is obtained by applying double negation to the **CHA P**$^+$ of hereditary subsets of $\mathbf{P} = (P, \sqsubseteq)$ (cf. §8.4). For each $S \in \mathbf{P}^+$, let S^* be $\neg\neg S$. Then **B**$_{\mathbb{T}}$ is $\{S^* : S \in \mathbf{P}^+\}$, the lattice of *regular* elements of **P**$^+$, and in general is a complete **BA** in which $\sqcap$ is set-theoretic intersection, $\bigsqcup X$ is $(\bigcup X)^*$, and $\neg$ is the Boolean complement (cf. Rasiowa and Sikorski [63], §IV.6). Since L is small, **B**$_{\mathbb{T}}$ is too, and **Sh(B**$_{\mathbb{T}}$**)** is a Grothendieck topos.

For each formula $\varphi \in L$, put

$$P(\varphi) = \{p \in P : \varphi \in \Gamma_p\},$$

and for each term t occurring in L, put

$$P(t) = \{p \in P : \text{every variable of } t \text{ occurs free in } P\}.$$

Then $P(\varphi)$ and $P(t)$ are hereditary in P. Let $[\![t]\!] = P(t)^*$.

If $\mathfrak{A}_{\mathbb{T}}(a)$ denotes the set of L-terms of sort a, then a **B**$_{\mathbb{T}}$-valued equality relation on $\mathfrak{A}_{\mathbb{T}}(a)$ is given by

$$[\![t \approx u]\!] = [\![t]\!] \cap [\![u]\!] \cap P(t \approx u)^*.$$

This makes $\mathfrak{A}_{\mathbb{T}}(a)$ into an object in the topos $\mathbf{B}_{\mathbb{T}}$-**Set** of $\mathbf{B}_{\mathbb{T}}$-valued sets (§11.9).

Similarly, for an n-placed relation symbol $\mathbf{R}\colon \langle a_1, \ldots, a_n \rangle$ of L, $\mathfrak{A}_{\mathbb{T}}(\mathbf{R})$ is defined as that function $\mathfrak{A}_{\mathbb{T}}(a_1) \times \cdots \times_{\mathbb{T}} (a_n) \to \mathbf{B}_{\mathbb{T}}$ given by

$$\mathfrak{A}_{\mathbb{T}}(\mathbf{R})\langle t_1, \ldots, t_n \rangle = [\![t_1]\!] \cap \ldots \cap [\![t_n]\!] \cap P(\mathbf{R}(t_1, \ldots, t_n))^*.$$

If $\mathbf{g}\colon \langle a_1, \ldots, a_n \rangle \to a_{n+1}$ is an n-placed operation symbol, then according to the definition of arrows and products in $\mathbf{B}_{\mathbb{T}}$-**Set** of §11.9, $\mathfrak{A}_{\mathbb{T}}(\mathbf{g})$ is to be a function from $\mathfrak{A}_{\mathbb{T}}(a_1) \times \cdots \times \mathfrak{A}_{\mathbb{T}}(a_{n+1})$ to $\mathbf{B}_{\mathbb{T}}$. The definition is

$$\mathfrak{A}_{\mathbb{T}}(\mathbf{g})\langle t_1, \ldots, t_n, u \rangle = [\![t_1]\!] \cap \ldots \cap [\![t_n]\!] \cap [\![\mathbf{g}(t_1, \ldots, t_n) \approx u]\!].$$

This construction defines a $\mathbb{T}$-model in $\mathbf{B}_{\mathbb{T}}$-**Set** such that for any sequent θ of formulae in L,

$$\mathfrak{A}_{\mathbb{T}} \vDash \theta \quad \text{iff} \quad \mathbb{T} \vdash \theta$$

([MR], §§4.1, 4.2, 5.1, 5.2). But from the work of Denis Higgs referred to in §14.7 we know that there is an equivalence between $\mathbf{B}$-**Set** and $\mathbf{Sh}(\mathbf{B})$, and this allows us to realise $\mathfrak{A}_{\mathbb{T}}$ as a model in $\mathbf{Sh}(\mathbf{B}_{\mathbb{T}})$ as desired.

With regard to the Theorem of Barr, the construction is applied to the case that $\mathbb{T}$ is the theory $\mathbb{T}_{\mathbf{C}}$ of a small site $\mathbf{C}$ to give a model $\mathfrak{A}_{\mathbb{T}_{\mathbf{C}}}\colon \mathscr{L}_{\mathbf{C}} \to \mathbf{Sh}(\mathbf{B}_{\mathbb{T}_{\mathbf{C}}})$ such that $\mathfrak{A}_{\mathbb{T}_{\mathbf{C}}} \vDash \mathbb{T}_{\mathbf{C}}$. Then $\mathfrak{A}_{\mathbb{T}_{\mathbf{C}}}$ is a continuous morphism from $\mathbf{C}$ to $\mathbf{Sh}(\mathbf{B}_{\mathbb{T}_{\mathbf{C}}})$. We may choose our small fragment L of $\mathscr{L}_{\mathbf{C}}^{\mathrm{g\check{\infty}}}$ to include the formula $\mathbf{cov}(C)$, i.e. $\bigvee \{ \exists v_x(f_x(v_x) \approx v)\colon x \in X \}$ for every set $C = \{ f_x\colon x \in X \}$ of $\mathbf{C}$-arrows with a common codomain. Then if $\{ \mathfrak{A}_{\mathbb{T}_{\mathbf{C}}}(f_x)\colon x \in X \}$ is epimorphic in $\mathbf{Sh}(\mathbf{B}_{\mathbb{T}_{\mathbf{C}}})$, we have $\mathfrak{A}_{\mathbb{T}_{\mathbf{C}}} \vDash \mathbf{cov}(C)$, and so $\mathbb{T}_{\mathbf{C}} \vdash \mathbf{cov}(C)$ by the above construction. Since $E_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{Sh}(\mathbf{C})$ is a model of $\mathbb{T}_{\mathbf{C}}$ in $\mathbf{Sh}(\mathbf{C})$, Soundness then implies that $E_{\mathbf{C}} \vDash \mathbf{cov}(C)$, which means that $\{ E_{\mathbf{C}}(f_x)\colon x \in X \}$ is epimorphic in $\mathbf{Sh}(\mathbf{C})$. By Theorem 16.2.3, it follows that the geometric morphism $\mathbf{Sh}(\mathbf{B}_{\mathbb{T}_{\mathbf{C}}}) \to \mathbf{Sh}(\mathbf{C})$ determined by $\mathfrak{A}_{\mathbb{T}_{\mathbf{C}}}$ is surjective.

16.5. Theories as sites

To derive Deligne's Theorem from Classical Completeness, a theory $\mathbb{T}_{\mathbf{C}}$ was associated with each finitary site $\mathbf{C}$. To make the converse derivation, the association will be reversed. Given a geometric theory $\mathbb{T}$ of finitary formulae, a site $\mathbf{C}_{\mathbb{T}}$ will be constructed such that models of $\mathbb{T}$ in a Grothendieck topos $\mathscr{E}$ correspond to continuous morphisms $\mathbf{C}_{\mathbb{T}} \to \mathscr{E}$. In particular, the canonical functor $E_{\mathbf{C}_{\mathbb{T}}}\colon \mathbf{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathbf{C}_{\mathbb{T}})$ becomes a $\mathbb{T}$-model

in $\mathbf{Sh}(\mathbf{C}_\mathbb{T})$ satisfying

$$E_{\mathbf{C}_\mathbb{T}} \vDash \varphi \quad \text{iff} \quad \mathbb{T} \vdash \varphi.$$

Application of Deligne's Theorem to $\mathbf{Sh}(\mathbf{C}_\mathbb{T})$ then yields Classical Completeness for $\mathbb{T}$.

The construction of $\mathbf{C}_\mathbb{T}$ is an elegant development of the "Lindenbaum algebra" notion outlined for propositional logic in §§6.5 and 8.3. To present the construction we will work from now within the class $\mathcal{L}^g$ of positive-existential finitary $\mathcal{L}$-formulae, where $\mathcal{L}$ is the language of the geometric theory $\mathbb{T}$.

