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ON THE GEOMETRIC STRUCTURE OF HYPERSURFACES OF CONULLITY TWO IN EUCLIDEAN SPACE

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Abstract. In this paper we introduce the notion of a semi-developable surface of codimension two as a generalization of the notion of a developable surface of codimension two. We give a characterization of the developable and semi-developable surfaces in terms of their second fundamental forms. We prove that any hypersurface of conullity two in Euclidean space is locally a foliation of developable or semi-developable surfaces of codimension two.

1. Introduction

The class of semi-symmetric spaces was first studied by Cartan [3] in connection with his research on locally symmetric spaces. All locally symmetric spaces and all two-dimensional Riemannian manifolds belong to this class. In 1968 Nomizu [7] conjectured that in all dimensions greater or equal to three every irreducible complete Riemannian semi-symmetric space is locally symmetric. His conjecture was refuted in 1972 by Takagi [11], who constructed a complete irreducible hypersurface in $\mathbb{E}^4$, which is semi-symmetric but is not locally symmetric, and by Sekigawa [8], who gave counterexamples of arbitrary dimensions. In 1982 Szabó [9] gave a local classification of Riemannian semi-symmetric spaces, dividing them into three basic classes: trivial, exceptional and typical. Semi-symmetric spaces of the typical class were studied also by Boeckx *et al* in [2] under the name Riemannian manifolds of conullity two.

In the present paper we study the class of the typical semi-symmetric hypersurfaces (hypersurfaces of conullity two) in Euclidean space $\mathbb{E}^{n+1}$, considering them with respect to their second fundamental form.

In Section 3 we introduce the notion of a semi-developable surface of codimension two as a generalization of the notion "developable surface" of codimension two and give a characterization of the developable and semi-developable surfaces of codimension two in terms of their second fundamental form.

In Section 4 we prove the following structure theorem:

Each hypersurface of conullity two in $\mathbb{E}^{n+1}$ *is locally a foliation (one-parameter system) of developable or semi-developable surfaces of codimension two.*

2. Preliminaries

For an n-dimensional Riemannian manifold (M^n, g) we denote by T_pM^n the tangent space to M^n at a point $p \in M^n$ and by $\mathfrak{X}M^n$ – the algebra of all vector fields on M^n. The associated Levi-Civita connection of the metric g is denoted by ∇, the Riemannian curvature tensor R of type $(1,3)$ is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \mathfrak{X}M^n$$

and the corresponding curvature tensor of type $(0,4)$ is given by

$$R(X,Y,Z,U) = g(R(X,Y)Z,U), \qquad X, Y, Z, U \in \mathfrak{X}M^n.$$

We remark that all manifolds, vector fields, differential forms, functions and surfaces are assumed to be smooth (i.e., of differentiability class C^∞).

A **semi-symmetric space** is a Riemannian manifold (M^n, g), whose curvature tensor R satisfies the identity

$$R(X,Y) \cdot R = 0$$

for all $X, Y \in \mathfrak{X}M^n$. According to Szabó's classification (using the terminology of [2]) every locally irreducible semi-symmetric space belongs to one of the following three classes:

1) "trivial" class, consisting of all locally symmetric spaces and all two-dimensional Riemannian manifolds
2) "exceptional" class of all elliptic, hyperbolic, Euclidean and Kählerian cones
3) "typical" class of all Riemannian manifolds foliated by Euclidean leaves of codimension two.

The trivial semi-symmetric manifolds are well-known and the exceptional ones are described and constructed explicitly in [9] and [10]. For the class of foliated semi-symmetric spaces Zsabó [10] had derived a system of non-linear partial differential equations, describing their metrics.

Foliated semi-symmetric spaces were studied by Boeckx *et al* [2] with respect to their metrics as Riemannian manifolds of conullity two.

A Riemannian manifold (M^n, g) is of **conullity two**, if at every point $p \in M^n$ the tangent space T_pM^n can be decomposed in the form

$$T_pM^n = \Delta_0(p) \oplus \Delta_0^{\perp}(p)$$

where $\dim \Delta_0(p) = n-2$, $\dim \Delta_0^{\perp}(p) = 2$ and $\Delta_0(p)$ is the nullity vector space of the curvature tensor R_p, i.e.,

$$\Delta_0(p) = \{X \in T_pM^n;\ R_p(X,Y)Z = 0,\ Y, Z \in T_pM^n\}.$$

The $(n-2)$-dimensional distribution $\Delta_0 : p \longrightarrow \Delta_0(p)$ is integrable and its integral manifolds are totally geodesic and locally Euclidean. So, (M^n, g) is foliated by Euclidean leaves of codimension two.

In [2] the metrics of the Riemannian manifolds of conullity two are described by systems of non-linear partial differential equations. For some classes of manifolds of conullity two the metrics are determined in explicit form.

We study hypersurfaces of conullity two (or foliated semi-symmetric hypersurfaces) in Euclidean space $\mathbb{E}^{n+1}$ with respect to their second fundamental form, considering them as one-parameter systems of geometrically determined surfaces of codimension two.

We denote the standard metric in $\mathbb{E}^{n+1}$ by g and its Levi-Civita connection by ∇'. Let ∇ be the induced connection on a hypersurface M^n in $\mathbb{E}^{n+1}$ and $h(X,Y) = g(AX,Y)$, $X, Y \in \mathfrak{X}M^n$ be the second fundamental tensor of M^n with corresponding shape operator A.

