PREFACE

No doubt there are as many reasons for writing books as there are people who write them. One function served by this particular work has been the edification of its author. Translations can sometimes create a sense of *explanation,* and this seemed to me particularly true of the alternative account of mathematical constructions being produced by category theory. Writing the book gave me a framework within which to confirm that impression and to work through its ramifications in some detail. At the end I knew a great deal more than when I began, so that the result is as much a recording as a reconstruction of the progress of my own understanding. And at the end it seemed to me that much that I had dwelt on had finally fallen into place.

As to the more public functions of the book – I hope that it provides others with the prospect of a similar experience. Less presumptiously, I have tried to write an exposition that will be accessible to the widest possible audience of logicians – the philosophically motivated as well as the mathematical. This, in part, accounts for the style that I have adopted. There is a tendency in much contemporary literature to present material in a highly systematised fashion, in which an abstract definition will typically come before the list of examples that reveals the original motivation for that definition. Paedogogically, a disadvantage of this approach is that the student is not actually *shown* the genesis of concepts – how and why they evolved – and is thereby taught nothing about the mechanisms of creative thinking. Apart from lending the topic an often illusory impression of completedness, the method also has the drawback of inflating prerequisites to understanding.

All of this seems to me particularly dangerous in the case of category theory, a discipline that has more than once been referred to as "abstract nonsense". In my experience, that reaction is the result of features that are not intrinsic to the subject itself, but are due merely to the style of some of its expositors. The approach I have taken here is to try to move always from the particular to the general, following through the steps of the abstraction process until the abstract concept emerges naturally. The starting points are elementary (in the "first principles" sense), and at the finish it would be quite appropriate for the reader to feel that (s)he had just arrived at the subject, rather than reached the end of the story.

As to the specific treatment of category theory, I have attempted to play down the functorial perspective initially and take an elementary (in the sense of "first-order") approach, using the same kind of combinatorial manipulation of algebraic structure that is employed in developing the basic theory of any of the more familiar objects of pure-mathematical study. In these terms categories as structures are no more rarified than groups, lattices, vector-spaces etc.

I should explain that whereas the bulk of the manuscript was completed around May of 1977, the sections 11.9, 14.7 and 14.8 were written a year later while I was on leave in Oxford (during which time I held a Travelling Fellowship from the Nuffield Foundation, whose assistance I am pleased to acknowledge). The additional material was simply appended to Chapters 11 and 14, since, although the arrangement is less than ideal, it was impractical at that stage to begin a major reorganisation. I imagine however that there will be readers interested in the construction of number-systems in 14.8 who do not wish to wade through the earlier material in Chapter 14 on Grothendieck topologies, elementary sites etc. In fact in order to follow the definition of Dedekind-reals in the topos of Ω-sets, and their representation as classical continuous real-valued functions, it would suffice to have absorbed the description of that topos given in 11.9. The full sheaf-theoretic version of this construction depends on the theory of Ω-sheaves developed in 14.7, but a sufficient further preparation for the latter would be to read the first few pages of 14.1, at least as far as the introduction of the axiom COM on page 362.

A point of terminology: – I have consistently used the word "categorial" where the literature uniformly employs "categorical". The reason is that while both can serve as adjectival forms of the noun "category", the second of them already has a different and long established usage in the domain of logic, one that derives from its ordinary-language meaning of "absolute". Logicians have known since the work of Gödel that set theory has no categorical axiomatisation. One function of this book will be to explain to them why it does have a categorial one.

There are a number of people who I would like to thank for their help in the production of the book. I am indebted to Shelley Carlyle for her skilful typing of the manuscript; to the Internal Research Committee and the Mathematics Department of the Victoria University of Wellington for substantially subsidising its cost; to Pat Suppes for responding favourably to it, and supporting it; and to Einar Fredriksson and Thomas van den

Heuvel for the expertise and cooperation with which they organised its editing and publishing.

My involvement with categorial logic gained impetus through working with Mike Brockway on his M.Sc. studies, and I have benefited from many conversations with him and access to his notes on several topics. In obtaining other unpublished material I was particularly helped by Gonzalo Reyes. Dana Scott, by his hospitality at Oxford, performed a similar service and provided a much appreciated opportunity to aquaint myself with his approach to sheaves and their logic. In preparing the material about the structure of the continuum I was greatly assisted by discussions with Scott, and also with Charles Burden.

Finally, it is a pleasure to record here my indebtedness to my teachers and colleagues in the logic group at VUW, particularly to my doctoral advisors Max Cresswell and George Hughes, and to Wilf Malcolm, for their involvement in my concerns and encouragement of my progress throughout the time that I have been a student of mathematical logic.