Two $\mathcal{L}^g$-formulae φ and ψ will be called *provably equivalent* relative to $\mathbb{T}$ in GL if $\mathbb{T} \vdash \varphi \supset \psi$ and $\mathbb{T} \vdash \psi \supset \varphi$. This defines an equivalence relation on $\mathcal{L}^g$, for which the equivalence class of φ will be denoted $[\varphi]$. In the Lindenbaum algebra, equivalence classes are partially ordered by putting

$$[\varphi] \sqsubseteq [\psi] \quad \text{iff} \quad \mathbb{T} \vdash \varphi \supset \psi,$$

but in the case of $\mathbf{C}_\mathbb{T}$, these equivalence classes are going to be *arrows*, rather than objects. The objects, on the other hand, are to be classes of formulae under the equivalence relation determined by "changes of variables". To define this relation, consider two variable-sequences $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$ and $\mathbf{v}' = \langle v'_1, \ldots, v'_m \rangle$ of the same length, with each v_i of the same sort as the corresponding v'_i. Let $\mathbf{v} \to \mathbf{v}'$ denote the function which associates v'_i with v_i. Acting on a formula $\varphi(\mathbf{v})$, $\mathbf{v} \to \mathbf{v}'$ produces the formula, denoted $\varphi(\mathbf{v}/\mathbf{v}')$ or simply $\varphi(\mathbf{v}')$, which is obtained by replacing every free occurrence of v_i in φ by v'_i. The *change of variables* $\mathbf{v} \to \mathbf{v}'$ is *acceptable* for $\varphi(\mathbf{v})$ if each v_i is free for v'_i in φ (cf. §11.3). An equivalence relation is then given by putting $\varphi \sim \psi$ iff ψ is the result of applying some acceptable change of variables to φ. The class $\{\psi : \varphi \sim \psi\}$ will be denoted $\{\varphi\}$. These classes are the objects of a category $\mathscr{C}_\mathbb{T}$. In dealing with these objects, it is useful to know that $\{\varphi\} \subseteq [\varphi]$, i.e. that if $\varphi \sim \psi$ then $\mathbb{T} \vdash \varphi \supset \psi$ and $\mathbb{T} \vdash \psi \supset \varphi$. Hence, by Soundness, if $\mathfrak{A} \vDash \mathbb{T}$ and $\varphi \sim \psi$, then $\mathfrak{A}(\varphi) = \mathfrak{A}(\psi)$.

The $\mathscr{C}_\mathbb{T}$-arrows from $\{\varphi(\mathbf{v})\}$ to $\{\psi(\mathbf{w})\}$ are the provable equivalence classes $[\alpha(\mathbf{v}', \mathbf{w}')]$ of formulae $\alpha(\mathbf{v}', \mathbf{w}')$, where $\mathbf{v}'$ and $\mathbf{w}'$ are disjoint sequences of variables having $\mathbf{v} \to \mathbf{v}'$ acceptable for $\varphi(\mathbf{v})$ and $\mathbf{w} \to \mathbf{w}'$ acceptable for $\psi(\mathbf{w})$, such that the following three geometric formulae are derivable from $\mathbb{T}$:

($\alpha 1$) $\alpha(\mathbf{v}', \mathbf{w}') \supset \varphi(\mathbf{v}') \vee \psi(\mathbf{w}')$,

($\alpha 2$) $\varphi(\mathbf{v}') \supset \exists \mathbf{w}' \alpha(\mathbf{v}', \mathbf{w}')$,

($\alpha 3$) $\alpha(\mathbf{v}', \mathbf{w}') \wedge \alpha(\mathbf{v}', \mathbf{w}'') \supset \mathbf{w}' \approx \mathbf{w}''$.

The notation being used here has $\exists\mathbf{v}\varphi$ abbreviating $\exists v_1 .\,. \exists v_m\varphi$, and $(\mathbf{v} \approx \mathbf{v}')$ abbreviating $(v_1 \approx v_1') \wedge .\,. \wedge (v_m \approx v_m')$, and so on. Note that since there are infinitely many variables of each sort, any two objects $\{\varphi(\mathbf{v})\}$ and $\{\psi(\mathbf{w})\}$ can be represented, by suitable changes of variables, in such a way that $\mathbf{v}$ and $\mathbf{w}$ are disjoint sequences, so that arrows can be taken in the form $[\alpha(\mathbf{v}, \mathbf{w})]$. This kind of relettering can be extended to all the objects and arrows of a finite diagram, and we will sometimes assume this has already been done in what follows.

To understand the definition of $\mathscr{C}_{\mathsf{T}}$-arrow, observe that in a **Set**-model $\mathfrak{A}$, $\mathfrak{A}(\alpha)$ will be the graph of a function from $\mathfrak{A}(\varphi)$ to $\mathfrak{A}(\psi)$. This interpretation motivates much of the structure of $\mathscr{C}_{\mathsf{T}}$. A formalisation of the interpretation is given in the next exercise, which we will make use of later on.

EXERCISE 1. Let $\mathfrak{A} : \mathscr{L} \to \mathscr{E}$ be a model in a topos, and let $\varphi(\mathbf{v})$, $\psi(\mathbf{w})$, $\alpha(\mathbf{v}, \mathbf{w})$ be formulae such that the formulae

$(\alpha 1)$ $\alpha(\mathbf{v}, \mathbf{w}) \supset \varphi(\mathbf{v}) \wedge \psi(\mathbf{w})$,

$(\alpha 2)$ $\varphi(\mathbf{v}) \supset \exists\mathbf{w}\alpha(\mathbf{v}, \mathbf{w})$,

$(\alpha 3)$ $\alpha(\mathbf{v}, \mathbf{w}) \wedge \alpha(\mathbf{v}, \mathbf{w}') \supset \mathbf{w} \approx \mathbf{w}'$

are true in $\mathfrak{A}$. Assume that $\mathbf{v}$ and $\mathbf{w}$ are disjoint, so that if $\mathbf{z}$ is the sequence $\mathbf{v}, \mathbf{w}$, then $\mathfrak{A}(\mathbf{z})$ can be identified with $\mathfrak{A}(\mathbf{v}) \times \mathfrak{A}(\mathbf{w})$.

(1) Show that the product arrow $h : \mathfrak{A}^{\mathbf{v}}(\varphi) \times \mathfrak{A}^{\mathbf{w}}(\psi) \to \mathfrak{A}(\mathbf{z})$ is monic, and so determines a subobject of $\mathfrak{A}(\mathbf{z})$.

(2) Use the $\mathfrak{A}$-truth of $(\alpha 1)$ to show that there is a monic $k : \mathfrak{A}^{\mathbf{z}}(\alpha) \rightarrowtail \mathfrak{A}^{\mathbf{v}}(\varphi) \times \mathfrak{A}^{\mathbf{w}}(\psi)$ factoring $\mathfrak{A}^{\mathbf{z}}(\alpha) \rightarrowtail \mathfrak{A}(\mathbf{z})$ through h.

(3) Let g be $pr \circ k$, where

$$\mathfrak{A}^{\mathbf{z}}(\alpha) \rightarrowtail \mathfrak{A}^{\mathbf{v}}(\varphi) \times \mathfrak{A}^{\mathbf{w}}(\psi)$$
$$\searrow{\scriptstyle g} \qquad \downarrow{\scriptstyle pr}$$
$$\mathfrak{A}^{\mathbf{v}}(\varphi)$$

pr is the projection. Use the truth of $(\alpha 2)$ and $(\alpha 3)$ to deduce that g is iso in $\mathscr{E}$.

(4) Using g^{-1}, construct an arrow $f_\alpha : \mathfrak{A}^{\mathbf{v}}(\varphi) \to \mathfrak{A}^{\mathbf{w}}(\psi)$ such that $\langle \mathbf{1}, f_\alpha \rangle : \mathfrak{A}^{\mathbf{v}}(\varphi) \rightarrowtail \mathfrak{A}^{\mathbf{v}}(\varphi) \times \mathfrak{A}^{\mathbf{w}}(\psi)$ and $\mathfrak{A}^{\mathbf{z}}(\alpha) \rightarrowtail \mathfrak{A}(\mathbf{z})$ are equal as subobjects of $\mathfrak{A}(\mathbf{z})$.