Foliated semi-symmetric hypersurfaces are characterized in terms of the second fundamental form as follows [5].

Proposition 1. *A hypersurface M^n in $\mathbb{E}^{n+1}$ is of conullity two if and only if its second fundamental form h is*

$$h = \lambda\omega \otimes \omega + \mu(\omega \otimes \eta + \eta \otimes \omega) + \nu\eta \otimes \eta, \qquad \lambda\nu - \mu^2 \neq 0$$

where ω and η are unit one-forms; λ, μ and ν are functions on M^n.

Here the Euclidean leaves of the foliation are the integral submanifolds of the distribution Δ_0, determined by the one-forms ω and η

$$\Delta_0(p) = \{X \in T_pM^n;\ \omega(X) = 0,\ \eta(X) = 0\}, \qquad p \in M^n.$$

We denote by $\Delta_0^{\perp}$ the distribution of M^n, orthogonal to Δ_0. Since the second fundamental form h of M^n is symmetric, then locally there exist two mutually orthogonal unit vector fields ξ_1, $\xi_2 \in \Delta_0^{\perp}$ with corresponding unit one-forms η_1 and η_2, respectively, such that

$$h = \nu_1\eta_1 \otimes \eta_1 + \nu_2\eta_2 \otimes \eta_2, \qquad \nu_1\nu_2 \neq 0 \tag{1}$$

where ν_1 and ν_2 are functions on M^n. The vector fields ξ_1 and ξ_2 determine the principal directions of the shape operator A of M^n.

Using the Codazzi equations for a hypersurface with second fundamental form h, satisfying (1), in Section 4 we obtain all involutive $(n-1)$-dimensional distributions, containing Δ_0, and prove that the integral surfaces of these distributions are developable or semi-developable surfaces of codimension two.

3. Developable and Semi-Developable Surfaces of Codimension Two

A $(k+1)$-dimensional surface M^{k+1} in Euclidean space $\mathbb{E}^{n+1}$, which is a one-parameter system $\{\mathbb{E}^k(s)\}$, $s \in J$ of k-dimensional linear subspaces of $\mathbb{E}^{n+1}$, defined in an interval $J \subset \mathbb{R}$, is said to be a *ruled* $(k+1)$*-surface* [4, 1]. The planes $\mathbb{E}^k(s)$ are called *generators* of M^{k+1}. A ruled surface M^{k+1} is said to be **developable** [1], if the tangent space T_pM^{k+1} at all regular points p of an arbitrary fixed generator $\mathbb{E}^k(s)$ is the same. A developable ruled hypersurface $M^n = \{\mathbb{E}^{n-1}(s)\}$, $s \in J$ in $\mathbb{E}^{n+1}$ is called a **torse**.

Now we shall consider a ruled $(n-1)$-surface $M^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ (ruled surface of codimension two). Let $\{N_1, N_2\}$ be a normal frame field of M^{n-1}, consisting of two mutually orthogonal unit vector fields. We denote by h_1 and h_2 the second fundamental forms of M^{n-1} corresponding to the vector fields N_1 and N_2, respectively and by A_1 and A_2 their corresponding shape operators, i.e.,

$$h_1(x,y) = g(A_1x, y), \qquad h_2(x,y) = g(A_2x,y), \qquad x, y \in \mathfrak{X}M^{n-1}.$$

If D is the normal connection of M^{n-1}, then the Gauss and Weingarten formulas imply

$$\begin{array}{ll} \nabla'_x y = \nabla_x y + h_1(x,y)N_1 + h_2(x,y)N_2, & x, y \in \mathfrak{X}M^{n-1} \\ \nabla'_x N_1 = -A_1 x + D_x N_1, & \nabla'_x N_2 = -A_2 x + D_x N_2. \end{array} \tag{2}$$

Let p be an arbitrary point of M^{n-1} and $\mathbb{E}^{n-2}(s)$ be the generator of M^{n-1} containing p. We denote by $\Delta_0(p)$ the subspace of T_pM^{n-1}, tangent to $\mathbb{E}^{n-2}(s)$ and by Δ_0 – the distribution $\Delta_0 : p \to \Delta_0(p)$. The unit vector field on M^{n-1}, orthogonal to Δ_0 and its corresponding one-form are denoted by W and ω, respectively (W is determined up to a sign). Since the integral submanifolds $\mathbb{E}^{n-2}(s)$ of the distribution Δ_0 are auto-parallel, then $\nabla'_{x_0} y_0 \in \Delta_0$ for all $x_0, y_0 \in \Delta_0$. Hence, the first equality in (2) implies

$$h_1(x_0, y_0) = h_2(x_0, y_0) = 0, \qquad x_0, y_0 \in \Delta_0. \tag{3}$$

Using the unique decompositions

$$x = x_0 + \omega(x)\, W, \qquad y = y_0 + \omega(y)\, W, \qquad x_0, y_0 \in \Delta_0$$

of arbitrary vector fields $x, y \in \mathfrak{X}M^{n-1}$, from (3) we get

$$\begin{aligned} h_1(x,y) &= p\,\omega(x)\,\omega(y) + \omega(x)\,h_1(y_0,W) + \omega(y)\,h_1(x_0,W) \\ h_2(x,y) &= q\,\omega(x)\,\omega(y) + \omega(x)\,h_2(y_0,W) + \omega(y)\,h_2(x_0,W) \end{aligned} \qquad (4)$$

where $p = h_1(W,W)$, $q = h_2(W,W)$.

