Eighth International Conference on
Geometry, Integrability and Quantization
June 9–14, 2006, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2007, pp 48–118

GEOMETRICALLY IMPLIED NONLINEARITIES IN MECHANICS AND FIELD THEORY

JAN JERZY SŁAWIANOWSKI

Institute of Fundamental Technological Research, Polish Academy of Sciences
21, Świętokrzyska Str., 00–049 Warsaw, Poland

Abstract. Here we discuss the concept of essential nonlinearity, i.e., one which cannot be meaningfully decomposed into well-defined linear background and "small" nonlinear correction. Therefore, the traditional perturbative techniques and asymptotic methods are non-effective then. Two well-established classes of essentially nonlinear field theories exist in the market: 1) The General Relativity and other generally covariant schemes, 2) The Born–Infeld type nonlinearity in traditional and generalized sense. The essential nonlinearity of 1) is intimately connected with the invariance under the very huge group $\mathrm{Diff}(M)$ of all space–time diffeomorphisms. The Born–Infeld scheme 2) is also geometrically motivated by the theory of scalar densities in manifolds. But there is no explicitly seen relationship between these two types of nonlinearities. Below we show that there exists however some hidden link between general covariance and Born–Infeld mechanism. The structure of the group of internal symmetries (target space symmetries) is also relevant. Roughly speaking, "huge" symmetry groups are intimately connected with essential strong nonlinearities. It is so even in finite-dimensional analytical mechanics. Let us remind our affinely-invariant models in mechanics of homogeneously deformable bodies [18–21,23,25–29]. There is no systematic theory, nevertheless some rough although convincing arguments do exist. This essay is just concentrated around the study of the interplay between (high) symmetries and (essential) nonlinearities. The examples quoted below confirm the idea and exhibit a kinship between general covariance and Born–Infeld paradigm. The special stress is laid on models which, by abuse of language, resemble the structured continua with affine geometry of degrees of freedom. These models are based on the bundles $\mathrm{L}M = T^1_1 M$ over the "space–time" manifold M, and FM, the principal bundle of linear frames. And the special stress is laid on scalar multiplets (trivial bundles over M).

1. Introduction

Our presentation is concentrated around the study of certain essentially nonlinear dynamical models and their invariance properties. The special stress is laid on nonlinearities the structure of which resembles the Born–Infeld electrodynamics. It turns out that there exists some relationship between this kind of nonlinearity and the symmetry group of the model. This is a special case of the more general, yet not completely understood "phenomenological" rule: mathematically and physically interesting nonlinear models turn out to be invariant under "large" and intuitively "natural" symmetry groups, sometimes "hidden" ones. Conversely, the demand of invariance of hypothetic models under such groups implies their essential nonlinearity. Of course, to some extent this is a rough qualitative statement. First of all, let us explain what we mean by "non-essential" and "essential" nonlinearity. Obviously, "non-essential" does not mean mathematically or physically trivial in any way and so perhaps the term "perturbative" would be more adequate. In such models there exists a well-defined linear background and nonlinearity appears as a "small" correction term, just "perturbation." Field theories used in elementary particle physics have such a structure. After quantization they work effectively when the perturbation techniques are used together with the renormalization procedure. Below we present some general discussion and review some geometrically distinguished models. The particular attention is devoted to generally covariant models for multiplets of scalar fields (they are related to things like strings membranes, p-branes, σ-models). By the way, there is an interesting class of scalar-valued models with $\mathrm{GL}(n,\mathbb{R})$ ($n = \dim M$) as the target space, i.e., with cross-sections of the trivial bundle $M \times \mathrm{GL}(n,\mathbb{R})$ as field variables. There exists an interesting kinship between models using the bundles $\mathrm{L}M$ (the bundle of mixed second order tensors on M), FM (the bundle of linear frames on M), $M \times \mathrm{GL}(n,\mathbb{R})$. All of them may be also considered as an alternative descriptions of gravitation or continua (both relativistic and non-relativistic) with internal degrees of freedom. There is some kinship between models discussed here and ones developed by Mladenov, Vassilev and Djondjorov in biomechanical models of cells [7,33]. There are also some similarities to models used in mechanics of engineering structures like plates, shells, etc. An essential part of this study is a formulation of open questions which in our opinion are worth of detailed study because of both geometrical and physical reasons. We start with some general remarks. Let us quote a few purely symbolic expressions. Physical situations will be denoted by elements Ψ of some linear space H. For example, when we deal with mechanical motion in a flat space, Ψ is a system of coordinates as functions of time, $\mathbb{R} \ni t \mapsto q^i(t) \in \mathbb{R}$, $i = 1,\dots,f$ (f is the number of degrees of freedom). In classical field theory Ψ is a vector-valued function on the physical space–time, $X \ni x \mapsto \Psi(x) \in V$, where the linear space V is the corresponding target space. Analytically, i.e., componentwise we use the symbols $\Psi^A(x^\mu)$ as field-theoretic counterparts of $q^i(t)$ ("fields" on the one-dimensional time axis). Let L be some linear operator on H and f some element of H. Homogeneous linear equations may be symbolically written as

$$L\Psi = 0 \tag{1}$$

and their linear non-homogeneous (affine, strictly speaking) counterparts have the form,

$$L\Psi = f \tag{2}$$

where f is physically interpreted either as a source or an external excitation term. In realistic applications which are of interest for us L is a differential operator of at most second order (in mechanics and classical field theory). In any case it is so in fundamental theories. In various branches of applied physics integral operators and higher-order differential operators are also used (e.g., fourth-order differential equations in shell theory). If L is a differential operator with constant coefficients, the homogeneous problem is in principle "rigorously solvable" in terms of Fourier transforms. In certain problems with variable coefficients also something may be done, e.g., with the use of Frobenius power series method or other means based on the function series expansions. Non-homogeneous equations are also treatable with the use of Green functions and variation-of-constants methods. Even if it is impossible to find a rigorous solution, approximate techniques like Galerkin and Ritz methods are in linear problems incomparably more efficient than in nonlinear ones. In weakly (perturbatively) nonlinear models the equations (1) and (2) are replaced respectively by

$$L\Psi + N(\varepsilon, \Psi) = 0 \tag{3}$$

$$L\Psi + N(\varepsilon, \Psi) = f \tag{4}$$

where $N(\varepsilon, \cdot)$ are nonlinear operators in H depending in a sufficiently smooth way on the real parameter ε and vanishing when ε vanishes

$$N(0, \Psi) = 0. \tag{5}$$

Therefore, for $\varepsilon = 0$ the problem becomes linear. Roughly speaking, the magnitude of ε controls the degree of nonlinearity. It is convenient to assume $N(\cdot, \Psi)$ to be analytic at $\varepsilon = 0$ and expand it into power series,

$$N(\varepsilon, \Psi) = \sum_{k=1}^{\infty} \varepsilon^k N_k(\Psi). \tag{6}$$

Obviously, in realistic and effective models $N(\varepsilon, \Psi)$ is a low-order polynomial of ε, i.e., $N_k = 0$ for $k > m$, where m is some fixed threshold. Very often, but obviously not always, $N(\varepsilon, \Psi) = \varepsilon N_0(\Psi)$, i.e., N is linear in ε. In fundamental theories based on at most second order differential equations the quantities $N_k(\Psi)$, and therefore the total $N(\varepsilon, \Psi)$ as well, are built in a pointwise algebraic way on Ψ and its at most second order derivatives,

$$(N_k(\Psi))(x) = \mathcal{N}_k \left(\Psi(x), \partial\Psi(x), \partial^2\Psi(x)\right). \tag{7}$$

The perturbative procedure consists in assuming that Ψ is in the form of power series of ε

$$\Psi = \sum_{n=0}^{\infty} \varepsilon^n \Psi_n \tag{8}$$

and substituting it into (3) or (4). The quantities Ψ_n are independent of ε. If N is non-polynomial in ε, i.e., $m = \infty$ (non-efficient, non-physical model), then one still can Taylor-expand the N_k-terms about Ψ_0. After substituting (8) into (3) or (4) one obtains on the left-hand side of these equations the infinite power series with respect to ε. To satisfy the resulting equation identically with respect to ε, one must put the ε-independent term to zero (respectively to f for non-homogeneous case) and also all coefficients at ε^n, $n > 0$, must vanish. In this way one obtains the infinite hierarchy of equations for coefficients Ψ_n. It implies that Ψ_0 is a solution of the background linear system (1) or (2). And this solution is assumed to be "known." For Ψ_n, $n > 1$, one obtains non-homogeneous equations with "source" ("excitation") terms built of the "earlier" Ψ_k-s, $k < n$. "Solving" this hierarchy and using the "known" solutions Ψ_0 of the background linear system one obtains "in principle" the total Ψ. Obviously, one can achieve this just only "in principle." And even if we manage to calculate all coefficients Ψ_n explicitly, it turns out that, as a rule, the resulting series (8) is merely an asymptotic one, usually divergent. Usually one terminates on determining the first order correction, i.e., Ψ_1. It is intuitively seen that this procedure is rather artificial, it is a kind of the "necessary evil." In a sense, it looks like a miracle that in quantum field theory the union of perturbative expansion and renormalization procedure is so effective at least in electrodynamics and electro-weak interactions. It fails however in strong interactions, where ε is "large." In fundamental field theories one derives differential equations from the variational principle. Lagrangian densities $\mathcal{L}_0$ of linear models are built in a local quadratic way of the pair $(\Psi, \partial\Psi)$. In specially-relativistic Poincare-invariant theories expressed in terms of pseudo-Euclidean coordinates the coefficients of the underlying quadratic forms are constant (x^μ-independent). Obviously, they are non-constant in curvilinear coordinates. In Lagrangian theories it is more convenient to introduce the perturbative nonlinearity by deforming $\mathcal{L}_0$ in the following way

$$\mathcal{L}(\Psi, \partial\Psi) = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{L}_n(\Psi, \partial\Psi) \tag{9}$$

where $\mathcal{L}_n$ are polynomials of $(\Psi, \partial\Psi)$ of the degree higher than 2 if $n > 0$.

Remarks:

i) The meaning of ε in (9) differs from that in (6), however, to avoid the crowd of symbols we do not change the notation.

ii) When n increases, the polynomial degree of $\mathcal{L}_n$ increases as well, however it need not and in general does not equal $n+2$, with the obvious exception of $n=0$.

Writing down the Euler–Lagrange equations for (9) we again obtain (3), (4) or (6) and "solve" it with respect to Ψ_n appearing in the Ansatz (8) (keeping in mind that the meaning of ε is now different). Let us remind that the quantum field-theoreticians use the terminology according to which the linear models, i.e., ones based on $\mathcal{L}_0$, are non-interacting. From this point of view any discrete or continuous system of harmonic oscillators is "non-interacting," i.e., "free." Obviously, literally this is not true, because the elements of such a system are mutually coupled by some "elastic strings." But the resulting system is trivial because in principle "already solved." Indeed, the normal modes coordinates turn it into a system of fictitious non-interacting one-dimensional harmonic oscillators. Anharmonic terms may qualitatively perturb this structure, generating mutual (and irreducible) interaction between modes. That is why for quantum field-theoreticians the linear background $\mathcal{L}_0$ is free and genuine interactions are introduced by the terms $\mathcal{L}_n$, $n>0$. Let us quote a few commonly known examples:

- Quartically-corrected charged Klein–Gordon field

$$\mathcal{L} = g^{\mu\nu}\partial_\mu\overline{\Psi}\partial_\nu\Psi\sqrt{|g|} - m^2\overline{\Psi}\Psi\sqrt{|g|} - \varkappa\left(\overline{\Psi}\Psi\right)^2\sqrt{|g|} \tag{10}$$

where g denotes the space–time metric tensor, and $|g|$ is an abbreviation for the absolute value of its determinant in given coordinates

$$|g| := |\det\left[g_{\mu\nu}\right]| \,. \tag{11}$$

Here it is $\varkappa$ that plays the role of the perturbation parameter ε. This toy model, especially with the real (neutral) field Ψ, was an important "theoretical laboratory" for studying quantum fields phenomena. Let us mention incidentally that in general the quartic Lagrangians, i.e., cubically-nonlinear discrete or continuous oscillator systems play an essential role in mechanics and field theory as they provide the simplest models of physically reasonable (e.g., reflections-invariant) nonlinearities. The most elementary model is that of one-dimensional cubically anharmonic oscillator in mechanics

$$L = \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{k}{2}x^2 - px^4 \tag{12}$$

where m, k, p are constants. Some toy models of deterministic chaos may be formulated in such terms. When speaking about Lagrangians algebraically quartic in the field Ψ (thus cubically-nonlinear) one should mention about Higgs models and their profound role in explaining the mass generation of gauge fields via the spontaneous symmetry breaking mechanism. This is however something physically else than (10), namely the constant m^2 is then negative and cannot be directly interpreted as the squared mass of the linear background.

- The coupled system: Maxwell and charged Klein–Gordon field,

$$\mathcal{L} = g^{\mu\nu} D_\mu \overline{\Psi} D_\nu \Psi \sqrt{|g|} - m^2 \overline{\Psi}\Psi \sqrt{|g|} - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \sqrt{|g|} \tag{13}$$

with the usual meaning of symbols

$$D_\mu \Psi = \partial_\mu \Psi - \mathrm{i}eA_\mu \Psi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{14}$$

where e is the coupling constant (elementary charge in natural units), and A_μ are the components of the covector potential of the electromagnetic field. In this minimal-coupling scheme e plays the role of the perturbation parameter ε and the nonlinear correction is a second degree polynomials of e. The terms linear and quadratic in e are respectively given by

$$\mathrm{i}eg^{\mu\nu} A_\mu \left(\overline{\Psi} \partial_\nu \Psi - \Psi \partial_\nu \overline{\Psi} \right) \sqrt{|g|} = g^{\mu\nu} A_\mu j_\nu = A_\mu j^\mu \tag{15}$$

$$e^2 g^{\mu\nu} A_\mu A_\nu \overline{\Psi}\Psi \sqrt{|g|}. \tag{16}$$

They are respectively cubic and quartic in the field system (Ψ, A).
Remark: j_μ on the right-hand side of (15) denotes the Noether $\mathrm{U}(1)$ current of Ψ, not the local $\mathrm{U}(1)$-gauge-invariant electric **four-current** $\Im_\mu$ appearing in Maxwell equations as the source term as the latter is obviously given by

$$\Im_\mu = \mathrm{i}e \left(\overline{\Psi} D_\mu \Psi - \left(D_\mu \overline{\Psi} \right) \Psi \right) \sqrt{|g|}.$$

- The coupled system: Maxwell and charged Dirac field,

$$\begin{aligned} \mathcal{L} = {} & \frac{\mathrm{i}}{2} e^\mu{}_A \left(\widetilde{\Psi} \gamma^A D_\mu \Psi - \left(D_\mu \widetilde{\Psi} \right) \gamma^A \Psi \right) \sqrt{|g|} - m \widetilde{\Psi}\Psi \sqrt{|g|} \\ & - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \sqrt{|g|} \end{aligned} \tag{17}$$

where the meaning of symbols is as follows: $e^\mu{}_A$ are the components of some g-orthonormal anholonomic reference frame,

$$g(e_A, e_B) = g_{\mu\nu} e^\mu{}_A e^\nu{}_B = \eta_{AB} \tag{18}$$

η is the standard **Minkowski metric** on $\mathbb{R}^4$

$$[\eta_{AB}] = \mathrm{diag}(1, -1, -1, -1) \tag{19}$$

$e^A{}_\mu$ are components of the dual co-tetrad of e^A,

$$\left\langle e^A, e_B \right\rangle = e^A{}_\mu e^\mu{}_B = \delta^A{}_B, \quad e^\mu{}_A e^A{}_\nu = \delta^\mu{}_\nu, \quad g = \eta_{AB} e^A \otimes e^B$$

and thus,

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu, \qquad g^{\mu\nu} = e^\mu{}_A e^\nu{}_B \eta^{AB} \tag{20}$$

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu, \qquad \eta^{AC}\eta_{CB} = \delta^A{}_B$$

$$\gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB} I_4 \tag{21}$$

$$\Gamma^A{}_{\bar{r}s} = \overline{\Gamma^A{}_{\bar{s}r}}, \qquad \Gamma^A{}_{\bar{r}s} = G_{\bar{r}z}\gamma^{Az}{}_s \tag{22}$$

where G is sesquilinear hermitian of neutral signature, e.g.,

$$[G_{\bar{r}s}] = \mathrm{diag}(1, 1, -1, -1). \tag{23}$$

Besides

$$\widetilde{\Psi}_r = \overline{\Psi}^{\bar{s}} G_{\bar{s}r} \tag{24}$$

is Dirac conjugate bispinor,

$$\omega_\mu = \frac{1}{8}\Gamma_{KL\mu}\left(\gamma^K\gamma^L - \gamma^L\gamma^K\right) \tag{25}$$

is the bispinor connection and

$$\Gamma_{KL\mu} = -\Gamma_{LK\mu} = \eta_{KM}\Gamma^M{}_{L\mu} \tag{26}$$

is the $\mathrm{SO}(1,3)$-ruled connection with

$$\Gamma^\alpha{}_{\beta\mu} = e^\alpha{}_A \Gamma^A{}_{B\mu} e^B{}_\beta + e^\alpha{}_A e^A{}_{\beta,\mu}. \tag{27}$$

Automatically $\Gamma^\alpha{}_{\beta\mu}$ is a **Riemann–Cartan connection**, i.e.,

$$\nabla_{[\Gamma]} g = 0 \tag{28}$$

and

$$D_\mu\Psi = \partial_\mu\Psi + \omega_\mu\Psi - \mathrm{i}eA_\mu\Psi = \nabla_\mu\Psi - \mathrm{i}eA_\mu\Psi \tag{29}$$

is the covariant differentiation of bispinors. This is a crowd of symbols, obscure when "telegraphically" quoted. However it simplifies remarkable in flat Minkowskian space when pseudo-Cartesian coordinates and their associated tetrad fields are used

$$e^A{}_\mu = \delta^A{}_\mu, \qquad g_{\mu\nu} = \eta_{\mu\nu}, \qquad \Gamma^\alpha{}_{\beta\mu} = 0, \qquad \Gamma^A{}_{B\mu} = 0. \tag{30}$$

The general message is that the nonlinear correction term is linear in the perturbation parameter e (coupling constant), Lagrangian is cubic in the field system (Ψ, A) and the field equations are quadratically nonlinear. The nonlinear correction term is given by

$$eA_\mu e^\mu{}_B \widetilde{\Psi}\gamma^B\Psi = A_\mu j^\mu. \tag{31}$$

The models quoted above are well established in physics, experimentally confirmed and theoretically efficient. The coupling constants controlling the nonlinear interaction terms appeared there as a low-degree polynomial. There are also other practically useful and theoretically important models like, e.g., sin-Gordon, sinh-Gordon, etc. Nevertheless, usually they have the same general structure: additively combined linear backgrounds of well-known properties and nonlinear corrections describing "true interactions." In spite of their practical utility, the above and other perturbative models (linear background plus nonlinear corrections) look rather artificial from the point of view of perspectives and philosophical foundations. One has the feeling that there is something provisional, non-essential in this kind of nonlinearity. It seems much more natural to search essentially nonlinear models without any distinguished linear background, when the perturbative procedures fail and the only reliable way is a kind of constructive analysis based on some geometric ideas, first of all on symmetry principles. This is just what we mean by essential nonlinearity. Obviously, the very necessity of nonlinear studies follows directly from experimental data and even from very rough phenomenological models. But there are also very deep, fundamental arguments. As mentioned, without nonlinearity there is no thermalization of energy in multiparticle and continuous systems like fields and radiation, no equipartition, etc. Let us also mention the classical problem of the relationship between field equations and equations of motion of field sources, e.g., the paradox of non-interacting charges in Maxwell electrodynamics. And even in models without canonical linear background in fundamental dynamical laws, the idea of linearization is non-reliable and often dangerous or tricky. For example, it happens that luckily some particular solution or a set of particular solutions may be found. Then it is a natural temptation to linearize the problem in a neighbourhood of the particular known solution. Namely, the unknown function is represented as a sum of this background solution and some "small" correction, substituted to the original field equation, and then only the terms linear in this correction are retained while all higher-order ones are dropped out. One obtains a linear equation for the perturbation terms. This is the so-called Jacobi field. In variational theories it is ruled by some effective variational principle based on the quadratic Lagrangian. However, as a rule, in strongly nonlinear field problems it is only solutions with large and geometrically well-established symmetry groups that (sometimes) may be found in an explicit analytical form. And the point is that in generally covariant field theories (infinite-dimensional symmetry group of the Lagrangian is essential here) solutions invariant under some Lie subgroups of transformations (do not confuse the symmetry group of the Lagrangian with its subgroup preserving a given solution) are often pathological and always suspected from the linearization point of view. The set of a priori admitted fields is an infinite-dimensional manifold. The particular structure details usually depend of some physical demands because there is even no natural, canonical topology in infinite dimension. One expects the general solution of field equations to be a differential submanifold in the mentioned variety of all "kinematically allowed" fields. It turns out however that some critical points-solutions may exist at which there is no well-defined tangent space and the submanifold structure breaks down there. And it is just solution invariant under Lie subgroups where it may happen. As linearization procedure consists just in moving infinitesimally along tangent vectors, it fails in such situations. Such "linearly non-perturbable" solutions are various "cusps" in the variety of fields given by the general solution. The assumed symmetry demands to facilitate remarkably the finding of some particular solutions, however, these "beautiful" solutions are just exceptional and have a good chance to be singular points and therefore this is some kind of qualitatively deep non-stability. So, there are two shortcoming of simplifying things on the basis of linearity idea – the linear background of dynamics seems artificial and the linearization procedure is non-reliable. One has the feeling that some link exists between essential nonlinearity and high-symmetry demand. There is an important message from the soliton theory where good, non-accidental nonlinearities lead to an infinite number of constants of motion, therefore, to "large" groups of hidden symmetries. But also conversely: roughly speaking, linearity implies that the action functional is quadratic in the system $(\Psi, \partial\Psi)$ or, more precisely, it becomes so if an appropriate coordinatization of the target space is chosen (usually one deals with vector bundles and a "proper" coordinatization is self-evident). Therefore, the Lagrangian is a local quadratic function of $(\Psi, \partial\Psi)$, possibly with x^{μ}-dependent coefficients. But this means that the dynamical model pre-assumes some fixed bilinear (or sesquilinear) scalar product as an absolute object. And to be able to construct a quadratic Lagrangian as a one-component geometric object (scalar W-density of weight one) one must have at disposal a fixed metric tensor in space-time. These pre-established quadratic forms (external with respect to the physical degrees of freedom of a given model) restrict the groups of dynamical symmetries to certain isometries (quadratic forms must be preserved), i.e., to relatively "small" groups. To avoid this restriction one should somehow avoid absolute objects replacing them by something dependent on degrees of freedom, i.e., on the fields Ψ themselves. But quadratic forms on a linear space V become non-quadratic expressions when their constant coefficients are replaced by some functions on V. The action functional becomes non-quadratic in $(\Psi, \partial\Psi)$ and leads to nonlinear field equations for Ψ. The natural demand of higher dynamical symmetry implies nonlinearity, in this case an essential one because geometrically motivated. Summarizing: "essential nonlinearity" implies large symmetry groups, and conversely, large symmetry group often just implies essential nonlinearity. To finish this philosophical introduction let us mention a profound master pattern. This is the General Relativity. When no matter is included, just only the pure gravitational field, it is the dynamical metric tensor g that is used as the above field quantity Ψ. It gives rise to the well-known sequence of concomitants, namely, the Levi-Civita affine connection, its curvature tensor, Ricci tensor and finally the scalar curvature $R[g]$. Variational principle of General Relativity is based on the **Hilbert Lagrangian**

$$\mathcal{L}_H[g] = \mathcal{L}_H\left(g, \partial g, \partial^2 g\right) = -\frac{1}{16\pi k} R[g]\sqrt{|g|} \tag{32}$$

possibly modified additively by the cosmological term,

$$\mathcal{L}_{\mathrm{cosm}}[g] = \Lambda\sqrt{|g|} \tag{33}$$

with Λ denoting the cosmological constant. As is well known, $\mathcal{L}_H$ depends on the second derivatives $\partial^2 g$ in an artificial way, namely, linearly with coefficients depending on g alone, not on ∂g. The corresponding term may be represented as a total divergence and removed from the Lagrangian. The main term of the Lagrangian is proportional to

$$g^{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} \partial_\mu g_{\alpha\beta} \partial_\nu g_{\gamma\delta}. \tag{34}$$

Because of this the leading second order differential term of field equations is given by

$$g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta}. \tag{35}$$

This is obviously a non-tensorial expression, nevertheless, obviously, field equations are tensorial

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{36}$$

where $R_{\mu\nu}$ are components of the Ricci tensor built of g. In the absence of matter this is obviously identical with

$$R_{\mu\nu} = 0. \tag{37}$$

There are no fixed absolute objects, tensor indices at g, ∂g are contracted just with the use of g itself, not something external with respect to g. Because of this the action functional is non-quadratic in $(g, \partial g)$ and the resulting field equations are nonlinear in g, although they are quasi-linear. And because of the absence of absolute objects, variational principle and field equations are invariant under the infinite-dimensional group Diff(M) of all diffeomorphisms of the space-time manifold M. The elements of this huge group are labelled by four arbitrary (up to smoothness demands) functions on the space-time manifold (coordinates of image-points as functions of argument-points). There is no well-defined linear background in the dynamics, nonlinearity is essential and non-perturbative. And really the very strong Diff(M)-invariance demand (general covariance) just implies the essential nonlinearity, moreover, up to cosmological term, it determines uniquely the corresponding nonlinear model. Let us also remind some models from the realm of mechanics of systems with a finite number of degrees of freedom. In our earlier papers [18–21, 25–29] we have discussed the so-called affinely-rigid bodies, i.e., roughly speaking, homogeneously deformable gyroscopes. The idea appeared also earlier in mechanics of structured continua, in molecular dynamics and the theory of molecular crystals, and also in certain astrophysical problems. Application of the model in macroscopic elasticity and in dynamics of inclusions and suspensions are also possible. Practically in all papers devoted to this topic the kinetic energy was quadratic with constant coefficients in generalized velocities. Therefore, the corresponding contribution to equations of motion was linear. Such a structure of kinetic energy implied that the corresponding symmetry groups in physical and material spaces were various subgroups of the corresponding orthogonal or rather isometry groups. This is qualitatively incompatible with the affine group which rules geometry of degrees of freedom and kinematics. The kinetic term of the corresponding Hamiltonian system with the affine or linear group as a configuration space is neither left- or right-invariant. Therefore, it does not belong to the category of invariant systems on groups as developed by Arnold, Hermann and others. And because of this theoretical and analytical profits from the group structure of degrees of freedom are rather limited in comparison with system with left or right (or both) invariant geodetic backgrounds. If we demand the kinetic energy form (i.e., the configuration metric tensor underlying it) to be affinely-invariant, the coefficients of the corresponding quadratic form become essentially non-constant, i.e., non-reducible to constants by any change of generalized coordinates, and therefore the configuration metric has a non-vanishing curvature tensor. Because of this one obtains an essential non-perturbative nonlinearity even before introducing any potential (interaction) term. In such a model with affinely-invariant geodetic background three very interesting new novelties appear:

i) the dynamics of the volume-preserving elastic vibrations may be encoded in a purely geodetic model, without any potential term. This encoding of the dynamics in the configuration space metric tensor resembles the Jacobi–Maupertuis variational principle. The resulting models of elastic vibrations shows a strong non-perturbative nonlinearity. In a sense this is an over-simplified finite-dimensional counterpart of the nonlinearity which appears in the General Relativity as a consequence of postulating the total diffeomorphism group as a physical symmetry. It is also a finite-dimensional model of the nonlinearity appearing in hydrodynamic equations of the ideal incompressible fluid (where the essential nonlinearity of Euler equations has to do with the huge group of the volume-preserving diffeomorphism of the material space).

ii) to some extent the above mentioned models may be explicitly solved in terms of the exponential matrix expressions. Just here the dynamical affine invariance enables one to use the standard analytical techniques applicable to systems on Lie groups.

iii) it turns out that the resulting equations shed some light on the dynamics of one-dimensional multiparticle chains.

