Eighth International Conference on
Geometry, Integrability and Quantization
June 9–14, 2006, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2007, pp 207–220

MORSE THEORY IN FIELD THEORY

PETER KOROTEEV

Institute for Theoretical and Experimental Physics
117259 Moscow, Russia

Abstract. We describe a geometrical interpretation of Topological Quantum Mechanics (TQM). Basics of the general topological theories are briefly discussed as well. The appropriate correspondence between objects of TQM and the algebraic topology is pointed out. It is proved that the correlators in TQM can be expressed via intersection numbers of some submanifolds of the target space with paths of steepest descent between critical points. Another correspondence is only conjectured, namely the correspondence between correlators and an integral of Massey products on cohomology classes of the target manifold.

Introduction

Topological Quantum Field Theories (TQFT) and Topological String Theories originating from the works of Witten *et al* [8–10] may be helpful in searches for the truly fundamental physical theory and in the treatment of important mathematical problems.

The main feature of topological theories is the independence of the correlation functions on metrics and coordinates [1]. In **Topological Field Theories** (TFT) there are no propagating (local) degrees of freedom, the vacuum expectation values of operators and transition amplitudes (both further referred to as "correlators") depend only on topology of the target manifold.

In this paper we employ for our purposes a simple example of Topological Field Theory – **Topological Quantum Mechanics** (TQM) with a BRST-like invariant action. It was already shown [2, 4] that in zero-dimensional analog of this theory partition function is equal to the Euler character of the target manifold.

We have two main aims: the first is to make manifest the correspondence between TFT and geometry of target manifold, and the second one is to study the correspondence between TFT and a differential graded algebra of cohomology classes on target manifold. The first aim is reached by providing a proof at a reasonable "physical" level of strictness, whereas the second is only conjectured and studied phenomenologically.

First we propose a geometrical interpretation of TQM developed earlier in [5, 6]. We prove that there is a correspondence between a special kind of observables and one-codimensional cycles on the target manifold. Moreover, transition amplitudes in the theory correspond to intersection indices of paths of steepest descent and cycles. This correspondence is proven using path integral representation of correlation function. Establishing a correspondence between TQM and the topology of the target manifold we find a geometrical interpretation of all quantities in the theory. It is also shown that the correlator can be introduced independently as an integral of pull-backs of forms corresponding to observables over the moduli space of graph embeddings into the target manifold.

In [6] it was shown that the correlators in TQM satisfy the so-called anticommutativity equation which is a general property of TQM. This allows us to conclude that the same equation holds for the intersection numbers. Thus an interesting mathematical fact is proven by "physical means."

Second, a conjecture concerning correlators in TQM and some algebraic operation (Massey product) on cohomology classes is formulated. The conjecture, together with the previous property of correlators, makes possible to relate the Massey products and the intersection numbers.

1. Topological Quantum Mechanics. An Overview

1.1. The Setup

Quantum Mechanics can be considered as the simplest version of TFT. TQM is based on the following set of axioms:

1. The Hamiltonian $H \in \mathrm{End}_{\mathbb{R}}(\mathcal{H})$ acting on a Hilbert space of states $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ can be represented as

$$H = [Q, G]$$

where $Q, G \in \mathrm{End}_{\mathbb{R}}(\mathcal{H})$ are odd nilpotent operators

$$Q^2 = 0, \qquad G^2 = 0$$

and the bracket stands for the supercommutator, i.e., for two operators A, B with parities a and b respectively, one has

$$[A, B] = AB - (-1)^{ab} BA.$$

The Hamiltonian annihilates the vacua space

$$H\mathcal{H}_0 = 0$$

which is postulated to be non-empty. H is positively defined on $\mathcal{H}_1$ and commutes with the parity operator $(-1)^F$, whereas Q anticommutes

$$H(-1)^F = (-1)^F H, \qquad Q(-1)^F = -(-1)^F Q.$$

Here F is a fermion number.

2. The observables $\mathcal{O}_i \in \mathrm{End}_{\mathbb{R}}\mathcal{H}$ in TQM form the algebra

$$\mathcal{O}_i \mathcal{O}_j = C_{ij}^k \mathcal{O}_k \tag{1}$$

where C_{ij}^k are its structure constants.