We denote by $\overline{\beta}_1$ and $\overline{\beta}_2$ the one-forms on M^{n-1}, defined by

$$\begin{aligned} \overline{\beta}_1(x_0) &= h_1(x_0,W), \qquad \overline{\beta}_1(W) = 0, \qquad x_0 \in \Delta_0 \\ \overline{\beta}_2(x_0) &= h_2(x_0,W), \qquad \overline{\beta}_2(W) = 0. \end{aligned}$$

Let β_1 and β_2 be their corresponding unit one-forms, i.e.,

$$\begin{aligned} \overline{\beta}_1 &= b_1\,\beta_1, \qquad b_1 = \|\overline{\beta}_1\| \\ \overline{\beta}_2 &= b_2\,\beta_2, \qquad b_2 = \|\overline{\beta}_2\|. \end{aligned}$$

Hence, the equalities (4) imply

$$h_1 = p\,\omega\otimes\omega + b_1(\omega\otimes\beta_1 + \beta_1\otimes\omega), \qquad h_2 = q\,\omega\otimes\omega + b_2(\omega\otimes\beta_2 + \beta_2\otimes\omega). \quad (5)$$

Let B_1 and B_2 be the unit vector fields on M^{n-1}, corresponding to the one-forms β_1 and β_2, respectively, i.e.,

$$\beta_1(x) = g(B_1,x), \qquad \beta_2(x) = g(B_2,x), \qquad x \in \mathfrak{X}M^{n-1}.$$

It is obvious that $B_1, B_2 \in \Delta_0$. We denote by θ the one-form on M^{n-1}, defined as follows

$$\theta(x) = g(\nabla'_x N_2, N_1), \qquad x \in \mathfrak{X}M^{n-1}.$$

Using that $g(\nabla'_x N_i, N_i) = 0$, $i = 1,2$, the Weingarten formulas and equalities (5), we obtain

$$\begin{aligned} \nabla'_x N_1 &= -b_1\omega(x)\,B_1 - b_1\beta_1(x)\,W - p\,\omega(x)\,W - \theta(x)\,N_2 \\ \nabla'_x N_2 &= -b_2\omega(x)\,B_2 - b_2\beta_2(x)\,W - q\,\omega(x)\,W + \theta(x)\,N_1. \end{aligned} \qquad (6)$$

The developable surfaces of codimension two in $\mathbb{E}^{n+1}$ are characterized [6] by

Lemma 1. *Let M^{n-1} be a surface in $\mathbb{E}^{n+1}$ with normal frame field $\{N_1, N_2\}$. Then, M^{n-1} is locally a developable surface of codimension two if and only if*

$$\begin{aligned} \nabla'_x N_1 &= -p\,\omega(x)W - \mu\omega(x)N_2, \qquad x \in \mathfrak{X}M^{n-1} \\ \nabla'_x N_2 &= -q\,\omega(x)W + \mu\omega(x)N_1 \end{aligned} \qquad (7)$$

where μ, p and q are functions on M^{n-1}, such that $p^2 + q^2 > 0$.

Remark. The planes $\mathbb{E}^{n-1}$ of codimension two can be considered as trivial developable surfaces of codimension two, for which $p = q = 0$ [5].

Now, let M^{n-1} be a developable surface of codimension two with normal frame field $\{N_1, N_2\}$, satisfying (7). If $\{\overline{N}_1, \overline{N}_2\}$ is another normal frame field of M^{n-1}, such that

$$\overline{N}_1 = \cos\varphi\, N_1 + \sin\varphi\, N_2, \quad \overline{N}_2 = -\sin\varphi\, N_1 + \cos\varphi\, N_2, \quad \varphi = \angle(N_1, \overline{N}_1)$$

then

$$\begin{aligned} \nabla'_x \overline{N}_1 &= -\overline{p}\,\omega(x)W - \overline{\mu}\omega(x)\overline{N}_2, \qquad x \in \mathfrak{X}M^{n-1} \\ \nabla'_x \overline{N}_2 &= -\overline{q}\,\omega(x)W + \overline{\mu}\omega(x)\overline{N}_1 \end{aligned}$$

where

$$\overline{p} = p\cos\varphi + q\sin\varphi, \qquad \overline{q} = -p\sin\varphi + q\cos\varphi, \qquad \overline{\mu} = \mu - \mathrm{d}\varphi(W).$$

The last equalities imply

$$\overline{\mu} - \mathrm{d}\arctan\frac{\overline{q}}{\overline{p}}(W) = \mu - \mathrm{d}\arctan\frac{q}{p}(W).$$

Consequently, the function $\kappa = \mu - \mathrm{d}\arctan\dfrac{q}{p}(W)$ does not depend on the choice of the normal frame field $\{N_1, N_2\}$ of M^{n-1}.

We call a developable surface M^{n-1} in $\mathbb{E}^{n+1}$ **planar**, if there exists a hyperplane $\mathbb{E}^n$ in $\mathbb{E}^{n+1}$, such that M^{n-1} lies in $\mathbb{E}^n$. The planar developable surfaces of codimension two are studied in [5] under the name torses of codimension two and are characterized as follows:

Lemma 2. *A developable surface of codimension two is planar iff* $\kappa = 0$.

