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ON THE POLARIZATION OF GRAVITATIONAL WAVES

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Abstract. Physical properties of gravitational waves, belonging to the larger class of exact solutions of Einstein field equations which are invariant for a non-Abelian two-dimensional Lie algebra of Killing fields, are described. It is shown that in the would be quantum theory of gravity they correspond to spin $-$ 1 massless particles. The gravitational interaction of two pencils of light is analyzed.

Introduction

The aim of this talk is to illustrate some interesting and, in a sense, surprising physical properties of special solutions of Einstein field equations, belonging to the large class of Einstein metrics invariant for a non-Abelian two-dimensional Lie algebra of symmetries, which throw new light on the following problem.

A long time ago Tolman, Ehrenfest and Podolsky [30] and later Wheeler [33] analyzed the gravitational field of light beams and the corresponding geodesics in the linear approximation of Einstein equations. They discovered that null rays behave differently according whether they propagate parallel or antiparallel to a steady, long, straight beam of light, but they did not provide a physical explanation of this fact. The result was clarified in part by Faraoni and Dumse [14] using an approach based on a generalization to null rays of the gravitoelectromagnetic Lorentz force of linearized gravity. They also extended the analysis to the realm of exact *pp*-wave solutions of the Einstein equations.

Since the problem of the gravitational interaction of two photons is still unsolved, it appears necessary to take into full account the nonlinearity of Einstein's equations when studying the generation of gravitational waves from strong sources [12, 29].

On the other hand, some decades ago, Belinski and Zakharov have shown [4] that there exist metrics such that the corresponding vacuum Einstein field equations reduce to a system of non-linear partial differential equations, whose *generalized Lax form* is characteristic for integrable systems. Then, by using a suitable generalization of the *Inverse Scattering Transform*, they were able to find *solitary waves solutions*.

A geometric inspection of mentioned metrics shows that they are invariant under translations along the x, y-axes, i.e., they admit two Killing fields, ∂_x and ∂_y, closing on an Abelian two-dimensional Lie algebra $\mathcal{A}_2$. Moreover, the distribution $\mathcal{D}$ generated by ∂_x and ∂_y is two-dimensional and the distribution $\mathcal{D}^{\perp}$ orthogonal to $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$.

Since a two-dimensional Lie algebra is either Abelian ($\mathcal{A}_2$) or non-Abelian ($\mathcal{G}_2$), it has been natural to consider [24–26] the problem of characterizing all gravitational fields g admitting a Lie algebra $\mathcal{G}$ of Killing fields such that

I the distribution $\mathcal{D}$, generated by vector fields of $\mathcal{G}$, is two-dimensional.

II the distribution $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$.

The condition of transversality can be relaxed [9,10], so that in order to distinguish the different cases, the notation $(\mathcal{G}, r)$ is used. Metrics satisfying the conditions *I* and *II* are called *of* $(\mathcal{G}, 2)$*-type*. Metrics satisfying conditions *I* and *II*, except the transversality condition, are called *of* $(\mathcal{G}, 0)$*-type* or *of* $(\mathcal{G}, 1)$*-type* according to the rank of their restriction the leaves of $\mathcal{D}$ which are also called *Killing leaves*.

All the possible situations corresponding to a two-dimensional Lie algebra of isometries, are described by the following Table 1 in which the cases indicated with bold letters are essentially solved [2, 9, 10, 24–27] where a non integrable two-

Table 1

	$\mathcal{D}^{\perp}, r = 0$	$\mathcal{D}^{\perp}, r = 1$	$\mathcal{D}^{\perp}, r = 2$
$\mathcal{G}_2$	**integrable**	**integrable**	**integrable**
$\mathcal{G}_2$	**semi-integrable**	**semi-integrable**	**semi-integrable**
$\mathcal{G}_2$	**non-integrable**	**non-integrable**	non-integrable
$\mathcal{A}_2$	**integrable**	**integrable**	integrable
$\mathcal{A}_2$	semi-integrable	semi-integrable	semi-integrable
$\mathcal{A}_2$	non-integrable	non-integrable	non-integrable

dimensional distribution has been called *semi-integrable* if it is part (i.e., a suitable restriction) of a three-dimensional integrable distribution.