Lemma 1.1 ([5]). *In the above setup all correlation functions*

$$\begin{aligned}\langle \mathcal{O}_{i_1}(t_1)\ldots\mathcal{O}_{i_m}(t_m)\rangle \\ = \mathrm{trace}(-1)^F \mathrm{e}^{-t_1 H}\mathcal{O}_{i_1}\mathrm{e}^{(t_1-t_2)H}\ldots\mathrm{e}^{(t_{m-1}-t_m)H}\mathcal{O}_{i_m}\mathrm{e}^{t_m H}\end{aligned} \tag{2}$$

are independent on coordinates if

$$[Q, \mathcal{O}_i] = 0 \tag{3}$$

is valid. The trace is taken over the Hilbert space $\mathcal{H}$.

Definition 1.1. *The operators $\mathcal{O}$ satisfying equation* (3) *are referred to as zero-observables.*

The correlator however may jump after interchanging of some observables, so their order should be preserved in such considerations [5].

1.2. Deformation and One-Observables

Let us deform the operator Q as

$$Q \to Q + \sum T^A \mathcal{O}_A = Q + \mathcal{O}$$

where T_A are parameters (coupling constants) and $\mathcal{O}_A$ are the zero-observables. Then the Hamiltonian becomes

$$H = [Q, G] \to [Q, G] + [\mathcal{O}, G] = H_0 + H_1. \tag{4}$$

Considering H_1 as an interaction Hamiltonian we can rewrite the evolution operator and the derivative of one-point correlator $\mathcal{O}_1(t)$ as

$$\begin{aligned}\partial_{T^A}\langle \mathcal{O}_1(t_1)\rangle &= -\int_0^{t_1} \mathrm{d}\tau\ \mathrm{trace}(-1)^F \mathrm{e}^{-(t_1-\tau)H}[\mathcal{O}_A, G]\mathrm{e}^{-\tau H}\mathcal{O}_1 \\ &= \int_0^{t_1} \langle \mathcal{O}_A^{(1)}(\tau)\mathcal{O}_1(0)\rangle\, \mathrm{d}\tau\end{aligned} \tag{5}$$

where $\mathcal{O}_A^{(1)} \equiv -[\mathcal{O}_A(t), G]\mathrm{d}t$.

Definition 1.2. *The one-form $\mathcal{O}_A^{(1)}$ on $\mathbb{R}$ is referred to as one-observable.*

1.3. Generating Function for Correlators

The following property holds for a correlator [6]

$$\langle \mathcal{O}_{A_i}\rangle^A_{B\,\text{deformed}} = \partial_{T^{A_i}}\langle \mathcal{O}\rangle^A_{B\,\text{deformed}} = \langle \mathcal{O}_{A_i} \mathrm{e}^{\int_{\mathbb{R}}[\mathcal{O}(t),G]\mathrm{d}t}\rangle^A_B. \tag{6}$$

Here $\langle ..\rangle_{\text{deformed}}$ denotes vacuum expectation value in an interacting (deformed) theory, $\langle ..\rangle$ – the same quantity in a free (non-deformed) theory.
We can expand the exponent in (6) in Taylor series in terms of the parameters T_A

$$\mathcal{F}^A_B(T) \equiv \left\langle \mathrm{T}\left\{\mathcal{O}_{A_1}\mathrm{e}^{\int_{\mathbb{R}}[\mathcal{O}(t),G]\,\mathrm{d}t}\right\}\right\rangle^A_B = \sum_{m=1}^{\infty} \mathcal{F}^A_{B;\,A_1,\dots,A_m} T^{A_2}\dots T^{A_m} \tag{7}$$

where the coefficients are expressed via

$$\begin{aligned}\mathcal{F}^A_{B;\,A_1\dots A_m} &\equiv \frac{1}{(m-1)!}\left\langle \mathrm{T}\left\{\mathcal{O}_{A_1}\left[\prod_{i=2}^{m}\int_{\mathbb{R}}[\mathcal{O}_{A_i}(t_i),G]\,\mathrm{d}t_i\right]\right\}\right\rangle^A_B \\ &= \int_{\mathbb{R}^{m-1}_+}\mathrm{d}\tau^1\dots\mathrm{d}\tau^{m-1}\langle \mathcal{O}_{A_1}G\mathrm{e}^{-\tau_1 H}\mathcal{O}_{A_2}(0)G\mathrm{e}^{-\tau_2 H}\dots\mathcal{O}_{A_m}(0)\rangle^A_B.\end{aligned} \tag{8}$$

Here the parameters T_A have the meaning of coupling constants and T{..} stands for the chronological ordering. The whole expression (7), if interpreted physically, corresponds to the vacuum expectation value of $\mathcal{O}_{A_1}$ in the theory with interaction H_1. If all $\mathcal{O}_{A_i} = \mathcal{O}$ are the same, and the operator $K = \int_0^{+\infty} G\mathrm{e}^{-H\tau}\,\mathrm{d}\tau$ being introduced, the above formula can be compactly rewritten as

$$\mathcal{F}^{(m)\,A}_B = \langle \mathcal{O}K\mathcal{O}\dots K\mathcal{O}\rangle^A_B \tag{9}$$

where $\mathcal{F}^{(m)}$ is a short notation for the value determined in (8).