It is easily seen that for each developable surface M^{n-1} of codimension two there exists locally a normal frame field $\{l_1, l_2\}$, with respect to which the equalities (7) take the form

$$\nabla'_x l_1 = -\nu_1\,\omega(x)W, \qquad \nabla'_x l_2 = -\nu_2\,\omega(x)W, \qquad x \in \mathfrak{X}M^{n-1}$$

where ν_1 and ν_2 are functions on M^{n-1}. Such a normal frame field is called a **canonical normal frame field** of M^{n-1}. It is determined up to a constant orthogonal matrix.

Now we shall consider non-developable ruled surfaces of codimension two. Let $M^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ be such a surface. If $\{N_1, N_2\}$ is an arbitrary normal frame field of M^{n-1}, then the equalities (6) hold good. As a generalization of the notion developable surface of codimension two we give the following

Definition 1. A ruled surface $M^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ in $\mathbb{E}^{n+1}$ is called **semi-developable**, if there exists a unit normal vector field N of M^{n-1}, which is constant along each fixed generator $\mathbb{E}^{n-2}(s)$, i.e., $\nabla'_{x_0} N = 0$, $x_0 \in \Delta_0$.

We shall prove the following

Proposition 2. *Let M^{n-1} be a non-developable ruled surface in $\mathbb{E}^{n+1}$ with normal frame field $\{N_1, N_2\}$. Then, M^{n-1} is semi-developable if and only if*

$$\theta(x_0) = \varepsilon\, \mathrm{d} \arctan \frac{b_2}{b_1}(x_0), \quad x_0 \in \Delta_0, \qquad b_1^2 + b_2^2 \neq 0 \tag{8}$$

where $\varepsilon = \pm 1$.

Proof: 1) Let M^{n-1} be a semi-developable surface of codimension two with normal vector field $N = \cos\varphi\, N_1 + \sin\varphi\, N_2$, $\varphi = \angle(N_1, N)$, which is constant along each generator and $N^{\perp}$ be the normal vector field on M^{n-1}, defined by $N^{\perp} = -\sin\varphi\, N_1 + \cos\varphi\, N_2$. Then the equalities (6) imply

$$\begin{aligned}
\nabla'_x N = & -b_1 \cos\varphi\omega(x) B_1 - b_2 \sin\varphi\omega(x) B_2 \\
& - (b_1 \cos\varphi\beta_1(x) + b_2 \sin\varphi\beta_2(x))W \\
& - (p\cos\varphi + q\sin\varphi)\,\omega(x)W - (\theta(x) - \mathrm{d}\varphi(x))N^{\perp} \\
\nabla'_x N^{\perp} = & \; b_1 \sin\varphi\omega(x) B_1 - b_2 \cos\varphi\omega(x) B_2 \\
& + (b_1 \sin\varphi\beta_1(x) - b_2 \cos\varphi\beta_2(x))W \\
& + (p\sin\varphi - q\cos\varphi)\,\omega(x)W + (\theta(x) - \mathrm{d}\varphi(x))N.
\end{aligned}$$

Using that $\nabla'_{x_0} N = 0$, $x_0 \in \Delta_0$, from the last equalities we get

$$b_1 \cos\varphi\beta_1(x_0) + b_2 \sin\varphi\beta_2(x_0) = 0, \quad \theta(x_0) - \mathrm{d}\varphi(x_0) = 0, \quad x_0 \in \Delta_0. \tag{9}$$

If we assume that $b_1 = b_2 = 0$, then the second equality of (9) implies $\theta(x) - \mathrm{d}\varphi(x) = \mu\omega(x)$, where $\mu = \theta(W) - \mathrm{d}\varphi(W)$. So, we obtain

$$\begin{aligned}
\nabla'_x N \; &= -\overline{p}\,\omega(x)W - \mu\omega(x)N^{\perp}, \qquad x \in \mathfrak{X}M^{n-1} \\
\nabla'_x N^{\perp} &= -\overline{q}\,\omega(x)W + \mu\omega(x)N
\end{aligned}$$

where $\overline{p} = p\cos\varphi + q\sin\varphi$, $\overline{q} = -p\sin\varphi + q\cos\varphi$. Then, according to Lemma 1, M^{n-1} is locally a developable surface of codimension two, which contradicts the condition that M^{n-1} is non-developable. Hence, $b_1^2 + b_2^2 \neq 0$.

In the case when $b_2 = 0$ (or $b_1 = 0$) the equalities (9) imply

$$\cos\varphi = 0 \quad (\text{or } \sin\varphi = 0), \qquad \theta(x_0) = 0, \quad x_0 \in \Delta_0.$$

Hence, the conditions (8) are fulfilled.

In the case when $b_1 b_2 \neq 0$, the first equality of (9) implies

$$\beta_1(x_0) = -\frac{b_2}{b_1} \tan\varphi\, \beta_2(x_0)$$

which shows that the one-forms β_1 and β_2 are collinear. Since β_1 and β_2 are unit one-forms, then $\beta_1 = \varepsilon\beta_2$, where $\varepsilon = \pm 1$. Hence, $\tan\varphi = -\varepsilon\, b_1/b_2$. Using the second equality of (9) we get (8).