Roughly speaking, passing over to the "large" affine group of symmetries is a finite-dimensional model of passing over to the invariance under the group of all diffeomorphisms. In both cases the remarkable extension of the symmetry group leads to essential, non-perturbative nonlinearities where the perturbative methods fail.

2. Born–Infeld-type Nonlinearities

It is our opinion that on the fundamental level the deepest and most promising nonlinearities are those based on some extension, generalization of the model of nonlinear electrodynamics formulated long ago by Born and Infeld [3–5,14,16,30]. It is interesting that such models are also useful in quite practical, almost engineering problems of shells and membranes. Incidentally, such intuitive, near to the common sense "engineering" concepts inspired also some models in fundamental physics like strings, p-branes, etc. At the same time, they turn out to be of interest for biophysics and biomechanics, e.g., in the dynamics of biological cells, erythrocytes, leucocytes, etc. Quite an interdisciplinary model covering fundamental fields of theoretical physics and children-toys-like soap bubbles. From some point of view the modified Born–Infeld nonlinearity is just optimal and most natural within the framework of theories based on seriously treated variational principles. And as yet one believes commonly that fundamental field and mechanical theories are structurally variational. This is particularly evident when one starts from quantum models as primary ones, because the path integration formalism just assumes Lagrangian as something fundamental. It turns out that there is a natural link between generalized Born–Infeld nonlinearities and symmetry principles, including also hidden symmetry groups. We mentioned above that nonlinearity of Einstein equations and the structure of Hilbert variational principle are particularly interesting as a model of essential, non-perturbative nonlinearity in fundamental field theories. Nevertheless, even this theory is based on some kind of a compromise with the linearity idea. Namely, it is quasi-linear, i.e., the second order (highest-order) derivatives of field quantities enter the field equations (Einstein equations) in a linear way with coefficients built algebraically of the fields themselves. In more details, in the Einstein theory the coefficients at $\partial_\alpha \partial_\beta g_{\mu\nu}$ are rational functions of g-quantities, simply $g^{\alpha\beta}$ are the components of the reciprocal contravariant metric. Unlike linearity, the quasi-linear structure is compatible with the demand of general covariance (invariance under the huge group of all smooth diffeomorphisms). The early ideas of nonlinear electrodynamics were motivated by two problems faced with in Maxwell theory:

i) infinite electromagnetic mass of the electron (of point sources in general)

ii) a bad relationship between field equations and equations of motion of its sources (their independence in a sense).

Infinite electromagnetic self-energy is not immediately seen as a direct consequence of linearity. It follows from the simple formula based on classical electrostatics and Coulomb potential. But in the Maxwell theory Coulomb potential is not assumed from outside as it describes a stationary spherically symmetric solution of Maxwell equations without extended sources. When physically reasonable "boundary conditions" at infinity are chosen, this solution is unique up to the multiplicative integration constant, just interpreted as a charge value of the hypothetic point source placed at the centre of spherical symmetry. Within the specially-relativistic framework this is the only solution (up to the mentioned conditions at infinity) invariant under the "small subgroup" of the Poincare group preserving a fixed time-like straight line (the world line of a freely moving "charge"). And the corresponding "infinite self-energy" appears when the Minkowski-orthogonal projection of the energy momentum tensor onto this world line is integrated over any three-dimensional spatial hyperplane orthogonal to the world line (synchronous hypersurface). Taking two world lines and superposing the corresponding solutions we obtain a new solution defined in the open region of space-time remaining after removing the world lines. This is a consequence of linearity of free Maxwell equations. And so it is for any finite or discrete system of time-like straight lines. But this solution represents a system of freely moving, non-interacting charged particles, thus something non-physical. Nonlinearity would prevent this kind of solutions. It is just the case in the General Relativity where, moreover, equations of the gravitational field imply in a sense equations of motion of particles. This was an additional motivation for the search of nonlinear electrodynamics. And after Dirac some hypothetic nonlinearity introduced by non-electromagnetic cohesive forces was expected to prevent the self-acceleration catastrophe in classical electrodynamics. The idea of the Born–Infeld electrodynamics was simple: to avoid divergences one must prevent the electromagnetic field to be "too strong," i.e., introduce some saturation mechanism. Some hint is suggested by relativistic mechanics. Namely, if some reference frame is fixed and we use the corresponding $3+1$ decomposition of Minkowski space–time, Lagrangian of the relativistic material point has the form

$$L = L_{\mathrm{kin}} - U(t, \bar{r}, \bar{v}) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - U(t, \bar{r}, \bar{v}) \tag{38}$$

with the standard meaning of symbols, i.e., $(t, \bar{r}, \bar{v})$ denote respectively the time variable, radius vector and velocity vector, v^2 is the squared absolute value of $\bar{v}$ and the label "kin" refers to the "kinetic" term of L and obviously, c is the velocity of the light. The crucial point is that L_{kin} is non-differentiable at $v = c$ and this luminal situation is a repulsive singularity. If the resulting Euler–Lagrange equations are written in the Newton form

$$m\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \bar{F}(t, \bar{r}, \bar{v}) \tag{39}$$

it is seen that the repulsive singularity of $\bar{F}$ at $v = c$ prevents the material point to exceed the velocity of light, independently of the shape of generalized potential U. Similarly, the electromagnetic Born–Infeld Lagrangian is given by

$$\mathcal{L} = b^2\left(1 - \sqrt{1 - \frac{2}{b^2}S - \frac{1}{b^4}P^2}\right)\sqrt{|g|} \tag{40}$$

where b is a constant and S, P are basic invariants of the electromagnetic field, respectively the scalar and pseudoscalar one

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} = \frac{1}{2}\left(\bar{E}^2 - \bar{B}^2\right) \tag{41}$$

$$P = -\frac{1}{4}F_{\mu\nu}\check{F}^{\mu\nu} = -\frac{1}{8\sqrt{|g|}}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \bar{E}\cdot\bar{B} \tag{42}$$

with the obvious meaning of symbols. Here ε is the totally skew-symmetric Ricci symbol with the convention

$$\varepsilon^{0123} = 1 \tag{43}$$

and therefore,

$$\varepsilon_{0123} = -1. \tag{44}$$

The Lagrangian (40) was the final model. The primary idea was

$$\mathcal{L} = b^2\left(1 - \sqrt{1 + \frac{1}{b^2}\left(\bar{B}^2 - \bar{E}^2\right)}\right)\sqrt{|g|}. \tag{45}$$

The model was originally formulated on the basis of the flat Minkowskian space–time in pseudo-Cartesian coordinates where

$$[g_{\mu\nu}] = \mathrm{diag}(1, -1, -1, -1), \qquad \sqrt{|g|} = 1.$$

The final version (40) may be written down as follows

$$\mathcal{L} = b^2\sqrt{|g|} - \sqrt{|\det\left[bg_{\mu\nu} + F_{\mu\nu}\right]|}. \tag{46}$$

Obviously, the first term, independent on the electromagnetic field (and constant in specially-relativistic theory formulated in pseudo-Cartesian coordinates) does not influence field equations and is chosen in such a way that both the Lagrangian and field energy (to be more precise, energy-momentum tensor) vanish when the field F vanishes. The parameter b in (40) and (46) fixes the saturation strength of the field. Saturation is attained when the expression under the square-root sign vanishes, i.e., when the tensor $bg + F$ has a singular coefficients matrix. Just like in the relativistic point mechanics one is faced with "repulsive singularity" and the field cannot attain some finite critical strength. Therefore, the field remains bounded in a neighbourhood of point sources. Namely, for the vacuum field (no external continuously distributed charges) the stationary spherically symmetric solution has, in an appropriate gauge, the vanishing magnetic vector potential and the scalar potential is given by

$$\varphi(r) = \int_r^\infty \frac{e\,\mathrm{d}x}{\sqrt{r_0^4 + x^4}}, \qquad r_0 = \sqrt{\frac{e}{b}} \tag{47}$$

where e is an integration constant interpretable physically as the value of the electric charge placed at $r = 0$. The non-essential additive constant is chosen as zero so that φ vanishes at infinity. If r/r_0 is large, (47) asymptotically approaches the Coulomb formula

$$\varphi(r) = \frac{e}{r} \tag{48}$$

following from the Maxwell theory. Obviously, the electric field is given by

$$\bar{E}(\bar{r}) = \frac{e}{\sqrt{r_0^4 + r^4}} \frac{\bar{r}}{r} \tag{49}$$

with $\bar{r}$ denoting the radius vector laid off from the symmetry centre. It is seen that $\bar{E}$, although bounded around the origin, is non-definite there, just as expected, because the scalar potential as a function on the three-dimensional space suffers non-differentiability at $\bar{r} = 0$. The reason is that the derivative of (47) as a function of one real variable r has a non-vanishing limit when $r \to +0$. The electric displacement (induction) vector $\bar{D}$ and the energy density $w = T_{00}$ are infinite at $r = 0$, nevertheless the total energy, i.e., the electromagnetic mass multiplied by c^2, is finite

$$\mathcal{E} = \int w \mathrm{d}_3\bar{r} < \infty \tag{50}$$

(the improper integral is convergent). For our purposes the most important message of (46) is its structure as the square root of the determinant of matrix components of some twice covariant tensor built in a simple way from the field (its first derivatives, to be more precise). This will be just the main hint for developing our models. However, before doing this we remind briefly some important features of historical Born–Infeld models. These features make the Born–Infeld paradigm promising and reliable at least just as a guiding idea. Various models of nonlinear electrodynamics were formulated, let us mention, e.g., one due to G. Mie. However, the Born–Infeld theory is characterized by the astonishing and amazing coincidence of a lot of very desirable things [3–5, 16]. Let us quote them as an evidence of exceptionality of this theory and its uniqueness in a sense:

- Born–Infeld model is gauge-invariant, unlike, e.g., Mie theory
- field energy is positively definite

- point charges have a finite electromagnetic mass (finite electrostatic self-energy)
- the energy current four-vector is not space-like
- there is no birefringence in vacuum
- there exist plane wave solutions imposed onto the background of the constant electromagnetic field. In particular, solitary waves do exist.

What concerns the peculiarity and exceptionality of (40) and (46) however, let us mention that (45) is equivalent to the final Born–Infeld model in all problems concerning stationary spherically symmetric solutions. And in both models the quantity r_0 may be interpreted in a sense as the classical radius of the electron. After some period of focusing the attention of physicists the Born–Infeld model lost for some time its attractive power, in spite of the advantages listed above. Activity of physicists concentrated mainly on quantum problems, in particular on quantum electrodynamics. Efficacy of renormalization techniques in QED reduced remarkably the motivation for fighting with infinities in classical electromagnetism, the more so that even the purely classical Dirac renormalization turned out to be relatively successful. One expected the evidence of nonlinearity in the spectra of superheavy atoms, but nothing has been found. And there are some intrinsic theoretical difficulties in the Born–Infeld model. The Lagrangian is non-polynomial, the nonlinearity is perfectly essential and non-perturbative, and, therefore, no easy success in quantization might be expected. Although the paradox of freely moving non-interacting point charges does not occur in Born–Infeld theory, there was no remarkable success in deriving equations of motion from the field equations. In this respect the analogy with the General Relativity is rather misleading. The point is that it is not a mere nonlinearity that is responsible for the generally-relativistic problem of motion. This is the very special kind of nonlinearity implied by the general covariance, i.e., invariance of the Hilbert Lagrangian with respect to the group Diff(M) of all diffeomorphisms of the space–time manifold M. Elements of this group are labelled by n arbitrary functions of n variables, where, obviously, $n = \dim M$ (physically $n = 4$). According to the Noether theory this implies n identities. Roughly speaking, they have to do with the four-momentum balance and in the case of purely mechanical sources they are essentially equivalent to the equations of motion. These equations need not be separately formulated and in variational principle there is no need to subject world lines to the variation procedure. Some difficulty is faced with when the "external" charged matter is to be taken into account. In a sense the primary motivation was monistic: the pure field was to be "materia prima" and charged particles were expected to appear as "non-singular singularities" of the field, thus some byproducts. Their "non-singularity" was due to the regularizing effect of nonlinearity. In the monistic treatment based on linear Maxwell electrodynamics they were true singularities. The Born–Infeld nonlinearity results in an effective smearing out the point charge. The divergence of the displacement (induction) field $\bar{D}$ is not proportional to the Dirac delta function. This mechanism replaces nonlinearity of the dualistic field-matter model like, e.g., (13). But nowadays it seems almost sure that it is impossible to eliminate either field or matter degrees of freedom. One should have both of them. But then the question arises what would be the Born–Infeld version of (13). The simplest hypothesis would seem to be

$$\mathcal{L} = b^2\sqrt{|g|} - \sqrt{|bg + F|} + g^{\mu\nu} D_\mu \overline{\Psi} D_\nu \Psi \sqrt{|g|} - m^2 \overline{\Psi}\Psi\sqrt{|g|} \tag{51}$$

with the obvious meaning of symbols. But such a model is very complicated because the Lagrangian mixes two kinds of irrational expressions. Therefore, the resulting field equations are irrational in field variables and their derivatives. Although the "monistic" Born–Infeld Lagrangian (46) is also irrational in fields and their derivatives, the corresponding Euler–Lagrange equations are rational, or, to be more precise, become so after multiplying by $\sqrt{|bg + F|}$. All physical quantities like, e.g., the energy-momentum tensor factorize into the products of rational expressions and the standard irrational term. Unlike this, the essentially irrational structure of (51) implies the model to be rather artificial, computationally very non-effective and because of this probably non-physical. Because of all these objections the Born–Infeld model for many years almost disappeared from the fundamental research. It was used, perhaps heuristically, in certain quasi-classical considerations concerning the light-light scattering represented in quantum field theory as shown in Fig. 1. Such a process with virtual electron and positron lines

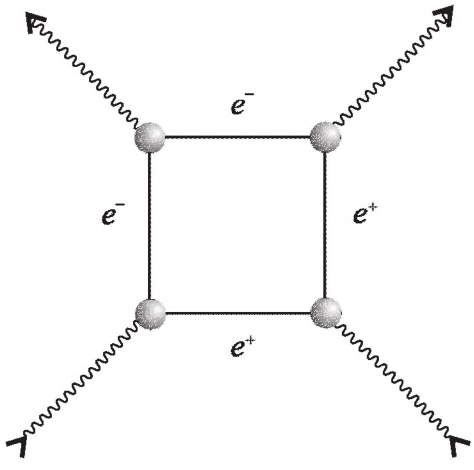

Figure 1

may be calculated on the basis of the quantized version of the classical model based on the Lagrangian (17). The underlying classical model is perturbatively nonlinear in (A, Ψ) and dualistic. It turns out that the above process may be relatively adequately described by the classical light self-interaction based on the monistic Born–Infeld model. From the point of view of quantum electrodynamics it is so as if the virtual electron-positron loop shrank to the point quartic vertex replacing the quadruple of cubic vortices. This is something like replacing the Salam–Glashow–Weinberg model of electroweak processes by the old Fermi model (again quartic instead of cubic). This is just eliminating some degrees of freedom and reobtaining their contribution by introducing stronger nonlinearity in the effective Lagrangian of remaining degrees of freedom. Predictions concerning the light-light scattering based on the classical Born–Infeld model are relatively acceptable.

But this was phenomenology done by hand which is a kind of Ersatz-Model. As an attempt of fundamental theory the Born–Infeld model for many years became a kind of historical curiosity. Recently the things are changed due to the advent of new theories like strings, p-branes and other field-theoretical ideas. And besides, the Born–Infeld paradigm was modified and extended so as to be based on deeper and more convincing geometric ideas. And in any case the exceptional features of the model, its uniqueness in a sense, seemed to indicate that it was based on good intuitions, in spite of certain shortcomings to be overcome. In the meantime a new interesting observation was done. Namely, the sourceless Maxwell electrodynamics is invariant under duality transformation which replaces $(\bar{E}, \bar{B})$ by $(\bar{B}, -\bar{E})$. In four-dimensional notation this is just the Hodge transformation [3–5], $F \mapsto *F$. Later on it was shown that as a matter of fact this is nothing but the very special case of the $\mathrm{SO}(2, \mathbb{R})$ group of internal symmetries,

$$(\bar{E}, \bar{B}) \mapsto (\bar{E}', \bar{B}') = (\bar{E}\cos\alpha + \bar{B}\sin\alpha, -\bar{B}\sin\alpha + \bar{E}\cos\alpha)\,. \tag{52}$$

In four-dimensional notation

$$F \mapsto F' = \cos\alpha\, F + \sin\alpha * F. \tag{53}$$

Obviously, the usual duality corresponds to $\alpha = \pi/2$. In non-Maxwellian (thus nonlinear) models this suggestive $\mathrm{SO}(2, \mathbb{R})$-invariance in general does not hold any longer. And again the Born–Infeld model is exceptional. It shares this symmetry with the Maxwell theory. In a sense, the Maxwell and Born–Infeld models are two opposite poles somehow distinguished by mysterious physical reasons. Let us summarize the final message of the above heuristic analysis:

i) Physically promising and natural nonlinearities are non-perturbative, i.e., do not possess a well-defined linear background
ii) Interesting nonlinearities have to do with "large" symmetry groups. In field theories it is natural to expect the general covariance and rich group of internal symmetries (the ones acting in target spaces)

iii) There is something deep and fundamental in the square-root structure of the Born–Infeld Lagrangians.

The above mentioned square-root structure is due to the very fundamental fact that the status of the Lagrangian $\mathcal{L}$ as a geometric object on M is that of the scalar W-density of weight one. If M is orientable it is equivalent to the differential n-form $\mathbf{L}$ locally given by

$$\mathbf{L}\,(\Psi, \partial\Psi) = \mathcal{L}\,(\Psi, \partial\Psi)\ \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n. \tag{54}$$

The standard way to construct a scalar W-density of weight one is to take some twice covariant tensor $\mathcal{L}_{\mu\nu}\,(\Psi, \partial\Psi)$ (referred to as the Lagrange tensor) and define

$$\mathcal{L}\,(\Psi, \partial\Psi) = \sqrt{|\mathcal{L}|} = \sqrt{|\det\left[\mathcal{L}_{\mu\nu}\,(\Psi, \partial\Psi)\right]|}. \tag{55}$$

Indeed $\det\left[\mathcal{L}_{\mu\nu}\right]$ is the scalar density of weight two. Taking it absolute value one obtains the W-density of weight two. Taking a square root of it one obtains the required W-density of weight one. An alternative way is to take some differential one-form $\lambda\,(\Psi, \partial\Psi)$ with values in some n-dimensional linear space V, i.e., analytically, when some basis in V is fixed, an n-tuple of usual ($\mathbb{R}$-valued) one-forms $\lambda^A\,(\Psi, \partial\Psi)$ locally represented as

$$\lambda^A\,(\Psi, \partial\Psi) = \lambda^A{}_\mu(\Psi, \partial\Psi)\,\mathrm{d}x^\mu, \qquad A = 1, \ldots, n. \tag{56}$$

Then $\mathcal{L}$ may be defined as

$$\mathcal{L}(\Psi, \partial\Psi) = \det\left[\lambda^A{}_\mu\right]. \tag{57}$$

The corresponding n-form is given by

$$\mathbf{L}(\Psi, \partial\Psi) = \lambda^1(\Psi, \partial\Psi) \wedge \ldots \wedge \lambda^n(\Psi, \partial\Psi) = \mathcal{L}(\Psi, \partial\Psi)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n. \tag{58}$$

In specially-relativistic theories, or more generally in field theories in pseudo-Riemannian manifolds (M, g) with the metric tensor $g_{\mu\nu}$ fixed once for all as an absolute object, Lagrangians are factorized in the usual way

$$\mathcal{L}(\Psi, \partial\Psi; g, \partial g) = \Lambda(\Psi, \partial\Psi; g, \partial g)\sqrt{|g|} \tag{59}$$

where Λ is a scalar function which sometimes is incorrectly referred to as a Lagrangian (in specially-relativistic field theories $\mathcal{L}$ and Λ numerically coincide when pseudo-Cartesian coordinates are used). The argument ∂g refers to the fact that the covariant derivatives of Ψ-fields may occur in $\mathcal{L}$, and the coefficients of the Levi-Civita connection depend (linearly) on ∂g. Perhaps it would be more concise to write

$$\mathcal{L}[\Psi; g] = \Lambda[\Psi; g]\sqrt{|g|}. \tag{60}$$

The same structure is used in generally-relativistic theories when the gravitation is taken into account, i.e., g becomes dynamic on the equal footing with other fields. Simply, the "matter" Lagrangian is linearly combined with the Hilbert

Lagrangian (32) for g and possibly with the cosmological term (33). The well-established viable models have just this multiplicative structure (60) with the scalar factor Λ built as simply as possible of its arguments $(\Psi, \partial\Psi)$. And the simplest models with the non-dynamical (absolute) metric g are just linear ones, when the scalar $\Lambda(\Psi, \partial\Psi)$ is quadratic in $(\Psi, \partial\Psi)$. To obtain realistic models with genuine interactions one introduces some perturbative non-quadratic terms in Λ just as described previously and these terms lead to non-linear corrections in field equations. When g is dynamical, i.e., relativistic gravitation is switched on, $\Lambda(\Psi, \partial\Psi; g, \partial g)$ contains the Hilbert term $\Lambda_{\mathrm{H}}(g, \partial g)$ proportional to the scalar curvature $R[g]$, and perhaps also the constant cosmological term. Λ_{H} is quadratic in the first derivatives ∂g with coefficients depending algebraically (more precisely, rationally) on g itself. The second derivatives $\partial^2 g$ enter Λ_{H} in an artificial way, linearly with coefficients given by rational functions of g. They may be gathered into a total divergence term and do not influence the field equations. These equations are second order quasi-linear in g while the coefficients at $\partial^2 g$ are rational functions of g. This is an essential non-perturbative nonlinearity following from the demand of general covariance (Diff(M)-invariance). In spite of that it is being well established by experimental data, the above factorization scheme (59), (60) looks somehow structurally artificial, even in spite of the essential nonlinearity of the Einstein–Hilbert gravitational sector. The metric g seems to be overestimated. Among all physical fields it is distinguished by its being a focus and at the same time a (claimly) necessary condition of the essential nonlinearity and general covariance. Moreover, it is also distinguished by its seemingly universal role in constructing scalar densities of weight one. The canonical prescription (55) for the scalar density of weight one together with the mentioned interesting features of the Born–Infeld model suggest us, however, some alternative methodology: It is perhaps neither the Lagrangian $\mathcal{L}$ nor its scalar factor Λ that is to be "simple," but rather the Lagrange tensor $\mathcal{L}_{\mu\nu}$, i.e., the "square-root" of $\mathcal{L}$. And for $\mathcal{L}_{\mu\nu}$ "simple" is presumably a low-order polynomial tensor function of $(\Psi, \partial\Psi)$, probably, at most quadratic in $\partial\Psi$. This is certainly the simplest model within the class of ones written as in (55). So, we have two alternative "poles of simplicity"

i) The traditional models (59) with Λ being a second or first order polynomial of derivatives. In particular, when Λ is a second order polynomial of its dynamical arguments, the theory is linear and may be used, e.g., as a linear background of perturbatively nonlinear models. And more generally, if Λ is quadratic in derivatives with coefficients algebraically built of the fields, the resulting theory is quasi-linear but in general its nonlinearity is non-perturbative.
ii) The modified Born–Infeld models (55) with the Lagrange tensor $\mathcal{L}_{\mu\nu}$ being polynomial of at most second degree in derivatives $\partial\Psi$. Such models are always essentially nonlinear and do not need any correction terms. Their physical nonlinearity is geometrically unified with the very idea of Lagrangians as

weight one W-densities. Roughly speaking, they are essentially nonlinear but at the same time structurally as similar to linear models as possible. Incidentally, let us remind in this connection that Maxwell and Born–Infeld models of electromagnetism are the only (exceptional) ones which are invariant under the extended duality (52), (53).