(5) Hence show that there is a *unique* arrow $\mathfrak{A}^{\mathbf{v}}(\varphi) \to \mathfrak{A}^{\mathbf{w}}(\psi)$ whose "graph" is $\mathfrak{A}^{\mathbf{z}}(\alpha) \rightarrowtail \mathfrak{A}(\mathbf{z})$. $\square$

To specify the structure of $\mathscr{C}_{\mathbb{T}}$ as a category, the identity arrow on $\{\varphi(\mathbf{v})\}$ is defined to be $[\varphi(\mathbf{v}) \wedge (\mathbf{v} \approx \mathbf{v}')] : \{\varphi(\mathbf{v})\} \to \{\varphi(\mathbf{v}')\}$. The composite of $[\alpha(\mathbf{v}, \mathbf{w})] : \{\varphi(\mathbf{v})\} \to \{\psi(\mathbf{w})\}$, and $[\beta(\mathbf{w}, \mathbf{z})] : \{\psi(\mathbf{w})\} \to \{\chi(\mathbf{z})\}$ is given by $[\exists \mathbf{w}(\alpha(\mathbf{v}, \mathbf{w}) \wedge \beta(\mathbf{w}, \mathbf{z}))]$ (assuming that $\mathbf{v}, \mathbf{w}$, and $\mathbf{z}$ have been chosen to be mutually disjoint).

EXERCISE 2. Verify the category axioms for $\mathscr{C}_{\mathbb{T}}$.

EXERCISE 3. Show that for any formula φ, $[\varphi]$ is the one and only arrow from $\{\varphi\}$ to $\{\top\}$.

EXERCISE 4. Show that $\{\varphi(\mathbf{v})\}$ and $\{\psi(\mathbf{w})\}$ have product object $\{\varphi(\mathbf{v}) \wedge \psi(\mathbf{w})\}$ with the projection to $\{\varphi(\mathbf{v}')\}$ being $[\varphi(\mathbf{v}) \wedge \psi(\mathbf{w}) \wedge (\mathbf{v} \approx \mathbf{v}')]$, and similarly for the projection to $\{\psi(\mathbf{w}')\}$.

EXERCISE 5. Show that arrows $[\alpha_i(\mathbf{v}_i, \mathbf{z})] : \{\varphi_i(\mathbf{v}_i)\} \to \{\chi(\mathbf{z})\}$, for $i = 1, 2$, have a pullback whose domain is $\{\exists \mathbf{z}(\alpha_1(\mathbf{v}_1, \mathbf{z}) \wedge \alpha_2(\mathbf{v}_2, \mathbf{z}))\}$.

EXERCISE 6. Show that $[\alpha(\mathbf{v}, \mathbf{w})]$ is monic iff $\mathbb{T} \vdash \alpha(\mathbf{v}, \mathbf{w}) \wedge \alpha(\mathbf{v}', \mathbf{w}) \supset \mathbf{v} \approx \mathbf{v}'$. $\qquad\square$

From Exercises 3 and 4 it follows that $\mathscr{C}_{\mathbb{T}}$ has a terminal object and pullbacks, and therefore has all finite limits. Note, by Exercise 4, that if $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$, then $\{\mathbf{v} \approx \mathbf{v}\}$, i.e. $\{(v_1 \approx v_1) \wedge \ldots \wedge (v_m \approx v_m)\}$, is the product of the objects $\{v_i \approx v_i\}$. If $\mathbf{v}$ is a sequence appropriate to φ, we write $\varphi^{\mathbf{v}}$ for the formula $\varphi \wedge (\mathbf{v} \approx \mathbf{v})$. We then have a subobject $\{\varphi^{\mathbf{v}}\} \rightarrowtail \{\mathbf{v} \approx \mathbf{v}\}$ of $\{\mathbf{v} \approx \mathbf{v}\}$ given by $[\varphi \wedge (\mathbf{v} \approx \mathbf{v}')]$. This subobject may be denoted simply as $\{\varphi^{\mathbf{v}}\}$, and whenever it is presented without naming the arrow, it will be the arrow just mentioned that is intended.

EXERCISE 7. Show that if $\mathbf{v}$ is appropriate to $(t_1 \approx t_2)$, and w is a variable of the same sort as t_1 and t_2, then $\{(t_1 \approx t_2)^{\mathbf{v}}\} \rightarrowtail \{\mathbf{v} \approx \mathbf{v}\}$ equalises the arrows $[(t_i \approx w) \wedge (\mathbf{v} \approx \mathbf{v})] : \{\mathbf{v} \approx \mathbf{v}\} \to \{w \approx w\}$, for $i = 1, 2$.

EXERCISE 8. Show that $\{(\varphi_1 \wedge \varphi_2)^{\mathbf{v}}\}$ is $\{\varphi_1^{\mathbf{v}}\} \cap \{\varphi_2^{\mathbf{v}}\}$. $\qquad\square$

To make $\mathscr{C}_{\mathbb{T}}$ into a site, a *finite* set $C = \{[\alpha_x(\mathbf{v}_x, \mathbf{w})] : x \in X\}$ of arrows, where $[\alpha_x]$ is of the form $\{\varphi_x(\mathbf{v}_x)\} \to \{\psi(\mathbf{w})\}$, is defined to be *provably epimorphic* if

$$\mathbb{T} \vdash \psi(\mathbf{w}) \to \bigvee \{\exists \mathbf{v}_x \alpha_x(\mathbf{v}_x, \mathbf{w}) : x \in X\}.$$

In particular, if C is the empty set, this means that $\mathbb{T} \vdash (\psi(\mathbf{w}) \supset \bot)$, i.e. $\mathbb{T} \vdash \sim\psi(\mathbf{w})$.

The reader may care to verify that the finite provably epimorphic families form a pretopology on $\mathscr{C}_{\mathbb{T}}$, and this gives us a *finitary* site $\mathbf{C}_{\mathbb{T}}$, and hence a *coherent* topos $\mathbf{Sh}(\mathbf{C}_{\mathbb{T}})$.

EXERCISE 9. Suppose that $\mathbf{v}$ is appropriate to $\varphi_1 \vee \varphi_2$. For $i = 1, 2$, let $f_i : \{\varphi_i^{\mathbf{v}}\} \to \{(\varphi_1 \vee \varphi_2)^{\mathbf{v}}\}$ be $[\varphi_i \wedge (\mathbf{v} \approx \mathbf{v}')]$. Show that $\{f_1, f_2\}$ is provably epimorphic. Hence show that if $F : \mathbf{C}_{\mathbb{T}} \to \mathscr{E}$ is a continuous morphism, where $\mathscr{E}$ is a Gorthendieck topos with its canonical pretopology, then $F(\{(\varphi_1 \vee \varphi_2)^{\mathbf{v}}\})$ is $F(\{\varphi_1^{\mathbf{v}}\}) \cup F(\{\varphi_2^{\mathbf{v}}\})$.

EXERCISE 10. Let $\mathbf{v}$ be appropriate to the formula $\exists w\varphi$, with w a variable not occurring in $\mathbf{v}$, and let $\mathbf{z}$ be the sequence $\mathbf{v}, w$. Let g_φ be the arrow $[\varphi^{\mathbf{z}} \wedge (\mathbf{v} \approx \mathbf{v}')] : \{\varphi^{\mathbf{z}}\} \to \{(\exists w\varphi)^{\mathbf{v}}\}$. Show that $\{g_\varphi\}$ is provably epimorphic, and hence that if F is as in Exercise 9, then $F(g_\varphi)$ is an epic arrow in $\mathscr{E}$.

EXERCISE 11. Let $\mathbf{v}$, w, φ, $\mathbf{z}$, and g_φ be as in the last Exercise. Show that the diagram

$$
\begin{array}{ccc}
\{\varphi^{\mathbf{z}}\} & \rightarrowtail & \{\mathbf{z} \approx \mathbf{z}\} \\
{\scriptstyle g_\varphi}\downarrow & & \downarrow{\scriptstyle pr} \\
\{(\exists w\varphi)^{\mathbf{v}}\} & \rightarrowtail & \{\mathbf{v} \approx \mathbf{v}\}
\end{array}
$$

commutes, where pr is the evident projection. Hence show that if $F : \mathbf{C}_{\mathbb{T}} \to \mathscr{E}$ is continuous, as in Exercise 9, then F takes this diagram to an epi-monic factorisation of the F-image of $\{\varphi^{\mathbf{z}}\} \rightarrowtail \{\mathbf{z} \approx \mathbf{z}\} \xrightarrow{pr} \{\mathbf{v} \approx \mathbf{v}\}$. Using the left exactness of F, deduce from this that

$$
F(\{(\exists w\varphi)^{\mathbf{v}}\}) = \exists_{F(pr)}(F\{\varphi^{\mathbf{z}}\}). \qquad \square
$$

These last exercises indicate that if $F : \mathbf{C}_{\mathbb{T}} \to \mathscr{E}$ is a continuous morphism from $\mathbf{C}_{\mathbb{T}}$ to a Grothendieck topos then F preserves some of the structure relevant to the interpretation of formulae. Indeed we can use F to define a $\mathbb{T}$-model $\mathfrak{A}_F : \mathscr{L} \to \mathscr{E}$, where $\mathscr{L}$ is the language of $\mathbb{T}$, as follows.