2) Let M^{n-1} be a non-developable ruled surface with normal frame field $\{N_1, N_2\}$, satisfying (6), such that the conditions (8) hold good. For the curvature tensor R' of ∇' from (6) we calculate

$$\begin{aligned} R'(x,y,N_1,N_2) = & -\mathrm{d}\theta(x,y) + b_1\, b_2(\beta_1 \wedge \beta_2)(x,y) \\ & + b_1\, q(\beta_1 \wedge \omega)(x,y) - b_2\, p(\beta_2 \wedge \omega)(x,y) \end{aligned} \tag{10}$$

where $x, y \in \mathfrak{X}M^{n-1}$. Using that $R' = 0$ and $\mathrm{d}\theta(x_0, y_0) = 0$, $x_0, y_0 \in \Delta_0$, from (10) we obtain

$$b_1\, b_2(\beta_1 \wedge \beta_2)(x_0, y_0) = 0, \qquad x_0, y_0 \in \Delta_0. \tag{11}$$

In the case when $b_2 = 0$ (or $b_1 = 0$) we get from (8) that $\theta(x_0) = 0$, $x_0 \in \Delta_0$. Hence, $\theta = \theta(W)\omega$. Denoting $\mu = \theta(W)$, we obtain from (6)

$$\nabla'_x N_2 = -q\,\omega(x) W + \mu\omega(x) N_1 \quad (\text{or } \nabla'_x N_1 = -p\,\omega(x) W - \mu\omega(x) N_2)$$

which implies $\nabla'_{x_0} N_2 = 0$ (or $\nabla'_{x_0} N_1 = 0$), $x_0 \in \Delta_0$. Consequently, M^{n-1} is a semi-developable surface.

In the case when $b_1\, b_2 \neq 0$, the equality (11) implies $(\beta_1 \wedge \beta_2)(x_0, y_0) = 0$, $x_0, y_0 \in \Delta_0$, which shows that $\beta_1 = \varepsilon \beta_2$ ($\varepsilon = \pm 1$). Setting $\varphi = -\varepsilon \arctan \dfrac{b_1}{b_2}$ and considering the normal frame field $\{N, N^\perp\}$ of M^{n-1}, defined by

$$N = \cos\varphi\, N_1 + \sin\varphi\, N_2, \qquad N^\perp = -\sin\varphi\, N_1 + \cos\varphi\, N_2$$

we get the formulas

$$\begin{aligned} \nabla'_x N &= -\overline{p}\,\omega(x)\, W - \mu\omega(x)\, N^\perp \\ \nabla'_x N^\perp &= -b\,\omega(x)\, B - b\,\beta(x)\, W - \overline{q}\,\omega(x)\, W + \mu\omega(x)\, N \end{aligned}$$

where $\beta = \beta_1 = \varepsilon\beta_2$, $B = B_1 = \varepsilon B_2$, $\overline{p} = p\,\cos\varphi + q\,\sin\varphi$, $\overline{q} = -p\,\sin\varphi + q\,\cos\varphi$, $b^2 = b_1^2 + b_2^2$, $\mu = \theta(W) - \mathrm{d}\varphi(W)$. Consequently, $\nabla'_{x_0} N = 0$, $x_0 \in \Delta_0$, i.e., M^{n-1} is a semi-developable surface. □

In the process of proving Proposition 2 we have obtained that for each semi-developable surface M^{n-1} there exists a normal frame field $\{N, N^\perp\}$, such that

$$\begin{aligned} \nabla'_x N &= -p\,\omega(x)\, W - \mu\omega(x)\, N^\perp \\ \nabla'_x N^\perp &= -b\,\omega(x)\, B - b\,\beta(x)\, W - q\,\omega(x)\, W + \mu\omega(x)\, N. \end{aligned} \tag{12}$$

We call such a normal frame field a **canonical normal frame field** of the semi-developable surface M^{n-1} and the normal vector field N we call **main normal vector field** of M^{n-1}. The main normal vector field N is determined up to a sign.

A semi-developable surface M^{n-1} in $\mathbb{E}^{n+1}$ is said to be **planar**, if there exists a hyperplane $\mathbb{E}^n$ in $\mathbb{E}^{n+1}$, such that M^{n-1} lies in $\mathbb{E}^n$. It is obvious, that if M^{n-1} is a planar semi-developable surface, lying in a hyperplane $\mathbb{E}^n \subset \mathbb{E}^{n+1}$ with normal

N, then N is the main normal vector field of M^{n-1}. The planar semi-developable surfaces of codimension two are characterized by

Lemma 3. *Let M^{n-1} be a semi-developable ruled surface in $\mathbb{E}^{n+1}$ with a canonical normal frame field $\{N, N^{\perp}\}$ satisfying* (12). *Then, M^{n-1} is planar iff $p = 0$.*

Proof: 1) Let M^{n-1} be a planar semi-developable surface, lying in a hyperplane $\mathbb{E}^n$ with normal vector field N. Then, $\nabla'_x N = 0$, $x \in \mathfrak{X}M^{n-1}$ and the first equality of (12) implies that $p = 0$, $\mu = 0$.
2) Let M^{n-1} be a semi-developable surface with normal frame field $\{N, N^{\perp}\}$ and $p = 0$. Then, the equalities (12) imply

$$R'(x, y, N, W) = b\,\mu(\beta \wedge \omega)(x, y), \qquad x, y \in \mathfrak{X}M^{n-1}.$$

Using that $R' = 0$, we get

$$b\,\mu(\beta \wedge \omega)(x, y) = 0, \qquad x, y \in \mathfrak{X}M^{n-1}.$$

Since M^{n-1} is non-developable, then $b \neq 0$ and $\beta \wedge \omega \neq 0$. So, the last equality gives $\mu = 0$. Hence, $\nabla'_x N = 0$, $x \in \mathfrak{X}M^{n-1}$, i.e., M^{n-1} is planar. □

The non-planar semi-developable surfaces of codimension two are characterized by

Lemma 4. *Let M^{n-1} be a surface in $\mathbb{E}^{n+1}$ with normal frame field $\{N_1, N_2\}$. Then, M^{n-1} is locally a non-planar semi-developable surface with canonical normal frame field $\{N_1, N_2\}$ if and only if*

$$\begin{aligned} \nabla'_x N_1 &= -p\,\omega(x)\,W - \mu\,\omega(x)\,N_2 \\ \nabla'_x N_2 &= -b\,\omega(x)\,B - b\,\beta(x) W - q\,\omega(x) W + \mu\omega(x) N_1 \end{aligned} \tag{13}$$

where b, p, q and μ are functions on M^{n-1}, such that $b \neq 0$, $p \neq 0$.