Where did topos theory come from? In the introduction to his recent book on the subject, Peter Johnstone describes two lines of development in the fields of algebraic geometry and category theory. It seems to me that a full historical perspective requires the teasing out of a third strand of events in the area of specific conern to this book, i.e. logic, especially model theory. We may begin this account with Cohen's work in 1963 on the independence of the continuum hypothesis et. al. His forcing technique proved to be the key to the universe of classical set theory, and led to a wave of exploration of that territory. But as soon as the method had been reformulated in the Scott–Solovay theory of Boolean-valued models (1965), the possibility presented itself of replacing "Boolean" by "Heyting" and thereby generalising the enterprise. Indeed Scott made this point in his 1967 lecture-notes and then took it up in his papers (1968, 1970) on the topological interpretation of intuitionistic analysis.

Meanwhile the notion of an *elementary topos* had independently emerged through Lawvere's attempts to axiomatise the category of sets. The two developments became linked together by the concept of a *sheaf:* the study of cartesian-closed categories with subobject classifiers (topoi) got under way in earnest once it was realised that they included all the Grothendieck sheaf-categories, while the topological interpretation was seen to have provided the first examples for a general axiomatic theory of sheaf-models over Heyting algebras that was subsequently devised by Scott and developed in association with Michael Fourman (cf. 14.7 and

14.8). In this latter context (many of whose ideas have precursors in the initial Boolean work), the earlier problem (Scott 1968, p. 208) of dealing with partially defined entities is elegantly resolved by the introduction of an *existence predicate*, whose semantical interpretation is a measure of the extent to which an individual is defined (exists). To complete the picture, and round out this whole progression of ideas, some unpublished work of Denis Higgs (1973) demonstrated that the category of sheaves over **B** (a complete Boolean algebra) is equivalent to the category of **B**-valued sets and functions in the original Scott–Solovay sense.

And what of the future? What, for instance, is the likely impact of the latest independence results to the effect that there exist topoi in which the Heine–Borel Theorem fails, the Dedekind-reals are not real-closed, complex numbers lack square-roots etc.? Predictions at this stage would I think be premature – after all today's pathology may well be dubbed "classical" by some future generation. The intellectual tradition to which topos theory is a small contribution goes back to a time when mathematics was closely tied to the physical and visual world, when "geometry" for the Greeks really had something to do with land-measurement. It was only relatively recently, with the advent of non-Euclidean geometries, that it became possible to see that discipline as having a quite independent existence and significance. Analogously, that part of the study of structure that is concerned with those structures called "logics" has evolved to a point that lies beyond its original grounding (the analysis of principles of reasoning). But the separation from this external authority has no more consequences as to the true nature of reasoning than does the existence of non-Euclidean geometries decide anything either way about the true geometrical properties of visual space.

The laws of Heyting algebra embody a rich and profound mathematical structure that is manifest in a variety of contexts. It arises from the epistemological deliberations of Brouwer, the topologisation (localisation) of set-theoretic notions, and the categorial formulation of set theory, all of which, although interrelated, are independently motivated. This ubiquity lends weight, not to the suggestion that the correct logic is in fact intuitionistic instead of classical, but rather to the recognition that thinking in such terms is simply inappropriate – in the same way that it is inappropriate to speak without qualification about *the* correct geometry.

At the same time, these developments have shown us more clearly than ever just how the properties of the structures we study depend on the principles of logic we employ in studying them. Particularly striking is the fine-tuning that has been given to the modern logical/set-theoretical articulation of the structure of the intuitively conceived continuum (which to Euclid was not a set of points at all, let alone an object in a topos). Indeed it seems that the deeper the probing goes the less will be the currency given to the definite article in references to "the continuum".

Other areas of mathematics (abstract algebra, axiomatic geometry) have long since become autonomous activities of mental creation, just as painting and even music have long since progressed beyond the representational to aquire substantial (in some cases all-consuming) subjective and intellectual components. A similar situation could be said to be arising in mathematical logic. In the absence of that external authority (the representation of things "out there") we may not so readily determine what is worthwhile and significant, just as it is no longer so easy to understand and make judgements about many contemporary aesthetic developments. Were we to identify the valuable with that whose value is lasting, a considerable period of winnowing might well be required before we could decide what is wheat and what is chaff. Looking back over the progress of the last two decades or so we see several strands that weave together to present the current interest in Heyting-valued structures as the natural product of the evolution of a substantial area of mathematical thought. Wherever it may be heading, we may already locate its permanent importance in the way it has brought a number of disciplines (logic, set theory, algebraic geometry, category theory) together under one roof, and in the contribution it has thereby made to our understanding of the house that we mentally build for ourselves to live in.

No doubt these remarks will be thought contentious by some. I hope that they will be found provocative as well. Should it inspire, or incite, anybody to respond to them, this book will have fulfilled one of its intended functions.

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