If a is an $\mathscr{L}$-sort, we choose a variable $v : a$, and put $\mathfrak{A}_F(a) = F(\{v \approx v\})$. Since $\{v \approx v\} = \{v' \approx v'\}$ whenever v' is any other variable of sort a, the definition is unambiguous. If $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$, with $v_i : a_i$, then $\mathfrak{A}_F(\mathbf{v})$ is $F(\{v_1 \approx v_1\}) \times \cdots \times F(\{v_m \approx v_m\})$. But F preserves products, so then $\mathfrak{A}_F(\mathbf{v})$ is $F(\{\mathbf{v} \approx \mathbf{v}\})$. Hence if $\mathbf{g} : \langle a_{i_1}, \ldots, a_{i_n} \rangle \to a$ is an n-placed

operation symbol, we can put $\mathfrak{A}_F(\mathbf{g}) = F(\bar{g})$, where $\bar{g}$ is $[\mathbf{g}(v_{i_1}, \ldots, v_{i_n}) \approx v] : \{\mathbf{v}' \approx \mathbf{v}'\} \rightarrow \{v \approx v\}$, where $v: a$, and $\mathbf{v}' = \langle v_{i_1}, \ldots, v_{i_n} \rangle$. If $\mathbf{c}$ is an individual constant of sort a, we take $\mathfrak{A}_F(\mathbf{c})$ to be $F([\mathbf{c} \approx v] : \{\top\} \rightarrow \{v \approx v\})$. Finally, if $\mathbf{R}: \langle a_{i_1}, \ldots, a_{i_n} \rangle$ is an n-placed relation symbol, we put $\mathfrak{A}_F(\mathbf{R}) = F(\{\mathbf{R}(v_{i_1}, \ldots, v_{i_n})\} \rightarrowtail \{\mathbf{v}' \approx \mathbf{v}'\})$, noting that F preserves monics.

EXERCISE 12. Suppose that $\mathbf{v}$ is appropriate to the term $t: a$, and let $w: a$. Show that $\mathfrak{A}_F^{\mathbf{v}}(t)$ is the F-image of

$$[(t \approx w) \wedge (\mathbf{v} \approx \mathbf{v})] : \{\mathbf{v} \approx \mathbf{v}\} \rightarrow \{w \approx w\}. \qquad \square$$

THEOREM 1. *If φ is in $\mathscr{L}^g$, then for any sequence $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$ appropriate to φ,*

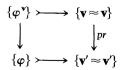$$\mathfrak{A}_F^{\mathbf{v}}(\varphi) = F(\{\varphi^{\mathbf{v}}\})$$

PROOF. By induction on the formation of φ.

(1) If φ is $\top$, $\mathfrak{A}_F^{\mathbf{v}}(\varphi)$ is the maximum subobject of $\mathfrak{A}_F(\mathbf{v})$, i.e. of $F(\{\mathbf{v} \approx \mathbf{v}\})$. But $\{\top^{\mathbf{v}}\} \rightarrowtail \{\mathbf{v} \approx \mathbf{v}\}$ is iso, and F preserves iso's, hence the result holds in this case.

(2) If φ is $\bot$, $\mathfrak{A}_F^{\mathbf{v}}(\varphi)$ is the minimum subobject $0 \rightarrowtail F(\{\mathbf{v} \approx \mathbf{v}\})$. But since $\top \vdash \bot^{\mathbf{v}} \supset \bot$, the empty set is provably epimorphic and covers $\{\bot^{\mathbf{v}}\}$ in $\mathbf{C}_{\top}$. By continuity of F then, the empty set covers $F(\{\bot^{\mathbf{v}}\})$ in $\mathscr{E}$. But canonical covers in $\mathscr{E}$ are effectively epimorphic, and so by Exercise 16.2.16, $F(\{\bot^{\mathbf{v}}\})$ is initial in $\mathscr{E}$, as desired.

(3) If φ is $(t_1 \approx t_2)$, then $\mathfrak{A}_F^{\mathbf{v}}(\varphi)$ equalises $\mathfrak{A}_F^{\mathbf{v}}(t_1)$ and $\mathfrak{A}_F^{\mathbf{v}}(t_2)$. Since F preserves equalisers, the result follows by Exercises 7 and 12.
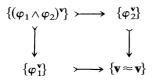
(4) If φ is $\mathbf{R}(v_{i_1}, \ldots, v_{i_n})$, then with $\mathbf{v}' = \langle v_{i_1}, \ldots, v_{i_n} \rangle$, there is a pullback in $\mathscr{C}_{\top}$ of the form

$$
\begin{array}{ccc}
\{\varphi^{\mathbf{v}}\} & \rightarrowtail & \{\mathbf{v} \approx \mathbf{v}\} \\
\downarrow & & \downarrow {\scriptstyle pr} \\
\{\varphi\} & \rightarrowtail & \{\mathbf{v}' \approx \mathbf{v}'\}
\end{array}
$$

But F preserves pullbacks, and the F-image of the bottom arrow is, by definition, $\mathfrak{A}_F(\mathbf{R})$. Hence the F-image of the top arrow is $\mathfrak{A}_F^{\mathbf{v}}(\varphi)$, by definition of the latter.

(5) If φ is $(\varphi_1 \wedge \varphi_2)$, and the result holds for φ_1 and φ_2, then $\mathfrak{A}_F^{\mathbf{v}}(\varphi)$ is

$F(\{\varphi_1^\mathbf{v}\}) \cap F(\{\varphi_2^\mathbf{v}\})$. But by Exercise 8 there is a pullback

$$
\begin{array}{ccc}
\{(\varphi_1 \wedge \varphi_2)^\mathbf{v}\} & \rightarrowtail & \{\varphi_2^\mathbf{v}\} \\
\Big\downarrow & & \Big\downarrow \\
\{\varphi_1^\mathbf{v}\} & \rightarrowtail & \{\mathbf{v} \approx \mathbf{v}\}
\end{array}
$$

in $\mathscr{C}_\mathbb{T}$, and F preserves pullbacks, and so $\mathfrak{A}_F^\mathbf{v}(\varphi)$ is $F(\{(\varphi_1 \wedge \varphi_2)^\mathbf{v}\})$.

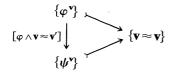(6) If φ is $\varphi_1 \vee \varphi_2$, use Exercise 9 in a similar manner to the previous case.

(7) If φ is $\exists w\psi$, then $\mathfrak{A}_F^\mathbf{v}(\varphi)$ is $\exists_p(\mathfrak{A}_F^\mathbf{z}(\psi))$, where $\mathbf{z}$ is $\mathbf{v}, w$ and $p : \mathfrak{A}_F(\mathbf{z}) \to \mathfrak{A}_F(\mathbf{v})$ is the projection. Hence if the Theorem holds for ψ, $\mathfrak{A}_F^\mathbf{v}(\varphi)$ is $\exists_p(\{\varphi^\mathbf{z}\})$. But by left exactness, p is $F(pr : \{\mathbf{z} \approx \mathbf{z}\} \to \{\mathbf{v} \approx \mathbf{v}\})$, and so the desired conclusion follows by Exercise 11.   □

COROLLARY 2. *For any geometric $\mathscr{L}$-formula θ,*

$$\mathbb{T} \vdash \theta \quad implies \quad \mathfrak{A}_F \vDash \theta.$$

In particular, $\mathfrak{A}_F \vDash \mathbb{T}$.