Proof: 1) It is obvious from the considerations above that for a non-planar semi-developable surface M^{n-1} with canonical normal frame field $\{N_1, N_2\}$ the formulas (13) hold true and $b \neq 0$, $p \neq 0$.
2) Let M^{n-1} be a surface in $\mathbb{E}^{n+1}$ with normal frame field $\{N_1, N_2\}$, satisfying (13) and $b \neq 0$, $p \neq 0$. Using that $R'(x, y, N_1) = 0$, from (13) we get

$$\begin{aligned} & p\,\omega(x)\nabla'_y W - p\,\omega(y)\nabla'_x W \\ & \qquad + (b\,\mu(\beta \wedge \omega)(x, y) - \mathrm{d}(p\,\omega)(x, y))W - \mathrm{d}(\mu\omega)(x, y)\,N_2 = 0 \end{aligned} \tag{14}$$

which implies that $\mathrm{d}\omega(x_0, y_0) = 0$, $x_0, y_0 \in \Delta_0$. Hence, the distribution Δ_0 is involutive. Consequently, for each point $p \in M^{n-1}$ there exists a unique maximal integral submanifold S_p^{n-2} of Δ_0 containing p. With the help of formulas (13) and (14) we obtain

$$g(\nabla'_{x_0} y_0, N_1) = 0, \quad g(\nabla'_{x_0} y_0, N_2) = 0, \quad g(\nabla'_{x_0} y_0, W) = 0, \quad x_0, y_0 \in \Delta_0$$

which imply that $\nabla'_{x_0} y_0 \in \Delta_0$ for all $x_0, y_0 \in \Delta_0$, i.e., the integral submanifold S_p^{n-2} of Δ_0 is totally geodesic. So, S_p^{n-2} lies on an $(n-2)$-dimensional plane $\mathbb{E}_p^{n-2}$. Hence, M^{n-1} is locally a one-parameter system $\{\mathbb{E}^{n-2}(s)\},\ s \in J$ of planes of codimension three, i.e., M^{n-1} is locally a ruled surface of codimension two. More over, the first equality of (13) implies $\nabla'_{x_0} N_1 = 0,\ x_0 \in \Delta_0$. Hence, locally M^{n-1} is a semi-developable surface with main normal vector field N_1. $\square$

4. Structure Theorem

In what follows we consider a hypersuface M^n of conullity two with unit normal vector field N (N is determined up to a sign). The second fundamental form h of M^n satisfies (1). The $(n-2)$-dimensional distribution, determined by the one-forms η_1 and η_2, is denoted by Δ_0. Applying the Codazzi equations for a hypersurface with second fundamental form defined by (1), we obtain the equalities

$$\begin{array}{ll} 1) & \nabla_{x_0}\xi_1 = -\gamma(x_0)\,\xi_2 \\ 2) & \nabla_{x_0}\xi_2 = \gamma(x_0)\,\xi_1 \\ 3) & g(\nabla_{\xi_1}\xi_1, x_0) = \mathrm{d}\ln\nu_1(x_0) \\ 4) & g(\nabla_{\xi_2}\xi_2, x_0) = \mathrm{d}\ln\nu_2(x_0) \\ 5) & g(\nabla_{\xi_1}\xi_2, x_0) = \dfrac{\nu_2-\nu_1}{\nu_2}\gamma(x_0) \\ 6) & g(\nabla_{\xi_2}\xi_1, x_0) = \dfrac{\nu_2-\nu_1}{\nu_1}\gamma(x_0) \\ 7) & (\nu_1-\nu_2)^2 g(\nabla_{\xi_1}\xi_1, \xi_2) = (\nu_1-\nu_2)\,\mathrm{d}\nu_1(\xi_2) \\ 8) & (\nu_1-\nu_2)^2 g(\nabla_{\xi_2}\xi_2, \xi_1) = -(\nu_1-\nu_2)\,\mathrm{d}\nu_2(\xi_1) \end{array} \tag{15}$$

where γ is a one-form on Δ_0, defined by $\gamma(x_0) = g(\nabla_{x_0}\xi_2, \xi_1),\ x_0 \in \Delta_0$.
We denote by Δ_{ξ_1} and Δ_{ξ_2} the $(n-1)$-dimensional distributions, orthogonal to the vector fields ξ_1 and ξ_2, respectively, i.e.,

$$\Delta_{\xi_1}(p) = \{x \in T_pM^n;\ \eta_1(x) = 0\} = \Delta_0 \oplus \mathrm{span}\{\xi_2\}, \quad p \in M^n$$
$$\Delta_{\xi_2}(p) = \{x \in T_pM^n;\ \eta_2(x) = 0\} = \Delta_0 \oplus \mathrm{span}\{\xi_1\}, \quad p \in M^n.$$