The study of $\mathcal{A}_2$-invariant Einstein metrics goes back to Einstein and Rosen [13], Kompaneyets [16], Geroch [15], Belinsky, Khalatnikov, Zakharov [3, 4], so that some exact solutions already known in the literature [28] have been rediscovered. Nevertheless, the geometric approach allows to perform in a natural way the choice of coordinates, i.e., the coordinates adapted to the symmetries of the metrics, even if they do not admit integrable $\mathcal{D}^{\perp}$ distribution. Usually, the standard techniques to find exact solutions assume, from the very beginning, that there exist natural vector fields, surfaces forming, which simplify the choice of the coordinates system. These assumptions are strong topological constraints on the space-time. The method developed in [2, 9, 10, 24–27] can be applied also when such topological assumptions do not hold.

The paper is organized as follows. In Section 1 gravitational fields invariant for a non Abelian two-dimensional Lie algebra, when the commutator of generators of the Lie algebra is of *light-type*, are characterized from a geometric point of view. In Section 2, the canonical and the Landau energy-momentum pseudo-tensors are introduced and a comparison with the linearized theory is performed. The role of (realistic) sources for such gravitational waves is also described. Eventually, the analysis of the polarization leads to the conclusion that these fields are spin $-$ 1 gravitational waves. In Section 3, spin $-$ 1 and spin $-$ 2 gravitational waves are compared from a gravitoelectromagnetic perspective. Section 4 is devoted to the analysis of the photon-photon gravitational interaction.

1. Geometric Aspects

Let g be a metric on the space-time $\mathcal{M}$ and $\mathcal{G}_2$ one of its Killing algebras whose generators X, Y satisfy $[X, Y] = sY$, $s = 0, 1$. The Frobenius distribution $\mathcal{D}$ generated by $\mathcal{G}_2$ is two-dimensional and in the neighborhood of a non-singular point *adapted coordinates* (x, y, p, q) exist ([9, 10, 24–27]) such that

$$X = \frac{\partial}{\partial p}, \qquad Y = \exp(sp)\frac{\partial}{\partial q}\cdot$$

In these coordinates, the general solution of vacuum Einstein equations, in the case in which the Killing field Y is of *light type*, is given by

$$g = 2f(\mathrm{d}x^2 \pm \mathrm{d}y^2) + \mu[(w(x,y) - 2sq)\mathrm{d}p^2 + 2\mathrm{d}p\mathrm{d}q] \tag{1}$$

where $\mu = A\Phi + B$ with $A, B \in \mathbb{R}$, $\Phi(x, y)$ is a non-constant harmonic function, $f = (\nabla\Phi)^2\sqrt{|\mu|}/\mu$, $w(x, y)$ is the solution of the *Euler-Darboux equation*

$$\Delta_{\pm}w + (\partial_x \ln|\mu|)\partial_x w \pm (\partial_y \ln|\mu|)\partial_y w = 0$$

where $\Delta_{\pm}$ is the Laplace (d'Alembert) operator in the (x, y)-plane. Metrics (1) are Lorentzian if the orthogonal leaves are conformally Euclidean, i.e., the positive sign is chosen, and Kleinian otherwise. Only the Lorentzian case will be analyzed and these metrics will be called of $(\mathcal{G}_2, 2)$-*isotropic type*.

In the particular case $s = 1$, $f = 1/2$ and $\mu = 1$, the above (Lorentzian) metrics are locally diffeomorphic [7] to a subclass of the vacuum Peres solutions [20, 28], that for later purpose we rewrite in the form

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + 2(\varphi_{,x}\mathrm{d}x + \varphi_{,y}\mathrm{d}y)\mathrm{d}u. \qquad (2)$$

The correspondence between (1) and (2) depends on the special choice of the function $\varphi(x, y, u)$, which, in general, is harmonic in x and y. In our case

$$x \to x, \quad y \to y, \quad u \to u, \quad v \to v + \varphi(x, y, u), \quad h = \varphi_{,u}.$$

In the case $\mu = \mathrm{const}$, the Euler-Darboux equation reduces to the Laplace equation. For $\mu = 1$, in the harmonic coordinates system (x, y, z, t) defined in [6], the above Einstein metrics take the particularly simple form

$$g = 2f(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}(w)\mathrm{d}(\ln|z - t|). \qquad (3)$$

This shows that, when w is constant, the Einstein metrics given by equation (3) are static and, under the further assumption $\Phi = x\sqrt{2}$, they reduce to the Minkowski one. Moreover, when w is not constant, gravitational fields (3) look like a *disturbance* propagating at light velocity along the z direction on the Killing leaves. In the following we will only consider the case $\Phi = x\sqrt{2}$.