PROOF. Let θ be $\varphi \supset \psi$, and let $\mathbf{v}$ be the sequence of all variables that have a free occurrence in θ. Then if $\mathbb{T} \vdash \theta$, we have $\mathbb{T} \vdash \varphi \supset \varphi \wedge \psi$ (by the Axiom and rules $R\bigwedge_2$ and $R\mathbb{T}$ of GL). From this it follows readily that

$$
\begin{array}{c}
\{\varphi^\mathbf{v}\} \\
[\varphi \wedge \mathbf{v} \approx \mathbf{v'}] \Big\downarrow \qquad \searrow \qquad \{\mathbf{v} \approx \mathbf{v}\} \\
\{\psi^\mathbf{v}\}
\end{array}
$$

commutes in $\mathscr{C}_\mathbb{T}$, i.e. $\{\varphi^\mathbf{v}\} \subseteq \{\psi^\mathbf{v}\}$. Since F preserves monics and commutative triangles, with the aid of the Theorem we then have $\mathfrak{A}_F^\mathbf{v}(\varphi) \subseteq \mathfrak{A}_F^\mathbf{v}(\psi)$ in $\mathscr{E}$, so that $\mathfrak{A}_F \vDash \varphi \supset \psi$, as desired.

Finally, since $\mathbb{T} \vdash \theta$ whenever $\theta \in \mathbb{T}$ (by the GL Axiom and $R\mathbb{T}$), this makes every member of $\mathbb{T}$ true in $\mathfrak{A}_F$.   □

Now the definition of $\mathfrak{A}_F$ can be applied in the case that F is the canonical continuous morphism $E_{\mathbf{C}_\mathbb{T}} : \mathbf{C}_\mathbb{T} \to \mathbf{Sh}(\mathbf{C}_\mathbb{T})$ to yield a model of $\mathbb{T}$ in $\mathbf{Sh}(\mathbf{C}_\mathbb{T})$, which we will denote $\mathfrak{A}_\mathbb{T}$ (so that the subscripting will not become ridiculous). Moreover, the pretopology defining $\mathbf{C}_\mathbb{T}$ is *precanonical* in the sense of §16.2. For if $C = \{[\alpha_x(\mathbf{v}_x, \mathbf{w})] \ x \in X\}$: is provably epimorphic, and $D = \{[\beta_x(\mathbf{v}_x, \mathbf{z})] : x \in X\}$ is a family that is compatible with

C, in the sense defined prior to Exercise 16.2.15, then

$$g = [\bigvee \{\exists \mathbf{v}_x(\alpha_x(\mathbf{v}_x, \mathbf{w}) \wedge \beta_x(\mathbf{v}_x, \mathbf{z})) \colon x \in X\}]$$

proves to be the one and only arrow

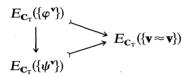$$\begin{array}{ccc} \{\varphi_x(\mathbf{v}_x)\} & \xrightarrow{\ [\beta_x]\ } & \\ {\scriptstyle[\alpha_x]}\Big\downarrow & \searrow & \{\chi(\mathbf{z})\} \\ \{\psi(\mathbf{w})\} & \dashrightarrow{g} & \end{array}$$

that factors each $[\beta_x]$ through the corresponding $[\alpha_x]$ (cf. Johnstone [77], p. 245). Thus each cover in $\mathbf{C}_{\mathbb{T}}$ is an effectively epimorphic family, so that the canonical functor $E_{\mathbf{C}_{\mathbb{T}}}$ is actually the Yoneda embedding, which thereby makes $\mathbf{C}_{\mathbb{T}}$ (isomorphic to) a full subcategory of $\mathbf{Sh}(\mathbf{C}_{\mathbb{T}})$. This allows us to sharpen the last Corollary in the case of $\mathfrak{A}_{\mathbb{T}}$.
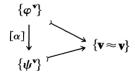
THEOREM 3. *For geometric θ,*

$$\mathbb{T} \vdash \theta \quad \textit{iff} \quad \mathfrak{A}_{\mathbb{T}} \vDash \theta.$$

PROOF. If $\mathfrak{A}_{\mathbb{T}} \vDash \varphi \supset \psi$, where $\varphi, \psi \in \mathscr{L}^{\mathrm{g}}$, then $\mathfrak{A}_{\mathbb{T}}^{\mathbf{v}}(\varphi) \subseteq \mathfrak{A}_{\mathbb{T}}^{\mathbf{v}}(\psi)$, so that, by Theorem 1, we have a factoring

$$\begin{array}{ccc} E_{\mathbf{C}_{\mathbb{T}}}(\{\varphi^{\mathbf{v}}\}) & \nearrow & \\ \Big\downarrow & \searrow & E_{\mathbf{C}_{\mathbb{T}}}(\{\mathbf{v} \approx \mathbf{v}\}) \\ E_{\mathbf{C}_{\mathbb{T}}}(\{\psi^{\mathbf{v}}\}) & \nearrow & \end{array}$$

in $\mathbf{Sh}(\mathbf{C}_{\mathbb{T}})$. But the Yoneda embedding is injective on objects, and bijective on hom-sets, and so this last diagram pulls back to a factoring

$$\begin{array}{ccc} \{\varphi^{\mathbf{v}}\} & \nearrow & \\ {\scriptstyle[\alpha]}\Big\downarrow & \searrow & \{\mathbf{v} \approx \mathbf{v}\} \\ \{\psi^{\mathbf{v}}\} & \nearrow & \end{array}$$

of $\{\varphi^{\mathbf{v}}\}$ through $\{\psi^{\mathbf{v}}\}$ in $\mathscr{C}_{\mathbb{T}}$. Applying acceptable reletterings $\mathbf{v} \to \mathbf{v}'$ and $\mathbf{v} \to \mathbf{v}''$ to $\varphi^{\mathbf{v}}$ and $\psi^{\mathbf{v}}$ respectively, the definition of composition in $\mathscr{C}_{\mathbb{T}}$ then implies that

$$\mathbb{T} \vdash \varphi(\mathbf{v}') \wedge (\mathbf{v}' \approx \mathbf{v}) \supset \exists \mathbf{v}''(\alpha(\mathbf{v}', \mathbf{v}'') \wedge \psi(\mathbf{v}'') \wedge (\mathbf{v}'' \approx \mathbf{v})),$$

from which we can obtain

$$\mathbb{T} \vdash \varphi(\mathbf{v}) \supset \psi(\mathbf{v}). \qquad \square$$

At last we are in a position to show that Deligne's Theorem implies the Classical Completeness Theorem for finitary geometric theories $\mathbb{T}$. For, if $\mathbb{T} \nvdash \varphi \supset \psi$, where $\varphi, \psi \in \mathscr{L}^g$, the last Theorem implies that $\mathfrak{A}_\mathbb{T}(\varphi) \nsubseteq \mathfrak{A}_\mathbb{T}(\psi)$ in the model $\mathfrak{A}_\mathbb{T}$ in the coherent topos $\mathbf{Sh}(\mathbf{C}_\mathbb{T})$. By Deligne's Theorem (and Exercise 16.2.20), there is therefore a point $p : \mathbf{Set} \to \mathbf{Sh}(\mathbf{C}_\mathbb{T})$ such that $p^*(\mathfrak{A}_\mathbb{T}(\varphi)) \nsubseteq p^*(\mathfrak{A}_\mathbb{T}(\psi))$. But then, as in §16.4, p gives rise to a model $p^*\mathfrak{A}_\mathbb{T} : \mathscr{L} \to \mathbf{Set}$ which, by Theorem 16.4.1(1), has $p^*\mathfrak{A}_\mathbb{T}(\varphi) \nsubseteq p^*\mathfrak{A}_\mathbb{T}(\psi)$. Moreover, as $\mathbb{T}$ consists of geometric formulae, 16.4.1(2) implies that $p^*\mathfrak{A}_\mathbb{T} \vDash \mathbb{T}$. Thus there exists a $\mathbb{T}$-model in $\mathbf{Set}$ in which $(\varphi \supset \psi)$ is not true.

<div align="right">□</div>

As a final, cautionary, note on this topic we observe that the above derivation is founded entirely on the structure of $\mathbf{C}_\mathbb{T}$, and hence ultimately on the properties of the relation of $\mathbb{T}$-derivability. In many cases one can most quickly confirm that $\mathbb{T} \vdash \varphi$ by observing that θ is true in all $\mathbf{Set}$ models of $\mathbb{T}$ and then appealing to Classical Completeness. But of course if we want to use this approach to prove Completeness, it has to be shown directly in each case that there is a proof sequence for θ within the axiom system in question.