Theorem 1.1 ([6]). *Let $\mathcal{F}$ be a matrix defined in* (7)*, then*

$$\mathcal{D}\mathcal{F} + \frac{1}{2}[\mathcal{F},\mathcal{F}] = 0 \qquad \textit{or} \qquad (\mathcal{D}+\mathcal{F})^2 = 0 \tag{10}$$

where $\mathcal{D} = C^K_{AB}T^AT^B\partial_{T^K}$ is BRST operator (Chevaller differential). The above equation is called anticommutativity equation.

Here $\mathcal{D}^2 = 0$ if C^K_{AB} is antisymmetric with respect to A and B.

2. Geometrical Interpretation of TQM

2.1. Path Integral Representation of TQM

In this Section we use a theory which is a particular case of TQM. We are going to proceed in a slightly unconventional way, namely, first defining the transition amplitudes and afterwards deriving the action functional from them. This will be done in order to make the geometrical interpretation of the theory more clear.

Let $\mathcal{M}$ be a smooth closed oriented Riemannian n-manifold, f a Morse function on it, v a gradient vector field constructed by means of this Morse function, and $\mathrm{CP}(\mathcal{M})$ be the space of its critical points. Let $A, B \in \mathrm{CP}(\mathcal{M})$ be critical points with indices $p+1$ and p, where $p = 0, 1, \ldots, n-1$ respectively, and Γ_B^A be a gradient curve initiating at A, terminating at B and satisfying the following set of ODEs

$$\dot{x}^i = v^i. \tag{11}$$

Their solutions are integral curves of v, or **paths of steepest descent** (PSD). The worldsheet of the theory is a line $\mathbb{R}$, target space is $\mathcal{M}$ and embeddings $x \in \mathrm{Map}_B^A$

$$\mathrm{Map}_B^A = \{x(t) \in C^\infty(\mathbb{R}^1, \mathcal{M});\ x(-\infty) = A,\ x(+\infty) = B\} \tag{12}$$

satisfies (11). So we embed a line into $\mathcal{M}$ with some fixed images of $\pm\infty$ and x are the local coordinates on the target space, requiring it to be one of the rigid paths of steepest descent between A and B (see Fig. 1).

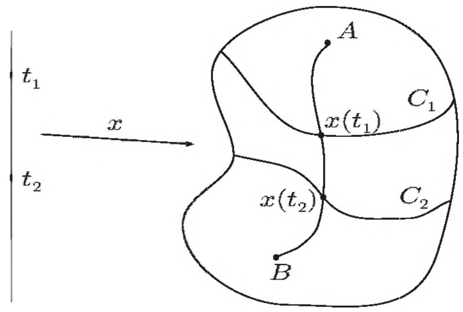

Figure 1. Worldsheet and target space of the theory

In our further considerations we will imply that the above boundary conditions are satisfied. The Map_B^A space is infinite-dimensional since parameterized paths are considered. For non-parameterized the following statement is valid

Lemma 2.1 ([3]).

$$\dim \mathrm{Map}_{B,\mathrm{nopar}}^A = \mathrm{ind}\, A - \mathrm{ind}\, B - 1. \tag{13}$$

To develop a quantum theory means to describe the states and the transition amplitudes among them. The transition amplitudes are given by the path integrals with appropriate boundary conditions. The key point in understanding the geometrical essence of TQM described below is that the transition amplitudes can be constructed from a purely geometric object. Indeed, let us consider the following path integral

Definition 2.1. *Let $V = \dot{x} - v$ be a vector field and $\delta[V]$ be a delta-functional. Then we define the transition amplitude as*

$$Z_{AB} = \int_{\mathrm{Map}_B^A} \mathcal{D}x\, \delta[V] \int_{\mathbb{R}} \mathrm{d}t \det(\nabla_i V^j) \tag{14}$$

where ∇ is the covariant derivative of the Levi-Civita connection.

In the above setup $V = 0$ corresponds to path of steepest descent. The determinant is a finite-dimensional determinant of the matrix $\int \mathrm{d}t \nabla_i V^j$. Further we would not specify the space of functional integration. The same boundary conditions for $x(t)$ will be imposed. We remind the reader (a mathematician) that in quantum field theory the functional integration is still ill defined, but we would not discuss such problems here. By construction this integral counts the number of PSDs with signs. This technique is an infinite dimensional generalization of the Mathai–Quillen method [7].