More precisely, the wave character and the polarization of gravitational fields (2) can be checked by using the covariant Pirani's criterion. To use this criterion the Weyl scalars must be evaluated according to the Petrov-Penrose classification [19, 21].

To perform the Petrov-Penrose classification, one has to choose a *tetrad* basis with two real null vector fields and two real spacelike (or two complex null) vector fields. Then, according to the Pirani's criterion, if the metric belongs to type **N** of the Petrov classification, it is a gravitational wave propagating along one of the two real null vector fields. Since ∂_u and ∂_v are null real vector fields and ∂_x and ∂_y are spacelike real vector fields, the above set of coordinates is the right one to apply for the Pirani's criterion.

Since the only nonvanishing components of the Riemann tensor, corresponding to the metric (2), are

$$R_{iuju} = -\partial^2_{ij}\partial_u\varphi, \qquad i, j = x, y$$

this gravitational fields belong to Petrov type **N** [11, 34]. Then, according to the Pirani's criterion, the metric (2) does indeed represent a gravitational wave propagating along the null vector field ∂_u.

2. Physical Properties

In the following, physical properties of metrics (1) will be analyzed only in the case of Lorentzian signature. In previous section, the wavelike nature of gravitational fields (1) has been checked [6] by using covariant criteria. Now, we will shortly review the most important properties of these waves which will turn out to have spin $-$ 1.

Let us remark that the definition and the meaning of spin or polarization for a theory, such as general relativity, which is non-linear and possesses a much bigger invariance than just the Poincaré one, deserve a careful analysis. It is well known that the concept of particle, together with its degrees of freedom like the spin, may be only introduced for linear theories. In these theories, when Poincaré invariant, the particles are classified in terms of the eigenvalues of two Casimir operators of the Poincaré group, P^2 and W^2 where P_μ are the translation generators and $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ is the **Pauli-Ljubanski polarization vector** with $M^{\mu\nu}$ Lorentz generators. Then, the total angular momentum $J = L + S$ is defined in terms of the generators $M_{\mu\nu}$ as $J^i = \frac{1}{2}\epsilon^{0ijk}M_{jk}$. The generators P_μ and $M_{\mu\nu}$ span the Poincaré algebra, ISO$(3,1)$. When $P^2 = 0$, $W^2 = 0$, W and P are linearly dependent of each other $W_\mu = \lambda P_\mu$; the constant of proportionality λ is given by $\lambda = \overrightarrow{P} \cdot \overrightarrow{J}/P_0$ and defines the *helicity* for massless particles like photons.

Let us turn now to the gravitational fields represented by equation (3). As it has been remarked, they represent gravitational waves moving at the velocity of light, that is, in the would be quantised theory, particles with zero rest mass. Thus, if a classification in terms of Poincaré group invariants could be performed, these waves would belong to the class of unitary (infinite-dimensional) representations of the Poincaré group characterized by $P^2 = 0$, $W^2 = 0$. Recall that, in order for such a classification to be meaningful P^2 and W^2 have to be invariants of the theory. This is not the case for general relativity, unless we restrict to a subset of transformations selected for example by some physical criterion or by experimental constraints. For the solutions of the linearized vacuum Einstein equations the choice of the harmonic gauge does the job [32]. There, the residual gauge freedom corresponds to the sole Lorentz transformations.

2.1. The Standard Linearized Theory

The standard analysis of linearized theory and the issue of the polarization will be analyzed. In particular, the usual transverse-traceless gauge in the linearized vacuum Einstein equations and the (usually implicit) assumptions needed to reduce to this gauge play an important role: the generality of the usual claim "the graviton has spin $-$ 2" (that, of course, is strictly related to the possibility of achieving this special gauge in any "reasonable" physical situation) is strictly related to these assumptions.