Obviously, the above models i) may be formally expressed like (55), and conversely, ii) admits the representation (59). But the "improper" representations are completely artificial and obscure. If ii) is expressed in terms of (59), then Λ is not a polynomial of fields and their derivatives If i) is represented in the form (55), then $\mathcal{L}_{\mu\nu}$ is not a polynomial either. For example, for the gravitational Hilbert Lagrangian we obtain the following disaster

$$L_{\mathrm{H}\mu\nu} = |R|^{2/n} g_{\mu\nu} = \sqrt{|R|} g_{\mu\nu} \tag{61}$$

in the academically general dimension n and in the physical one $n = 4$. In this way the distinction between the mentioned two "poles of simplicity" is obvious.

Remark: More precisely, the Hilbert Lagrangian $\mathcal{L}_{\mathrm{H}}$ is proportional to

$$\operatorname{sign} R \sqrt{|\det\left[\mathcal{L}_{\mathrm{H}\mu\nu}\right]|}. \tag{62}$$

This is an additional example of the distinction between models i) and ii) above. There is a very interesting and delicate point concerning the charge-free Born–Infeld electrodynamics. Namely, one could try to think about simplifying (46) by removing the metric tensor completely and putting

$$\mathcal{L} = -\sqrt{|\det\left[F_{\mu\nu}\right]|}. \tag{63}$$

And indeed such a conjecture was formulated, although some strange features of the "model" are seen from the very beginning. Namely, the repulsive differential singularity occurs when $\det[F_{\mu\nu}] = 0$, i.e., when $\bar{E} \cdot \bar{B} = 0$. So, it does not lead to the saturation of the field strength, but prevents the orthogonality of the electric vector $\bar{E}$ and the magnetic pseudovector $\bar{B}$. This would be completely exotic, but there is something else completely unacceptable in (63) as a physical model. Namely, the resulting Euler–Lagrange equations are nothing else but the trivial identity $0 = 0$ and do not impose any restrictions on the field F. The reason is that (63) may be represented as a total divergence

$$\sqrt{|\det\left[F_{\alpha\beta}\right]|} = \frac{D}{Dx^{\nu}} \left(\frac{2}{n} A_{\mu} \widetilde{F}^{\mu\nu} \sqrt{|\det\left[F_{\alpha\beta}\right]|} \right) \tag{64}$$

where, obviously, $n = \dim M$ (physically, $n = 4$) and $\widetilde{F}$ denotes the contravariant inverse of F

$$\widetilde{F}^{\mu\varkappa} F_{\varkappa\nu} = \delta^{\mu}{}_{\nu}. \tag{65}$$

Obviously, this "pathological" structure cannot be accidental. The point is that the "Lagrangian" (63) does not contain any absolute (or controlling) geometric objects, therefore, it is invariant under Diff(M) (generally-covariant) and so are the resulting "field equations." Diff(M) is "parameterized" by n arbitrary functions of n variables (physically, $n = 4$), and therefore, among field quantities there are n purely gauge variables, which, roughly speaking, may be given any a priori given form by an appropriate choice of coordinates. But in electrodynamics there are just n field variables A_μ, so from the point of view of Diff(M) they are all gauge variables in the "model" (63). Therefore, the theory is either trivial (all fields are solutions) or empty, intrinsically inconsistent (there are no solutions at all). In the case of (63) the first situation occurs. There is a natural temptation to construct the Born–Infeld versions of other field theories. It turns out that because of certain geometric reasons electromagnetism is exceptional in that that it admits $\mathcal{L}_{\mu\nu}$ being first order polynomial of derivatives $\partial_\mu A_\nu$. For the real scalar field the only natural possibility is the second order $\mathcal{L}_{\mu\nu}$ given by

$$\mathcal{L}_{\mu\nu} = b g_{\mu\nu} + \partial_\mu \Psi \partial_\nu \Psi. \tag{66}$$

The corresponding Lagrangian by analogy to (46) has the form

$$\mathcal{L} = b^2 \sqrt{|g|} - \sqrt{|\det [\mathcal{L}_{\mu\nu}]|}. \tag{67}$$

This is the "Born–Infeld'ization" of the linear d'Alembert model

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \sqrt{|g|} \tag{68}$$

$$\Box \Psi = g^{\mu\nu} \nabla_\mu \nabla_\nu \Psi = 0. \tag{69}$$

It is perhaps surprising that in spite of the quadratic dependence of (66) on derivatives the corresponding stationary isotropic solutions in Minkowski space coincide with (47), namely,

$$\Psi(r) = \int_r^\infty \frac{e\, \mathrm{d}x}{\sqrt{r_0^4 + x^4}} \tag{70}$$

with the same meaning of symbols. Let us remind that there are problems in optics (when polarization phenomena are non-essential), which are in a satisfactory way treatable within the framework of the scalar theory of light (68), (69). Therefore, the compatibility of (47) and (70) really confirms that (66) is the Born–Infeld counterpart of (68). Incidentally, it is geometrically impossible to invent anything else. And in fact, the model (66), (67) was successfully used in certain problems of nonlinear optics. Once successfully appearing, expressions quadratic in derivatives may seem acceptable and sometimes just unavoidable terms of the Lagrange tensor $\mathcal{L}_{\mu\nu}$. When discussing this idea one is faced with some interesting, not solved, even not yet touched, questions. The first of them concerns the matter-free electromagnetism, just the domain of the original model. It is exceptional in admitting an affine dependence of $\mathcal{L}_{\mu\nu}$ on the field derivatives $\partial_\alpha A_\beta$ (via $F_{\alpha\beta}$). The quadratic terms are not necessary, but are they admissible or not? The most natural hypothesis would be

$$\mathcal{L}_{\mu\nu} = \alpha g_{\mu\nu} + \beta F_{\mu\nu} + \gamma g^{\varkappa\lambda} F_{\mu\varkappa} F_{\nu\lambda} + \delta g^{\varkappa\rho} g^{\lambda\sigma} F_{\varkappa\lambda} F_{\rho\sigma} g_{\mu\nu} \tag{71}$$

with α, β, γ, δ being real constants. The structure of the last two terms is not exotic as the symmetric energy-momentum tensor of the Maxwell electrodynamics is just their linear combination. The question: is (71) a viable model? It is evident that for weak fields (71) has correct Maxwell asymptotics as (46). It is interesting what are stationary isotropic solutions when they are bounded or not, and is their energy finite or not? As yet nobody tried to answer such questions. It is also unknown which canonical properties of the Born–Infeld model are lost by (71) and to which extent they are lost. Obviously, it would be rather too speculative to think about higher-order polynomial dependence of $\mathcal{L}_{\mu\nu}$ on $F_{\alpha\beta}$, although formally it is possible. Let us quote, e.g., the terms like

$$g_{\mu\varkappa} \left(\widehat{F}^p\right)^{\varkappa}{}_{\nu}, \qquad \operatorname{Tr}\left(\widehat{F}^p\right) g_{\mu\nu}$$

where p is natural number, $\widehat{F}$ is the **mixed tensor**

$$\widehat{F}^\alpha{}_\beta := g^{\alpha\gamma} F_{\gamma\beta}$$

and $\widehat{F}^p$ is its p-th power. Analogy with linear, quasi-linear and linearly "backgrounded" models i) suggests us to stop at quadratic terms. There is no regulative idea, rather one is lost in jungle when going higher. And the qualitative essential nonlinearity is attained already at the stage of second order polynomials. The hypothetic model (71) has to do with the idea of Born–Infeld Lagrangians for the gauge fields [3–5]. Let G be a Lie group underlying some gauge field, $\mathfrak{g}$ is its Lie algebra and $\mathbf{A}$ is the gauge potential, i.e., $\mathfrak{g}$-valued differential one-form on M. The field strength will be denoted by $\mathbf{F}$ which is represented by a $\mathfrak{g}$-valued differential two-form on M. Analytically, when some bases in $\mathfrak{g}$ and coordinates in M are fixed, one uses the symbols $A^K{}_\mu$, $F^K{}_{\mu\nu}$, where

$$F^K{}_{\mu\nu} = \partial_\mu A^K{}_\nu - \partial_\nu A^K{}_\mu + g C^K{}_{RS} A^R{}_\mu A^S{}_\nu \tag{72}$$

g is the coupling constant and $C^K{}_{RS}$ are the structure constants of $\mathfrak{g}$ with respect to the fixed basis (geometrically C is a tensor in $\mathfrak{g}$, once contravariant and twice covariant-skew-symmetric). Let h denote the Killing tensor on $\mathfrak{g}$, analytically

$$h_{KL} = C^R{}_{SK} C^S{}_{RL}. \tag{73}$$

If G is simple, then the natural counterpart of (71) is

$$\mathcal{L}_{\mu\nu} = \alpha g_{\mu\nu} + \gamma h_{RS} F^R{}_{\mu\varkappa} F^S{}_{\nu\lambda} g^{\varkappa\lambda} + \delta h_{RS} F^R{}_{\varkappa\lambda} F^S{}_{\rho\sigma} g^{\varkappa\rho} g^{\lambda\sigma} g_{\mu\nu} \tag{74}$$

with the same as previously provisos concerning the higher than second polynomial terms. There is no counterpart of the linear in F β-controlled term in (71). Such term would contradict the gauge invariance because in simple Lie algebras there are no Ad-invariant directions. If $\mathfrak{g}$ is semi-simple, thus splits into a direct sum of N simple ideals

$$\mathfrak{g} = \oplus_{p=1}^{N} \mathfrak{g}_{(p)} \tag{75}$$

then $\mathbf{A}$, $\mathbf{F}$ are represented by N-tuples of $\mathfrak{g}_p$-valued differential forms $\mathbf{A}_{(p)}$, $\mathbf{F}_{(p)}$, there are N coupling constants $\mathfrak{g}_{(p)}$, N systems of structure constants C_p and N Killing tensors h_p in simple ideals $\mathfrak{g}_{(p)}$. Then the second and third terms in (74) split into sums of N terms with the corresponding coefficients $\gamma_{(p)}$, $\delta_{(p)}$. If $\mathfrak{g}$ is a direct sum of the one-dimensional centre $\mathfrak{g}_{(0)} \simeq \mathbb{R}$ and the complementary semi-simple Lie algebra $\mathfrak{g}$, then one can introduce into (74) an additional term analogous to the linear β-term in (71), namely,

$$\beta F_{(0)\mu\nu}$$

where $F_{(0)}$, $A_{(0)}$ are the "components" of $\mathbf{F}$ and $\mathbf{A}$ in $\mathfrak{g}_{(0)} \subset \mathfrak{g}$, thus $\mathfrak{g}_{(0)}$-valued ($\mathbb{R}$-valued) differential forms. The gauge group of electroweak interactions $\mathrm{U}(1) \times \mathrm{SU}(2)$ has just this structure, thus, as expected, the traditional term linear in $\mathbf{F}$ may appear in $\mathcal{L}_{\mu\nu}$. Finally, it turns out that Lagrange tensors quadratic in derivatives of the field variables may reconcile the Born–Infeld electromagnetism with the "external" charged matter and to avoid the artificial and certainly non-useful model (51). First of all let us notice that the Born–Infeld electrodynamics with massive photons, i.e., "Born–Infeld'ization" of the Proca theory would be based on the Lagrange tensor

$$\mathcal{L}_{\mu\nu} = bg_{\mu\nu} + F_{\mu\nu} - \varkappa A_\mu A_\nu \tag{76}$$

i.e., on the Lagrangian

$$\mathcal{L} = b^2\sqrt{|g|} - \sqrt{|\det\left[bg_{\mu\nu} + F_{\mu\nu} + \varkappa^2 A_\mu A_\nu\right]|}. \tag{77}$$

The constant $\varkappa$ here is proportional to the "photon mass" as seen from the weak field expansion of (77) around the "vacuum" $A = 0$ and one obtains then the usual Proca Lagrangian. The corresponding "scalar Born–Infeld–Proca" electrodynamics, i.e., the massive version of (66), (67) is given by

$$\mathcal{L}_{\mu\nu} = bg_{\mu\nu} + \partial_\mu\Psi\partial_\nu\Psi - \varkappa^2\Psi^2 g_{\mu\nu} \tag{78}$$

$$\mathcal{L} = b^2\sqrt{|g|} - \sqrt{|\det\left[\mathcal{L}_{\mu\nu}\right]|}. \tag{79}$$

Its weak field expansion is just the Klein–Gordon Lagrangian

$$\mathcal{L} = \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - \frac{\varkappa^2}{2}\Psi^2\right)\sqrt{|g|}. \tag{80}$$

Let us now consider the massive and complex scalar field, i.e., the quasi-classical description of the coherent quantum charged matter. Its quadratic Lagrangian is given by

$$\mathcal{L} = \left(g^{\mu\nu}\partial_\mu\overline{\Psi}\partial_\nu\Psi - m^2\overline{\Psi}\Psi\right)\sqrt{|g|} \tag{81}$$

and reversing the above transition from (78), (79) to (80) we obtain the following "Born–Infeld'ization" of (81)

$$\mathcal{L} = b^2\sqrt{|g|} - \sqrt{\left|\det\left[bg_{\mu\nu} + \partial_\mu\overline{\Psi}\partial_\nu\Psi - m^2\overline{\Psi}\Psi g_{\mu\nu}\right]\right|} \tag{82}$$

i.e.,

$$\mathcal{L}_{\mu\nu} = bg_{\mu\nu} + \partial_\mu\overline{\Psi}\partial_\nu\Psi - m^2\overline{\Psi}\Psi g_{\mu\nu}. \tag{83}$$

The parameter m plays the role of mass and in the limit of weak fields the essentially nonlinear model (83) asymptotically approaches that based on (81). Let us observe that $\mathcal{L}_{\mu\nu}$ is complex, nevertheless hermitian

$$\mathcal{L}_{\mu\nu} = \overline{\mathcal{L}}_{\nu\mu} \tag{84}$$

and, therefore, its determinant is real. Another possibility is to postulate

$$\begin{aligned}\mathcal{L}_{\mu\nu} &= bg_{\mu\nu} + \partial_{(\mu}\overline{\Psi}\partial_{\nu)}\Psi - m^2\overline{\Psi}\Psi g_{\mu\nu} \\ &= \operatorname{Re}\left(bg_{\mu\nu} + \partial_\mu\overline{\Psi}\partial_\nu\Psi - m^2\overline{\Psi}\Psi g_{\mu\nu}\right).\end{aligned} \tag{85}$$

So, (83) and (85) are two alternative "Born–Infeld'ization" of the linear Klein–Gordon model for the charged matter field. Unifying this expression with (46) we obtain the simplest Born–Infeld model of the mutually interacting charged scalar field (matter) and the electromagnetic field

$$\mathcal{L}_{\mu\nu} = bg_{\mu\nu} + F_{\mu\nu} + D_\mu\overline{\Psi}D_\nu\Psi - m^2\overline{\Psi}\Psi g_{\mu\nu} \tag{86}$$

with $D_\mu\Psi$ given by (14). Just as previously, we can use also another form, the real one

$$\mathcal{L}_{\mu\nu} = bg_{\mu\nu} + F_{\mu\nu} + D_{(\mu}\overline{\Psi}D_{\nu)}\Psi - m^2\overline{\Psi}\Psi g_{\mu\nu}. \tag{87}$$

In the limit of weak fields this reduces asymptotically to

$$\mathcal{L} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\sqrt{|g|} + g^{\mu\nu}D_\mu\overline{\Psi}D_\nu\Psi\sqrt{|g|} - m^2\overline{\Psi}\Psi\sqrt{|g|}. \tag{88}$$

In analogy to (10) one can also introduce into (88) the quartic correction term $-\varkappa\left(\overline{\Psi}\Psi\right)^2\sqrt{|g|}$ and the Born–Infeld form (86) is then corrected by

$$-\varkappa\left(\overline{\Psi}\Psi\right)^2 g_{\mu\nu}. \tag{89}$$

The field equations are now rational in fields and their derivatives, so there is no longer conflict between the Born–Infeld electromagnetism and extra extended charges. Obviously, $\mathcal{L}_{\mu\nu}$ might be corrected by introducing into $\mathcal{L}_{\mu\nu}$ additional terms quadratic in F, however we do not quote the corresponding explicit formula. It is obvious that the Lagrangian

$$\mathcal{L} = b^2\sqrt{|g|} - \sqrt{|\det\left[\mathcal{L}_{\mu\nu}\right]|} \tag{90}$$

with $\mathcal{L}_{\mu\nu}$ given by (86) is compatible with the Born–Infeld paradigm and unifies in a smooth way the electromagnetic field with external charged matter so as to result in rational field equations, which are gauge invariant under the local $\mathrm{U}(1)$-group.

3. A General Covariance

We have stressed two particular mechanisms of the essential and geometrically-implied nonlinearity in field theory: the general covariance and the generalized Born–Infeld structure based on geometry of scalar densities. There is no automatic link between them and besides some conceptual gap between the General Relativity and the traditional Born–Infeld model is obvious. However, some interesting relationship does exist. Before going into details let us express a few remarks concerning the general covariance. There is a popular and incorrect view that there is no general covariance without the Hilbert–Einstein metric channel. Nevertheless, there are also other fields which admit Diff(M)-invariant Lagrangians. And it turns out that these Lagrangians have generalized Born–Infeld structure and are free of the artificial universal splitting (59). As mentioned before, in an n-dimensional manifold M only fields with $N > n$ components may admit generally-covariant Lagrangians free of controlling absolute quantities. Because of this (63) was bad. Similarly, the (n-component) contravariant vector density of weight one $\mathcal{A}^\mu$ is not viable as an autonomous field although it admits a Diff(M)-invariant prescription for the scalar density of weight one built of the first derivatives. Indeed, it is a divergence

$$\mathcal{L} = \partial_\mu \mathcal{A}^\mu \tag{91}$$

and nothing else does exist just because of the component number n. Historically the twice covariant symmetric nonsingular tensor (and automatically its contravariant inverse) was the first known object viable in this sense (cf. the Hilbert Lagrangian (32), (33)). This is an irreducible tensorial object. Obviously, the general twice covariant tensor (without defined symmetry properties) would be also good. These objects have respectively $n(n+1)/2$ and n^2 independent components, much more than necessary to admit Diff(M)-invariant Lagrangians. Quite a natural question arises as to the existence of generally covariant Lagrangians for mixed second order tensors. The question is perhaps a little academic because as yet the mixed tensors did not find any applications as fundamental physical fields. Nevertheless it is an interesting question because of the geometric meaning of once contravariant and once covariant tensors. If X is such a field, then for any $p \in M$

the tensor X_p is a linear mapping of the tangent space T_pM into itself

$$X_p \in T_pM \otimes T_p^*M \simeq L(T_pM).$$

They produce vectors from vectors (dually their conjugates produce covectors from covectors) just like twice covariant tensors produce covectors from vectors and twice contravariant tensors produce vectors from covectors. One can hope that second order mixed tensor fields may find applications in certain studies concerning gauge models of gravitation and other alternative treatments. Compare in this respect certain toy models of internal degrees of freedom discussed in our papers concerning affinely-rigid bodies [18–21,23,25–27]. A priori it might seem not very likely that the mixed second order tensor X admits a Diff(M)-invariant variational principle, although having $n^2 > n$ components it satisfies the necessary condition. Nevertheless such a Lagrangian (does not matter if physically promising) really exists and its construction is based on the Nijenhuis torsion. Let us remind that the **Nijenhuis torsion** $S(X, Y)$ assigned to the pair of mixed second order tensor fields X, Y is defined as a once contravariant and twice covariant-antisymmetric tensor field given by [10]

$$\begin{aligned} S(X,Y)^{\mu}{}_{\nu\lambda} := {}& X^{\rho}{}_{\nu}\partial_{\rho}Y^{\mu}{}_{\lambda} + Y^{\rho}{}_{\nu}\partial_{\rho}X^{\mu}{}_{\lambda} - X^{\rho}{}_{\lambda}\partial_{\rho}Y^{\mu}{}_{\nu} - Y^{\rho}{}_{\lambda}\partial_{\rho}X^{\mu}{}_{\nu} \\ & - X^{\mu}{}_{\rho}\partial_{\nu}Y^{\rho}{}_{\lambda} - Y^{\mu}{}_{\rho}\partial_{\nu}X^{\rho}{}_{\lambda} + X^{\mu}{}_{\rho}\partial_{\lambda}Y^{\rho}{}_{\nu} + Y^{\mu}{}_{\rho}\partial_{\lambda}X^{\rho}{}_{\nu}. \end{aligned} \tag{92}$$

It is obvious that $S(X, Y)$ is skew-symmetric in the lower-case indices

$$S(X,Y)^{\mu}{}_{\nu\lambda} = -S(X,Y)^{\mu}{}_{\lambda\nu} \tag{93}$$

and symmetric in the tensors X, Y

$$S(X,Y) = S(Y,X) \tag{94}$$

do not confuse these two symmetries. The formula (94) involves only the tensors X, Y and their partial derivatives without using anything like affine connection, etc., and so, without appealing to some general theory it is quite not obvious that it really defines the tensor field. It is so however as the non-tensorial terms in the transformation rule mutually cancel. Let us remind also the coordinate-free definition. Being a T^1_2-type tensor, $S(X, Y)$ may be identified with some prescription producing vectors from pairs of vectors. If A, B are two vector fields, then the evaluation of $S(X, Y)$ on the pair A, B, locally

$$S(X,Y)\cdot(A,B) = S(X,Y)^{\mu}{}_{\nu\lambda}A^{\nu}B^{\lambda}\frac{\partial}{\partial x^{\mu}} \tag{95}$$

is given by

$$\begin{aligned} S(X,Y)\cdot(A,B) = {}& [XA,YB] + [YA,XB] + XY[A,B] + YX[A,B] \\ & - X[A,YB] - X[YA,B] - Y[A,XB] - Y[XA,B] \end{aligned} \tag{96}$$

where $[A, B]$ denotes the Lie bracket, and locally

$$[A, B]^{\mu} = A^{\lambda}\partial_{\lambda}B^{\mu} - B^{\lambda}\partial_{\lambda}A^{\mu} \tag{97}$$

in which XY is the algebraic composition of linear mappings in the tangent spaces

$$(XY)^{\mu}{}_{\nu} = X^{\mu}{}_{\alpha}Y^{\alpha}{}_{\nu}. \tag{98}$$

In (96) again the problem with derivatives appears, i.e., if $S(X, Y)$ is to be a T^1_2 tensor, this expression must depend algebraically on A, B, but it is seen that the first order derivatives of A, B enter this formula. However, they mutually cancel and this is a fact that is not immediately seen unless one works within the framework of a wider theory. In particular, putting $Y = X$ we can construct the quantity

$$S(X) := S(X, X) \tag{99}$$

i.e., some T^1_2-type tensor field built algebraically of X and its derivatives ∂X. One can also consider the objects like

$$S^{k,l}(X) := S\left(X^k, X^l\right) = S^{l,k}(X) \tag{100}$$

where k, l are naturals and the powers X^k are meant as pointwise compositions of linear mappings (98). When X is non-singular, the integer negative powers k, l may be also used. Obviously, the zeroth order power is simply

$$X^0 = \mathrm{Id}, \qquad \left(X^0\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} \tag{101}$$

but of course

$$S^{0,1}(X) = S^{1,0}(X) = 0. \tag{102}$$

Let us summarize: we have at our disposal a family of third order tensors $S^{k,l}(X)$, $k > 0$, $l > 0$, built algebraically of X and its first derivatives ∂X. Their dependence on derivatives is linear. The simplest and most natural of them is $S(X) := S(X, X) = S^{1,1}(X)$. It is linear not only in ∂X but also in X itself. Being free of any absolute (controlling) object, the assignment $X \mapsto S(X)$ is generally covariant, i.e., for any diffeomorphism $\varphi : M \to M$ the following holds

$$S(\varphi_* X) = \varphi_* S(X) \tag{103}$$

i.e., the assignment is φ-transparent. Therefore, $S(X)$ may be interpreted as invariantly defined derivative of X (just like the exterior derivative is an invariant differentiation of differential forms). Having first order derivatives one can wonder what would be generally-covariant Lagrangians built of X. The only possibilities are based on the Born–Infeld scheme (55). The simplest and most natural among them are those with the Lagrange tensor $\mathcal{L}(X, \partial X)$ which is quadratic in $S(X)$, thus also quadratic in derivatives ∂X

$$\mathcal{L}(X, \partial X)_{\mu\nu} = AS^{\lambda}{}_{\mu\varkappa}S^{\varkappa}{}_{\nu\lambda} + BS^{\lambda}{}_{\mu\lambda}S^{\varkappa}{}_{\nu\varkappa} + CS^{\lambda}{}_{\varkappa\lambda}S^{\varkappa}{}_{\mu\nu} \tag{104}$$

where A, B, C are real constants. As a matter of fact, the constant C at the third, skew-symmetric term may be also purely imaginary, but then $\mathcal{L}$ is hermitian and its determinant is real. The first two terms are symmetric, in particular, the first of them has the suggestive Killing structure

$$G_{\mu\nu} = G_{\nu\mu} = S^{\lambda}{}_{\mu\varkappa} S^{\varkappa}{}_{\nu\lambda}. \tag{105}$$