In general, Δ_{ξ_1} and Δ_{ξ_2} are not involutive. We shall find all involutive $(n-1)$-dimensional distributions of M^n containing Δ_0. An arbitrary unit vector field $\xi \in \Delta_0^\perp$ is decomposed in the form $\xi = \cos\varphi\,\xi_1 + \sin\varphi\,\xi_2$, where $\varphi = \angle(\xi_1, \xi)$. Let $\xi^\perp$ denotes the unit vector field in $\Delta_0^\perp$, orthogonal to ξ, i.e., $\xi^\perp = -\sin\varphi\,\xi_1 + \cos\varphi\,\xi_2$. Then, the distribution Δ_ξ, orthogonal to ξ, is presented by

$$\Delta_\xi(p) = \{x \in T_pM^n;\ g(\xi, x) = 0\} = \Delta_0 \oplus \mathrm{span}\{\xi^\perp\}, \quad p \in M^n.$$

Proposition 3. *Let M^n be a hypersurface of conullity two in $\mathbb{E}^{n+1}$ with principal directions ξ_1, ξ_2 and $\xi = \cos\varphi\,\xi_1 + \sin\varphi\,\xi_2$ be a vector field in $\Delta_0^\perp$. Then, the* *distribution Δ_ξ, which is orthogonal to ξ, is involutive if and only if the function φ satisfies*

$$\mathrm{d}\varphi(x_0) = \left(\frac{1}{k}\cos^2\varphi + k\sin^2\varphi\right)\gamma(x_0) - \frac{1}{k}\sin\varphi\cos\varphi\,\mathrm{d}k(x_0) \tag{16}$$

where $x_0 \in \Delta_0$ and $k = \dfrac{\nu_1}{\nu_2}$.

Proof: Since Δ_0 is an involutive distribution ($[x_0, y_0] \in \Delta_0$ for all $x_0, y_0 \in \Delta_0$), then the distribution Δ_ξ is involutive if and only if $[x_0, \xi^\perp] \in \Delta_\xi$ for all $x_0 \in \Delta_0$, i.e.,

$$g(\nabla'_{x_0}\xi^\perp, \xi) - g(\nabla'_{\xi^\perp}x_0, \xi) = 0, \quad x_0 \in \Delta_0$$

or equivalently

$$g(\nabla_{x_0}\xi^\perp, \xi) + g(\nabla_{\xi^\perp}\xi, x_0) = 0, \quad x_0 \in \Delta_0.$$

The vector fields $\nabla_{x_0}\xi^\perp$ and $\nabla_{\xi^\perp}\xi$ can be expressed as follows

$$\begin{aligned}\nabla_{x_0}\xi^\perp &= (\gamma(x_0) - \mathrm{d}\varphi(x_0))\,\xi\\ \nabla_{\xi^\perp}\xi &= \sin\varphi\cos\varphi\,(\nabla_{\xi_2}\xi_2 - \nabla_{\xi_1}\xi_1) - \sin^2\varphi\,\nabla_{\xi_1}\xi_2 + \cos^2\varphi\,\nabla_{\xi_2}\xi_1\\ &\quad + (\cos\varphi\,\mathrm{d}\varphi(\xi_2) - \sin\varphi\,\mathrm{d}\varphi(\xi_1))\,\xi^\perp.\end{aligned}$$

Using the last equalities and (15) we obtain that Δ_ξ is an involutive distribution if and only if φ satisfies (16). □

As a corollary we obtain

Corollary 1. *Let M^n be a hypersurface of conullity two in $\mathbb{E}^{n+1}$ with principal directions ξ_1 and ξ_2. The distributions Δ_{ξ_1} and Δ_{ξ_2} are involutive if and only if $\gamma = 0$.*

Since the distribution Δ_0 of M^n is involutive, then locally there exist parameters $u, v, w^1, \ldots, w^{n-2}$ on M^n, such that $\Delta_0 = \mathrm{span}\left\{\dfrac{\partial}{\partial w^\alpha}\right\}_{\alpha=1,\ldots,n-2}$. We denote

$$\varphi_\alpha = \mathrm{d}\varphi\left(\frac{\partial}{\partial w^\alpha}\right), \quad k_\alpha = \mathrm{d}k\left(\frac{\partial}{\partial w^\alpha}\right), \quad \gamma_\alpha = \gamma\left(\frac{\partial}{\partial w^\alpha}\right).$$

So, the equalities (16) can be written in the form

$$\varphi_\alpha = \left(\frac{1}{k}\cos^2\varphi + k\sin^2\varphi\right)\gamma_\alpha - \frac{1}{k}\sin\varphi\cos\varphi\,k_\alpha, \quad \alpha = 1, \ldots, n-2$$

or equivalently

$$\varphi_\alpha = \frac{1-k^2}{2k}\gamma_\alpha\cos 2\varphi - \frac{k_\alpha}{2k}\sin 2\varphi + \frac{1+k^2}{2k}\gamma_\alpha, \quad \alpha = 1, \ldots, n-2. \tag{17}$$

Setting $a_\alpha = \dfrac{1-k^2}{2k}\gamma_\alpha, \;\; b_\alpha = -\dfrac{k_\alpha}{2k}, \;\; c_\alpha = \dfrac{1+k^2}{2k}\gamma_\alpha$, we rewrite (17) in the form

$$\varphi_\alpha = a_\alpha \cos 2\varphi + b_\alpha \sin 2\varphi + c_\alpha, \qquad \alpha = 1, \dots, n-2. \tag{18}$$

Now, if we fix (u, v) the equalities (18) can be considered as a system of partial differential equations for the unknown function $\varphi(w^1, \dots, w^{n-2}, u, v)$, where u and v are parameters. We shall prove that the integrability conditions

$$\varphi_{\alpha\beta} = \varphi_{\beta\alpha}, \qquad \alpha, \beta = 1, \dots, n-2$$

for the system (18) are fulfilled.