Classifying topoi

What is the relationship between a finitary geometric theory $\mathbb{T}$ and the theory $\mathbb{T}_{\mathbf{C}_\mathbb{T}}$ of its associated finitary site $\mathbf{C}_\mathbb{T}$? Introducing the notation "$\mathfrak{A} : \mathbb{T} \to \mathscr{E}$" to mean "$\mathfrak{A}$ is a model of $\mathbb{T}$ in $\mathscr{E}$", we can say from our earlier work (Theorem 16.3.2) that models $\mathbb{T}_{\mathbf{C}_\mathbb{T}} \to \mathscr{E}$ in a Grothendieck topos $\mathscr{E}$ correspond precisely to continuous morphisms $\mathbf{C}_\mathbb{T} \to \mathscr{E}$. We have also seen in this Section (Corollary 2) that such morphisms determine models $\mathbb{T} \to \mathscr{E}$ of $\mathbb{T}$ in $\mathscr{E}$. We will now show that the converse is true, i.e. that every $\mathbb{T} \to \mathscr{E}$ arises in this way from a unique continuous $\mathbf{C}_\mathbb{T} \to \mathscr{E}$. In this sense, the theories $\mathbb{T}$ and $\mathbb{T}_{\mathbf{C}_\mathbb{T}}$ have exactly the same models in Grothendieck topoi.

Given a model $\mathfrak{A} : \mathscr{L} \to \mathscr{E}$ such that $\mathfrak{A} \vDash \mathbb{T}$, we define a continuous morphism $F_\mathfrak{A} : \mathbf{C}_\mathbb{T} \to \mathscr{E}$. For each $\mathscr{L}^g$-formula φ, $\mathfrak{A}(\varphi)$ is a subobject of $\mathfrak{A}(\mathbf{v})$, where $\mathbf{v}$ is the list of free variables of φ. Identifying $\mathfrak{A}(\varphi)$ with its domain, so that we can regard it as an $\mathscr{E}$-object, we put $F_\mathfrak{A}(\{\varphi\}) = \mathfrak{A}(\varphi)$ (strictly speaking this determines $F_\mathfrak{A}$ "up to isomorphism" only). Note that if $\{\varphi\} = \{\psi\}$, then $[\varphi] = [\psi]$ and so, as $\mathfrak{A} \vDash \mathbb{T}$, we have $\mathfrak{A}(\varphi) = \mathfrak{A}(\psi)$ by Soundness. Hence $F_\mathfrak{A}$ is unambiguously defined on objects.

If $[\alpha(\mathbf{v}, \mathbf{w})] : \{\varphi(\mathbf{v})\} \to \{\psi(\mathbf{w})\}$ is a $\mathscr{C}_\mathbb{T}$-arrow, then the geometric formulae $(\alpha 1)$, $(\alpha 2)$, and $(\alpha 3)$, whose $\mathbb{T}$-derivability is implied by the

definition of "$\mathscr{C}_{\mathbb{T}}$-arrow", must be true in $\mathfrak{A}$. Exercise 1 then yields a unique $\mathscr{E}$-arrow $f_\alpha : \mathfrak{A}(\varphi) \to \mathfrak{A}(\psi)$ whose graph is $\mathfrak{A}(\alpha)$. We put $F_{\mathfrak{A}}([\alpha]) = f_\alpha$. Again a Soundness argument confirms that if $[\alpha] = [\beta]$ then $f_\alpha = f_\beta$. It is left to the reader to verify that $F_{\mathfrak{A}}$ preserves identities and commutative triangles, and so is a functor from $\mathscr{C}_{\mathbb{T}}$ to $\mathscr{E}$.

EXERCISE 13. Given $[\alpha(\mathbf{v}, \mathbf{w})]$ and f_α as above, show that $\operatorname{im} f_\alpha : f_\alpha(\mathfrak{A}(\varphi)) \rightarrowtail \mathfrak{A}(\psi)$ is equal, as a subobject of $\mathfrak{A}(\psi)$, to $\mathfrak{A}(\exists \mathbf{v}\alpha(\mathbf{v}, \mathbf{w})) \rightarrowtail \mathfrak{A}(\psi)$, where the latter monic derives, via $(\alpha 1)$ and $R\exists_2$, from the fact that $\mathfrak{A} \vDash \exists \mathbf{v}\alpha(\mathbf{v}, \mathbf{w}) \supset \psi$. $\qquad \square$

Now if $C = \{[\alpha_x(\mathbf{v}_x, \mathbf{w})] : x \in X\}$ is a provably epimorphic family in $\mathbf{C}_{\mathbb{T}}$, with $[\alpha_x] : \{\varphi_x(\mathbf{v}_x)\} \to \{\psi(\mathbf{w})\}$, then in the $\mathbb{T}$-model $\mathfrak{A}$ we have

$$\mathfrak{A} \vDash \psi(\mathbf{w}) \supset \bigvee \{\exists \mathbf{v}_x \alpha_x(\mathbf{v}_x, \mathbf{w}) : x \in X\},$$

and so

$$\mathfrak{A}(\psi) \subseteq \bigcup \{\mathfrak{A}(\exists \mathbf{v}_x \alpha_x(\mathbf{v}_x, \mathbf{w})) : x \in X\}.$$

From the last Exercise it then follows that

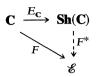$$\mathfrak{A}(\psi) \subseteq \bigcup \{f_{\alpha_x}(\mathfrak{A}(\varphi_x)) : x \in X\},$$

so that $\{f_{\alpha_x} : x \in X\}$ is a canonical cover in $\mathscr{E}$. Thus $F_{\mathfrak{A}}$ preserves covers. Since $\mathfrak{A}(\top)$ is 1 (by definition), and $\{\top\}$ is terminal in $\mathbf{C}_{\mathbb{T}}$ (Exercise 3), $F_{\mathfrak{A}}$ preserves terminals. Finally, we leave it to the reader once more to confirm that $F_{\mathfrak{A}}$ preserves pullbacks, and hence complete the proof that $F_{\mathfrak{A}}$ is a continuous morphism.

If we now use $F_{\mathfrak{A}}$ to construct a model $\mathfrak{A}_{F_{\mathfrak{A}}} : \mathscr{L} \to \mathscr{E}$ as above, then Theorem 1 implies that $\mathfrak{A}_{F_{\mathfrak{A}}}(\varphi) = F_{\mathfrak{A}}(\{\varphi\}) = \mathfrak{A}(\varphi)$. Indeed for any $\mathscr{L}$-sort a, if $v : a$ then $\mathfrak{A}_{F_{\mathfrak{A}}}(a)$ is $F_{\mathfrak{A}}(\{v \approx v\})$, i.e. $\mathfrak{A}(v \approx v)$, which we identify with the domain of the identity arrow on $\mathfrak{A}(a)$. If $\mathbf{g}$ is an operation symbol, and $[\alpha(\mathbf{v}, \mathbf{w})]$ is $[\mathbf{g}(\mathbf{v}) \approx w]$, then $\mathfrak{A}(\alpha)$ is the graph of $\mathfrak{A}(\mathbf{g})$, so that $\mathfrak{A}(\mathbf{g}) = F_{\mathfrak{A}}([\alpha]) = \mathfrak{A}_F(\mathbf{g})$. Similarly, $\mathfrak{A}_{F_{\mathfrak{A}}}(\mathbf{R})$ is the same subobject of $\mathfrak{A}(\mathbf{v})$ as is $\mathfrak{A}(\mathbf{R})$, and so $\mathfrak{A}_{F_{\mathfrak{A}}}$ and $\mathfrak{A}$ prove to be the same model.
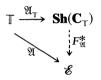
On the other hand, starting from F we find that $F(\{\varphi\}) = \mathfrak{A}_F(\varphi) = F_{\mathfrak{A}_F}(\{\varphi\})$. Since $\mathfrak{A}_F(\varphi)$, as a object, is only defined up to isomorphism, we find that the functors F and $F_{\mathfrak{A}_F}$ are naturally isomorphic. In this sense we obtain an exact correspondence between $\mathscr{E}$-models of $\mathbb{T}$ and continuous morphisms $\mathbf{C}_{\mathbb{T}} \to \mathscr{E}$.