Below it will be made clear that there is a one-to-one correspondence between critical points of the Morse function on the manifold and the vacua in TQM – a path integral over even functional variables $\mathcal{D}x\mathcal{D}p$ and odd functional variables $\mathcal{D}\psi\mathcal{D}\pi$, and the measure $\mathcal{D}x\mathcal{D}\psi$ being measure with fixed endpoints, while the measure $\mathcal{D}p\mathcal{D}\pi$ is measure with arbitrary endpoints.

Lemma 2.2. *The formula* (14) *can be rewritten as follows*

$$Z_{AB} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}p\mathcal{D}\pi\, \mathrm{e}^{-S[x(t),p(t),\psi(t),\pi(t)]} \tag{15}$$

where

$$\begin{aligned} S &= \int_{-\infty}^{+\infty} \mathrm{d}t\, \left[p_a\left(\dot{x}^a - v^a\right) - \psi^i\left(\nabla_i v^a(x)\right)\pi_a + \epsilon\psi^i\psi^j F_{ij\ a}^{b}\pi_c\eta^{ac} + \epsilon p_a p_b \eta^{ab}\right] \\ &= \int p_a\, \mathrm{d}x^a + \pi_a\, \mathrm{d}\psi^a - [Q,G]\, \mathrm{d}t = \int P\, \mathrm{d}Q - H\, \mathrm{d}t \end{aligned} \tag{16}$$

is the action functional, A is an affine connection constructed by means of the Levi-Civita connection, and the operators

$$Q = \psi^i\nabla_i + p_a\frac{\partial}{\partial\pi_a} + \psi^i\psi^j F_{ij}{}^a_b\pi_a\frac{\partial}{\partial p_b}, \qquad G = \pi_a v^a + \epsilon\pi_a p_b \eta^{ab} \tag{17}$$

where $F_{ij}{}^a_b = \partial_i A_j{}^a_b + A_i{}^a_c A_j{}^c_b$ is the curvature tensor.

Accordingly, $x(t)$, $p(t)$, $\psi(t)$, $\pi(t)$ are identified with the dynamic variables, Z_{AB} with the transition amplitude and ϵ plays the role of coupling constant.

Critical points A, B are identified with the vacua of the theory due to the following reason: in Lagrangian formalism $V(x) \sim (v^i(x))^2$ has the meaning of potential (this relation becomes apparent after Gauss integration by p), due to positive-definiteness, its zeros are its minima; the zeros of a gradient vector field are critical points of the corresponding Morse function.

We provide the following Table of correspondence.

Abstract TQM	PI representation of TQM	Morse theory
vacua space $\mathcal{H}_0$, $\|A\rangle$	minima A of the potential	critical points $\mathrm{CP}(f, \mathcal{M})$, A
observables $\mathcal{O}$	δ-functions (one-forms) on cycles	cycles $C \subset \mathcal{M}$
amplitude $\langle A\|B\rangle$	amplitude $\langle A\|B\rangle$	number of PSDs from A to B
operator Q	$\psi^i \nabla_i + p_a \frac{\partial}{\partial \pi_a} + \psi^i \psi^j F_{ij}{}^a_b \pi_a \frac{\partial}{\partial p_b}$	de-Rham differential d
operator G	$\pi_a v^a + \epsilon \pi_a p_b \eta^{ab}$	vector field substitution ι_v

2.2. Morse Theory, Witten Complex and n-Matrices

In our theory the space $\mathcal{H}_0$ of vacua corresponds to the space $\mathrm{CP}(f, \mathcal{M})$ of critical points of function f on $\mathcal{M}$. So the vacua $|A\rangle, |B\rangle$ correspond to critical points A and B and the transition between the initial and the final state corresponds to motion of the point on the manifold from A to B along a rigid path Γ^A_B.

Let CP^i be the linear space of formal linear combinations of all critical points of $\mathcal{M}$ of index i. The following complex of chains CP^i

$$\ldots \longrightarrow \mathrm{CP}^{k-1} \longrightarrow \mathrm{CP}^k \longrightarrow \mathrm{CP}^{k+1} \longrightarrow \ldots \tag{18}$$

is said to be the Witten complex [3,10]. Here n ($n^2 = 0$) is a coboundary operator which increases grading in the complex that is given by the explicit formula

$$n|B\rangle = \sum_{\Gamma^A_B} \operatorname{sign} \Gamma^A_B |A\rangle \tag{19}$$

where $|B\rangle \in \mathrm{CP}^i$, $|A\rangle \in \mathrm{CP}^{i+1}$ and the sign Γ^A_B in the formula (19) means the sign for each PSD from A to B defined in Witten's paper [10]. If there are several

PSDs then their signs are summed up so that we can rewrite the formula in the following more convenient matrix notation

$$n|B\rangle = \sum_A n_B^A |A\rangle \tag{20}$$

where n_B^A is a matrix element, which is equal to the number of PSDs computed with signs, initiating at A and terminating at B.