The gravitational field is said to be *weak* (in M') if there exists a (harmonic) coordinates system and a region $M' \subset M$ of space-time in which the following conditions hold

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \qquad |h_{\mu\nu,\alpha}| \ll 1. \tag{4}$$

As it is known, in the weak field approximations in a harmonic coordinates system the Einstein equations read

$$\Box h_{\mu\nu} = 0. \tag{5}$$

The choice of the harmonic gauge plays a key role in deriving equation (5). No other special assumption either on the form or on the analytic properties of the perturbation h has been done. It is commonly believed that, with a suitable gauge transformation preserving the harmonicity of the coordinate system and the "weak character" of the field, one can always kill the "spin $-$ 1" components of the gravitational waves. However, even if not explicitly declared, the standard textbook analysis of the polarization is performed for globally *square integrable* solutions of the wave-equation (5) (that is, solutions which are square integrable everywhere on M) but, as we will see in the following, some very interesting solutions do not belong to this class.

What is lacking in our case is, obviously, the global *square integrability* due to the presence of the harmonic function solution of the two-dimensional Laplace equation. Therefore, non-globally square integrable spin $-$ 1 perturbations are not pure gauge because they cannot be killed by infinitesimal diffeomorphisms. Even if global *square integrability* is lacking, there exist solutions of this form that far away the singularities are perfectly well-behaved. In other words, spin $-$ 1 perturbations which are square integrable on a submanifold $M' \subset M$ of the whole spacetime can be found: thus, in order to exist, spin $-$ 1 perturbations necessarily need some singularities and/or some region with non trivial topology.

A transparent method to determine the spin of a gravitational wave is to look at its physical degrees of freedom, i.e., the components which contribute to the energy. One should use the Landau-Lifshitz (pseudo)-tensor t^{μ}_{ν} which, in the asymptotically flat case, agrees with the Bondi flux at infinity [8].

It is worth to remark that the canonical and the Landau-Lifchitz energy-momentum pseudo-tensors are tensors for Lorentz transformations. Thus, any Lorentz transformation will preserve the form of these tensor; this allows to perform the analysis according to the Dirac procedure. A globally square integrable solution $h_{\mu\nu}$ of the wave equation is a function of $r = k_\mu x^\mu$ with $k_\mu k^\mu = 0$. With the choice $k_\mu = (1, 0, 0, -1)$, we get for the energy density t^0_0 and the energy momentum t^3_0 the following result

$$16\pi t^0_0 = \frac{1}{4}(u_{11} - u_{22})^2 + u^2_{12}, \qquad t^0_0 = t^3_0$$

where $u_{\mu\nu} \equiv \mathrm{d}h_{\mu\nu}/\mathrm{d}r$. Thus, the physical components which contribute to the energy density are $h_{11} - h_{22}$ and h_{12}. These amplitudes are eigenvectors of the infinitesimal rotation generator $\mathcal{R}$, in the plane x–y belonging to the eigenvalues $\pm 2\mathrm{i}$. Thus, the components of $h_{\mu\nu}$ which contribute to the energy correspond to spin $-$ 2.

In the case of the prototype of spin $-$ 1 gravitational waves (3), for $f = 1/2$, we have

$$\tau_0^0 \sim c_1(h_{0x,x})^2 + c_2(h_{0y,x})^2, \qquad t_0^0 = t_0^3$$

where c_1 and c_2 are constants, so that the physical components of the metric are h_{0x} and h_{0y}. Following the usual analysis one can see that these two components are eigenvectors of $\mathrm{i}\mathcal{R}$ belonging to the eigenvalues ± 1. In other words, metrics (3), which are not pure gauge since the Riemann tensor is not vanishing, represent spin $-$ 1 gravitational waves propagating along the z-axis at light velocity.

This is related to the harmonic function of the transverse coordinates: in order to have an asymptotically flat wave, singularities or some sort of non triviality in the spacetime topology are necessary. The question is, can reasonable sources be found to smooth out the singularities? The answer is positive as we will see in more details in the next sections. Now we will show a simple and interesting example of such solutions.