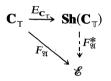Let us now return to the co-universal property of the canonical

morphism $E_{\mathbf{C}} : \mathbf{C} \to \mathbf{Sh}(\mathbf{C})$ of a small site $\mathbf{C}$, as expressed by the diagram

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\;E_{\mathbf{C}}\;} & \mathbf{Sh}(\mathbf{C}) \\
 & {}_{F}\searrow & \Big\downarrow{\scriptstyle F^{*}} \\
 & & \mathscr{E}
\end{array}
$$

(cf. Theorem 16.2.2). The diagram tells us that every continuous morphism defined on $\mathbf{C}$ extends along $E_{\mathbf{C}}$ to a continuous morphism on $\mathbf{Sh}(\mathbf{C})$ that is unique up to a natural isomorphism. This property gives rise to the following diagram of $\mathbb{T}$-models

$$
\begin{array}{ccc}
\mathbb{T} & \xrightarrow{\;\mathfrak{A}_{\mathbb{T}}\;} & \mathbf{Sh}(\mathbf{C}_{\mathbb{T}}) \\
 & {}_{\mathfrak{A}}\searrow & \Big\downarrow{\scriptstyle F^{*}_{\mathfrak{A}}} \\
 & & \mathscr{E}
\end{array}
$$

This diagram conveys that for any $\mathbb{T}$-model $\mathfrak{A}$ there is a unique (up to natural isomorphism) continuous $F^{*}_{\mathfrak{A}} : \mathbf{Sh}(\mathbf{C}_{\mathbb{T}}) \to \mathscr{E}$, given by

$$
\begin{array}{ccc}
\mathbf{C}_{\mathbb{T}} & \xrightarrow{\;E_{\mathbf{C}_{\mathbb{T}}}\;} & \mathbf{Sh}(\mathbf{C}_{\mathbb{T}}) \\
 & {}_{F_{\mathfrak{A}}}\searrow & \Big\downarrow{\scriptstyle F^{*}_{\mathfrak{A}}} \\
 & & \mathscr{E}
\end{array}
$$

such that the $\mathscr{E}$-model $F^{*}_{\mathfrak{A}}\mathfrak{A}_{\mathbb{T}}$, defined as for Theorem 16.4.1, is $\mathfrak{A}$ itself.

This characteristic property of $\mathfrak{A}_{\mathbb{T}}$ is what is meant by the notion of a *classifying topos* for a (possibly infinitary) geometric theory $\mathbb{T}$. To define this concept in general we fix a "base" Grothendieck topos $\mathscr{E}$, and consider pairs $(\mathscr{F}, \mathfrak{A})$ consisting of a Grothendieck $\mathscr{E}$-topos $\mathscr{F}$ and an $\mathscr{F}$-model $\mathfrak{A}$ of $\mathbb{T}$ (we call $\mathfrak{A}$ a $\mathbb{T}$-model *over* $\mathscr{E}$). Then we say that $(\mathscr{E}[\mathbb{T}], \mathfrak{A}_{\mathbb{T}})$ is a *classifying $\mathscr{E}$-topos for* $\mathbb{T}$ with *generic* model $\mathfrak{A}_{\mathbb{T}}$, if it is universal among such pairs, i.e. if for any pair $(\mathscr{F}, \mathfrak{A})$ as above, there is a geometric morphism $f : \mathscr{F} \to \mathscr{E}[\mathbb{T}]$ over $\mathscr{E}$ unique up to natural isomorphism such that $\mathfrak{A} = f^{*}\mathfrak{A}_{\mathbb{T}}$. Thus a generic $\mathbb{T}$-model over $\mathscr{E}$ has the property that every other $\mathbb{T}$-model over $\mathscr{E}$ arises by pulling the generic model back along some (unique) geometric morphism. Since inverse image functors preserve geometric formulae, it follows that the geometric formulae true in $\mathfrak{A}_{\mathbb{T}}$ are precisely those that are true in all $\mathbb{T}$-models over $\mathscr{E}$. The notation $\mathscr{E}[\mathbb{T}]$ is intended to convey the idea that the classifying $\mathscr{E}$-topos for $\mathbb{T}$ is generated by "adjoining a generic $\mathbb{T}$-model to $\mathscr{E}$".

If $\mathbf{C}$ is a small site, then models of the (infinitary) theory $\mathbb{T}_{\mathbf{C}}$ in

Grothendieck topoi correspond to geometric morphisms into **Sh(C)**, and so by the first of the above three diagrams, $E_{\mathbf{C}}: \mathbf{C} \to \mathbf{Sh(C)}$ is a generic $\mathbb{T}_{\mathbf{C}}$-model, making **Sh(C)** a classifying topos for $\mathbb{T}_{\mathbf{C}}$. Since Grothendieck topoi are all defined over **Set**, we can thus express **Sh(C)** as **Set[$\mathbb{T}_{\mathbf{C}}$]**. Hence each Grothendieck topos arises by adjoining to **Set** a generic model of a geometric theory.

In the converse direction, if $\mathbb{T}$ is a finitary geometric theory, then the construction of $\mathfrak{A}_{\mathbb{T}}$ from $E_{\mathbf{C}_{\mathbb{T}}}: \mathbf{C}_{\mathbb{T}} \to \mathbf{Sh(C_{\mathbb{T}})}$ provides a generic $\mathbb{T}$-model, making **Sh($\mathbf{C}_{\mathbb{T}}$)** the classifying **S**-topos **Set[$\mathbb{T}$]** for $\mathbb{T}$. Hence the coherent topoi are *precisely* the classifying **S**-topoi for finitary geometric theories. This analysis can be extended to any infinitary geometric theory $\mathbb{T}$, to show that **Set[$\mathbb{T}$]** exists, but this requires a great deal more work. Amongst other things, the category $\mathscr{C}_{\mathbb{T}}$ has to be "enlarged" to include coproducts and "quotients of equivalence relations". However that is a story that we shall have to leave for the reader to pursue in Chapters 8 and 9 of [MR].

The conclusion of this work is that the concepts of "Grothendieck topos" and "classifying topos of a geometric theory" are coextensive. This has particular relevance in Algebraic Geometry, where some of the most important categories ("Zariski" topos, "Etale" topos) which form the focus of the work of the Grothendieck school turn out to be the classifying topoi for certain naturally occurring algebraic theories (cf. [MR], Chapter 9, Wraith [79]).
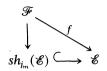
Forcing topologies

Let $\mathscr{L}$ be a language that has altogether finitely many sorts, relation and operation symbols, and constants, and let $\mathbb{T}$ be a finite geometric $\mathscr{L}$-theory. Then it can be shown that for any elementary topos $\mathscr{E}$ with a natural numbers object there exists a classifying topos $\mathscr{E}[\mathbb{T}]$ for models of $\mathbb{T}$ in $\mathscr{E}$-topoi. The proof of this is given by Tierney [76] (cf. also Johnstone [77], 6.56). We will not attempt to reproduce it here, but will briefly discuss an aspect of the construction which uses topologies $j: \Omega \to \Omega$ in an interesting way to produce $\mathbb{T}$-models.

If $I \rightarrowtail \Omega$ is a subobject of Ω in an elementary topos $\mathscr{E}$ then some work of Diaconescu [75] (cf. Johnstone [77], 3.58) shows that there is a smallest subobject $J \rightarrowtail \Omega$ containing I such that the characteristic arrow $\Omega \to \Omega$ of J is a topology. This characteristic arrow is called the topology *generated by I*. If $m: a \rightarrowtail b$ is any $\mathscr{E}$-monic, and $I_m \rightarrowtail \Omega$ the image of $\chi_m: b \to \Omega$, we denote the topology generated by I_m by j_m. Then the
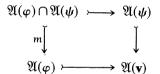
inclusion functor $sh_{j_m}(\mathscr{E}) \hookrightarrow \mathscr{E}$ has the property that for any geometric morphism $f : \mathscr{F} \to \mathscr{E}$ there exists a factorisation of the form

$$
\begin{array}{ccc}
\mathscr{F} & & \\
\Big\downarrow & \searrow^{f} & \\
sh_{j_m}(\mathscr{E}) & \overset{\hookrightarrow}{\longrightarrow} & \mathscr{E}
\end{array}
$$

if, and only if, $f^*(m)$ is iso in $\mathscr{F}$ (Tierney [76], p. 212; Johnstone [77], 4.19). In particular the sheafification functor $\mathscr{S}\hskip-2pt h_{j_m} : \mathscr{E} \to sh_{j_m}(\mathscr{E})$ makes $\mathscr{S}\hskip-2pt h_{j_m}(m)$ iso. In view of this universal property, Tierney calls j_m the topology that *forces m* to be iso.