The tensor $G_{\mu\nu}$, or more generally the symmetric part of (104), certainly with the non-vanishing A (which, e.g., by convention is put equal to one in appropriate units) might be perhaps physically interpreted as a kind of metric tensor. $\mathcal{L}_{\mu\nu}$ is homogeneous-quadratic in derivatives, thus the corresponding Lagrangian $\mathcal{L}$ is homogeneous of degree n in derivatives. This resembles the Finsler structures corresponding to the homogeneous variational formalism in mechanics (then $n = 1$, $M = \mathbb{R}$) [31]. Expression (104) is the simplest class of generally-covariant Lagrangians for mixed second order tensors. One can try to complicate it by replacing the constants A, B, C by some scalars built of $(X, \partial X)$. The simplest of them are built of X alone in a purely algebraic way and may be expressed as functions of n basic invariants I_p, $p = 1, \ldots, n$

$$I_p = \operatorname{Tr}\left(X^p\right). \tag{106}$$

Obviously, according to the Cayley–Hamilton theorem, for any $k < 0$ and any $k > n$, $\operatorname{Tr}\left(X^k\right)$ may be expressed through (106). One can also use other systems of basic invariants, e.g., the coefficients $c_k(X)$ of the eigenvalue equation

$$\det\left(X - \lambda I\right) = \sum_{k=0}^{n} c_k \lambda^k = 0 \tag{107}$$

except of the standard coefficient at λ^n, $c_n = (-1)^n$. Another possibility are just the eigenvalues $\lambda_i(X)$, taken, e.g., in an increasing order which is well-defined in the generic case of the simple spectrum. There are also invariants built of derivatives of X, i.e., of the tensor $S(X)$. For example, if G in (105) is nondegenerate, then using its contravariant inverse $G^{\mu\nu}$ $(G^{\mu\alpha}G_{\alpha\nu} = \delta^{\mu}{}_{\nu})$ we can construct scalars built according to the "Wietzenböck scheme" [13, 15, 24] as follows

$$J_1 = G_{\mu\alpha} G^{\nu\beta} G^{\lambda\gamma} S^{\mu}{}_{\nu\lambda} S^{\alpha}{}_{\beta\gamma} \tag{108}$$

$$J_3 = G^{\mu\nu} S^{\alpha}{}_{\mu\alpha} S^{\beta}{}_{\nu\beta}. \tag{109}$$

Obviously, J_2 will be trivial

$$J_3 = G^{\mu\nu} S^{\alpha}{}_{\mu\beta} S^{\beta}{}_{\nu\alpha} = G^{\mu\nu} G_{\mu\nu} = G^{\mu\nu} G_{\nu\mu} = \delta^{\mu}{}_{\mu} = n. \tag{110}$$

Similarly one can construct more S-factors and G-factors. But it is easy to see that all of them are homogeneous of degree zero in S, i.e., homogeneous of degree zero in ∂X. The reason is that G is not "external," but just built of S. All such scalars and their functions f are generally covariant, i.e.,

$$f[\varphi_* X] = \varphi_* f[X] = f[X] \circ \varphi^{-1} \tag{111}$$

for any $\varphi \in \mathrm{Diff}(M)$. One can show that any scalar first-jet function $f[X] = f(X, \partial X)$ built of X in a generally-covariant way satisfy (111), depends on ∂X, thus also on the torsion $S(X)$, homogeneously of degree zero, i.e., for any $\lambda > 0$

$$f(X, \lambda \partial X) = f\{X, \lambda S(X)\} = \lambda f(X, \partial X) = \lambda f\{X, S(X)\}. \tag{112}$$

This is one of the consequences of the Noether identity following from the general covariance. Let us observe that when we put coefficients A, B, C to be functions of the basic algebraic invariants (106), then they play the role of something like the potential energy in Maupertui variational principle of analytical mechanics [1, 2]. This example, although as yet academic, is very interesting in showing how the demand of general covariance just implies the Born–Infeld structure. It is also interesting in that it is free of any introduced by hand geometry in the target space, i.e., in the bundle $T^1_1 M$ of mixed tensors over M. In the last respect it is similar to the General Relativity based on the bundle $T^0_2 M$, or isomorphically $T^2_0 M$, when one restricts (as necessary in the General Relativity) to non-degenerate tensors. However, the standard formulation of the General Relativity has nothing to do with the Born–Infeld scheme. Nevertheless, the two paradigms seem to converge if one tries to use the formulation based on the bundle of affine connections over M. Namely, many years ago Schrödinger [17] tried to develop the theory where the gravitational field was to be described by the symmetric affine connection $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$ without any use of the metric tensor. Namely, Γ gives rise to the curvature tensor $R^\lambda{}_{\mu\nu\gamma}$ and then to the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$. Unlike the curvature scalar, these are intrinsic purely affine objects which "do not know" the metric tensor. In particular, $R[\Gamma]_{\mu\nu}$ is a (symmetric) twice covariant tensor built in an affine (first order polynomial) way of derivatives $\partial_\mu \Gamma^\lambda{}_{\nu\varkappa}$. Because of this Schrödinger suggested the model (55) with the Lagrange tensor

$$\mathcal{L}(\Gamma, \partial\Gamma)_{\mu\nu} \simeq R[\Gamma]_{\mu\nu}. \tag{113}$$

This was expected to be an alternative model of the gravitation. As shown recently [9] there is an interpretation based on the standard concepts of the General Relativity. The metric tensor appears there as a quantity built of canonical momenta conjugated to Γ and automatically satisfies Einstein equations in virtue of Euler–Lagrange equations. But of course both theories may differ on the quantization level. It is well known that formulations of the same model based on different Lagrangians may lead to non-equivalent quantum theories. Incidentally, it is quite not clear what would result from the attempts of analyzing (113) in terms of momenta conjugate to $X^\mu{}_\nu$. There is another open question: why not to try something like the Palatini scheme with a priori independent field variables $g_{\mu\nu}$, $\Gamma^{\lambda}{}_{\mu\nu}$ (metric and affine connections) and postulate something like, e.g.,

$$\mathcal{L}(g;\Gamma,\partial\Gamma)_{\mu\nu} = \alpha g_{\mu\nu} + \beta R(\Gamma,\partial\Gamma)_{\mu\nu}. \tag{114}$$

And if so, why not to use the term with the scalar curvature, thus

$$\mathcal{L}(g;\Gamma,\partial\Gamma)_{\mu\nu} = \alpha g_{\mu\nu} + \beta R(\Gamma,\partial\Gamma)_{\mu\nu} + \gamma R(g;\Gamma,\partial\Gamma) g_{\mu\nu}. \tag{115}$$

This does not violate the paradigm of the first order variational principles because now g, Γ are a priori independent dynamical variables

$$R(g;\Gamma,\partial\Gamma) = g^{\mu\nu} R(\Gamma,\partial\Gamma)_{\mu\nu}. \tag{116}$$

Obviously, usual Lagrangians for the electromagnetic, Proca gauge and scalar (both real and complex) fields may be subject to the "Born–Infeld'ization" procedure (55) according to the prescriptions (46) or perhaps (71) – (electromagnetism), (74) – (gauge), (76) – (Proca), (66), (78), (83) – (scalars). Their mutually-interacting versions are given by expressions like (86), (87), (88) and similar ones. The suggested procedure of inserting all the fields into one Lagrange tensor $\mathcal{L}_{\mu\nu}$ depending in at most second order polynomial way on derivatives leads to reasonably looking and in principle analytically treatable models. For example, as mentioned, one reconciliates then the Born–Infeld scheme with external continuously distributed electric charges. The resulting field equations do not involve strange irrational terms as those appearing in a seemingly natural scheme (51). Nevertheless, these generalized Born–Infeld models are not generally covariant and contain a controlling (absolute) quantity, namely the metric tensor g. Schrödinger dynamics (113) of the affine connection is generally covariant and has evidently the Born–Infeld structure. Nevertheless, its status is not clear. One obtains something relatively exotic when coupling $\Gamma^{\lambda}{}_{\mu\nu}$ with other physical fields and trying to interpret the Ricci tensor $R_{\mu\nu}$ as a kind of metric. When interpreting it according to Kijowski [9], where the metric tensor appears as a byproduct of canonical momenta conjugated to $\Gamma^{\lambda}{}_{\mu\nu}$, one recovers in principle the Einstein theory with its quasi-linear, although non-perturbatively nonlinear structure. The very hypothetic models (114) and (115) with their status of $g_{\mu\nu}$, $\Gamma^{\lambda}{}_{\mu\nu}$ as a priori independent dynamical variables ("Palatini-like" schemes) are generally-covariant and retain the usual concept of metric, nevertheless, it is completely unclear whether they may be physically interpretable. The model (104) with its further modifications is as yet rather academic, nevertheless, it unifies the general covariance and the Born–Infeld structure in a very natural way, they both are simply implied by the very geometry of degrees of freedom (just as the Einstein theory with a possible cosmological terms is simply implied by geometry of metrical degrees of freedom $g_{\mu\nu}$). The natural question arises as to the possibility of reconciliation of the scalar Born–Infeld models, starting from the simplest one (66), (67) with the paradigm of general covariance. The question is interesting and non-academic because as we saw there are some quite physical applications of (66), (67) in nonlinear scalar optics (for example, when discussing the interaction of laser beams with the matter) and in other problems of the electrodynamics of continuous media. Various models of scalar fields and their multiplets are also essential for the theory of fundamental interactions (various σ-models and Higgs sector of gauge theories). Obviously, generally covariant models of multiplets of N real scalar fields (some or all of them may be real and imaginary parts of complex scalar fields) exist only if $N > n$, i.e., if there are more real dependent variables than non-dependent ones (as we always stress, although in well-established field theory $n = 4$, it is more convenient to admit formally the general n). Let us discuss this framework in more details. We begin with traditional models involving fixed metric tensor $g_{\mu\nu}$. So, let (M, g) be a physical (pseudo-)Riemannian space-time, or more generally, some n-dimensional manifold M endowed with the metric tensor g considered as an absolute (controlling) element of the theory. Besides we are given the target space (W, η), where W is a differentiable manifold of the real dimension N (in a moment it needs not be confined by the condition $N > n$) endowed with some internal geometry usually given by some twice covariant tensor field η on W. As a rule, this tensor will be symmetric or hermitian (in the case of a complex W), but more general situations are also possible, e.g., symplectic target geometry. Having the "arena" (M, g) and the "target" (W, η) we can consider dynamical variables given by mappings $\phi : M \rightarrow W$. If x^{μ}, y^{A} denote respectively local coordinates in M, W, the mappings ϕ are analytically represented by the functions $y^{A} = \phi^{A}(x^{\mu})$. The simplest possible Lagrangians have the form

$$\mathcal{L}\left(\phi(x), \partial\phi(x)\right) = \frac{1}{2}\eta_{AB}(\phi(x))\partial_{\mu}\phi^{A}(x)\partial_{\nu}\phi^{B}(x)g^{\mu\nu}(x)\sqrt{|g(x)|}. \tag{117}$$

Here exceptionally the dependence on x is carefully inserted for the sake of full clarity. Apparently (117) has the "d'Alembert" structure, however this concerns only the quadratic dependence on $\partial\phi$. If W is a manifold and η is a general tensor field, then, as a rule, the algebraic dependence of $\mathcal{L}$ on ϕ is irreducible and field equations are in general nonlinear, although quasi-linear. More realistic are the models involving some "potential" term $U : W \rightarrow \mathbb{R}$, then

$$\mathcal{L}\left[\phi\right] = \frac{1}{2}\eta_{AB}\partial_{\mu}\phi^{A}\partial_{\nu}\phi^{B}g^{\mu\nu}\sqrt{|g|} - U(\phi)\sqrt{|g|}. \tag{118}$$

As a rule, the possibility of constructing U is based on some additional structures. If, e.g., W is a linear space as it is often in realistic models and η is "flat" in the sense of being constant in linear coordinates, one can consider, e.g., **multi-component Klein–Gordon models**

$$\mathcal{L} = \frac{1}{2}\eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B g^{\mu\nu}\sqrt{|g|} - \frac{m^2}{2}\eta_{AB}\phi^A\phi^B\sqrt{|g|} \tag{119}$$

or more general ***U*-models**

$$\mathcal{L} = \frac{1}{2}\eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B g^{\mu\nu}\sqrt{|g|} - U(\phi)\sqrt{|g|} \tag{120}$$

with $U(\phi)$ given by

$$U(\phi) = f\left(\eta_{AB}\phi^A\phi^B\right) \tag{121}$$

and where f is some real-valued function on $\mathbb{R}$, e.g.,

$$f(x) = -\frac{m^2}{2}x, \qquad f(x) = \alpha(x-\lambda)^2 \tag{122}$$

and so on, that covers the Klein–Gordon models, quartically perturbed Klein–Gordon models and mass-generating Higgs terms. Obviously, in the above formulas ϕ is real and η-symmetric (and as a rule positively definite). For multiplets of complex fields one must use Hermitian sesquilinear target metrics

$$\eta_{\bar{A}B} = \overline{\eta}_{\bar{B}A} \tag{123}$$

and then

$$\mathcal{L} = \frac{1}{2}\eta_{\bar{A}B}\partial_\mu\overline{\phi}^{\bar{A}}\partial_\nu\phi^B g^{\mu\nu}\sqrt{|g|} - \frac{m^2}{2}\eta_{\bar{A}B}\overline{\phi}^{\bar{A}}\phi^B\sqrt{|g|} \tag{124}$$

or

$$U(\phi) = f\left(\eta_{\bar{A}B}\overline{\phi}^{\bar{A}}\phi^B\right). \tag{125}$$

Let us observe that (119), (120) and (121) are invariant under isometries (M,g) (e.g., Poincare group when (M,g) is Minkowskian) and under the group of internal symmetries $\mathrm{O}(W,\eta)$ (η-orthogonal groups). In the complex case (124), (125) the internal symmetry group becomes unitary (or perhaps pseudo-unitary) one $\mathrm{U}(W,\eta)$. It is clear that the "Born–Infeld'ization" of the above models is given by the Lagrangians

$$\mathcal{L}_{\mu\nu} := \alpha g_{\mu\nu} + \beta\eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B + \gamma\eta_{AB}\phi^A\phi^B g_{\mu\nu} \tag{126}$$

$$\mathcal{L}_{\mu\nu} := \alpha g_{\mu\nu} + \beta\eta_{\bar{A}B}\partial_\mu\overline{\phi}^{\bar{A}}\partial_\nu\phi^B + \gamma\eta_{\bar{A}B}\overline{\phi}^{\bar{A}}\phi^B g_{\mu\nu} \tag{127}$$

respectively for the real and complex scalar multiplets in which α, β, γ are real constants. It is clear that in the limit of weak fields Ψ one obtains as an asymptotics the quadratic expressions (119), (124). If W is a general manifold endowed only with the tensor field η but no additional structure like, e.g., linear space with translationally invariant η (more precisely with the constant $\eta \in W^* \otimes W^*$), then it is impossible to introduce any non-arbitrary potential term U, in particular, the

γ-controlled "mass" term. But the first two terms obviously continue to be well-defined, because geometrically $\partial_\mu \phi^A$ are nothing else but the matrix elements of tangent mappings

$$T\phi_p : T_pM \to T_{\phi(p)}W, \qquad T\phi_p \in \mathrm{L}\left(T_pM, T_{\phi(p)}W\right)$$

and their matrix elements are meant in the sense of local coordinates x^μ, y^A in some neighbourhoods of $p \in M$, $\phi(p) \in W$. Therefore, in the "β-term" we are dealing just with the well-defined ϕ-pull-back of η locally represented as

$$\left(\phi^* \cdot \eta\right)_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} \tag{128}$$

or, in absolute terms

$$\left(\phi^* \cdot \eta\right)_p (u, v) = \eta\left(T_{\phi(p)} \cdot u, T_{\phi(p)} \cdot v\right) \tag{129}$$

for any vectors $u, v \in T_pM$ attached at p.

Remark: In principle, in both (119), (124) and (126), (127) we could used two different "η-s," let us say η and $\varkappa$ which are both twice covariant tensor fields on W. This would mean that the mass term would be based on some "tensor of mass" and we would be dealing with some spectrum of masses for various modes (scalar invariants of the pair $(\eta, \varkappa)$). But such details here are evidently outside our main scope. What is essential here is that the above Born–Infeld models (126), (127) are "imperfect" in that the space–time metric tensor $g_{\mu\nu}$ is an absolute quantity "taken from the sky" and non-subject to the variational procedure, just as it was in the traditional Born–Infeld electrodynamics (46). But now this is no longer necessary. Namely, we can simply assume that it is just the pull-back (128) that will play the role of the spatio-temporal metric tensor

$$g[\phi]_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB}. \tag{130}$$

By its very construction it is symmetric. Obviously, it may be non-degenerate only if $N > n$. It will be normal-hyperbolic only if η itself is hyperbolic and one restricts ourselves to such ϕ-s that at the pull-back procedure $T\phi$ reproduces the $(+---)$ or $(---+)$ system of signs. One can interpret this procedure in the following "philosophical" terms: (W, η) is a proper, ideal "world" endowed with the fixed prescribed metric η. And the "real" world of our experience is just the "membrane" $\phi(M) \subset W$ endowed with the restricted metric $g\|\phi(M)$. Both the "membrane" and its induced metric are dynamical, non-fixed. If W has no other geometry than that given by η, then practically the only possible Lagrange tensor $\mathcal{L}[\phi]_{\mu\nu}$ is just $g[\phi]_{\mu\nu}$ itself

$$\mathcal{L}[\phi]_{\mu\nu} = g[\phi]_{\mu\nu}. \tag{131}$$

Geometrically this means that when some region $\Omega \subset M$ is chosen and boundary conditions $\phi|_{\partial\Omega}$ are fixed, then in W we consider all possible n-dimensional membranes $\phi(\Omega)$ spanned on the fixed $(n-1)$-dimensional boundary $\partial(\phi(\Omega)) =$

$\phi(\partial\Omega)$. The action of ϕ over Ω equals then the (pseudo-)Riemannian η-volume of $\phi(\Omega)$ and the variational principle consists in that this volume

$$I[\phi, \Omega] = \mathrm{Vol}_{\eta}\phi(\Omega) \tag{132}$$

has a stationary value in the class of all possible membranes spanned on the fixed $\phi(\partial\Omega)$. If η was a positively-definite Riemannian metric, the resulting $\phi(\Omega)$ are simply the minimal surfaces limited by $\phi(\partial\Omega) = \partial(\phi(\Omega))$. Obviously, everything is based on the assumption that ϕ is an injection, and therefore, locally

$$\mathrm{rank}\left[\partial_{\mu}\phi^{A}\right] = n. \tag{133}$$

Using a toy example one deals with "soap films" arising on the "wire loops" when put into a proper soap solvent. From the point of view of M the prescription producing the Lagrange tensor $\mathcal{L}[\phi]_{\mu\nu}$ from ϕ is generally-covariant, and the model based on the Lagrangian

$$\mathcal{L}[\phi] = \sqrt{\left|\det\left[\mathcal{L}[\phi]_{\mu\nu}\right]\right|} = \sqrt{\left|\det\left[g[\phi]_{\mu\nu}\right]\right|} \tag{134}$$

is Diff(M)-invariant. It is also invariant under the isometry group Diff$(W, \eta) \subset$ Diff(W), i.e., under transformations of W preserving the target metric η. Therefore, the total symmetry group is Diff$(M) \times$ Diff(W, η). If we do not demand the internal symmetry Diff(W, η) or restrict it to some proper subgroup, then the class of admissible Lagrangians may be extended by introducing some potential terms, e.g.,

$$\mathcal{L}[\phi] = K(\phi)\sqrt{\left|\det\left[g[\phi]_{\mu\nu}\right]\right|} = \sqrt{F(\phi)\left|\det\left[g[\phi]_{\mu\nu}\right]\right|} = \sqrt{\left|\det\left[U(\phi)g[\phi]_{\mu\nu}\right]\right|}. \tag{135}$$

These are of course various ways of writing the same and the scalar quantities K, F, U built algebraically of ϕ play the role of some potentials, just like in mechanical Jacobi–Maupertuis principle. Obviously, formally one could claim that there is no essential distinction between (134) and (135) and that everything is just the definition of η. Nevertheless, one can realize situations when η is somehow physically fixed and K, F, U are extra introduced additional self-interaction models. The general covariance in M is simply the freedom of reparameterization of of $\phi(M)$ as a "surface" in W. The Lagrangians (134) and (135) are homogeneous of degree n in derivatives $\partial\phi$, just like the one corresponding to (104) for the mixed tensor field X. This is an n-dimensional counterpart of the Finsler geometry and homogeneous variational principles in mechanics (where $n = 1$). If W is a linear space and η-constant in Cartesian coordinates, $\eta \in W^{*} \otimes W^{*}$, then again we have at disposal the class of scalars built of the η-norm of ϕ

$$f(\phi) = h\left(\|\phi\|^{2}\right) = h\left(\eta_{AB}\phi^{A}\phi^{B}\right). \tag{136}$$

They may be used as the above "potentials" K, F, U. In this special case one can also use another quite natural representation of the Lagrange tensor, namely,

$$\mathcal{L}_{\mu\nu} = \omega\eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B + \varkappa\lambda_\mu\lambda_\nu \tag{137}$$

where ω, $\varkappa$ are scalar functions of $\|\phi\|$ and

$$\lambda_\mu = \frac{1}{2}\partial_\mu\|\phi\|^2 = \eta_{AB}\phi^A\partial_\mu\phi^B. \tag{138}$$

Obviously, this again might be identified with (134) in which the original η is replaced by

$$\widetilde{\eta}_{AB} := \omega\eta_{AB} + \varkappa\eta_{AC}\eta_{BD}\phi^C\phi^D = \omega\eta_{AB} + \varkappa\phi_A\phi_B \tag{139}$$

the moving of capital indices meant in the sense of η. However, just as previously one can reasonably admit situations where both terms have some well-defined physical individualities and both ω, $\varkappa$ encode some interaction. One might also think about admitting some more general corrections like, e.g.,

$$\widetilde{\eta}_{AB} := \omega\eta_{AB} + \varkappa\lambda_A\lambda_B, \qquad \lambda_A = \nu_{AD}\phi^D. \tag{140}$$

However in such models the original $\mathrm{O}(U,\eta)$ internal symmetry would be broken. It is an important and interesting fact that there exists some relationship between the generally-covariant minimal surface Lagrangian (134), the usual metric-controlled d'Alembert Lagrangian (117) and the Palatini procedure generalizing the one from the General Relativity. Namely, let us consider the field system with degrees of freedom given by two a priori independent things: the metric tensor $G_{\mu\nu}$ in M and the multiplet of (real, let us assume) scalar fields ϕ^A on M, so our field quantity is the pair (G,ϕ). And now let us consider the dynamical model based on the following Lagrangian

$$\mathcal{L}[G,\phi] = \frac{1}{2}G^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB}\sqrt{|G|} + C\sqrt{|G|} \tag{141}$$

with the obvious meaning of $\sqrt{|G|}$. Variational procedure is performed with respect to both $G_{\mu\nu}$ and ϕ^A just like in the Palatini principle in the General Relativity, so C is something like the "cosmological constant." And just like in Palatini principle G enters $\mathcal{L}[G,\phi]$ in a purely algebraic way, derivatives $\partial_\varkappa G_{\mu\nu}$ do not occur. It turns out that for $n \neq 2$, in particular for the realistic space–time dimension $n = 4$, the variation with respect to the metric $G_{\mu\nu}$ implies that

$$G^{\mu\nu} = \frac{2C}{2-n}g[\phi]^{\mu\nu}, \quad G_{\mu\nu} = \frac{2-n}{2C}g[\phi]_{\mu\nu} = \frac{2-n}{2C}\partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB} \tag{142}$$

where the upper-case indices refer to the contravariant inverses of $G_{\mu\nu}$, $g_{\mu\nu}$

$$G^{\mu\alpha}G_{\alpha\nu} = \delta^\mu{}_\nu, \qquad g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu. \tag{143}$$

In particular, $g^{\mu\nu}$, $G^{\mu\nu}$ are not meant respectively as

$$G^{\mu\alpha} G^{\nu\beta} g_{\alpha\beta}, \qquad g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta} \tag{144}$$

instead, the G-subsystem (142) of the Euler–Lagrange equations implies that

$$g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta} = \frac{(2-n)^2}{4C^2} G^{\mu\nu}. \tag{145}$$

Thus, the two things coincide only for two special values of C

$$C = \pm \left(\frac{n}{2} - 1 \right) \tag{146}$$

and in particular

$$C = \pm 1 \tag{147}$$

for the physical value of dimension $n = 4$. Substituting (142) into the ϕ-subsystem of the Euler–Lagrange equations one obtains exactly the Euler–Lagrange equations following from (134). For the "pathological" dimension $n = 2$ the G-subsystem of the Euler–Lagrange equations is inconsistent for non-vanishing values of C. So, we put $C = 0$ and then one obtains that

$$G_{\mu\nu} = f g[\phi]_{\mu\nu} = f \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} \tag{148}$$

where f is an arbitrary (nowhere vanishing) function, thus the metrics $g[\phi]$, G are conformal to each other. This resembles the well-known peculiarities of the two-dimensional real manifolds, somehow related to the properties of complex analytic functions. Substituting (148) to the variational derivatives of (141) with respect to ϕ we nevertheless again obtain the Euler–Lagrange equations following from (134). The above reasoning seems to suggest that in some cases there is no essential physical distinction between quadratic in derivatives Lagrangians and their Born–Infeld "mutations." There are nevertheless some subtle points on the quantization level. Namely, classically equivalent Lagrangians may lead to non-equivalent quantum theories with numerically different predictions. The difference may be particularly drastic when one uses different configuration spaces, even with different numbers of degrees of freedom, just as above, when either ϕ itself or the pair (G, ϕ) were systems of generalized coordinates. Let us mention that the "pathological" model with $n = 2$ is just interesting in string models, when M is $\mathbb{R}^2$ or some strip in $\mathbb{R}^2$, and Minkowski space (W, η) is used as a target manifold (in "usual" field theories it occurs as a manifold of "independent variables"). The configurations ϕ are then assumed to be such that the induced metric $g[\phi] = \phi^* \cdot \eta$ is (normal-) hyperbolic. Physically this corresponds to the "world tube" of the string. It is know that in quantum theory the above models are non-equivalent (like the ones investigated by Polyakov). To explain the essence of the relationship between traditional Born–Infeld pattern and the general covariance and internal symmetry we quote some facts concerning the Euler–Lagrange equations following from (134). After some calculations it turns out that if W is a real manifold and η is a symmetric non-degenerate tensor field on W, then the field equations may be written in the following concise form

$$g[\phi]^{\mu\nu}\nabla_{[g[\phi]]\mu}\nabla_{[g[\phi]]\nu}\phi^A = 0, \qquad A = 1, \ldots, N \tag{149}$$

where $\nabla_{[g[\phi]]}$ denotes the covariant differentiation in the sense of the Levi-Civita affine connection built of $g[\phi]$. This is formally similar to the system of d'Alembert equations

$$G^{\mu\nu}\nabla_{[G]\mu}\nabla_{[G]\nu}\phi^A = 0, \qquad A = 1, \ldots, N \tag{150}$$

following from the Lagrangian

$$\mathcal{L}[\phi] = \frac{1}{2}G^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB}\sqrt{|G|} \tag{151}$$

with the fixed non-dynamical metric G on M and (W, η). Let us observe however that these equations are structurally completely different. Equations (150) are linear and mutually independent for different values of A. Unlike this equations (149) form a (strongly) coupled system of essential nonlinear second order equations. The reason is that the metric $g[\phi]$, the corresponding Levi-Civita connection and covariant differentiation depend themselves on the total multiplet ϕ. The system (149) is not even quasi-linear because the coefficients $g[\phi]^{\mu\nu}$ at the second (thus highest) derivatives $\partial_\mu\partial_\nu\phi^A$ of field variables are not algebraically built of ϕ itself alone. Instead they are rational functions of ϕ and its first order derivatives $\partial\phi$. Expression (149) has a geometric concise form. When explicitly written in terms of partial derivatives it becomes