As a simple example let us consider perturbations, as in equation (3), of the form $h = \mathrm{d}w(x,y) \cdot \mathrm{d}f(z-t)$ which are not globally square integrable. The metric

$$g = \eta + \mathrm{d}w(x,y) \cdot \mathrm{d}f(u), \qquad u = z - t, \qquad \left(\partial_x^2 + \partial_y^2\right) w = 0 \qquad (6)$$

being spatially asymptotically flat for a wide choice of harmonic functions w. Indeed, it represents a physically interesting gravitational field: gravitational waves propagating along the z-axis at light velocity. Besides to be a solution of the linearized Einstein equations on flat background, it is an exact solution of Einstein equations too.

It is trivial to verify that metric (6) is written in harmonic coordinates and has an *off-diagonal* form, that is, the perturbation h has only one index in the plane x–y orthogonal to the propagation direction z. For this reason the above gravitational wave has spin equal to one and is not a pure gauge [6]. With a suitable transformation it is possible to bring the above gravitational wave in the standard *transverse-traceless* form, however one can check that the new coordinates are not harmonic anymore.

Summarizing: *globally square integrable spin*-1 *gravitational waves propagating on a flat background are always pure gauge. Spin* $-$ 1 *gravitational waves which are not globally square integrable are not pure gauge.*

2.2. Asymptotic Flatness and Matter Sources

In the vacuum case, the coordinates (x, y, z, t) of the metrics (6) are harmonic. Being z the propagation direction, the physical effects manifest themselves in the x–y planes orthogonal to the propagation direction. In order these metrics be *asymptotically Minkowski* for $x^2 + y^2 \to \infty$, the function w is required to satisfy the condition

$$\lim_{x^2+y^2\to\infty} (w - c_1 x + c_2 y - c_3) = 0$$

where c_1, c_2 and c_3 are arbitrary constants and the behaviour of w can be easily recognized by looking at the Riemann tensor of the metrics (6)

$$R_{uiuj} = f_{,u} w_{,ij} \tag{7}$$

which depends on the second derivatives of the harmonic function w.

Therefore, to have an asymptotically Minkowski metric, the function w must be asymptotically close to a linear functions. But, due to standard results in the theory of linear Partial Differential Equations, this is impossible unless w is a linear function everywhere and this would imply the flatness of the metrics (6). However, if we admit δ-like singularities in the x–y planes, non trivial spatially asymptotically Minkowski vacuum solutions with $w \neq \text{const}$ can exist [7]. Of course, it is not necessary to consider δ-like singularities: it is enough to take into account matter sources. For example, in the presence of an electromagnetic wave propagating along the z axis, with energy density equal to ρ which vanishes outside a compact region of the x–y planes, the exact non vacuum Einstein equations for metrics (6) read (see, for example [7])

$$f_{,u} \left(\partial_x^2 + \partial_y^2 \right) w = \kappa \rho$$

where κ is the gravitational coupling constant. Thus, one can have non-singular spin $-$ 1 gravitational waves by considering suitable matter sources which smooth out the singularities.

From the phenomenological point of view, it is worth to note that these kind of wave-like gravitational fields, unlike standard spin $-$ 2 gravitational waves which can be singularities free even in the vacuum case, have to be coupled to matter sources in order to represent reasonable gravitational fields. The observational consequence of this fact is that spin $-$ 1 gravitational waves are naturally weaker than spin $-$ 2 gravitational waves [18]. Typically, if the characteristic velocity of the matter source is v, the spin $-$ 1 wave is suppressed by factors $(v/c)^n$ with respect to a spin $-$ 2 wave. It is worth to note that a gravitational field may also have a repulsive character. For instance, a Kerr black hole is "more repulsive" than a Schwarzschild black hole with the same mass. This is obviously related to the angular momentum. Roughly speaking, this effect may be attributed to the "gravitomagnetic" part of the Kerr metric which, in our terminology, is the "spin $-$ 1" part. On the other hand, the Kaluza-Klein mechanism allows to construct in pure five-dimensional gravity, solutions with spin $-$ 1 excitations (which in four dimensions may be interpreted as electromagnetic and, therefore, *repulsive*-degrees of freedom. Of course, the Kaluza-Klein mechanism also works when reducing from four to three dimensions. Solutions we are calling *spin* $-$ 1 *gravitational waves* when reduced to three dimensions (considering as extra dimension the propagation direction of the wave) give rise to purely electromagnetic fields.