Along with n_B^A we can introduce $n_B^A(C_1, \ldots, C_m)$ – a number of PSDs from A to B intersecting the cycles $C_1, \ldots, C_m \subset \mathcal{M}$. Here a transversal intersection of one-dimensional cycles and curve is assumed. Also we consider the case in which each cycle intersects the PSD only once.

Definition 2.2. *Under the above assumptions,*

$$n_B^A(C_1, \ldots, C_m) = \sum_\Gamma \prod_{i=1}^m \mathrm{ind}(C_i, \Gamma_B^A) \tag{21}$$

is said to be higher Morse differential (or n-matrices). Here $\mathrm{ind}(C_i, \Gamma_B^A)$ *is an intersection index of the objects into parentheses.*

Eventually we have a family of operators, represented by the matrices n_B^A, where $n_B^A(C_1, \ldots, C_m) \in \mathrm{Hom}(\mathrm{CP}^i, \mathrm{CP}^{i+1})$ and as all these objects are nilpotent one can consider a complex for each operator $n_B^A(C_1, \ldots, C_m)$ which is analogous to Witten complex for n_B^A.

2.3. Correlator via Intersection Numbers

Here we are going to introduce an explicit formula for the correlator in the Morse theory version of TQM. It will be introduced as a definition but in the next subsection we will show that the given expression really can be expressed via some path integral.

First we take an embedding $x \in \mathrm{Map}_B^A$ for A and B as critical points of certain Morse function and obtain the images of the points $t_1, \ldots, t_m$ by this map $x(t_1)$, $\ldots$, $x(t_m)$. One can treat this situation as the evaluation map

$$ev : \mathbb{R}^1 \times \mathrm{Map}_B^A \longrightarrow \mathcal{M}, \qquad (t, x) \longmapsto x(t).$$

We will employ the pull-back map of differential forms

$$ev^* : \Omega^\bullet(\mathcal{M}) \longrightarrow \Omega^\bullet(\mathbb{R}^1 \times \mathrm{Map}_B^A). \tag{22}$$

For some form $\omega \in \Omega^\bullet(\mathcal{M})$ this map is the following

$$ev^* \omega_I(x)\, \mathrm{d}x^I = \omega_I(x(t))(\dot{x}^I(t)\, \mathrm{d}t + \mathrm{d}\varphi^I) \tag{23}$$

where the differentials $\mathrm{d}\varphi^I$ belong to Map_B^A space.

We construct a set of transversal cycles $C_1, \ldots, C_m$ to the path AB on $\mathcal{M}$ so that $x(t_i) \in C_i$, $\operatorname{codim} C_i = 1$. Our next step here is to build closed differential forms on $\mathcal{M}$ which are delta-functions on the cycle.

Definition 2.3. *The form $\omega(x)$ is said to be a delta-form on the cycle C and denoted respectively as*

$$\omega(x) = \delta_C$$

if

$$\int_{\mathcal{M}} \omega \wedge \delta_C = \int_C \omega. \tag{24}$$

Definition 2.4. *The pull-back of the delta-form*

$$\mathcal{O}_i = ev^* \omega_i$$

is referred to as an observable in TQM.

Now we will construct axiomatically the correlation functions in TQM.

Definition 2.5. *Let $\mathfrak{M} \equiv \mathrm{Map}_B^A \times \mathbb{R}^1$. The correlator in TQM is represented as follows*

$$\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_m} \rangle := \int_{\mathfrak{M} \times \mathbb{R}_+^{m-1}} ev^* \delta_{i_1} \wedge \cdots \wedge ev^* \delta_{i_m}. \tag{25}$$

Here $\mathbb{R}_+^{m-1}$ is the moduli space the of embeddings of the graphs $(-\infty)$—t_1—t_2—$\cdots$—t_m—$(+\infty)$ into all paths of steepest descent between A and B.

The substantial statement is the following

Lemma 2.3.

$$\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_m} \rangle = n_B^A (C_1 \ldots C_m). \tag{26}$$

Proof: Obvious by construction. □

If there is no an intersection of the cycle with PSD the answer is zero for the whole integral above and equals one if each cycle has an intersection with the curve.