This notion can be extended to finitely many monics $m_1, \ldots, m_n$: the topology generated by $I_{m_1} \cup .. \cup I_{m_n} \rightarrowtail \Omega$ forces all of $m_1, \ldots, m_n$ to be iso.

Now let $\mathfrak{A} : \mathscr{L} \to \mathscr{E}$ be an $\mathscr{L}$-model in $\mathscr{E}$. Then a geometric $\mathscr{L}$-formula $(\varphi \supset \psi)(\mathbf{v})$ is true in $\mathfrak{A}$ iff the monic m in the following pullback is iso

$$
\begin{array}{ccc}
\mathfrak{A}(\varphi) \cap \mathfrak{A}(\psi) & \rightarrowtail & \mathfrak{A}(\psi) \\
{\scriptstyle m}\Big\uparrow\Big\downarrow & & \Big\downarrow \\
\mathfrak{A}(\varphi) & \rightarrowtail & \mathfrak{A}(\mathbf{v})
\end{array}
$$

Thus if $m_1, \ldots, m_n$ are all these monics corresponding to the members of $\mathbb{T}$, and $j_{\mathbb{T}}$ is the topology that forces $m_1, \ldots, m_n$ to be iso, then $j_{\mathbb{T}}$ forces $\mathfrak{A}$ to become a $\mathbb{T}$-model in $sh_{j_{\mathbb{T}}}(\mathscr{E})$. For any geometric morphism $f : \mathscr{F} \to \mathscr{E}$, f factors through $sh_{j_{\mathbb{T}}}(\mathscr{E}) \hookrightarrow \mathscr{E}$ iff $f^*\mathfrak{A}$ is a model of $\mathbb{T}$ in $\mathscr{F}$.

This forcing construction is not special to models of first-order languages. Tierney observes that "given any diagram D in a topos $\mathscr{E}$, we can force any appropriate finite configuration (or even not necessarily finite if properly indexed over a base topos) in D to become a limit or colimit".

Rings and fields

We end this chapter by pointing the reader in the direction of literature that applies logical aspects of geometric morphisms to some familiar algebraic notions.

In classical algebra, a *commutative ring with unity* (henceforth simply called a *ring*) can be defined as a structure $(R, +, 0, \times, 1)$, consisting of a set R carrying two commutative binary operations $+$ and $\times$, and two

distinguished elements 0 and 1, such that

   (i) $(R, +, 0)$ is a group;

   (ii) $(R, \times, 1)$ is a monoid; and

   (iii) $x \times (y + z) = (x \times y) + (x \times z)$, for all $x, y, z \in R$.

Standard examples of rings are of course the number systems $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, with $+$, $\times$, 0, 1 having their usual arithmetical meanings. The notion of a ring can be expressed in the first-order language having symbols for $+$, $\times$, 0, 1 by a finite set of *equations*.

A ring is a *field* if $R - \{0\}$ is a group under $\times$ with identity 1. This requires that $0 \neq 1$ (a geometric condition), and that any $x \neq 0$ has an inverse under $\times$. Writing $U(x)$ to mean that $\exists y (x \times y = 1)$, then in **Set** this last condition can be expressed by any of the following three classically equivalent assertions.

(1) $(x = 0) \vee U(x)$;

(2) $\sim(x = 0) \supset U(x)$;

(3) $\sim U(x) \supset x = 0$.

These conditions are not generally equivalent for rings (i.e. models of the ring axioms) in non-Boolean topoi. Since (1) is expressed by a geometric formula, rings satisfying it are called *geometric fields*. (2) and (3) define, respectively, the notions of "field of fractions" and "residue field" (Johnstone [77], 6.64).

Another possible field axiom, considered by Kock [76], is

(4) $\sim\left(\bigwedge_{i=1}^{n} (x_i = 0)\right) \supset \bigvee_{i=1}^{n} U(x_i)$.

This in turn implies the geometric condition

(5) $U(x + y) \supset U(x) \vee U(y)$,

which defines the notion of a *local ring*. In general, (5) is weaker than (4), but Kock proves the significant fact that the generic local ring is a field in the sense of (4). If $\mathbb{T}_l$ is the geometric theory consisting of the ring axioms together with (5), then by the generic local ring is meant the generic model $\mathfrak{A}_{\mathbb{T}_l}$ in the classifying **S**-topos for $\mathbb{T}_l$. Now the geometric formulae true in all local rings in **S**-topoi are precisely those true in $\mathfrak{A}_{\mathbb{T}_l}$, i.e. those deducible from $\mathbb{T}_l$. Since, as Kock proves, $\mathfrak{A}_{\mathbb{T}_l}$ satisfies (4), the Soundness Theorem then yields the following metalogical principle:

If θ is geometric and $\mathbb{T}_l$, (4) $\vdash \varphi$, then $\mathbb{T}_l \vdash \varphi$.

Thus in deriving a geometric consequence of $\mathbb{T}_l$ we can invoke the assistance of the stronger, non-geometric, condition (4). Indeed (4) can be replaced in this argument by any axiom which is satisfied by the generic local ring. Further results along these lines, and a detailed analysis of a dozen or so possible axioms for fields and local rings, are given by Johnstone [77i] and [77], §§6.5, 6.6.

There is now in existence a vast literature about the *representation* of rings, and other algebraic structures, by global sections of sheaves (cf. Pierce [67], Dauns and Hofmann [68], Hofmann [72], Hofmann and Liukkonen [74]). Suppose that $p : A \to I$ is a sheaf (local homeomorphism) over a topological space I such that each stalk $p^{-1}(\{i\})$ is a ring in its own right under operations $+_i$, $\times_i$, and identities 0_i and 1_i. If $f, g : I \to A$ are global sections of the sheaf, then we can define sections $f + g$, $f \times g$ by putting

$$f + g(i) = f(i) +_i g(i),$$
$$f \times g(i) = f(i) \times_i g(i), \quad \text{all } i \in I.$$

If $f + g$, $f \times g$, and the sections 0 and 1 having $0(i) = 0_i$, $1(i) = 1_i$, are continuous whenever f and g are continuous, then p is called a *sheaf of rings* over the space I. In this situation, and with these definitions, the set of continuous global sections of p forms a ring. The aim of representation theory is to show that a given ring is isomorphic to the ring of continuous global sections of some sheaf of rings. An important result in this direction concerns *regular* rings, which are those satisfying

$$\forall x \exists y (x \times y \times x = x).$$

Regular rings include fields (let y be the $\times$-inverse of x). But every regular ring can be represented as the ring of continuous sections of a sheaf in which each stalk is a field! (Pierce [67], §10). This phenomenon gives rise to "transfer principles", in which properties of fields are shown to hold for regular rings by showing that they are preserved by the representation. An early paper on this theme is MacIntyre [73], concerned with transferring a property called "model completeness".

More generally we can study sheaves whose stalks are all **Set**-models of some theory $\mathbb{T}$, and seek to show that a Set-model $\mathfrak{A}$ of some other theory $\mathbb{T}_0$ is isomorphic to the structure of continuous sections of a sheaf of $\mathbb{T}$-models over some space I. In this situation $\mathfrak{A}$ may also be regarded as a model in the topos **Top**(I) of sheaves over I. Its behaviour as a **Top**(I)-model may differ from that which it exhibits as a **Set**-model. In particular, any geometric formula true in each stalk will be true in the

Top(I)-model 𝔄 (Fourman and Scott [79], 6.9). Thus if a regular ring R is represented by a sheaf of fields over I, then R becomes a **Top**(I)-model of the geometric field axioms, i.e. the regular ring R "is" a field from the point of view of the mathematical universe **Top**(I) (cf. Fourman and Scott [79], p. 367).

· This theme is taken up in the thesis of Louillis [79], who adapted some of the work of classical model theory to categories of sheaves. The papers of Coste [79], Bunge and Reyes [81] and Bunge [81] present major advances in the use of geometric morphisms to transfer model-theoretic properties from the theory of the stalks to the theory of the global continuous sections of sheaves. A survey of more classical applications of model theory in sheaves is given by Burris and Werner [79].