3. A Gravitoelectromagnetic Perspective

A different point of view, which is useful in clarifying the nature of spin $-$ 1 gravitational waves is provided by the *gravitoelectromagnetism*, henceforth GEM (see, for example, [17]). In this scheme one tries to exploit as much as possible the similarities between the Maxwell and the linearized Einstein equations. To make this analogy evident it is enough to write a weak gravitational field fulfilling conditions (4) in the GEM form (see, for example, [17,22])

$$\mathrm{d}s^2 = c^2\left(1+2\frac{\Phi_{(g)}}{c^2}\right)\mathrm{d}t^2 + \frac{4}{c}\left(\mathbf{A}_{(g)}\cdot \mathrm{d}\mathbf{x}\right)\mathrm{d}t - \left(1-2\frac{\Phi_{(g)}}{c^2}\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad (8)$$

with

$$h_{00} = \frac{4\Phi_{(g)}}{c^2}, \qquad h_{0i} = -\frac{4A_{(g)i}}{c^2}$$

(in this section the speed of light c will be explicitly written). Hereafter, the spatial part of four-vectors will be denoted in bold and the standard symbols of three-dimensional vector calculus will be adopted. In terms of $\Phi_{(g)}$ and $\mathbf{A}$, the harmonic gauge condition reads

$$\frac{1}{c}\frac{\partial \Phi_{(g)}}{\partial t} + \frac{1}{2}\nabla\cdot\mathbf{A}_{(g)} = 0 \quad (9)$$

and, once the gravitoelectric and gravitomagnetic fields are defined in terms of GEM potentials, as

$$\mathbf{E}_{(g)} = -\nabla\Phi_{(g)} - \frac{1}{2c}\frac{\partial \mathbf{A}_{(g)}}{\partial t}, \qquad \mathbf{B}_{(g)} = \nabla\wedge\mathbf{A}_{(g)} \quad (10)$$

one finds that the linearized Einstein's equations resemble the Maxwell equations. Consequently, being the dynamics fully encoded in Maxwell-like equations, the GEM formalism describes the physical effects of the vector part of the gravitational field. The situations which are usually described in this formalism are, typically, static. In fact, when this assumption is dropped, GEM gravitational waves are also possible.

The energy-momentum content of gravitational fields of form in (8) is well described, in the asymptotically flat case, by the Landau-Lifshitz pseudo-tensor $t^{\mu\nu}$ [17].

Spin $-$ 1 gravitational waves, as one could expect on the basis of the Kaluza-Klein theory, can be put in an "almost" GEM form. This can be understood recalling that they have only one index in the plane transversal to the propagation direction. A generic spin $-$ 1 gravitational wave, propagating along the z axis on flat space-times, has the form in equation (2) with $u = z - t$ and $v = z + t$, and this is almost the same form of equation (8) provided one replaces the time-like index 0 with the light-like index u representing the propagation direction of the wave. The analogous of the gravitomagnetic potential reads in this case $\mathbf{A}_{(g)}(x, y, u) = (\varphi_{,x}, \varphi_{,y}, 0)$ and the harmonic gauge condition is

$$\nabla \cdot \mathbf{A}_{(g)} = \left(\partial_x^2 + \partial_y^2\right) \varphi = 0. \tag{11}$$

Thus, the analogue of the gravitoelectric and magnetic fields are

$$\mathbf{E}_{(g)} = -\frac{1}{2c}(\varphi_{,xu}, \varphi_{,yu}, 0), \qquad \mathbf{B}_{(g)} = (\varphi_{,yu}, -\varphi_{,xu}, 0) \tag{12}$$

and the Einstein equations reduce to

$$\nabla \cdot \mathbf{E}_{(g)} = -4\pi G\rho \tag{13}$$

so that, outside the matter sources, the harmonic gauge condition implies the vacuum field equations.

4. Back to Tolman-Ehrenfest-Podolsky-Wheeler Problem

A steady light beam lying along the z-axis is described by electromagnetic field $F_{\mu\nu}$ whose non vanishing components are

$$E_x = -F_{01} = E_0 \cos(kz - \omega t) = B_y = F_{31}.$$

The only non vanishing components of-the energy momentum tensor $T_{\mu\nu}$ are

$$T_{00} = T_{33} = -T_{03} = -T_{30} = E_0 2 \cos 2(kz - \omega t)/4\pi.$$

Taking the time average over a time grater than ω^{-1} and localizing the waves in a beam, we get

$$T_{00} = T_{33} = -T_{03} = -T_{30} = E_0 2\delta(x)\delta(y)/8\pi$$

where the Dirac delta-function δ has been introduced.