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi^A + \partial_\nu\phi^A\left(\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\partial_\mu g_{\alpha\beta} = 0 \tag{152}$$

where for brevity we write g instead of $g[\phi]$. In this form the term with highest (second) derivatives is explicitly pointed out. Geometrically the equations (149) and (152) express the fact that the submanifold $\phi(M) \subset W$ has the vanishing mean curvature in the (pseudo-)Riemannian space (W, η) that we consider the general situation $N > n$, not necessarily the hypersurface case $n = N - 1$. Therefore, the second curvature form at any $p \in \phi(M)$ is a symmetric bilinear mapping $\mathcal{K}_p$ defined on the tangent space $T_p\phi(M) \subset T_pW$ and taking values in its η-orthogonal complement $T_p^\perp\phi(M) \subset T_pW$

$$\mathcal{K}_p : T_p\phi(M) \times T_p\phi(M) \to T_p^\perp\phi(M).$$

Analytically, when $\phi(M)$ is parameterized by coordinates x^μ on M, and e_r, $r = n+1, \ldots, N$ is a fixed system of mutually orthonormal and orthogonal to $\phi(M)$

vector fields defined on $\phi(M)$, K_p is represented by the system of $(N-n)$ twice covariant symmetric tensors $\mathcal{K}^r_p : T^*_p\phi(M) \otimes T^*_p\phi(M)$ with coordinates $\left(\mathcal{K}^r_p\right)_{\mu\nu}$

$$\mathcal{K}_p = \mathcal{K}^r_p e_{rp} \tag{153}$$

when p runs over the submanifold $\phi(M)$, one obtains the system of $(N-n)$ tensor fields $\mathcal{K}^r$. The pull-back by ϕ identifies $\mathcal{K}^r$ with tensor fields

$$K^r = \phi^* \mathcal{K}^r. \tag{154}$$

We use also the symbols $\mathcal{K}^r[\phi]$, $K^r[\phi]$ to denote explicitly the dependence of K, $\mathcal{K}$ on the injection ϕ. Obviously, $K^r[\phi]$ are built algebraically on ϕ and its first and second order derivatives

$$K^r[\phi] = K^r\left(\phi, \partial\phi, \partial^2\phi\right). \tag{155}$$

The tensor fields $K^r[\phi]$ depend on the choice of basic orthonormal vectors, nevertheless they characterize the intrinsic geometric object $K[\phi]$. Their system is usually referred to as the vector of second quadratic forms of the injection ϕ. Having the pairs of twice covariant tensors $g[\phi]$, $K^r[\phi]$ one can construct their systems of invariants, according to the general prescription. The system of mean curvatures is given by

$$H^r[\phi] := g[\phi]^{\mu\nu} K^r[\phi]_{\mu\nu}, \qquad r = n+1, \ldots, N. \tag{156}$$

These are the only invariants built linearly of the second quadratic form $K[\phi]$. Their collection is called the mean curvature vector in $\mathbb{R}^{N-n}$. It is an $\mathbb{R}^{N-n}$-valued field on M or $\phi(M) \subset W$. Using more geometric terms one can introduce the quantity

$$H[\phi] := H^r[\phi] e_r. \tag{157}$$

It does not depend on the choice of the system of vectors e_r, $r = n+1, \ldots, N$. As a geometric object $H[\phi]$ is a field defined on $\phi(M)$ (equivalently on M) which assigns to any point $p \in \phi(M)$ some vector $H[\phi]_p$ orthogonal (in the sense of $g[\phi]$) to $T_p\phi(M)$. It is obvious how to interpret $H[\phi]$ as a cross-section of an appropriate vector bundle over $\phi(M)$ – the fibre over p is just the $(N-n)$-dimensional linear space $T^\perp_p\phi(M) \subset T_p\phi(M)$. Equations (149) mean exactly that $\phi(M)$ has the vanishing mean curvature

$$H[\phi] = 0. \tag{158}$$

Therefore any solution over some domain Ω with the fixed boundary conditions $\phi(\partial\Omega) = \partial(\phi(\Omega))$ minimizes, or, to be more precise, "stationarizes" the n-dimensional surface-volume

$$I[\Psi, \Omega] = \int_\Omega \sqrt{|\det[g[\Psi]_{\mu\nu}]|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \tag{159}$$

among the class of all possible maps $\Psi : \Omega \to W$ with the boundary values

$$\Psi|_\Omega = \phi|_\Omega. \tag{160}$$

If our Lagrangian contains some potential like in (135), this is no longer the case, nevertheless it becomes true for the modified metric tensor

$$g_U[\phi] := U(\phi)g[\phi]. \tag{161}$$

This is just the idea of the mechanical Jacobi–Maupertuis variational principle which encodes the potential in an appropriately modified geodetic model. It is clear that analytically (158) is a system of $(N-n)$ equations on the N-tuple of field variables ϕ^A. This over-determinacy is just due to the general covariance of the Lagrangian and the resulting field equations. Among N fields ϕ^A there are n purely gauge ones corresponding to the n arbitrary functions labelling the elements of the group Diff(M). To obtain an effective system one should eliminate them by imposing n purely non-tensorial conditions and then separate the system of $(N-n)$ equations for $(N-n)$ gauge-free variables labelling the physical degrees of freedom. It is seen from (152) that the most natural way is to fix such a system of coordinates x^μ in which the second term does vanish and this is a correct procedure because this non-tensorial expression is independent on highest-order (second) derivatives. So, one assumes coordinates in which

$$\frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\partial_\mu g_{\alpha\beta} = 0. \tag{162}$$

It is seen that due to the scheme of contraction these coordinate conditions resemble the transversal Lorentz conditions like $G^{\mu\nu}\partial_\mu A_\nu = 0$, $\partial_\mu G^{\mu\nu} = 0$, and so on, known from electrodynamics and the General Relativity. With this type of gauging the field equations reduce to

$$g[\phi]^{\mu\nu}\partial_\mu\partial_\nu\phi^A = 0, \qquad A = 1, \dots, N. \tag{163}$$

These equations are still redundant unless one eliminates from them n quantities with the help of the n-th conditions (162). This may be done very easily locally and the procedure is also correct globally unless the topology of M and W creates some obstacles. Namely, one can identify the first n-tuple of fields ϕ^A with coordinates in M

$$x^\mu = \phi^\mu, \qquad \mu = 1, \dots, n.$$

It is easily seen that then (162) becomes an identity. The gauge-free fields, i.e., true degrees of freedom are then represented by the $(N-n)$-tuple ϕ^r, $r = n+1, \dots, N$, and have to satisfy the equations

$$g[\phi]^{\mu\nu}\partial_\mu\partial_\nu\phi^r = 0, \qquad r = n+1, \dots, N \tag{164}$$

whereas the first n-tuple of (152), (163) is an identity as well. In this way one obtains an effective system of $(N-n)$ second order differential equations for $(N-n)$ field variables ϕ^r, $r = n+1, \dots, N$. The system (164) is still strongly nonlinear and mutually coupled because of the way the quantities ϕ^r, $\partial_\mu\phi^r$ enter in the pull-back metric $g_{\mu\nu}$. The effective Lagrange tensor for (164) $\mathcal{L}^{\text{eff}}_{\mu\nu}$ is identical with the pull-back metric $g[\phi]_{\mu\nu}$ with the above coordinate conditions substituted

$$\mathcal{L}^{\text{eff}}_{\mu\nu} = \eta_{\mu\nu} + \eta_{r\mu}\partial_\nu\phi^r + \eta_{\nu r}\partial_\mu\phi^r + \eta_{rs}\partial_\mu\phi^r\partial_\nu\phi^s \tag{165}$$

i.e., if one takes into account the symmetry of η

$$\mathcal{L}^{\text{eff}}_{\mu\nu} = \eta_{\mu\nu} + 2\eta_{r(\mu}\partial_{\nu)}\phi^r + \eta_{rs}\partial_\mu\phi^r\partial_\nu\phi^s. \tag{166}$$

Let us remind that the summation over r, s is extended over the "gauge-fixed" range $r = n+1, \ldots, N$. If W is linear and η is constant, $\eta \in \mathrm{Sym}\,(W^* \otimes W^*)$, then one can always chose such a basis in W that η reduces to the block form and $\eta_{r\mu} = 0$ while the term linear in derivatives is then absent in (165) and (166). But even then it may happen that the block structure is not always the most convenient one and that it is better to admit expressions linear in $\partial\phi$. Let us now compare (165) and (166) with the expressions like (46), (66), (71), (74), (78), (126), (127), etc. It is seen that the simplest Born–Infeld scalar models are just implied by the demand of general covariance and their particular structure is shaped by some assumptions concerning internal symmetry (target space transformations). Roughly speaking, the traditional Born–Infeld structure with the external spatio-temporal metric in the space–time M (either absolute or Generally Relativistic one) is a byproduct of general covariant of square-root models without such metrics but with higher-dimensional target spaces W. As mentioned before, when the target space (W, η) is flat and the block representation is used for $\eta \in W^* \otimes W^*$, then the effective Lagrangian is given by

$$\mathcal{L}^{\text{eff}} = \sqrt{|\det\left[\eta_{\mu\nu} + \partial_\mu\phi^r\partial_\nu\phi^s\eta_{rs}\right]|} \tag{167}$$

where the summation convention over r, s is restricted to the range $r = n+1$, $\ldots, N$. Therefore, in the fixed-gauge description the originally internal target metric η_{AB} plays a double role. First, the system $[\eta_{\mu\nu}]$, $\mu, \nu = 1, \ldots, n$, acts as an effective space–time metric, and, second, $[\eta_{rs}]$, $r, s = n+1, \ldots, N$, represents the effective internal geometry of gauge-free state variables. And in this way one comes back to the starting point of our Born–Infeld'ization programme (126), (127). The scalar Born–Infeld optics (66), (67) may be interpreted in generally-covariant terms as the dynamics of a four-dimensional membrane-minimal hypersurface living in a five-dimensional target universe ($n = 4$, $N = 5$) with the pseudo-Euclidean geometry of the normal-hyperbolic signature

$$[\eta_{\mu\nu}] = \mathrm{diag}(1, -1, -1, -1), \qquad \eta_{55} = \eta > 0, \qquad \eta_{\mu 5} = \eta_{5\mu} = 0.$$

It is evident that the Euler–Lagrange equations implied by (167) have particular solutions given by affine injections

$$\phi^r = C^r{}_\mu x^\mu + C^r \tag{168}$$

where ${C^r}_\mu$, C^r are constants, thus

$$g_{\mu\nu} = \eta_{\mu\nu} + \eta_{rs}{C^r}_\mu {C^s}_\nu. \tag{169}$$

If ϕ^r are to be physical fields, they must satisfy appropriate boundary conditions at infinity. If ϕ occurs in the Lagrangian only through its derivatives $\partial\phi$, that means that at infinity ϕ must be asymptotically constant, thus

$${C^r}_\mu = 0, \qquad \phi^r = C^r. \tag{170}$$

Let us consider small perturbations of the "vacuum" (170)

$$\phi^r = C^r + f^r, \qquad f^r \approx 0.$$

Performing the linearization procedure we find that up to higher-order terms f^r satisfy the variational Jacobi equations

$$\eta^{\mu\nu}\partial_\mu\partial_\nu f^r = 0, \qquad r = n+1, \ldots, N \tag{171}$$

i.e., just the usual d'Alembert equations following from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu f^r \partial_\nu f^s \eta_{rs}\sqrt{|\eta|}. \tag{172}$$

Let us finish this multiscalar topic with a few examples, some of them with mutually overlapping:

i) $n = 1$, N is arbitrary. We put $M = \mathbb{R}$, while (W, η) is taken to be a (pseudo-) Riemannian manifold. The "space–time" is one-dimensional and just plays the role of the structureless parameters. Minimal or rather stationary one-dimensional "surfaces" are geodetic curves, e.g., world lines of relativistic particles.
ii) $n = 1$, N is arbitrary, $M = \mathbb{R}$ (just parameters again), (W, η) is Riemannian and we take (135) with $F = 2(E - V)$, where E denotes a fixed energy value and $V : W \to \mathbb{R}$ is a potential energy function on the configuration space W. The corresponding scheme is nothing else but the Jacobi–Maupertuis variational principle in mechanics.
iii) $n = 1$, $N = 4$ and (W, η) being the Minkowski space. This is the special case of i), and

$$\mathcal{L}^{\mathrm{eff}} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} \tag{173}$$

with the obvious meaning of symbols is just the "non-relativistically written" Lagrangian of the freely moving relativistic particle. Let us remind that just this Lagrangian with its "saturation effect" was one of the motivations for the Born–Infeld electrodynamics.

iv) $n = 2$, $N = 3$ with (W, η) the usual Euclidean space. Here one stucks on the effects like soap films, rubber films, and so on, spanned, e.g., on the wire loop. Other possibility: "deformed minimal surfaces" described by (135), the "potentials" K, F, U representing, e.g., the effect of "wind."

v) $n = 2$, $N = 4$, $M = \mathbb{R}^2$ or $S^1 \times \mathbb{R} = \mathrm{U}(1) \times \mathbb{R}$, and (W, η) being the Minkowski space. If we restrict ourselves to such $\phi : M \to W$ so that $\phi^* \eta$ is normal-hyperbolic, the resulting objects are 't Hooft–Polyakov strings.

For multiplets of scalars it is necessary to use some absolute geometry in the target space. This may look perhaps disappointing from the point of view of the amorphous philosophy. Nevertheless such models are evidently related to realistic physical theories via concepts like strings, p-branes, etc. At the same time they contain a good deal of geometry like the theory of minimal surfaces. In this respect the academic model (104) and the other ones based on it have a completely different status. As yet there is no evidence for their practical utility. But there is a very interesting convolution of convincing geometric ideas in such models:

i) They are generally covariant in M, just like the above scalar multiplet models.

ii) They do not assume any metric or any kind of absolute geometry in the target space $W = T^1_1 M$ either. This is something completely new in comparison with the scalar models. The only used geometry of W is the intrinsic one, namely that of the corresponding fibre bundle over M. Incidentally, this induces us to admit only the cross-sections of $T^1_1 M$ over M as field configurations, not the general injections of M into $W = T^1_1 M$ (nevertheless, there are good reasons to check the usefulness of general injections in some hypothetic theory of structured continuous media).

iii) When one has at disposal only the $T^1_1 M$-degrees of freedom, then the general covariance exactly implies the Born–Infeld-type dynamics as the only possibility. In this respect it is so like in the scalar multiplet models, however now there is no extra introduced target space metric.

In spite of the above mentioned doubtful or at least non-clear perspectives of physical applications of the model one can try to speculate about what might seem to be expected. Element of the set $T^1_1 M$ are linear mappings of linear spaces $T_p M$ into themselves. These are degrees of freedom of affine bodies [18–22, 24–27]. So, if M is a four-dimensional space-time manifold, one can think about something like the relativistic micromorphic continuum which somehow unifies gravitational field with the "cosmic substratum." If M is the three-dimensional space, one obtains something like equilibrium problems for this kind of continua, description of internal (residual) stresses, etc. Physically these are rather speculative ideas. But even on the purely mathematical level nothing is known about solutions of (104). The main and intriguing curiosity of (104) is that it is the simplest scheme where the general covariance just implies the Born–Infeld type nonlinearity and no internal geometry is to be introduced by hand. With scalar multiplets the first was true but the second evidently false. It seems that the second (no necessity of extra introduced target metric) may be possible only for fields having some "external" spatio-temporal indices. They are usually cross-sections of some vector bundles over M with fibres $\mathcal{F}_p M$ over $p \in M$ containing factors like $T_p M$ and $T_p^* M$. In the model (104) we have

$$\mathcal{F}_p M = \mathrm{L}(T_p M) \simeq T_p M \otimes T_p^* M$$

and the fibres are purely M-tensorial. However, as we shall see, there are other interesting examples where $\mathcal{F}_p M$ is the tensor product of $T_p M$ or $T_p^* M$ with some auxiliary internal space V geometrically independent of M. Physically interesting are models where V is a real linear space of dimension $n = \dim M$, and $\mathcal{F}_p M = T_p M \otimes V^*$ or $V \otimes T_p^* M$. Field components are then labelled respectively as $e^{\mu}{}_{A}$, $e^{A}{}_{\mu}$ and if $\det\left[e^{\mu}{}_{A}\right] \neq 0$, one assumes that

$$\left\langle e^{A}, e_{B} \right\rangle = e^{A}{}_{\mu} e^{\mu}{}_{B} = \delta^{A}{}_{B}$$

i.e., one deals with the mutually dual fields of frames (tetrads) and co-frames (co-tetrads). Models based on such degrees of freedom are used in mechanics of structured continua, both non-relativistic and relativistic. More precisely, one deals then with the continuum of infinitesimal affine bodies, i.e., homogeneously deformable gyroscopes. Internal degrees of freedom are represented just by "legs" e_A or "co-legs" e^A of the frame or co-frame [24]. This is a phenomenological mechanical model. On the level of fundamental physics one deals with the so-called tetrad models of the gravitation. There are both similarities and differences between models with kinematics based on the bundles $\mathrm{L}M$ and F^*M (or FM) with fibres $\mathrm{L}(T_p M) \simeq T_p M \otimes T_p^* M$ and $F_p^* M = \mathrm{L}(T_p M, V) \simeq V \otimes T_p^* M$ (or $F_p M = T_p M \otimes V \simeq \mathrm{L}(V, T_p M)$). The fibres in both bundles have dimension n^2 and consist of linear mappings from $T_p M$ into (onto) something n-dimensional and linear, respectively $T_p M$ and V. In the second case this "something" is fixed and independent of V but we can simply put $\mathbb{R}^n$ after some choice of basis in V. However, in the case of $\mathrm{L}M$ all generally covariant models have automatically the Born–Infeld structure. It is perfectly based on the local paradigm and there is nothing like the fixed target metric. When we deal with FM there is no direct implication as general covariance $\Rightarrow$ Born–Infeld. And indeed in the tetrad models of gravitation there exists an infinity of models with Lagrangians quadratic in derivatives ∂e. All they, being generally covariant lead to essentially nonlinear, nevertheless quasi-linear field equations. And they must assume some internal metric in V, $\eta \in V^* \otimes V^*$. And they have the internal symmetry group $\mathrm{O}(V, \eta) \subset \mathrm{GL}(V)$, the η-orthogonal subgroup of $\mathrm{GL}(V)$. In tetrad models of gravitation $\dim V = \dim M = 4$ and η is normal-hyperbolic (Minkowskian).

There are also linear conformal models invariant under the Weyl group $\mathbb{R}^+\mathrm{O}(V,\eta)$ preserving η up to a constant linear factor. However, as shown in [24] there are maximally amorphous generally covariant models free of anything like η and invariant under $\mathrm{GL}(V)$. And just those ones with maximal available symmetries (thus maximally amorphous) automatically have the Born–Infeld structure and the resulting field equations are essentially nonlinear and even non quasi-linear.

4. Natural Examples

The above discussion of essentially nonlinear (non-perturbatively nonlinear) field-theoretic models showed the existence of some relationship between general covariance, Born–Infeld type nonlinearity and "large" groups of internal symmetries. It was seen that purely scalar models may be generally covariant only within the Born–Infeld scheme and that some target metric or other target structure (hermitian, symplectic etc.) had to be assumed there as something absolute. The (as yet academic) $T^1_1 M = \mathrm{L}(M)$-model is also distinguished by the property that its general covariance is intimately connected with the Born–Infeld nonlinearity, but moreover, the theory is completely amorphous. No internal geometry for field values was assumed additionally. And finally, "tetrad" models with fibres $V \otimes T^*_p M$ (or $T_p M \otimes V^*$) may be generally covariant (thus essentially nonlinear) without having necessarily the Born–Infeld structure, but then, just as with scalar models, the internal metric in V was necessary. However, the demand of higher $\mathrm{GL}(V)$-internal symmetry again implies the Born–Infeld structure. The status of bundles with fibres $V \otimes T^*_p M$ where the dimension of V is general (but higher than one) is not yet clear in this respect. In any case, the necessity of using in some models things like fixed internal geometries, or absolute elements, without a deeper motivation is rather dissatisfying. Thus the natural question arises about the possible target spaces with intrinsic geometries somehow motivated by more fundamental structures. Let us quote some natural examples, applicable ones.

1) Self-dual linear spaces. These are linear spaces of the form

$$W := U \times U^*$$

where U is a linear space and U^* is its dual. W by its very structure is endowed with the natural bilinear pairing θ given by

$$\theta\left((q_1, p_1), (q_2, p_2)\right) := \langle p_1, q_2\rangle. \tag{174}$$

It is evidently degenerate, however its symmetric and skew-symmetric parts are non-singular. Let us denote

$$\eta\left((q_1, p_1), (q_2, p_2)\right) := \langle p_1, q_2\rangle + \langle p_2, q_1\rangle \tag{175}$$

$$\Gamma\left((q_1, p_1), (q_2, p_2)\right) := \langle p_1, q_2\rangle - \langle p_2, q_1\rangle. \tag{176}$$

Obviously $2\theta = \eta + \Gamma$. If U is linear over reals, then the symmetric scalar product $\eta : W \times W \to \mathbb{R}$ has the neutral pseudo-Euclidean signature $(m+, m-)$, where obviously $m = \dim U$. So, if (W, η) is used as a target space, then images of injections $\phi : M \to W$ a priori may have various signatures, depending on the dimension of M. Obviously, Γ is a symplectic form on W. We did not investigate scalar multiplets with the skew-symmetric scalar product in the target space. It seems, rather few if anything is known about such models mathematically, the more physically. So, ϕ may be considered as a pair of scalar multiplets

$$Q := M \to U, \qquad P := M \to U^*$$

$\phi = (Q, P)$. Using dual bases $(\ldots, \varepsilon_a, \ldots)$, $(\ldots, \varepsilon^a, \ldots)$ in U, U^* we can represent ϕ as follows

$$\phi = (Q^a \varepsilon_a, P_a \varepsilon^a) .$$

Then, analytically

$$g[\phi]_{\mu\nu} = \partial_\mu Q^a \partial_\nu P_a + \partial_\nu Q^a \partial_\mu P_a \tag{177}$$

$$\gamma[\phi]_{\mu\nu} := (\phi^* \Gamma)_{\mu\nu} = \partial_\mu Q^a \partial_\nu P_a - \partial_\nu Q^a \partial_\mu P_a . \tag{178}$$

Both might be candidates for $\mathcal{L}[\phi]_{\mu\nu}$ but the second, skew-symmetric one is rather exotic. There is a natural monomorphism of the full linear group $\mathrm{GL}(U)$ into the η-orthogonal group $\mathrm{O}(W, \eta)$, namely, for any $A \in \mathrm{GL}(U)$ we define $\bar{A} \in \mathrm{O}(W, \eta)$ as follows

$$\bar{A}(q, p) := \left(Aq, p \circ A^{-1} \right) . \tag{179}$$

$\bar{A}$ preserves also the symplectic form Γ (it is an extended point transformation of (W, Γ) being a special kind of canonical transformations). Obviously, $\bar{A}$ preserves also the form θ and as a matter of fact, these are the most general linear transformations preserving θ.

