LOCALIZED INDUCTION EQUATION AND TRANSPORT PROPERTIES FOR STRETCHED VORTEX FILAMENT

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Abstract. The localized induction equation for the stretched vortex filament is reviewed. The transport of the momentum and the angular momentum carried by the stretching vortex filament is studied. The generalizations of the equation are considered.

1. Introduction

In this paper we would like to review our previous works on the generalizations of the **localized induction equation** (LIE) for the stretched vortex filament [4, 5, 6] and to study the momentum and the angular momentum transport by the stretched vortex filament.

Motion of the thin vortex filament is one of the important subject in fluid motion. Arms and Hama [1] derived the LIE for the filament as

$$R_t = \frac{R_s \times R_{ss}}{|R_s|^3} \tag{1}$$

by using the localized induction approximation. Here s is the parameter along the filament and t is the time. If $|R_s| = 1$, then (1) becomes

$$R_t = R_s \times R_{ss}. \tag{2}$$

In this case s denotes the arclength along the filament. The condition $|R_s| = 1$ means that the filament has no stretch [7]. Then, we call, hereafter, (2) as a LIE and (1) the stretching LIE. It is known that LIE is an integrable equation and has N soliton solution. Konno and Kakuhata [4] have found that the stretching LIE is also integrable. So we will discuss the connection between (1) and (2) and study the transport properties of the stretched vortex filament.

The transport properties of the vortex filament were considered by Kimura [3] and Fukumoto [2], where the references related to the transport properties can be found. Fukumoto proved that the linear momentum and the angular momentum are conserved quantities for LIE. We will discuss the transport properties for the stretched vortex filament.

In the next section we will discuss the relationship between LIE and the stretching LIE. In Section 3 the transport properties of the vortex filament will be discussed. Generalizations of the stretching LIE will be presented in Section 4. The last section presents conclusion.

2. Relationship Between LIE and Stretching LIE

In order to distinguish the stretching LIE from LIE, we rewrite the equations by introducing the independent variables s and r as follows

$$S_t = S_s \times S_{ss} \tag{3}$$

for LIE and

$$R_t = \frac{R_r \times R_{rr}}{|R_r|^3} \tag{4}$$

for the stretching LIE.

We can prove that $|S_s|$ and $|R_r|$ are independent of t by taking the inner products such as

$$\frac{\partial S}{\partial s} \cdot \frac{\partial}{\partial s}\left(\frac{\partial S}{\partial t}\right) = 0 \quad \text{and} \quad \frac{\partial R}{\partial r} \cdot \frac{\partial}{\partial r}\left(\frac{\partial R}{\partial t}\right) = 0$$

with the equations (3) and (4). Then $|S_s|$ and $|R_r|$ are functions only of s, respectively r.

Let us consider the transformation between the two equations generated by the relation

$$ds = g\,dr \tag{5}$$

where g is a metric defined by

$$g = \left|\frac{\partial R}{\partial r}\right|.$$

Obviously g is a function of r and represents the local stretch. With this metric we see that LIE is transformed into the stretching LIE.

From (5) we have

$$s = f(r). \tag{6}$$

We can obtain a solution for the stretching LIE by substituting $f(r)$ for s in the solution of LIE such as

$$R(r,t) = S(f(r),t).$$

In this way from N soliton solution S, we can obtain the respective R.

Introducing the inverse function $r = h(s)$ such as

$$f(h(s)) = s$$

we see that a solution of the stretching LIE becomes such for LIE.

One-vortex soliton of LIE is explicitly given by

$$S_x = \frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \sin 2(\lambda_R s - 2\omega_R t) \operatorname{sech} 2(\lambda_I s - 2\omega_I t)$$

$$S_y = -\frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \cos 2(\lambda_R s - 2\omega_R t) \operatorname{sech} 2(\lambda_I s - 2\omega_I t) \qquad (7)$$

$$S_z = s - \frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \tanh 2(\lambda_I s - 2\omega_I t)$$

where ω is given by $\omega = 2\lambda^2$ with $\lambda = \lambda_R + i\lambda_I$. From (6) we obtain the corresponding one-vortex soliton solution of the stretching LIE as

$$R_x = \frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \sin 2(\lambda_R f(r) - 2\omega_R t) \operatorname{sech} 2(\lambda_I f(r) - 2\omega_I t)$$

$$R_y = -\frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \cos 2(\lambda_R f(r) - 2\omega_R t) \operatorname{sech} 2(\lambda_I f(r) - 2\omega_I t) \qquad (8)$$

$$R_z = f(r) - \frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \tanh 2(\lambda_I f(r) - 2\omega_I t).$$