2.4. Correlator via Path Integral

Now we will convince the reader that $\mathcal{F}_B^{A\,(m)}$ in (8) is equal to $n_B^A(\underbrace{C \ldots C}_m)$ provided an appropriate correspondence for the abstract operator $G\mathcal{O}_{A_i}$ is specified in the path integral formulation.

Theorem 2.1. *Let v be a smooth vector field on $\mathcal{M}$ and $\mathcal{O}_{A_i}^{(1)}$ be one-observable in TQM. Let G be a vector field substitution operator $G = \iota_v$. Then the following equality between correlator and intersection numbers holds*

$$\mathcal{F}_{B;\,A_1 \ldots A_m}^A = n_B^A (C_1, \ldots, C_m). \tag{27}$$

Proof: One can see that the following correspondence arises from the theorem

$$G\mathcal{O}_{A_i} = \delta(x^n(0) - x^n(t_i))v^n(x(t_i)). \tag{28}$$

Here local coordinates are chosen in such a way that vector field v on $\mathcal{M}$ in the vicinities of the intersection points with a path of steepest descent has the only one nonzero component v^n, x^n is the n-th component of coordinate x, and $x(t_i(\tau))$ are images of the points $t_1, \dots, t_m$ which depends on τ by embedding x. Here by τ we imply $\tau_2, \dots, \tau_m$. One can see that after applying vector field substitution operator ι_v to one observable the unpleasant differential $\mathrm{d}\varphi^I$ vanishes.

Representation of the correlator (8) via the path integral yields

$$\mathcal{F}^{(m)} = \int_{\mathbb{R}_+^{m-1}} \mathrm{d}\tau_2 \dots \mathrm{d}\tau_m \int \mathrm{e}^{-S} \mathcal{O}_{A_1} \mathcal{O}_{A_2}^{(1)} \dots \mathcal{O}_{A_m}^{(1)}. \tag{29}$$

Here $\tau_i = |t_i - t_{i-1}|$, $i = 2, \dots, m$, parameterize the moduli space of embeddings of $\mathbb{R}$ with marked points $t_1, \dots, t_m$ into target manifold $\mathcal{M}$ (see subsections 2.2 and 2.3).

Then the correlator expansion coefficient can be expressed via the path integral

$$\begin{aligned} \mathcal{F}_B^{(m)\,A} = &\int \mathrm{d}\tau_2 \dots \mathrm{d}\tau_m \int \mathcal{D}x\mathcal{D}p\mathcal{D}\psi\mathcal{D}\pi \\ &\times \exp\left[-\int_{-\infty}^{\infty} \mathrm{d}t \left(p_a(v^a(x) - \dot{x}^a) + \pi_a\dot{\psi}^a - \psi^i(\nabla_i v^a)\pi_a\right)\right] \\ &\times \delta(x^n(0) - x^n(t_1(\tau))) \prod_{i=2}^m \delta(x^n(0) - x^n(t_i(\tau)))\, v^i(x^n(t_i(\tau))). \end{aligned} \tag{30}$$

As the operators contain no dependence on Grassmann fields $\psi(t)$, $\pi(t)$, one can integrate them out, resulting in $\det(\partial_\tau\delta_i^j - \partial_i v^j)$ in the numerator. The integral over $p(t)$ can also be easily done, simply by the definition of the delta-functional. Therefore,

$$\begin{aligned} \mathcal{F}_B^{(m)\,A} = &\int \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_m \int \mathcal{D}x \det(\partial_t\delta_i^j - \partial_i v^j)\, \delta[\dot{x}(t) - v(x)] \\ &\times \delta(x^n(0) - x^n(t_1(\tau))) \prod_{i=2}^m \delta(x^n(0) - x^n(t_i(\tau)))v^n(x(t_i(\tau))). \end{aligned} \tag{31}$$

One can take the integral over $\mathcal{D}x$ away by virtue of the delta-functional, replacing $x(t)$ with a solution $\mathrm{X}(t)$ of the classical Lagrange–Euler equations (11). However, special cares should be taken due to the presence of zero modes in these solutions. Hence, an integral over the space of collective coordinates λ remains after integrating the infinite-dimensional $\mathcal{D}x$ integral [1]. Geometrically λ corresponds to a shift of all points $t_1, \dots, t_m$ along $\mathbb{R}$ keeping the distances between them constant and parameterizes second multiplier in the definition of $\mathfrak{M}$ (see subsection 2.3). Re-expressing the delta-functional