2) Linear spaces of endomorphisms. Again let U be an arbitrary linear space and take $W := \mathrm{L}(U) \simeq U \otimes U^*$, the space of linear mappings of U into itself. There is a natural class of scalar products, i.e., bilinear forms, on $\mathrm{L}(U)$

$$\eta(X, Y) = \lambda \operatorname{Tr}(XY) + \mu \operatorname{Tr} X \operatorname{Tr} Y \tag{180}$$

where λ, μ are constants. Obviously, we cannot put $\lambda = 0$ without the catastrophic destruction of the non-singularity of η as the matrix underlying the μ-term has the rank one. Therefore, the second term is a mere auxiliary correction. If U is real, then η is pseudo-Euclidean and has the signature

$$\left(\frac{1}{2} m(m-1)-, \frac{1}{2} m(m+1)+ \right) .$$

Obviously, $\mathrm{L}(U)$ is canonically identical with $\mathfrak{gl}(u)$, the Lie algebra of $\mathrm{GL}(U)$. The special choice $\lambda = 2m$, $\mu = -2$ corresponds just to its Killing form

$$(X|Y) := \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y) \tag{181}$$

where $\mathrm{ad}_X : W \to W$ is given as usual by the rule

$$\mathrm{ad}_X \cdot A := [X, A]. \tag{182}$$

Obviously, (181) is degenerate because $\mathrm{GL}(U)$ is not semisimple and the singularity due to dilatations. However, (181) is non-degenerate on $\mathfrak{sl}(u)$, i.e., on the subspace of $\mathrm{L}(U)$ consisting of traceless mappings (the Lie algebra of $\mathrm{SL}(U)$). Just as previously, the group $\mathrm{GL}(U)$ is mapped monomorphically into the group $\mathrm{O}(W, \eta)$ in the sense

$$\mathrm{GL}(U) \ni B \mapsto \mathrm{Ad}_B \in \mathrm{GL}(W), \qquad \mathrm{Ad}_B X := BXB^{-1}. \tag{183}$$

But now this is not a monomorphism because the kernel consists of dilatations. For any (λ, μ) such that $\lambda/\mu \neq -m$ (thus for any but a measure-zero closed subset), (180) is non-degenerate. This is true in particular for $\mu = 0$, i.e., for the main term. It is obvious that

$$\begin{aligned} g[\phi]_{\mu\nu} &= \lambda\, \mathrm{Tr}\,(\partial_\mu \phi \partial_\nu \phi) + \mu\, \mathrm{Tr}\, \partial_\mu \phi\, \mathrm{Tr}\, \partial_\nu \phi \\ &= \lambda \partial_\mu \phi^A{}_B \partial_\nu \phi^B{}_A + \mu \partial_\mu \phi^A{}_A \partial_\nu \phi^B{}_B \end{aligned} \tag{184}$$

where, obviously, $\phi^A{}_B(x)$ are matrix elements of $\phi(x) \in \mathrm{L}(U)$.

3) For many reasons the example above is of particular interest of us, nevertheless $\mathfrak{sl}(u)$ is but the special case of the situation when W is a semisimple Lie algebra. Then it is a Killing metric η

$$\eta(A, B) = \mathrm{Tr}\,(\mathrm{ad}_A \mathrm{ad}_B)\,, \qquad \mathrm{ad}_A X := [A, X]. \tag{185}$$

And again the transformations Ad_g are isometries of (W, η) for any group element g, whereas their "logarithms" ad_A are infinitesimal isometries. Using some base in W we have obviously

$$\eta_{rs} = C^z{}_{tr} C^t{}_{zs} = \eta_{sr} \tag{186}$$

where $C^t{}_{zs}$ are structure constants with respect to the given basis. Then

$$g[\phi]_{\mu\nu} = \partial_\mu \phi^r \partial_\nu \phi^s \eta_{rs}. \tag{187}$$

The above examples were motivated mainly by models where the target space was a real manifold endowed with some pseudo-Riemannian metric, and usually just the pseudo-Euclidean vector space. However, and it was briefly mentioned, in applications one deals very often with complex linear spaces endowed with hermitian internal metrics, i.e., with (pseudo-)unitary geometry. Let us do some comments concerning scalar multiplets with such targets. So W is a complex vector space and $h : W \times W \to \mathbb{C}$) is a sesquilinear form with the convention: half-linear ("antilinear") in the first argument, and so, analytically

$$\eta(w, z) = \eta_{\bar{A}B}\bar{w}^{\bar{A}} z^B. \tag{188}$$

As mentioned, in applications it is usually hermitian

$$\eta(w, z) = \overline{\eta(z, w)}, \qquad \eta_{\bar{A}B} = \overline{\eta}_{\bar{B}A}. \tag{189}$$

The configuration mapping $\phi : M \to W$ enables one to perform the pull-back

$$g[\phi] = \phi^* \cdot \eta, \qquad g_{\mu\nu} = \partial_\mu \overline{\phi}^{\bar{A}} \partial_\nu \phi^B \eta_{\bar{A}B}. \tag{190}$$

There are however some subtle points. Namely, the tangent mappings $T\phi_x : T_x M \to W$ are linear over the real field $\mathbb{R}$ with respect to their arguments. One can extend them to the complexified tangent spaces ${}^{\mathbb{C}}T_x M = \mathbb{C} \otimes T_x M$ (tensor product performed over reals). However, one does not need this in usual field theory, and because of this the usual real space notation will be used, i.e., $g_{\mu\nu}$, not $g_{\overline{\mu}\nu}$ (bar over μ omitted). For any $x \in M$, g_x is an $\mathbb{R}$-bilinear machine which produces complex numbers $g_x(u, v) \in \mathbb{C}$ from real vectors $u, v \in T_x M$. This prescription is hermitian, and the corresponding coefficient matrix is a analytically hermitian

$$g_x(u, v) = \overline{g_x(v, u)}, \qquad g_{\mu\nu} = \overline{g}_{\nu\mu}. \tag{191}$$

But one must be aware, it is not quite clear what a prescription is to be used for constructing the Lagrangian. First of all, because the matrix $[g[\phi]_{\mu\nu}]$ is hermitian, its determinant is real (and if η is positively definite, this determinant is also positive). Therefore, the Lagrange tensor may be defined as

$$\mathcal{L}[\phi] := g[\phi]. \tag{192}$$

Some "potential" terms are also possible like in (135). But one can also think in the following way: $g[\phi]$ itself is not the space–time metric. Indeed, the "squared norm" $g(u, u)$ and "interval" $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ feel only the symmetric, i.e., autonomically real part of $g[\phi]$

$$\operatorname{Re} g_{\mu\nu} = \frac{1}{2}\left(g_{\mu\nu} + \overline{g}_{\mu\nu}\right) = \frac{1}{2}\left(g_{\mu\nu} + g_{\nu\mu}\right) = g_{(\mu\nu)}. \tag{193}$$

Imaginary part is skew-symmetric

$$\operatorname{Im} g_{\mu\nu} = \frac{1}{2\mathrm{i}}\left(g_{\mu\nu} - \overline{g}_{\mu\nu}\right) = \frac{1}{2\mathrm{i}}\left(g_{\mu\nu} - g_{\nu\mu}\right) = -\mathrm{i} g_{[\mu\nu]}. \tag{194}$$

And it is just

$$G[\phi]_{\mu\nu} = \operatorname{Re} g_{\mu\nu} = g_{(\mu\nu)} \tag{195}$$

which is a good candidate for the metric-like tensor ϕ-induced from (W, η). So, perhaps one should rather expect that

$$\mathcal{L}[\phi] := G[\phi]? \tag{196}$$

And by the way – why not the exotic skew-symmetric tensor

$$\mathcal{L}[\phi]_{\mu\nu} := g[\phi]_{[\mu\nu]}? \tag{197}$$

These are open questions to be considered carefully. Within the complex-hermitian framework there are some counterparts of (177) and (178).

4) We assume here again that there is an auxiliary linear space U, this time over the complex field $\mathbb{C}$, and the target space W is its byproduct of some specific structure resulting in the existence of a distinguished hermitian metric η. The space U gives rise to the quadrupole of complex linear spaces of the same dimension built on U itself, its complex conjugate $\overline{U}$, dual U^* and anti-dual $\overline{U}^* = \overline{U^*}$. Obviously, U^* consists of $\mathbb{C}$-valued $\mathbb{C}$-linear functions on U, the anti-dual $\overline{U}^*$ consists of antilinear (half-linear) functions on U, and the complex conjugation $\overline{U}$ consists of antilinear functions on the dual U^*. These definitions apply only to finite dimensional spaces, just as the identifications between $\overline{U}^*$ and $\overline{U^*}$ (similar to the finite dimensional identification of U^{**} and U). Now the most natural analogue of the real self-dual space is the following target model

$$W := U \times \overline{U^*} = U \times \overline{U}^*.$$

We would also use something like $\overline{W} = \overline{U} \times U^*$ and so on, obviously, there is noting essentially new in such modifications (the net of canonical isomorphisms). In analogy to (174) W is endowed with the sesquilinear form $\theta : W \times W \to \mathbb{C}$ which is degenerate and neither hermitian nor antihermitian,

$$\theta\left((q_1, p_1), (q_2, p_2)\right) = \overline{p_1(q_2)}. \tag{198}$$

We can decompose it into hermitian and anti-hermitian parts respectively $\eta/2$ and $\Gamma/2$, where

$$\eta\left((q_1, p_1), (q_2, p_2)\right) := \overline{p_1(q_2)} + p_2(q_1) \tag{199}$$

$$\Gamma\left((q_1, p_1), (q_2, p_2)\right) := \overline{p_1(q_2)} - p_2(q_1) \tag{200}$$

thus,

$$\theta = \frac{1}{2}(\eta + \Gamma) = \frac{1}{2}\eta + \frac{\mathrm{i}}{2}\varkappa, \qquad \varkappa := -\mathrm{i}\Gamma.$$

If $(\ldots, \varepsilon_a, \ldots)$ and $(\ldots, \varepsilon^a, \ldots)$ are mutually dual bases in U, U^* and $(\ldots, \overline{\varepsilon^a}, \ldots)$ is the corresponding anti-dual basis in $\overline{U^*}$, $\overline{\varepsilon^a}(\varepsilon_b) = \delta^a{}_b$, then with respect to the basis

$$(\ldots, \varepsilon_a, \ldots; \ldots, \overline{\varepsilon^a}, \ldots) \tag{201}$$

the sesquilinear hermitian forms θ, $\varkappa$ have the following representation via the $2m \times 2m$ matrices

$$[\eta_{\bar{r}s}] = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad [\varkappa_{\bar{r}s}] = \mathrm{i} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad [\Gamma_{\bar{r}s}] = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{202}$$

where obviously 0, I are respectively zero and identity $m \times m$ matrices. Concerning the notation used in (201) let us remind that for any $f \in U^*$, $\overline{f} \in \overline{U^*}$ is defined as follows

$$\overline{f}(u) := \overline{f(u)} = \overline{\langle f, u \rangle}. \tag{203}$$

Both hermitian tensors η and $\varkappa$ on W have the same neutral signature $(m+, m-)$. Representing $\phi : M \to W$ in terms of two mappings $Q : M \to U$, $P : M \to \overline{U^*}$, we have the following expressions for the metric-Lagrange tensors

$$g[\phi, \eta]_{\mu\nu} = \partial_\mu Q^a \partial_\nu \bar{P}_a + \partial_\nu \bar{Q}^{\bar{a}} \partial_\mu P_{\bar{a}} \tag{204}$$

$$g[\phi, \varkappa]_{\mu\nu} = -\mathrm{i}\partial_\mu Q^a \partial_\nu \bar{P}_a + \mathrm{i}\partial_\nu \bar{Q}^{\bar{a}} \partial_\mu P_{\bar{a}}. \tag{205}$$

Any of them may be used and both have the neutral signature. Therefore, both the elliptic (equilibrium) and hyperbolic (evolution-type, relativistic) applications are possible. It is interesting to mention about some very special and at the same time very important special case of intrinsic target structures. It has to do with spinors and bispinors in the four-dimensional space-time. Namely, let U be a two-dimensional complex linear space. According to the Finkelstein–Penrose–Weizsäcker philosophy dimension “two” is not accidental – it is just the linear shell of the two-element set corresponding to the elementary yes-no dichotomy of quantum measurements. The very complex dimension two implies some intrinsic structures. Namely, $\Lambda^2 U$ and $\Lambda^2 U^*$ are one-dimensional, so U carries the canonical, unique, conformal-symplectic structure (based on some bilinear skew-symmetric, not sesquilinear anti-hermitian form – as an essential difference). And similarly, linear spaces of scalar densities and skew-symmetric twice contravariant and twice covariant tensor densities (of any integer weight) are one-dimensional, thus conformally-unique. Physical interpretation: U is the space of the Weyl spinors (antineutrino), $\overline{U}^*$ is the space of the anti-Weyl spinors (neutrino) and $W = U \times \overline{U^*}$ represents the Dirac bispinors (massive fermions) – all of them exist in the four-dimensional space-time of course. Then η is used for constructing the Dirac conjugation and the mass term in the Dirac Lagrangian. In commonly used representations the matrix of η coincides numerically with that of γ^0, but of course it is a total mistake to confuse η and γ^0. The latter, as any γ^μ is a linear mapping of W into itself, not a sesquilinear form on W. The numerical coincidence of their matrices is an accidental property of certain choices of bases in W. Raising the first index of $\varkappa$ with the help of η one obtains the operator $\gamma^5 \in \mathrm{L}(W)$ in the Dirac theory. And similarly all $\gamma^\mu \in \mathrm{L}(W)$ are η-hermitian (but their matrices are not literally hermitian). Another peculiarity of $\dim U = 2$ is that the space $\mathrm{Herm}\left(\overline{U} \otimes U\right)$ of hermitian tensors in U, and automatically also $\mathrm{Herm}\left(\overline{U}^* \otimes U^*\right)$ is endowed with the canonical symmetric scalar product of the normal hyperbolic signature. More precisely, it is unique up to normalization. The complex conformal-symplectic structure $\Lambda^2 U$ generates the Lorentz-conformal structure of $\mathrm{Herm}\left(\overline{U} \otimes U\right)$. Therefore, being four-dimensional over $\mathbb{R}$, $\mathrm{Herm}\left(\overline{U} \otimes U\right)$ is Minkowskian and appears as a model space of the physical space–time. The above mentioned conformal-Minkowskian structure has to do with the target geometry of the gravitational (co-)tetrad. In this way, according to Finkelstein–Penrose–Weizsäcker approach the fundamental quantum ideas imply the normal-hyperbolic structure of the space–time (no doubt, the hermitian forms on U, i.e., elements of $\mathrm{Herm}\left(\overline{U}^* \otimes U\right)$ are close to quantum ideas). The above target spaces were linear. There are also other differentiable manifolds with intrinsic target metrics implied by some more fundamental geometry.

5) Lie groups as target spaces. It is convenient to use the language of linear groups to avoid the crowd of artificially sophisticated symbols. Incidentally, the only non-linear groups one is faced with in the physical studies are $\widetilde{\mathrm{GL}(V)}$, $\widetilde{\mathrm{SL}(V)}$, the universal covering groups of $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, where V is a real vector space. So let U be an auxiliary linear space, $\mathrm{GL}(U)$ its linear group and $G \in \mathrm{GL}(U)$ some Lie subgroup. More precisely, the target space W may be not the group itself, but its group space, which can be viewed as a homogeneous space with trivial isotropy groups. Also more general homogeneous spaces are of interest, however here we concentrate simply on the special case $W = \mathrm{GL}(U)$. Being a group, G possesses a large group of target transformations, e.g.,

$$G \ni X \to AXB^{-1} = \left(L_A \circ R_{B^{-1}}\right)(X) = \left(R_{B^{-1}} \circ L_A\right)(X) \tag{206}$$

where $A, B \in G$ and taking the inverse B^{-1} is obviously non-essential. The point is only that the assignment $G \times G \ni (A, B) \mapsto L_A \circ R_B^{-1}$ is a homomorphism of $G \times G$ into the group of transformations of G. Obviously, this homomorphism in general is not a monomorphism, for example,

$$L_A \circ R_{A^{-1}} = \mathrm{Id}_G \tag{207}$$

if A is an element of the centre of G. This holds for dilatations if $G = \mathrm{GL}(U)$. An important subgroup of (206) consists of inner automorphisms

$$\mathrm{Int}_A := L_A \circ R_{A^{-1}} = R_{A^{-1}} \circ L_A, \quad X \mapsto AXA^{-1}. \tag{208}$$

If G is not semisimple, then its automorphism group Aut_G is wider than Int_G and it is reasonable then to admit the total group generated by $L_G R_G$ and Aut_G. It is natural to expect that particulary interesting for applications will be the target metrics η that are invariant under appropriate subgroups of the above transformations groups. The most natural candidates are the metrics that are invariant under left, or right, or both regular translations, L_G, R_G, $L_G R_G$. To obtain them one should take some algebraic metrics N on the Lie algebra $\mathfrak{g} = T_e G$ (e denoting the identity element of G), $N \in \mathfrak{g}^* \otimes \mathfrak{g}^*$, and then extend them respectively with the use of left or right regular translations to metric tensor fields η defined all over the manifold G. Let us explain this in more detail. We construct differential form Ω, $\widehat{\Omega}$ on $\mathrm{GL}(U)$ with values in the commutator Lie algebra $\mathrm{L}(U) \simeq \mathfrak{gl}(u)$

$$\Omega := \mathrm{d}LL^{-1}, \qquad \widehat{\Omega} := L^{-1}\,\mathrm{d}L = L^{-1}\Omega L. \tag{209}$$

They are respectively right- and left-invariant in the sense of regular translations. And under the left and right translations (206) they obey to the following adjoint rules

$$\Omega \mapsto A\Omega A^{-1}, \qquad \widehat{\Omega} \mapsto B\widehat{\Omega} B^{-1}.$$

If $G \subset \mathrm{GL}(W)$ is a non-trivial submanifold, then Ω, $\widehat{\Omega}$ are restricted to points $y \in G$ and vectors $u \in T_g G \subset \mathrm{L}(W)$ tangent to G. Therefore, we use $\mathfrak{g}$-valued one-forms

$$\Omega_G := \Omega \| G, \qquad \widehat{\Omega}_G := \widehat{\Omega} \| G. \tag{210}$$

Nevertheless, to avoid the crowd of symbols, we continue to denote them simply by Ω, $\widehat{\Omega}$, if there is no danger of confusion. If $(\dots, \varepsilon_r, \dots)$ and $(\dots, \varepsilon^r, \dots)$ are dual bases in $\mathfrak{g}$, $\mathfrak{g}^*$, we expand

$$\begin{aligned} \Omega &= \Omega^r \varepsilon_r, & \widehat{\Omega} &= \widehat{\Omega}^r \varepsilon_r \\ \Omega^r &= \langle \varepsilon^r, \Omega \rangle, & \widehat{\Omega}^r &= \left\langle \varepsilon^r, \widehat{\Omega} \right\rangle. \end{aligned} \tag{211}$$

Then, we can express the mentioned Lie-algebraic metric in terms of the basis

$$N = N_{rs} \varepsilon^r \otimes \varepsilon^s, \qquad N_{rs} = \langle \varepsilon_r \otimes_s, N \rangle. \tag{212}$$

Then the most general left- and right-invariant metrics on $W = G$ are given respectively by the tensor fields on G

$$\eta_l[N] = N_{rs} \widehat{\Omega}^r \otimes \widehat{\Omega}^s, \qquad \eta_r[N] = N_{rs} \Omega^r \otimes \Omega^s. \tag{213}$$

Particularly interesting is the special case when the target metric on G is simultaneously left- and right-invariant. This happens of course when G is Abelian (thus $\Omega = \widehat{\Omega}$) and in the quite opposite case, when G is semisimple and the Killing metric is used, i.e.,

$$N_{rs} = C^{z}{}_{tr}C^{t}{}_{zs} = N_{sr} \tag{214}$$

where C are the structure constants

$$[\varepsilon_r, \varepsilon_s] = \varepsilon_t C^{t}{}_{rs}. \tag{215}$$

Obviously, then

$$\eta_l[N] = \eta_r[N] = \eta[N]. \tag{216}$$

If G is the total $\mathrm{GL}(U)$, then, choosing mutually dual bases $(\ldots, e_A, \ldots)$ and $\left(\ldots, e^A, \ldots\right)$ in U and U^* and the corresponding basis $(\ldots, \varepsilon_r, \ldots) = \left(\ldots, e^A \otimes e_B\right)$, we can write analytically

$$\eta_l[N] = N_A{}^B{}_C{}^D \widehat{\Omega}^A{}_B \otimes \widehat{\Omega}^C{}_D, \quad \eta_r[N] = N_A{}^B{}_C{}^D \Omega^A{}_B \otimes \Omega^C{}_D \tag{217}$$

where

$$\Omega = \Omega^A{}_B e^B \otimes e_A, \qquad \widehat{\Omega} = \widehat{\Omega}^A{}_B e^B \otimes e_A.$$

For the doubly-invariant model we have

$$\eta = A\widehat{\Omega}^K{}_L \otimes \widehat{\Omega}^L{}_K + B\widehat{\Omega}^K{}_K \widehat{\Omega}^L{}_L = A\Omega^K{}_L \otimes \Omega^L{}_K + B\Omega^K{}_K \Omega^L{}_L. \tag{218}$$

These expressions were used in our papers concerning systems with affine degrees of freedom [18–21, 25–27]. It was also mentioned there about two interesting metrics η_l, η_r, namely differing from (218) by additional terms given respectively by

$$I\delta_{KM}\delta^{LN}\widehat{\Omega}^K{}_L \widehat{\Omega}^M{}_N, \qquad I\delta_{KM}\delta^{LN}\Omega^K{}_L \Omega^M{}_N. \tag{219}$$

Obviously, in (218), (219) A, B, I are constants. It is clear that left and right regular translations in $W = \mathrm{GL}(U)$, $L_{\mathrm{GL}(U)}, R_{\mathrm{GL}(U)}$ are isometries in $(\mathrm{GL}(U), \eta)$ with η given by (218). On the other hand, corrections (219) and the corresponding total expressions for η_l, η_r are invariant respectively under the groups $L_{\mathrm{GL}(U)} R_{\mathrm{O}(U,\eta)}$ and $L_{\mathrm{O}(U,\eta)} R_{\mathrm{GL}(U)}$, no longer under the total $L_{\mathrm{GL}(U)} R_{\mathrm{GL}(U)}$ (unless $I = 0$, of course). Here $\mathrm{O}(U, \eta)$ denotes the subgroup of $\mathrm{GL}(U)$ preserving η, the η-orthogonal group. Let us write down explicitly the expression for the induced metric tensors $g[\phi, \eta] = \phi^* \eta$. It is convenient to introduce the auxiliary $\mathfrak{g}$-valued differential one-forms on M, just the pull-backs of Ω, $\widehat{\Omega}$

$$\Omega[\phi] = \phi^* \Omega = \Omega[\phi]^r \varepsilon_r, \qquad \widehat{\Omega}[\phi] = \phi^* \widehat{\Omega} = \widehat{\Omega}[\phi]^r \varepsilon_r. \tag{220}$$

Using local coordinate expressions

$$\begin{aligned}\Omega[\phi] &= \Omega[\phi]_\mu \, \mathrm{d}x^\mu = \Omega[\phi]^r{}_\mu \varepsilon_r \otimes \mathrm{d}x^\mu \\ \widehat{\Omega}[\phi] &= \widehat{\Omega}[\phi]_\mu \, \mathrm{d}x^\mu = \widehat{\Omega}[\phi]^r{}_\mu \varepsilon_r \otimes \mathrm{d}x^\mu\end{aligned} \tag{221}$$

$$\Omega[\phi]_\mu = \partial_\mu \phi \phi^{-1}, \qquad \widehat{\Omega}[\phi]_\mu = \phi^{-1} \partial_\mu \phi. \tag{222}$$

Then (213) lead to the formulas

$$g_l[\phi, N]_{\mu\nu} = N_{zs} \widehat{\Omega}^z{}_\mu \widehat{\Omega}^s{}_\nu, \quad g_r[\phi, N]_{\mu\nu} = N_{zs} \Omega^z{}_\mu \Omega^s{}_\nu. \tag{223}$$

In particular, when N is the Killing metric on $\mathfrak{g}$

$$g[\phi, N]_{\mu\nu} = N_{zs} \widehat{\Omega}^z{}_\mu \widehat{\Omega}^s{}_\nu = N_{zs} \Omega^z{}_\mu \Omega^s{}_\nu. \tag{224}$$

If $G = \mathrm{GL}(U)$, then (218) becomes

$$\begin{aligned} g[\phi]_{\mu\nu} &= A \operatorname{Tr} \left(\widehat{\Omega}_\mu \widehat{\Omega}_\nu \right) + B \operatorname{Tr} \widehat{\Omega}_\mu \operatorname{Tr} \widehat{\Omega}_\nu \\ &= A \operatorname{Tr} \left(\Omega_\mu \Omega_\nu \right) + B \operatorname{Tr} \Omega_\mu \operatorname{Tr} \Omega_\nu. \end{aligned} \tag{225}$$

Analytically, the auxiliary terms based on (219) have the forms

$$I \operatorname{Tr} \left(\widehat{\Omega}^T_\mu \widehat{\Omega}_\nu \right), \qquad I \operatorname{Tr} \left(\Omega_\mu \Omega_\nu \right). \tag{226}$$

All these expressions were used in our papers on mechanics of affine bodies [18–21, 25–27]. The corresponding scalar multiplet models resemble those described by (104) which are based on the bundle $\mathrm{L}(M) = T^1_1 M$ and the Born–Infeld models based on $V \otimes T^* M$. In all cited models one deals with fields which have internal degrees of freedom ruled by the linear group, i.e., with continua of infinitesimal homogeneously deformable gyroscopes. Obviously, if G is non-Abelian, the above metrics $g[\phi, N]$ have non-vanishing curvature tensors.

Remark: When the target space W has the structure of $G \subset \mathrm{GL}(U)$, then we are in the position to construct scalar functions on W in a completely invariant way. This is particularly suggestive, e.g., when $G = \mathrm{GL}(U)$ or $G = \mathrm{SL}(U)$. The corresponding basic scalars invariant under (208) have the form

$$I_p(X) = \operatorname{Tr} \left(X^p \right). \tag{227}$$

Functions on G built analytically of these I_p-invariants may be used as potentials $K(\phi)$, $F(\phi)$, $U(\phi)$ in (135) with values of ϕ substituted as X and with $g[\phi]$ defined as above, e.g., (225).