3. Momentum and Angular Momentum Transport

Here we consider the linear momentum and the angular momentum transport for the vortex soliton of the stretching LIE. The momentum P and the angular momentum M are defined by

$$P = \frac{\rho}{2} \int u \, dV \qquad \text{and} \qquad M = \frac{\rho}{3} \int X \times u \, dV \qquad (9)$$

where X denotes the position vector of a point on the filament such as S in (3) or R in (4), u is the velocity and ρ is the density of the fluid. By making use of the vorticity ω the velocity can be expressed as

$$u = X \times \omega$$

and then, P and M are given by

$$P = \frac{\rho}{2} \int X \times \omega \, dV \qquad \text{and} \qquad M = \frac{\rho}{3} \int X \times (X \times \omega) \, dV. \qquad (10)$$

For the case of the vortex filament of LIE the vorticity is proportional to the tangent vector along the filament, i.e.,

$$\omega = \alpha \frac{\partial S}{\partial s}. \qquad (11)$$

We can replace the volume element dV by $dS ds$ where dS denotes the area element of the cross-section perpendicular to the vortex filament and then (10) reduces to

$$P = \frac{\sigma}{2} \int S \times \frac{\partial S}{\partial s} \, ds \quad \text{and} \quad M = \frac{\sigma}{3} \int S \times \left(S \times \frac{\partial S}{\partial s} \right) \, ds \quad (12)$$

where

$$\sigma = \rho \int \alpha \, dS.$$

We see that P and M are conserved quantities on the interval curves of LIE (3). We calculate them for the one-vortex soliton (7) as

$$P = \frac{\sigma \lambda_R \lambda_I}{(\lambda_R^2 + \lambda_I^2)^2} e_z \quad \text{and} \quad M = -\frac{\sigma \lambda_I (3\lambda_R - \lambda_I)}{6(\lambda_R^2 + \lambda_I^2)^3} e_z. \quad (13)$$

For the case of the vortex filament for the stretching LIE, and if we assume that the vorticity is proportional to the tangent vector along the filament as

$$\omega = \beta \frac{\partial R}{\partial r} \quad (14)$$

we have respectively

$$P = \frac{\sigma}{2} \int R \times \frac{\partial R}{\partial r} \, dr, \quad M = \frac{\sigma}{3} \int R \times \left(R \times \frac{\partial R}{\partial r} \right) \, dr \quad (15)$$

where

$$\sigma = \rho \int \beta \, dS.$$

Then we can prove that P and M are conserved quantities under the evolution specified by the stretching LIE (4) for any $f(r)$ in (6). We can calculate them for the one-vortex soliton (8) and this gives

$$P = \frac{\sigma \lambda_R \lambda_I}{(\lambda_R^2 + \lambda_I^2)^2} e_z, \quad M = -\frac{\sigma \lambda_I (3\lambda_R - \lambda_I)}{6(\lambda_R^2 + \lambda_I^2)^3} e_z. \quad (16)$$

It is interesting that the local momentum and angular momentum densities such as $X \times \partial X / \partial s$ and $X \times (X \times \partial X / \partial s)$ give different values in virtue of the effect of stretch, but the global momentum and angular momentum such as (13) and (16) take the same values for any $f(r)$ in (6).

4. Generalizations of Stretching LIE

If we generalize the metric by taking

$$ds = g^n \, dr$$

then, we obtain a generalized localized induction equation

$$\boldsymbol{R}_t = \frac{\boldsymbol{R}_r \times \boldsymbol{R}_{rr}}{|\boldsymbol{R}_r|^{3n}}$$

which is still an integrable equation.

In accordance with to the above results, we find a further generalization of LIE by introducing the independent variable transformation as

$$s = f(r)$$

such that

$$\mathrm{d}s = \frac{\mathrm{d}f(r)}{\mathrm{d}r}\mathrm{d}r \equiv g\mathrm{d}r.$$

If we assume that

$$\boldsymbol{R}(r,t) = \boldsymbol{S}(f(r),t)$$

then LIE reduces to

$$\boldsymbol{R}_t = \frac{\boldsymbol{R}_r \times \boldsymbol{R}_{rr}}{g^3}.$$

5. Conclusion

We have reviewed the relationship between LIE and the stretching LIE by using the metric $g(r)$ and we have shown how to obtain N-vortex soliton solution of the stretching LIE by using N-soliton solution of LIE. We have explicitly given one-soliton solution for the stretched vortex filament. We have proved that the momentum and the angular momentum are conserved quantities even for the stretching LIE and calculated explicitly these for the one-soliton solution which does not depend on $f(r)$ in (6). Further generalizations of LIE have been found where the integrability of the reduced equation has been preserved.

For the momentum and the angular momentum there is no difference in the transport properties between LIE and the stretching LIE under the conditions (11) and (14). We expect some differences for the energy transport and the helicity which will be discussed in further publication.

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