$$\delta[\dot{x}^i(0) - v^i(x)] = \sum_\Gamma \frac{\delta[x^i(0) - \mathrm{x}(t)]}{|\det(\partial_t \delta^j_k - \partial_k v^j)|}$$

one cancels determinants (as it should be in a supersymmetric theory) up to sign $(-1)^a = \mathrm{sign}(\det(\partial_t \delta^j_k - \partial_k v^j))$ and obtains

$$\begin{aligned} \mathcal{F}^{(m)\,A}_B = \sum_\Gamma \int \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_m \int \mathrm{d}\lambda\, (-)^a \delta(x^{(cl)\,n}_\Gamma(0) - \mathrm{x}(t_1(\tau), \lambda)) \\ \times \prod_{i=2}^m \delta(\mathrm{x}(0) - \mathrm{x}(t_i(\tau), \lambda)) v^n(\mathrm{x}(t_i(\tau))). \end{aligned} \tag{32}$$

Integral over λ plays a crucial role here. It allows us to integrate out all the delta-functions, so that a regular expression remains. The latter integral possesses structure absolutely similar to that of the integral (25), which was obtained within a purely geometric construction of subsection 2.3. Indeed, sum over Γ and the integral over the zero mode λ are equivalent to integration over $\mathfrak{M}$, whereas the integrals over τ_i are taken over the same manifolds $\mathbb{R}^{m-1}_+$. One can make sure that the following integral

$$\int \mathrm{d}\tau_i\, (-)^a\, v^n(\mathrm{x}(t_i(\tau)))\, \delta(\mathrm{x}(0) - \mathrm{x}(t_i(\tau))) = \mathrm{ind}(\Gamma, C_i) \tag{33}$$

is an intersection index between Γ and C_i. Therefore, one gets using (33) the following expression

$$\mathcal{F}^{(m)\,A}_B = \sum_\Gamma \prod_{i=1}^m \mathrm{ind}(\Gamma, C_i) = n^A_B(C \ldots C). \tag{34}$$

Trivially generalizing this result, we thus have proven that for an arbitrary number of cycles

$$\mathcal{F}^A_{B;\,A_1 \ldots A_m} = n^A_B(C_1, \ldots, C_m). \tag{35}$$

□

2.5. Generating Function for N-Matrices

The correspondence described in the previous subsection is very useful and has interesting consequences. Indeed as $\mathcal{F}^{(m)} = n^A_B(C_1 \ldots C_m)$ one can rewrite for intersection numbers all relations valid for the correlators as well. First we mean the anticommutativity equation. As before we construct a generating function, namely, the whole matrix of them

$$N_B^A(T) := n_B^A + n_B^A(C_i)T^i + n_B^A(C_j, C_k)T^jT^k + \dots \\ + n_B^A(C_p, \dots, C_q)T^p \dots T^q + \dots = \sum_k n_B^A(C^{(k)})T^{(k)} \tag{36}$$

where t is a parameter. Actually, several nonequivalent cycles are admitted so one needs introducing the same number of parameters. The above construction is an element of the space $\mathrm{M}_{N\times N} \otimes \mathbb{R}[T^1 \dots T^l]$. Here l is a number of nonequivalent cycles on the manifold.

As anticommutativity equation holds for N ($\mathcal{D}$ is taken from (10))

$$[\mathcal{D} + N, \mathcal{D} + N] = 0 \qquad \text{or} \qquad \mathcal{D}N + \frac{1}{2}[N, N] = 0 \tag{37}$$

we obtain interesting relations for the intersection numbers for any order in T. These equations are indeed very interesting relations in the intersection theory.

3. Algebraic Interpretation of TQM

Massey product is defined as described in the following

Definition 3.1. *Let $\alpha \in H^p(\mathcal{M})$, $\beta \in H^q(\mathcal{M})$, $\gamma \in H^r(\mathcal{M})$ and $\alpha\beta = 0$, $\beta\gamma = 0$. Then the Massey product $\mathrm{MP}(\alpha, \beta, \gamma)$ is an element of the following quotient space*

$$H^{p+q+r-1}(\mathcal{M})/[\alpha \smile H^{q+r-1}(\mathcal{M}) + H^{p+q-1}(\mathcal{M}) \smile \gamma]. \tag{38}$$

Let the cocycles a, b, c be representatives of α, β, γ and the cochains u, v are such that $\mathrm{d}u = ab$ and $\mathrm{d}v = bc$. Then the cochain $-uc + (-1)^p av$ is a cocycle and its cohomological class represents $\mathrm{MP}(\alpha, \beta, \gamma)$.