6) Manifolds of scalar products as target spaces. Analytically, these are also manifolds of matrices, however, their geometrical meaning is completely different. The starting point is again some auxiliary linear space U. In the previous class of examples W was defined as some Lie subgroup G of $\mathrm{GL}(U)$, i.e., a manifold of mixed second order tensors in U (compare

the situation with the fibre bundle $LM = T^1_1 M$ consisting of mixed tensors in M, i.e., elements of $T_pM \otimes T^*_pM$). One can wonder what would be a natural geometry in manifold of non-mixed second order tensors, i.e., in submanifold of $W^* \otimes W^*$ and $W \otimes W$ (compare with the fibre bundles $T^0_2 M$ and $T^2_0 M$ consisting respectively of elements of $T^*_pM \otimes T^*_pM$ and $T_pM \otimes T_pM$ and the General Relativity with its symmetric second order nondegenerate tensors belongs here). The simplest situation is when U is a real vector space of dimension m and W is either the manifold of non-degenerate twice covariant symmetric tensors or the manifold of non-degenerate twice contravariant symmetric tensors. In this way W is an open submanifold of $\mathrm{Sym}\,(U^* \otimes U^*)$ or $\mathrm{Sym}\,(U \otimes U)$. Their complements to the total linear spaces $\mathrm{Sym}\,(U^* \otimes U^*)$, $\mathrm{Sym}\,(U \otimes U)$ consist of degenerate tensors and are closed subsets. Strictly speaking, the subsets of non-degenerate tensors are not connected as their connected submanifolds differ in signature. Any choice of the dual bases $(\dots, e_A, \dots)$ and $(\dots, e^A, \dots)$ in U and U^* gives rise to the bases $(\dots, e_A \otimes e_B, \dots)$, $(\dots, e^A \otimes e^B, \dots)$ respectively in $U^* \otimes U^*$ and $U \otimes U$. In subspaces $\mathrm{Sym}\,(U^* \otimes U^*)$, $\mathrm{Sym}\,(U \otimes U)$ we have then the bases composed respectively of $e^A \otimes e^B$, $A \leq B$, $e_A \otimes e_B$, $A \leq B$, or more convenient redundant "bases"

$$\begin{aligned} e^{(A} \otimes e^{B)} &= \frac{1}{2}\left(e^A \otimes e^B + e^B \otimes e^A\right) \\ e_{(A} \otimes e_{B)} &= \frac{1}{2}\left(e_A \otimes e_B + e_B \otimes e_A\right). \end{aligned} \tag{228}$$

The metrics $g \in \mathrm{Sym}\,(U^* \otimes U^*)$, $g \in \mathrm{Sym}\,(U \otimes U)$ are analytically expressed as

$$U^* \otimes U^* \ni g = g_{AB} e^A \otimes e^B = g_{AB} e^{(A} \otimes e^{B)} \tag{229}$$

$$U \otimes U \ni g = g^{AB} e_A \otimes e_B = g^{AB} e_{(A} \otimes e_{B)} \tag{230}$$

where

$$g_{AB} = g_{BA}, \qquad g^{AB} = g^{BA}. \tag{231}$$

Obviously, independent coordinates are given by g_{AB}, g^{AB}, $A \leq B$ (or with the reversed sign). It is clear that there exists a canonical diffeomorphism of non-singular $\mathrm{Sym}\,(U^* \otimes U^*)$ onto non-singular $\mathrm{Sym}\,(U \otimes U)$, it is analytically given by the reciprocal matrix formula

$$[g_{AB}] \mapsto \left[g^{AB}\right], \qquad g^{AC} g_{CB} = \delta^A{}_B. \tag{232}$$

Metric tensors on $\mathrm{Sym}\,(U^* \otimes U^*)$ and $\mathrm{Sym}\,(U \otimes U)$ are analytically represented as

$$\eta = \eta^{ABCD}(g)\, \mathrm{d}g_{AB} \otimes \mathrm{d}g_{CD} \tag{233}$$

$$\eta = \eta_{ABCD}(g)\, \mathrm{d}g^{AB} \otimes \mathrm{d}g^{CD} \tag{234}$$

where

$$\eta^{ABCD} = \eta^{CDAB} = \eta^{BACD} = \eta^{ABDC} \tag{235}$$

$$\eta_{ABCD} = \eta_{CDAB} = \eta_{BACD} = \eta_{ABDC} \tag{236}$$

(symmetry within the first and second bi-index and symmetry with respect to the mutual exchange of bi-indices). The structure of submanifolds of non-singular tensors distinguishes the two parameter class of metrics on any connected component of W

$$\eta^{ABCD} = \frac{\lambda}{2} g^{AC} g^{BD} + \frac{\lambda}{2} g^{BC} g^{AD} + \mu g^{AB} g^{CD} \tag{237}$$

$$\eta_{ABCD} = \frac{\lambda}{2} g_{AC} g_{BD} + \frac{\lambda}{2} g_{BC} g_{AD} + \mu g^{AB} g^{CD} \tag{238}$$

where λ, μ are constants. Some formal similarity of (233), (237) and (234), (238) to (218) is obvious, nevertheless also misleading. The apparently "Killing" structure in (237), (238) is something else than that in group manifolds. One should stress that above geometry in the manifolds of symmetric real scalar products is interesting in itself and to the best of our knowledge not yet understood. This is still an open problem in geometry. Let us stress that the metrics (237), (241) are by their very definition defined only on the (non-connected) manifold of non-singular tensors. And up to the arbitrariness of λ, μ they are completely intrinsic. Obviously, the λ-term is the dominant one and must be non-vanishing, e.g., it can put equal to one by convention. On the other hand, the μ-term is an auxiliary correction, rather qualitatively non-essential. The corresponding pull-back metrics in M are obviously given by the following expressions

$$g[\phi]_{\mu\nu} = \eta^{ABCD}(\phi) \partial_\mu \phi_{AB} \partial_\nu \phi_{CD} \tag{239}$$

$$g[\phi]_{\mu\nu} = \eta_{ABCD}(\phi) \partial_\mu \phi^{AB} \partial_\nu \phi^{BD} \tag{240}$$

$$g[\phi]_{\alpha\beta} = \lambda \phi^{AC} \phi^{BD} \partial_\alpha \phi_{AB} \partial_\beta \phi_{CD} + \mu \phi^{AB} \phi^{CD} \partial_\alpha \phi_{AB} \partial_\beta \phi_{CD} \tag{241}$$

$$g[\phi]_{\alpha\beta} = \lambda \phi_{AC} \phi_{BD} \partial_\alpha \phi^{AB} \partial_\beta \phi^{CD} + \mu \phi_{AB} \phi_{CD} \partial_\alpha \phi^{AB} \partial_\beta \phi^{CD} \tag{242}$$

respectively for (233), (234), (237), (238). Obviously, the group $\mathrm{GL}(U)$ acts on the target variables g according to the rule

$$L \in \mathrm{GL}(U) : g \mapsto L^* g \tag{243}$$

where

$$(L^* g)_{AB} = g_{CD} L^{-1C}{}_A L^{-1D}{}_B, \qquad (L^* g)^{AB} = L^A{}_C L^B{}_D g^{CD} \tag{244}$$

are the respective actions on the targets $\mathrm{Sym}\,(U^* \otimes U^*)$, $\mathrm{Sym}\,(U \otimes U)$ and as a matter of fact for the total tensors spaces $U^* \otimes U^*$, $U \otimes U$. It is clear that they preserve the signature, therefore, every connected component of the target W is separately invariant under (244). And it is obvious that the transformations (244) are isometries of the internal metrics (237), (238). And conversely, (237), (238) are the only metrics on the manifolds of U-metrics W invariant under (243), (244). Because of this they are symmetries of the generally covariant Lagrangians

$$L = \sqrt{|\det\left[g[\phi]_{\mu\nu}\right]|}. \tag{245}$$

An interesting difference between targets $G \subset \mathrm{GL}(U)$ and $\mathrm{Sym}\,(U^* \otimes U^*)$, $\mathrm{Sym}\,(U \otimes U)$ (or rather their open subsets consisting of non-degenerate symmetric tensors) is that without any fixed absolute object in U, there is no possibility to construct something like scalars built of the target variables g. Therefore, it is also impossible to introduce to (245) "potentials" like $K(\phi)$, $F(\phi)$, $U(\phi)$ in (135). In this respect the $U \otimes U$- and $U^* \otimes U^*$-targets are different than $\mathrm{GL}(U)$- or $T_pM \otimes T_p^*M$-type targets, although all of these targets consist of some second order tensors in something. And also, in this sense the targets $\mathrm{Sym}\,(U \otimes U)$, $\mathrm{Sym}\,(U^* \otimes U^*)$ are similar to the targets $\mathrm{Sym}\left(T_p^*M \otimes T_p^*M\right)$, $\mathrm{Sym}\,(T_pM \otimes T_pM)$ used in the General Relativity (more precisely, again the open submanifolds of non-degenerate symmetric tensors). The above scheme of contravariant or covariant second order tensors in some U may be easily modified and some more complicated but nevertheless (at least geometrically) interesting models may be constructed. There are a few natural lines of such modifications. First of all, instead spaces of symmetric scalar products $\mathrm{Sym}(U^* \otimes U^*)$, $\mathrm{Sym}(U \otimes U)$ (or rather manifolds of the corresponding non-degenerate tensors) we can admit the total manifolds of non-degenerate elements of $U^* \otimes U^*$, $U \otimes U$ as target spaces W. The general scheme is like in (233), (234), (237), (238), however for such general targets there are fewer symmetry demands for η and more admissible terms in (237), (238). So, if η is to be a pseudo-Riemannian metric on $U^* \otimes U^*$, $U \otimes U$, then only the symmetry with respect to transpositions of bi-indices survives

$$\eta^{ABCD} = \eta^{CDAB}, \qquad \eta_{ABCD} = \eta_{CDAB}. \tag{246}$$

And instead, e.g., (237) we have

$$\begin{aligned}\eta^{ABCD} &= \alpha g^{AB} g^{CD} + \frac{\delta}{2}(g^{AC} g^{BD} + g^{CA} g^{DB}) \\ &+ \beta g^{AD} g^{CB} + \frac{\varepsilon}{2}(g^{AD} g^{BC} + g^{DA} g^{CB}) \\ &+ \gamma g^{DA} g^{BC} + \frac{\varphi}{2}(g^{CA} g^{BD} + g^{AC} g^{DB})\end{aligned} \tag{247}$$

and similarly instead (238) with the lower-case indices instead the upper-case ones. Obviously, α, β, γ, δ, ε, φ are constants. Substituting the fields ϕ^{AB} instead g^{AB} one obtain as usual the Lagrange tensor

$$\mathcal{L}[\phi]_{\mu\nu} = \eta^{ABCD}(\phi)\partial_\mu \phi_{AB} \partial_\nu \phi_{CD}.$$

Another modifications of this target geometry is to take the anti-symmetric tensor manifolds Asym $(U^* \otimes U^*) = U^* \wedge U^*$ and Asym $(U \otimes U) = U \wedge U$, or anti-symmetric η, etc. There are also intrinsic metrics of some geometric and perhaps physical interest on the target spaces like $W = U \times (U^* \otimes U^*)$. For example, if we use some fixed bases in U and the corresponding coordinates u^A, g_{AB} on $U \times$ Sym $(U^* \otimes U^*)$ we can use the following class of intrinsic metrics η on W

$$\begin{aligned}\eta_{(u,g)} &= \nu g_{AB}\, \mathrm{d}u^A \otimes \mathrm{d}u^B + \lambda g^{AK} g^{BL}\, \mathrm{d}g_{AB} \otimes \mathrm{d}g_{KL} \\ &+ \mu g^{AB} g^{KL}\, \mathrm{d}g_{AB} \otimes \mathrm{d}g_{KL}.\end{aligned} \tag{248}$$

One can develop further such models as it was previously suggested by replacing again Sym $(U^* \otimes U^*)$, Sym$(U \otimes U)$ by the general $U^* \otimes U^*$, $U \otimes U$, or to replace the manifolds of metrics by the manifolds of symplectic structures $U^* \wedge U^*$, $U \wedge U$. Another class of interesting models may have to do with nonlinearity ideas in quantum mechanics. Namely, one can use some complex linear space U as an auxiliary tool. Then it is natural to construct the target space consisting, e.g., of twice covariant hermitian tensors on U, thus $W :=$ Herm $\left(\overline{U}^* \otimes U^*\right)$, or rather the manifold of non-degenerate scalar products on U. Let us represent analytically the scalar products $h \in W$ by their hermitian matrices $h_{\bar{A}B}$ ($h_{\bar{A}B} = \overline{h_{\bar{B}A}}$)

$$h(u, v) = h_{\bar{A}B} \bar{u}^{\bar{A}} v^B, \quad u = u^A e_A, \quad v = v^A e_A.$$

Without going into too much details we easily see that for the scalar multiplets of the type

$$\phi := M \to \mathrm{Herm}\left(\overline{U}^* \otimes U^*\right)$$

with non-degenerate values the analogue of (241) reads

$$g[\phi]_{\alpha\beta}=\lambda\phi^{A\bar{C}}\phi^{D\bar{B}}\partial_\alpha\phi_{\bar{B}A}\partial_\beta\phi_{\bar{C}D}+\mu\phi^{A\bar{B}}\phi^{D\bar{C}}\partial_\alpha\phi_{\bar{B}A}\partial_\beta\phi_{\bar{C}D}$$
$$+\lambda\overline{\phi}^{\bar{A}C}\phi^{D\bar{B}}\partial_\alpha\overline{\phi}_{B\bar{A}}\partial_\beta\phi_{\bar{C}D}+\mu\overline{\phi}^{\bar{A}B}\phi^{D\bar{C}}\partial_\alpha\overline{\phi}_{B\bar{A}}\partial_\beta\phi_{\bar{C}D} \tag{249}$$

and analogously for scalar mulitplets of the type

$$\phi := M \to \mathrm{Herm}\left(U\otimes\overline{U}\right).$$

In these multiplets simply the upper-case and lower-case indices are mutually interchanged. By the way, let us remind that contravariant upper case indices are meant in the sense

$$\phi^{A\bar{C}}\phi_{\bar{C}B}=\delta^A{}_B,\qquad \phi_{\bar{A}C}\phi^{C\bar{B}}=\delta_{\bar{A}}{}^{\bar{B}}.$$

In analogy to (248) one also consider target spaces of the form

$$W=U\times\mathrm{Herm}\left(\overline{U}^*\otimes U^*\right) \tag{250}$$

and the corresponding fields of the form

$$\phi=(\Psi,G),\quad \Psi:M\to W,\quad G:M\to\mathrm{Herm}\left(\overline{U}^*\otimes U^*\right). \tag{251}$$

Analytically we are dealing with the complex scalar multiplets consisting of $\mathbb{C}$-valued fields Ψ^A, $G_{\bar{A}B}$. The target W is endowed with the natural hermitian (but non-flat) metric analogous to (248). The corresponding metric-Lagrangian tensor induced on M by $\Psi : M \to W$ is given by

$$g[\phi]_{\alpha\beta}=\nu G_{\bar{A}B}\partial_\alpha\overline{\Psi}^{\bar{A}}\partial_\beta\Psi^B+\lambda G^{A\bar{C}}G^{D\bar{B}}\partial_\alpha G_{\bar{B}A}\partial_\beta G_{\bar{C}D}$$
$$+\mu G^{A\bar{B}}G^{D\bar{C}}\partial_\alpha G_{\bar{B}A}\partial_\beta G_{\bar{C}D}. \tag{252}$$

Let us mention that models of this kind may be somehow related to hypothetic nonlinearities in quantum mechanics. Namely, when $\dim M=1$, i.e., M is just the time axis, we can consider a pair of time-dependent quantities, $\left(\Psi^A(t),G_{\bar{A}B}(t)\right)$, the "state vector" Ψ of the N-level quantum system and the dynamical "scalar product" G. This scalar product is not fixed once for all, but, in analogy with the General Relativity, taken together with the wave function Ψ, it satisfies a closed system of equations, so there is a mutual interaction between Ψ and G. The evolution of the total system $\phi=(\Psi,G)$ is nonlinear and must be so if its equations are to be derivable from some reasonable variational principle. There is no place here for going into more details, but let us mentioned that there are Lagrangians for $(\Psi(t),G(t))$ which are structurally similar to (252). The same may be done for the "usual" quantum-mechanical system, when $\Psi(t)$ is a usual, perhaps multicomponent wave function on some configuration space Q, so that the index A becomes a pair (a,q) in which one has a discrete index and continuous classical configuration. So, we are dealing then with $\Psi^a(t,q)$ instead of $\Psi^A(t)$. The summation over A

becomes the summation over a and integration over q. The scalar product matrix is modified in a similar way. The corresponding effective nonlinearity with respect to the total pair (Ψ, G) might perhaps have to do with the well-known quantum difficulties like the reduction, decoherence, measurement and the dualism of "micro" (quantum) and "macro" (classical).

5. Final Remarks

We have discussed above essentially nonlinear models for fields which had only spatio-temporal indices, i.e., they were cross-sections of some tensor bundles over M, and essentially non-linear models for scalar multiplets in M, i.e., cross-sections of some trivial bundles $M \times W$ over M. We witnessed some link between general covariance and Born–Infeld type nonlinearity. In the case of mixed tensor bundles $\mathrm{L}M = T^1_1 M$ and trivial bundles the Born–Infeld structure was just implied by the general covariance. And some interesting similarities and differences between $\mathrm{L}M = T^1_1 M$-models and $M \times W \simeq M \times \mathrm{GL}(U) \simeq M \times \mathrm{GL}(n, \mathbb{R})$ models were observed $(\dim W = n^2 = (\dim M)^2)$. Both this models describe systems with affine degrees of freedom attached to space–time points. We must finish with tensor objects having both spatio-temporal and scalar-multiplet indices, first of all with

$$F^*M = \bigcup_{x \in M} F^*_x M \subset \bigcup_{x \in M} (T^*_x M)^n \simeq \bigcup_{x \in M} (T_x M)^n \supset FM$$

i.e., with (co-)frame ((co-)tetrad when $n = 4$) fields where the corresponding analytical symbols traditionally used for ϕ are $e^A{}_\mu$, $e^\mu{}_A$. This is just the physically most convincing model of "micromorphic continuum," i.e., system with internal affine degrees of freedom. Models of this kind were used in "infinitely many" tetrad models of gravitation [13, 15, 24]. Obviously, the fibres in bundles $\mathrm{L}M$, $M \times \mathrm{GL}(n, \mathbb{R})$, $F^*M \simeq FM$ are all n^2-dimensional and have kinematics ruled by the n-dimensional real linear group. And obviously, any cross-section of $F^*M \simeq FM$, i.e., any field of (co-)frames establishes uniquely some diffeomorphisms between LM and $M \times \mathrm{GL}(U) \simeq M \times \mathrm{GL}(n, \mathbb{R})$. Analytically the diffeomorphism given by the field of (co-)frames is given by

$$\phi^\mu{}_\nu = e^\mu{}_A \phi^A{}_B e^B{}_\nu, \qquad \phi^A{}_B = e^A{}_\mu \phi^\mu{}_\nu e^\nu{}_B. \tag{253}$$

It enables one to translate any (in particular any generally-covariant) Lagrangian on LM onto one on $M \times \mathrm{GL}(U)$ and conversely, however, just for the price of using an additional field e with its own Lagrangian. So, quite independently of any tetrad-gravitational motivation, Lagrangians for the field of linear (co-)frames are of particular interest, the more so the bundle F^*M (FM) is the principal bundle underlying the theory of all linear geometric objects in M (they are cross-sections of the corresponding associate bundles). In the principal bundles F^*M, (FM), the full linear group $\mathrm{GL}(n,\mathbb{R})$ acts in the standard way

$$F^*M \ni e = (\ldots, e_A, \ldots) \mapsto eL = \left(\ldots, e_B L^B{}_A, \ldots\right) \tag{254}$$

and dually

$$FM \ni \widetilde{e} = \left(\ldots, e^A, \ldots\right) \mapsto \widetilde{e}L = \left(\ldots, L^{-1A}{}_B e^B, \ldots\right). \tag{255}$$

Essential remark: This is just the action of the group $\mathrm{GL}(n,\mathbb{R})$, not $\mathrm{GL}(T_xM)$ identified with $\mathrm{GL}(n,\mathbb{R})$ by some choice of a fixed reference frame. Models with F^*M/FM-degrees of freedom were intensively studied in other papers [21, 24, 29]. Here we will concentrate on the link between the general covariance and the Born–Infeld structure. The first step towards constructing generally-covariant Lagrangians for the field e, i.e., for the cross-section of the principal fibre bundle F^*M, or equivalently FM is to define its invariant first order derivative. The general covariance, i.e., invariance of the Lagrangian $\mathcal{L}[e] = \mathcal{L}\,(e, \partial e)$ under the group $\mathrm{Diff}\,(M)$ consisting of elements labelled by n arbitrary functions of n variables

$$\mathcal{L}\,[\varphi^* e] = \varphi^* \mathcal{L}[e] \tag{256}$$

implies certain identities, the so-called generalized Bianchi identities. This is a consequence of the Noether theorem. At the stage when the Lagrangian is not yet explicitly defined, these identities may be looked upon as certain conditions, as a matter of fact differential equations imposed on the Lagrangian. As shown in some of our earlier papers (see [24] and references therein), these identities have the form

$$H_A{}^{\mu\nu} + H_A{}^{\nu\mu} = 0 \tag{257}$$

$$\frac{D}{Dx^\nu} t^\nu{}_\mu - \mathcal{L}^\nu{}_A \partial_\mu e^A{}_\nu = 0 \tag{258}$$

$$j^\nu{}_A - \left(t^\nu{}_\mu - H_B{}^{\varkappa\nu} \partial_\varkappa e^B{}_\mu\right) e^\mu{}_A = 0 \tag{259}$$

where the meaning of symbols is as follows

$$H_A{}^{\mu\nu} := \frac{\partial \mathcal{L}}{\partial e^A{}_{\mu,\nu}} \tag{260}$$

$$t^\nu{}_\mu := H_A{}^{\varkappa\nu} \partial_\mu e^A{}_\varkappa - \mathcal{L}\delta^\nu{}_\mu = H_\lambda{}^{\varkappa\nu} e^\lambda{}_A \partial_\mu e^A{}_\varkappa - \mathcal{L}\delta^\nu{}_\mu \tag{261}$$

$$j^\nu{}_A := -\frac{\partial \mathcal{L}}{\partial e^A{}_\nu}. \tag{262}$$

Using the familiar qualitative terms:

- H_A are field momenta, from the point of view of electromagnetic analogy, H_A form the multiplet of $(\bar{D}, \bar{H})$-fields, $\mathrm{d}e^A$ form the multiplet of $(\bar{E}, \bar{B})$-fields, whereas e^A themselves form the multiplect of covector potentials

- $t^{\nu}{}_{\mu}$ is the canonical "energy–momentum complex" (not the tensor, neither the tensor density, of course)
- $\mathcal{L}^{\nu}{}_{A}$ are components of the Euler–Lagrange variational derivative $\delta I/\delta e^{A}{}_{\nu}$ with I denoting the action

$$I = \int \mathcal{L}[\Psi]\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

- D/Dx^{ν} is the total derivative of quantities depending on x, $e(x)$, $\partial e(x)$, with respect to the variable x^{ν}
- $j^{\nu}{}_{A}$, the multiplet of contravariant vector densities of weight one, is a system of self-interaction currents.

The simplest identity (257) is just the statement of the fact that $\mathcal{L}$ depends on ∂e through the exterior differentials $\mathrm{d}e^A$, analytically

$$\left(\mathrm{d}e^A\right)_{\mu\nu} = \partial_{\mu} e^{A}{}_{\nu} - \partial_{\nu} e^{A}{}_{\mu}. \tag{263}$$

This was in any case a priori evident in a bare manifold. When dealing with the $LM = T^1_1 M$ bundle we used the tensorial object $S[x]^{\lambda}{}_{\mu\nu}$ (99), (102) containing the information about ∂X via some expression built algebraically of both ∂X and X. Incomparably simpler this may be done for the e-object, namely we take the tensor

$$S[e]^{\lambda}{}_{\mu\nu} := e^{\lambda}{}_{A} \left(\partial_{\nu} e^{A}{}_{\mu} - \partial_{\mu} e^{A}{}_{\nu}\right). \tag{264}$$

Just like in (102) it is linear in derivatives ∂X. The prescription $e \mapsto S[e]$ is generally covariant in M

$$\varphi \in \mathrm{Diff}(M) : S\left[\varphi^* e\right] = \varphi^* S[e]. \tag{265}$$

No external objects are involved. The prescription has also another suggestive property, namely is invariant under the target group $\mathrm{GL}(n, \mathbb{R})$

$$S[eA] = S[e], \qquad A \in \mathrm{GL}(n, \mathbb{R}). \tag{266}$$

By the way, as $\mathbb{R}^n$ carries the plenty of "parasite" structures which may be misleading, it is more reasonable to replace it by the auxiliary "target" space U, just linear space without any additional structure. Then the group $\mathrm{GL}(U)$ is the only natural candidate for internal symmetry. And the frames $e_x \in F^*_x M$, $\widetilde{e}_x \in F_x M$ are simply replaced by linear isomorphism $e_x \in \mathrm{LI}(T_x M, U)$, $\widetilde{e}_x = e_x^{-1} \in \mathrm{LI}(U, T_x M)$. Then the former expressions are obtained by simply introducing some bases in U, but are independent on the choice of basis and globally $\mathrm{GL}(U)$-invariant.