Thus, it is natural to consider a metric perturbation whose non vanishing components are

$$h_{00} = h_{33} = -h_{03} = -h_{30}, \qquad h^{\mu}_{\mu} = 0, \qquad \partial_t h_{\mu\nu} = \partial_z h_{\mu\nu} = 0.$$

Then, the gravitoelectric and the gravitomagnetic components of the metric are given by

$$E_{(g)\mu} = F_{(g)\mu 0}, \qquad B_{(g)_m} = -\varepsilon^{\mu ab} F_{(g)\alpha\beta}/2$$

where

$$F_{(g)}\mu\nu = \partial_\mu A_{(g)\nu} - \partial\nu A_{(g)\mu}, \qquad A_{(g)\mu} = -h_{0\mu}/4 = (\Phi_{(g)}, A_{(g)}).$$

It turns out that

- the first order geodesic motion for a *massive particle* in the light beam gravitational field is determined by the *force*

$$\mathbf{f}_{(g)} = -2\mathbf{E}_{(g)} - 4\mathbf{v} \wedge \mathbf{B}_{(g)}$$

  where $\mathbf{v}$ is the velocity of the particle.
- the first order geodesic motion for a *photon* propagating, in the light beam gravitational field, parallel(anti) to z-axis ($u_j = \pm\delta_{j3}$) is lightly different

$$\mathbf{f}_{(g)} = -4\left(\mathbf{E}_{(g)} + \mathbf{v} \wedge \mathbf{B}_{(g)}\right).$$

In previous section, we have seen that a gravitational wave generated by the light is described by the exact Einstein metric

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + wu^{-2}\mathrm{d}u^2.$$

In that case, the perturbation is given by

$$h_{00} = h_{33} = -h_{03} = -h_{30} = wu^{-2}$$

and we have

$$\mathbf{E}_{(g)} = -\frac{1}{4}\left(w_x, w_y, \frac{w}{u}\right)u^{-2}, \qquad \mathbf{B}_{(g)} = \frac{1}{4}\left(w_y, -w_x, \frac{w}{u}\right)u^{-2}.$$

The gravitational force acting over a massless particle is given by

$$\mathbf{f}_{(g)} = -[w_x(1 - v_z)\mathbf{i} + w_y(1 - v_z)\mathbf{j} + (w_x v_x + w_y v_y)\mathbf{k}]/4u^2.$$

If the photon propagates parallel to the light beam, $\mathbf{v} = (0, 0, 1)$, then

$$\mathbf{f}_{(g)} = 0$$

and there is not attraction or repulsion. It is worth to address that this result holds at first order approximation; the analysis in the strong gravity regime will be exposed in a forthcoming paper [31].

Thus, if the photon propagates antiparallel to the light beam $\mathbf{v} = (0, 0, -1)$, then

$$\mathbf{f}_{(g)} = -\nabla w/2u^2$$

and the force turns out to be attractive.

However, it is known from Quantum Field Theory that one consequence of spin $-$ 1 messengers is that particles with the same orientation repel and particles with opposite orientation attract.

Thus, the apparent lacking of attraction must be ascribed to the linear approximation since, according to our results, photons generate spin $-$ 1 gravitational waves and, as a consequence, two photons with same helicity must repel one another [31].

Are these effects observable?

It can be seen that the transversal acceleration per unit length for two laser beams in *VIRGO interferometer* ($W = 1$ watt, separation $d = 10\,\mathrm{cm}$) is only $\mathrm{d}v/\mathrm{d}l = 2.10^{-110}\,\mathrm{cm}^{-1}$ which is too small to be detected with the actual technology.

For a gravitational wave coming from *Virgo cluster* with dimensionless amplitude $h = 10^{-21}$ and frequency $\nu = 1\,\mathrm{KHz}$, it turns out that

$$\mathrm{d}v/\mathrm{d}l = h\nu/c = 3.3.10^{-29}\,\mathrm{cm}^{-1}.$$

Thus, even if the effects cannot be certainly observed in the Laboratory, they may be relevant at cosmic scale.

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