Higher Massey products are defined inductively via products of lower order Massey products. However, for higher order Massey products to exist it is necessary that the lower order Massey products are trivial. Massey product enables us to determine homotopic class of the manifold up to a torsion group.

One can see that the above construction is not well defined. Nevertheless, this problem can be solved introducing the so-called modified Massey product. We need $\alpha\beta$, $\beta\gamma$ vanished in cohomologies. If they are nonzero the above definition fails. So we introduce the following operator

$$K = \mathrm{d}^{-1} \circ (\mathrm{id} - \mathrm{Pr}_H) \tag{39}$$

where Pr_H is a projection operator on de-Rham cohomology groups. So, for each form ω one has $(\mathrm{id} - \mathrm{Pr}_H)\omega = \mathrm{d}\chi$ being exact, the operator d^{-1} is well defined and $K\omega = \chi$. But another problem arises here. The form χ is not closed anymore.

3.1. Conjecture

As there is an embedding of the space of critical points of the Morse function $\mathrm{CP}^\bullet \hookrightarrow H^\bullet(\mathcal{M})$ into de Rham cohomology groups of $\mathcal{M}$ (as linear spaces) one can make the following

Conjecture 3.1. *If $A \in \mathrm{CP}^{p+1}$ and $B \in \mathrm{CP}^p$ be critical points of indices $p+1$ and p respectively and $|A\rangle \in \mathcal{H}_0^{p+1}$, $|B\rangle \in \mathcal{H}_0^p$ be the representatives of the vacua space. The conjecture is:*
There is a correspondence between observables in TQM and forms in de Rham cohomology groups of $\mathcal{M}$

$$\mathrm{End}(\mathcal{H}_0) \ni \mathcal{O}_{A_i} \longleftrightarrow \omega_i \in H^1(\mathcal{M}) \tag{40}$$

such that

$$\begin{aligned}\int_{\mathbb{R}_+^{m-1}} \mathrm{d}^m\tau \,\langle \mathcal{O}_{\{A_1} G\mathrm{e}^{-\tau_2 H}\mathcal{O}_{A_2}\dots G\mathrm{e}^{-\tau_m H}\mathcal{O}_{A_m\}}\rangle_B^A \\ = \int_{\mathcal{M}} \tilde{\omega}^A \wedge \mathrm{MP}(\omega_1,\dots,\omega_m;\omega_B)\end{aligned} \tag{41}$$

where the tilde stands for Poincaré duality.

So, if the conjecture is valid, then we have an equality of three objects of very different nature – correlator in TQM, intersection matrix $n_B^A(C_1,\dots,C_m)$ and the expression in the r.h.s. of the above formula.

4. Conclusion

Here we have presented the geometrical pattern of TQM – a toy model of TFT. We expressed the correlators via the intersection numbers on target manifold and made a conjecture that they can be expressed via integral of Massey product. Now the main problem is to prove this statement.

Quantum mechanics in the setup described in this paper is a string theory with string length equal to zero. Generalization of this theory to topological sigma–model is a very interesting problem that has to be solved.

Acknowledgements

The author is grateful to his colleagues A. Zayakin, A. Gorsky, D. Mathcalotov, D. Volin, A. Chervov, M. Libanov, A. Litvinov and especially A. Losev for fruitful discussions. This work is supported in part by RFBR 01–02–17227 grant.

References

[1] Birmingham B., Blau M., Rakowski M. and Thompson G., *Topological Field Theory*, Phys. Rep. **209** (1991) 129–340.

[2] Cohen L. and Norbury P., *Morse Field Theory*, e-Print Archive math.GT/0509681 (2005).

[3] Fukaya K., *Morse Homotopy and its Quantization*, AMS/IP Studies in Advanced Math. **2** (1997) 409–440.

[4] Labastida J., *Morse Theory Interpretation of Topological Quantum Field Theories*, Commun. Math. Phys. **123** (1989) 641–658.

[5] Losev A. and Polyubin I., *Topological Quantum Mechanics for Physicists*, JETP Lett. **82** (2005) 373–380.

[6] Lysov V., *Anticommutativity Equation in Topological Quantum Mechanics*, JETP Lett. **76** (2002) 724–727.

[7] Mathai V. and Quillen D., *Superconnections, Thom Classes and Equivariant Differential Forms*, Topology **25** (1986) 85–110.

[8] Witten E., *Topological Quantum Field Theory*, Comm. Math. Phys. **117** (1988) 353–386.

[9] Witten E., *Topological Sigma Model*, Comm. Math. Phys. **118** (1988) 411–449.

[10] Witten E., *Supersymmetry and Morse Theory*, J. Diff. Geom. **17** (1982) 661–692.