Remark: There is no local $\mathrm{GL}(U)$ invariance, i.e., the A in (266) cannot be x-dependent as it is always rigidly fixed all over M, not a field $A : M \to \mathrm{GL}(U)$ like in gauge treatments. Now the geometric interpretation of $S[e]$ is much simpler that of the Nijenhuis objects in $\mathrm{L}M$. Namely, $S[e]$ is the torsion of the teleparallelism connection $\Gamma[e]$ built of e. This connection is uniquely defined by the demand that e is parallel under the corresponding covariant differentiation

$$\nabla e_A = 0, \qquad \text{i.e.,} \qquad \nabla e^A = 0. \tag{267}$$

It is obvious also that

$$\Gamma[e]^\lambda{}_{\mu\nu} = e^\lambda{}_A \partial_\nu e^A{}_\mu. \tag{268}$$

The parallel $\Gamma[e]$-transport of any tensor is path-independent and just consists in taking at another point the object with the same non-holonomic e-coordinates like at the original point. The curvature of $\Gamma[e]$ evidently vanishes and the torsion (264) of (268) is algebraically e-equivalent to the non-holonomy object $\Omega^A{}_{BC}[e]$ of e. Let us mention that unifying with the use of e the multiplet of currents j_A into a mixed second order tensor density

$$j^\mu{}_\nu := j^\mu{}_A e^A{}_\nu \tag{269}$$

we can easily understand the structure of the non-tensorial character of the complex t

$$j^\mu{}_\nu = t^\mu{}_\nu + H_\lambda{}^{\mu\varkappa} \Gamma_{\text{tel}}{}^\lambda{}_{\nu\varkappa}. \tag{270}$$

Perhaps j is a better candidate for the "energy–momentum" than t itself (we remember that the concept of the energy–momentum in generally covariant models is delicate if not doubtful). The general covariance implies the following "Bianchi identities"

$$\frac{D}{Dx^\mu}\left(e^A{}_\nu \mathcal{L}^\mu{}_A\right) - \mathcal{L}^\mu{}_A \partial_\nu e^A{}_\mu = 0. \tag{271}$$

We have also strong conservation laws

$$\frac{D}{Dx^\mu}\left(t^\mu{}_\nu + e^A{}_\nu \mathcal{L}^\mu{}_A\right) = 0 \tag{272}$$

and "improper" weak conservation laws

$$\frac{D}{Dx^\mu} t^\mu{}_\nu = 0 \tag{273}$$

which, unlike the strong ones, assume that the field equations are satisfied. Obviously, (273) is equivalent to the "continuity equation" for the current j_A

$$\frac{D}{Dx^\mu} j^\mu{}_A = 0. \tag{274}$$

Obviously, it would be to try to find the general solution of (257)–(262) as a system of differential equations for the Lagrangian $\mathcal{L}$ as a function of $e^A{}_\mu$ and $\xi^A{}_{\mu\nu} = \partial_\nu e^A{}_\mu$. One should rather use some direct methods. The simplest models are known from the tetrad models of gravitation. Those models preassumed some pseudo-Euclidean target metric $\eta \in \mathrm{Sym}\,(U^* \otimes U^*)$ in U. In the usual gravitation theory in realistic four-dimensional space–time, η is normally hyperbolic with signature $(+---)$ (or $(-+++)$ depending on the individual taste). Obviously, this is the target geometry, η has directly nothing to do with the spatio-temporal metric of special relativity. However, any (co-)frame field e given rise to some metric tensor $g[e,\eta]$ on M

$$g[e,\eta]_x = e^*_x \cdot \eta, \qquad g[e,\eta]_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu. \tag{275}$$

It is automatically normal-hyperbolic, although in general curved, unless e is holonomic, i.e., $\mathrm{d}e^A = 0$, $A = 1, \ldots, n$, or equivalently $S[e] = 0$. As all schemes with different hyperbolic metrics η in U are essentially identical, the only absolute element here is just the signature itself. Let us observe that unlike the prescription $e \mapsto S[e]$, the prescription $e \mapsto g[e,\eta]$ is invariant under the orthogonal internal subgroup $\mathrm{O}(U,\eta) \subset \mathrm{GL}(U)$, not under the total $\mathrm{GL}(U)$

$$g\left[eA,\eta\right] = g[e,\eta] \qquad \text{iff} \qquad A \in \mathrm{O}(U,\eta), \quad A^* \cdot \eta = \eta \tag{276}$$

i.e., analytically

$$\eta_{CD} A^C{}_K A^D{}_L = \eta_{KL}. \tag{277}$$

But, as the relationship (275) is purely algebraic, the invariance (276) is valid locally, i.e., for x-dependent A

$$A : M \to \mathrm{O}(U,\eta).$$

There are three basic scalars which may be built of g, S in a way quadratic in S, thus, quadratic in the derivatives ∂e. These are the so-called **Weitzenböck invariants** [24]

$$J_1 = g_{\mu\alpha} g^{\nu\beta} g^{\varkappa\delta} S^\mu{}_{\nu\varkappa} S^\alpha{}_{\beta\delta}, \; J_2 = g_{\mu\nu} S^\alpha{}_{\beta\mu} S^\beta{}_{\alpha\nu}, \; J_3 = g_{\mu\nu} S^\alpha{}_{\alpha\mu} S^\beta{}_{\beta\nu}. \tag{278}$$

Therefore, the most general $\mathrm{Diff}(M)$-invariant Lagrangian quadratic in derivatives ∂g (but with g-dependent coefficients) has the form

$$L = c_1 L_1 + c_2 L_2 + c_3 L_3 = \left(c_1 J_1 + c_2 J_2 + c_3 J_3\right) \sqrt{|g|}. \tag{279}$$

The resulting field equations are quasi-linear, i.e., linear in the highest (second) derivatives with coefficients algebraically depending on g. They are $\mathrm{Diff}(M)$-invariant in the space–time manifold and $\mathrm{O}(U,\eta)$- (Lorentz-) invariant in the internal space U (let us stress again: one has not to confuse the target Minkowskian metric in U with non-existing one in M). There are some rather delicate points concerning the status of the Einstein General Relativity and the Hilbert Lagrangian. Namely, the latter one (32) may be written down as follows

$$\begin{aligned} R[g]\sqrt{|g|} &= \left(J_1 + 2J_2 - 4J_3\right)\sqrt{|g|} + 4\nabla_\mu \left(S^\alpha{}_{\alpha\beta} g^{\beta\mu} \sqrt{|g|}\right) \\ &= \left(J_1 + 2J_2 - 4J_3\right)\sqrt{|g|} + 4\partial_\mu \left(S^\alpha{}_{\alpha\beta} g^{\beta\mu} \sqrt{|g|}\right) \end{aligned} \tag{280}$$

where ∇ denotes the covariant differentiation in the sense of Levi-Civita connection assigned to $g[e,\eta]$. And, as well-known, the covariant divergence of any contravariant vector density of weight one with respect to any symmetric affine connection, is identical with the usual divergence in the partial-derivative sense. Therefore, when one uses the (co-)frames ((co-)tetrads) as field quantities, then the Hilbert Lagrangian (279) may be equivalently replaced with the coefficients which are related as follows

$$c_1 : c_2 : c_3 = 1 : 2 : -4 \tag{281}$$

(obviously, up to subtle points concerning fixing derivatives ∂e on the boundary). The resulting equations are just the usual Einstein equations with g expressed through e like in (275). But once e is used as a fundamental field quantity, then at least formally we can manipulate the constants c_1, c_2, c_3. It turns out that in some range of their ratios the resulting field equations are compatible with the experiment provided that the Einstein–Hilbert ratio (281) is fulfilled. In this way the frame ("tetrad") model of degrees of freedom offers more possibilities for the field dynamics than the metric g itself. And one can wonder whether some models non-quadratic in S (in derivatives) could not be useful. In fact such general models were studied, e.g., by Plebański, Möller, Pellegrini and others [13, 15]. The Lagrangian is then expressed as a density-valued function of the tensors S and g, e.g., in the following form

$$L[e] = l\left(S[e], g[e,\eta]\right)\sqrt{|g[e,\eta]|} \tag{282}$$

l being some scalar function built of tensors $S[e]$, $g[e,\eta]$. For example, l may be some function of the Weitzenböck invariants

$$l(S,e) = f(J_1, J_2, J_3). \tag{283}$$

Obviously, when f is linear, we go back to the original teleparallelism models (281) with quadratic in the derivatives ∂e Lagrangian and quasi-linear second order differential equations. The idea of f nonlinear in J-s, i.e., f non-quadratic in S (in the derivatives ∂e) was motivated by attempts of avoiding singularities appearing in General Relativity and its simple generalizations. This motivation, at least in its present form, is rather old-fashionable because it is just typical that some interesting physics may be deduced from black holes theory. What is important now is that nothing qualitatively new may be obtained with the simple manipulations with the shape of f in (283), in particular, with ones replacing linearity in J by higher-order polynomials or rational functions. And there is yet no link between general covariance and the promising (but at the same time mysterious) Born–Infeld-type nonlinearity. This might seem a little bit disappointing, because this link was so "canonical" and convincing in theory of scalar multiplets and in the bundle $\mathrm{L}M = T^1_1 M$. But just now the symmetry idea enables one to get even more. The proper hint is just our above mentioned dissatisfaction with absolute target metrics.

If we once admit the non-Einsteinian ratio $c_1 : c_2 : c_3$ in (279), then the infinite-dimensional group of internal symmetries (276) with x-dependent Lorentz transformations A becomes drastically reduced to the $n(n-1)/2$-dimensional (physically six-dimensional) group of global, x-independent Lorentz transformations in the target space U. But whereas the local Lorentz symmetry is physically interpretable (it is g that is fundamental field and e is an auxiliary tool, reference frame), the global one is rather obscure and non-motivated. If global (x-independent) internal symmetries are once admitted, it is a tempting idea to try removing anything like the metric η from U. And if U is to be amorphous, it is natural to seek models with Lagrangians as amorphous as possible, i.e., not only $\mathrm{Diff}(M)$-invariant in M, but also $\mathrm{GL}(U)$-invariant (analytically $\mathrm{GL}(n,\mathbb{R})$-invariant) in internal degrees of freedom. This is impossible with scalar multiplets, but turns out to be possible with F^*M (FM) degrees of freedom. Namely, the Lagrangian must be built algebraically of S. And the simplest possibility is just one suggested by (104), i.e., the Lagrangian tensor

$$\mathcal{L}[e]_{\mu\nu} = AS^{\lambda}{}_{\mu\varkappa}S^{\varkappa}{}_{\nu\lambda} + BS^{\lambda}{}_{\mu\lambda}S^{\varkappa}{}_{\nu\varkappa} + CS^{\lambda}{}_{\varkappa\lambda}S^{\varkappa}{}_{\mu\nu} \tag{284}$$

where A, B, C are constants. This Lagrange tensor is very nicely quadratic in derivatives (quadratic in S). The Lagrangian itself is homogeneous of degree n in S (in derivatives). In F^*M (FM) models the total symmetry group $\mathrm{Diff}M \times \mathrm{GL}(U)$ (general covariance in space–time M and affine invariance in the target U) just implies the Born–Infeld nonlinearity. This is similar to implying the Born–Infeld structure by the general covariance of $\mathrm{L}M = T^1_1 M$-models. Because there is no counterpart of the global target U independent of $x \in M$, in this case there is even no possibility of discussing in $\mathrm{L}M$ something like the target symmetry. The nice Killing structure of

$$G_{\mu\nu} = S^{\lambda}{}_{\mu\varkappa}S^{\varkappa}{}_{\nu\lambda} = G_{\nu\mu} \tag{285}$$

suggests it to be a candidate for the metric tensor of M. It is important that $G[e]$ unlike $g[e,\eta]$ is built of e in a non-algebraic, namely, the first order differential way. This enables one to avoid the use of the target metric η. Moreover, a priori even the signature of $G[e]$ is not fixed. And in general the frame η need not be η-orthonormal

$$G[e]_{\mu\nu}e^{\mu}{}_{A}e^{\nu}{}_{B} \neq \eta_{AB} = g[e,\eta]_{\mu\nu}e^{\mu}{}_{A}e^{\nu}{}_{B}. \tag{286}$$

Another, a little more general, candidate for the metric tensor is the symmetric part of (284)

$$\mathcal{L}[e]_{\mu\nu} = AS^{\lambda}{}_{\mu\varkappa}S^{\varkappa}{}_{\nu\lambda} + BS^{\lambda}{}_{\mu\lambda}S^{\varkappa}{}_{\nu\varkappa}. \tag{287}$$

But obviously the Killing term is a dominant one and the B-term is merely some correction. Just like in the $\mathrm{L}M$ ($T^1_1 M$) model one can introduce some kind of "potentials" into the Lagrangian. There is a difference, however, namely in $\mathrm{L}M$-models they might be "true" potentials built algebraically of X alone (although the dependence on derivatives ∂X was also possible). Now, in $F^{*}M$ (FM) models there is no possibility to built scalars in an intrinsic way without the use of derivatives. There is no place here for a more detailed discussion but let us mention only two possible prescription for scalars. Namely, if the tensor $G[e]$ is non-degenerate, we can construct something similar to Weitzenböck invariants

$$G_{\alpha\beta}G^{\mu\varkappa}G_{\nu\lambda}S^{\alpha}{}_{\mu\nu}S^{\beta}{}_{\varkappa\lambda} \tag{288}$$

$$G^{\mu\alpha}S^{\alpha}{}_{\mu\alpha}S^{\beta}{}_{\nu\beta} \tag{289}$$

where $G^{\mu\alpha}G_{\alpha\nu} = \delta^{\mu}{}_{\nu}$. Let us notice however that there is nothing like the "second Weitzenböck invariant" because by definition it is a constant

$$G^{\mu\nu}S^{\alpha}{}_{\mu\beta}S^{\beta}{}_{\nu\alpha} = n = \dim M. \tag{290}$$

By a similar procedure one can construct more complicated scalars. All of them are homogeneous functions of degree zero in S (in derivatives). One can introduce some derivative-dependant "potentials," e.g., by putting A, B, C to be some functions of the basic scalars. Obviously, the resulting $\mathcal{L}[e]_{\mu\nu}$, $L[e]$ loose then their "Born–Infeld beauty," become terribly complicated and non-useful, probably also non-physical. It does not matter what scalars are used as the resulting Lagrangians $L[e] = L(e, \partial e)$ are always homogeneous of degree n in S, i.e., in velocities. This is some kind of "multivector Finsler geometry." The homogeneity of degree n is a direct consequence of the $\mathrm{Diff}(M) \times \mathrm{GL}(n, \mathbb{R})$-invariant. For any Lagrangian L satisfying

$$L\left[\varphi^{*}eA\right] = \varphi^{*}L[eA], \qquad \varphi \in \mathrm{Diff}(M), \qquad A \in \mathrm{GL}(U) \tag{291}$$

the identities following from Noether theorems just imply

$$S^{\lambda}{}_{\mu\nu}\frac{\partial L}{\partial S^{\lambda}{}_{\mu\nu}} = nL. \tag{292}$$

Because of its very nature as a "double quantity" with indices in $T_{x}M$ and $U \simeq \mathbb{R}$, $e^{A}{}_{\mu}$, the (co-)frame e intermediates between the two affine models of targets, $\mathrm{GL}(T_{x}M)$ and $\mathrm{GL}(U)$, i.e., between the bundles

$$LM = T^{1}_{1}M, \qquad M \times \mathrm{GL}(W)$$

in the sense of obvious formulas

$$X^{\mu}{}_{\nu}(x) = e^{\mu}{}_{A}(x)\phi^{A}{}_{B}(x)e^{B}{}_{\nu}(x) \tag{293}$$

and conversely

$$\phi^{A}{}_{B}(x) = e^{A}{}_{\mu}(x)X^{\mu}{}_{\nu}(x)e^{\nu}{}_{B}(x). \tag{294}$$

It would be interesting to investigate in some details the kinship between these three n^{2}-component fields, relationships between their Born–Infeld nonlinearities and their hypothetic physical applications.

6. Suggesting Born–Infeld Version of Willmore, Polyakov–Kleinert and Helfrich Functionals

We have discussed above the Born–Infeld-type Lagrangians (134) for multiplets of scalar fields. Geometrically their extremals (stationary points) are minimal n-dimensional surfaces in N-dimensional (pseudo-)Riemannian manifolds (W, η). Physically they have to do with alternative gravitation theories, σ-models, strings, membranes, shells, etc. There are also interesting models with some "potential" terms introduced to $\mathcal{L}[\phi]_{\mu\nu}$. Being minimal surfaces in (W, η) they have vanishing mean curvature. What about models which would possess (among other ones) solutions of constant, not necessarily vanishing curvature? Such models do exist in fact and have their origin in the so-called Willmore functional [7, 33]. This functional and certain modifications developed later on are useful in some mechanical-engineering problems and in biophysics, e.g., in the theory of cell membranes. Obviously, in such applications (W, η) is the usual three-dimensional Euclidean space, and the parameter manifold M is either an open subset of $\mathbb{R}^2$ or the two-dimensional unit sphere $S^2(0,1)$. Incidentally, depending on the assumed topology of considered surfaces in W, one can use also some other models of the "material space" M, e.g., the n-dimensional torus $T^n = \left(S^1(0,1)\right)^n$, sphere with "handles," etc. Without any changes everything remains literally valid for the general (W, η), $N = \dim W$, and the hypersurface situation $n = N - 1$. Natural generalizations for other values of n, when one deals with the mean curvature vector, also may be easily formulated, however, they need some additional comments. Let us concentrate on the simplest hypersurface case. The original Willmore functional is given by [7, 33]

$$\mathcal{W} = \int_M H^2 \, \mathrm{d}A = \int_M H^2 \sqrt{|\det\left[g[\phi]\right]|} \, \mathrm{d}x^1 \ldots \mathrm{d}x^n \tag{295}$$

where $H[\phi] = \mathcal{H}\left(\phi, \partial\phi, \partial^2\phi\right)$ is the field of the mean curvature of $\phi(M) \subset N$ (we identify objects on $\phi(M)$ with their pull-backs to M). H depends linearly on second derivatives of ϕ, therefore the Lagrangian itself

$$\mathcal{L}[\phi] = H^2 \sqrt{|g[\phi]|} \tag{296}$$

is a nonlinear (quadratic) function of $\partial^2\phi$ and the resulting Euler–Lagrange equations

$$\Delta H + 2\left(H^2 - K\right) H = 0 \tag{297}$$

are fourth-order partial differential equations, just like in the theory of elastic beams and shells (by the way, similarity is non-accidental). In (297) $\triangle$ denotes the Laplace–Beltrami operator on $\phi(M)$ (on M in the sense of the metric $g[\phi] = \phi^* g$), and K is the Gauss curvature of $\phi(M)$. Combining additively (134) and (296) one obtains the Polyakov–Kleinert string action based on the Lagrangian

$$\mathcal{L} = \left(aH^2 + b\right)\sqrt{|g[\phi]|} \tag{298}$$

where a, b are constants. Obviously, for $b = 0$ one obtains (296) and choosing $a = 0$ (134) results. The Helfrich functional of the bending energy of vesicle membrane is based on the Lagrangian [7,33]

$$\mathcal{L} = \left(aH^2 + bK\right)\sqrt{|g[\phi]|} \tag{299}$$

where again a, b are constants and K is the Gaussian curvature. All the above models are very strongly nonlinear. And again the natural temptation appears to construct their Born–Infeld counterparts without the artificial scalar-density factorization of L. A priori the most natural candidates for the Lagrange tensor are ones of the form

$$\mathcal{L}[\phi]_{\mu\nu} = \mathcal{L}[\phi]\left(\phi, \partial\phi, \partial^2\phi\right)_{\mu\nu} = ag_{\mu\nu} + bH_{\mu\nu} + cg^{\alpha\beta}H_{\mu\alpha}H_{\beta\nu} \tag{300}$$

where a, b, c are constants or, more generally, some simple functions of the mean curvature and Ricci curvature. It is seen that for week fields one can obtain (298), (299) as asymptotically equivalent to Lagrangians based on Lagrange tensors (300) with appropriately chosen scalars a, b, c. It is not clear yet if the models based on (300), especially the simplest ones with $c = 0$, may offer something physically new, computationally simple and geometrically interesting. This is one of the open questions, formulation of which was one of the purposes of this treatise.

Acknowledgements

This essay is based on the series of lectures presented by the author during the Eighth International Conference on Geometry, Integrability and Quantization, June 9–14, 2006, Varna, Bulgaria. The cordial thanks are due to Ivaïlo M. Mladenov for his hospitality and interest in the topic of these lectures. Both the visit and the scientific work underlying it were elements of our collaboration within the project #23/2006 “Group Structures Behind Two- and Three-Dimensional Elastic Structures.” This project is an element of the agreement between Bulgarian and Polish Academies of Sciences (Institute of Biophysics of BAS and Institute of Fundamental Technological Research of PAS) for the period 2006–2008 and we are grateful to the both institutions for supporting the visits and research program. Thanks are also due to other Bulgarian colleagues, first of all to Vassil Vassilev, Peter Djondjorov and Mariana Hadzilazova.

References

[1] Abraham R. and Marsden J., *Foundations of Mechanics* (2nd ed.), Benjamin–Cummings, London, 1978.

[2] Arnold V., *Mathematical Methods of Classical Mechanics*, Springer, New York, 1978.

[3] Chruściński D., *Point Charge in Born–Infeld Principle*, Phys. Lett. A **240** (1998) 8–14.

[4] Chruściński D., *Canonical Formalism for the Born–Infeld Particle*, J. Phys. A: Math. Gen. **31** (1998) 5775–5786.

[5] Chruściński D., *Strong Field Limit of the Born–Infeld p-form Electrodynamics*, Phys. Rev. D **62** (2000) 105007/1–7.

[6] Dirac P., *Interacting Gravitational and Spinor Fields*, In: Recent Developments in General Relativity, Pergamon, New York, PWN – Polish Scientific Publishers, Warsaw, 1962.

[7] Djondjorov P. and Vassilev V., *Symmetries and Conservation Laws of Plates and Shells Interacting with Fluid Flow*, In: Geometry, Integrability and Quantization VI, I. Mladenov and A. Hirshfeld (Eds), SOFTEX, Sofia 2005, pp 183–190.

[8] Fomenko A., *Variational Methods in Topology* (in Russian), Nauka, Moscow, 1982.

[9] Kijowski J., *On a New Variational Principle in General Relativity and the Energy of the Gravitational Field*, General Relativity and Gravitation **9** (1978) 857.

[10] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry I*, Interscience, New York, 1963.

[11] Liebscher D., *Relativistische Theorie und Metrische Raumstruktur*, In: Gravitation and Cosmos. Beiträge zur Problem der Allgemeinen Relativitäts Theorie, R. Wahsner (Ed), Akademie-Verlag, Berlin, 1982.

[12] Mischenko A. and Fomenko A., *Lectures on Differential Geometry and Topology* (in Russian), Moscow State University, Moscow, 1980.

[13] Möller C., *Energy–Momentum Complex in the General Relativity Theory*, Danske Videnskab. Selsk Mat.-Fys. Meddr. **31** (1959) 14.

[14] Ogawa N., *On the Chaplygin Gas and Brane*, In: Geometry, Integrability and Quantization VIII, I. Mladenov and M. de León (Eds), SOFTEX, Sofia, 2007, 100–110.

[15] Pellegrinni C. and Plebański J., *Tetrad Fields and Gravitational Fields*, Mat.-Fys. Skr. Danske Videnskab. Selsk **2** (1963) 1–39.

[16] Rzewuski J., *Field Theory, Classical Theory*, PWN – Polish Scientific Publishers, Warsaw, 1964.

[17] Schrödinger E., *Space–Time Structure*, Cambridge University, Cambridge, 1950.

[18] Sławianowska A. and Sławianowski J., *Quantization of Affinely Rigid Body in N Dimensions*, Rep. Math. Phys. **29** (1991) 297–320.

[19] Sławianowski J., *Affinely Rigid Body and Hamiltonian Systems on $GL(n,\mathbb{R})$*, Rep. Math. Phys. **26** (1988) 73–119.

[20] Sławianowski J., *Classical and Quantum Collective Dynamics of Deformable Objects. Symmetry and Integrability Problems*, In: Geometry, Integrability and Quantization V, I. Mladenov and A. C. Hirshfeld (Eds), SOFTEX, Sofia, 2004, pp 81–108.

[21] Sławianowski J., *Linear Frames in Manifolds, Riemannian Structures and Description of Internal Degrees of Freedom*, Rep. Math. Phys. **51** (2003) 345–369.

[22] Sławianowski J., *Classical and Quantized Affine Models of Structured Media*, Meccanica **40** (2005) 365–387.

[23] Sławianowski J., *Search for Fundamental Models with Affine Symmetry: Some Results, Some Hypotheses and Some Essay*, In: Geometry, Integrability and Quantization VI, I. Mladenov and A. Hirshfeld (Eds), SOFTEX, Sofia 2005, pp 126–172 (and references therein).

[24] Sławianowski J., $GL(n,\mathbb{R})$ *as a Candidate for Fundamental Symmetry in Field Theory*, Il Nuovo Cimento B **106** (1991) 645–668.

[25] Sławianowski J. and Kovalchuk V., *Invariant Geodetic Problems on the Affine Group and Related Hamiltonian Systems*, Rep. Math. Phys. **51** (2003) 371–379.

[26] Sławianowski J. and Kovalchuk V., *Classical and Quantized Affine Physics: A Step Towards It*, J. Nonlinear Math. Phys. **11** Supplement (2004) 157–166.

[27] Sławianowski J., Kovalchuk V., Sławianowska A., Gołubowska B., Martens A., Rożko E. and Zawistowski Z., *Invariant Geodetic Systems on Lie Groups and Affine Models of Internal and Collective Degrees of Freedom*, IFTR Reports **7**, Warsaw, 2004.

[28] Sławianowski J., Kovalchuk V., Sławianowska A., Gołubowska B., Martens A., Rożko E. and Zawistowski Z., *Affine Symmetry in Mechanics of Collective and Internal Modes. Part I. Classical Models*, Rep. Math. Phys. **54** (2004) 373–427.

[29] Sławianowski J., Kovalchuk V., Sławianowska A.., Gołubowska B., Martens A., Rożko E. and Zawistowski Z., *Affine Symmetry in Mechanics of Collective and Internal Modes. Part II. Quantum Models*, Rep. Math. Phys. **55** (2005) 1–45.

[30] Soper D., *Classical Field Theory*, Wiley, New York, 1976.

[31] Synge J. *Classical Dynamics*, Springer, Berlin, 1976.

[32] Terazawa H. and Akama K., *Dynamical Subquark Model of Pregauge and Pregeometric Interactions*, Phys. Lett. B **96** (1980) 276–278; *Subquark Pregeometry with Spontaneously Broken Conformal Invariance*, Phys. Lett. B **97** (1980) 81–83.

[33] Vassilev V. and Mladenov I., *Geometric Symmetry Groups, Conservation Laws and Group-Invariant Solutions of the Willmore Equation*, In: Geometry, Integrability and Quantization V, I. Mladenov and A. Hirshfeld (Eds), SOFTEX, Sofia, 2004, pp 81–108.