It is easy to calculate that

$$\begin{aligned} \varphi_{\alpha\beta} - \varphi_{\beta\alpha} = & \,(a_{\alpha\beta} - a_{\beta\alpha} + 2b_\alpha c_\beta - 2b_\beta c_\alpha)\cos 2\varphi \\ & + (b_{\alpha\beta} - b_{\beta\alpha} + 2c_\alpha a_\beta - 2c_\beta a_\alpha)\sin 2\varphi \\ & + (c_{\alpha\beta} - c_{\beta\alpha} + 2b_\alpha a_\beta - 2b_\beta a_\alpha) \end{aligned}$$

where

$$\begin{aligned} a_{\alpha\beta} &= \frac{\partial}{\partial w^\beta}\left(\frac{1-k^2}{2k}\gamma_\alpha\right) = -\frac{k^2+1}{2k^2}k_\beta\,\gamma_\alpha + \frac{1-k^2}{2k}\gamma_{\alpha\beta} \\ b_{\alpha\beta} &= \frac{\partial}{\partial w^\beta}\left(-\frac{k_\alpha}{2k}\right) = -\frac{1}{2}\frac{\partial}{\partial w^\beta}(\ln k)_\alpha = -\frac{1}{2}\frac{\partial}{\partial w^\alpha}(\ln k)_\beta = b_{\beta\alpha} \\ c_{\alpha\beta} &= \frac{\partial}{\partial w^\beta}\left(\frac{1+k^2}{2k}\gamma_\alpha\right) = \frac{k^2-1}{2k^2}k_\beta\,\gamma_\alpha + \frac{1+k^2}{2k}\gamma_{\alpha\beta} \\ \gamma_{\alpha\beta} &= \frac{\partial}{\partial w^\beta}(\gamma_\alpha). \end{aligned} \tag{19}$$

Using the fact that the one-form γ is closed and taking into account (19) we calculate

$$\varphi_{\alpha\beta} - \varphi_{\beta\alpha} = 0, \qquad \alpha, \beta = 1, \dots, n-2.$$

So, the integrability conditions for the system (18) are fulfilled. Hence, if $\varphi_0(u, v)$ is a given function, then there exists a unique solution $\varphi(w^1, \dots, w^{n-2}, u, v)$ of (18), defined for each $|w^\alpha| < \varepsilon, \; (u, v) \in D_0$, where $\varepsilon > 0, \; D_0 \subset \mathbb{R}^2$, satisfying $\varphi(0, \dots, 0, u, v) = \varphi_0(u, v)$.

Consequently, locally there exists a vector field $\xi = \cos\varphi\,\xi_1 + \sin\varphi\,\xi_2$, whose orthogonal distribution Δ_ξ is involutive. The integral submanifolds of Δ_ξ determine M^n locally as a one-parameter system of $(n-1)$-dimensional surfaces, i.e., locally M^n is a foliation of surfaces of codimension two. Moreover, each function $\varphi_0(u, v)$ determines an involutive distribution Δ_ξ of M^n, which generates the corresponding foliation of M^n.

Now we shall prove that the integral submanifolds M_ξ^{n-1} of each involutive distribution Δ_ξ of M^n are developable or semi-developable surfaces of codimension two.

Theorem 1 (Structure theorem). *Each hypersurface of conullity two in $\mathbb{E}^{n+1}$ is locally a foliation (one-parameter system) of developable or semi-developable surfaces of codimension two.*

Proof: First we shall consider the case $\nu_1 = \nu_2$. Setting $\nu := \nu_1 = \nu_2$ the equalities (15) imply

$$\begin{array}{ll} 1)\ \nabla_{x_0}\xi_1 = -\gamma(x_0)\,\xi_2 & 4)\ g(\nabla_{\xi_2}\xi_2, x_0) = \mathrm{d}\ln\nu(x_0) \\ 2)\ \nabla_{x_0}\xi_2 = \gamma(x_0)\,\xi_1 & 5)\ g(\nabla_{\xi_1}\xi_2, x_0) = 0 \\ 3)\ g(\nabla_{\xi_1}\xi_1, x_0) = \mathrm{d}\ln\nu(x_0) & 6)\ g(\nabla_{\xi_2}\xi_1, x_0) = 0. \end{array} \tag{20}$$

We denote $p = g(\nabla_{\xi_1}\xi_1, \xi_2),\ q = g(\nabla_{\xi_2}\xi_2, \xi_1)$. Applying Proposition 3, we get that the vector field $\xi = \cos\varphi\,\xi_1 + \sin\varphi\,\xi_2$ determines an involutive distribution Δ_ξ if and only if $\varphi_\alpha = \gamma_\alpha,\ \alpha = 1, \dots, n-2$, i.e., $\mathrm{d}\varphi(x_0) = \gamma(x_0),\ x_0 \in \Delta_0$. Each integral submanifold M_ξ^{n-1} of Δ_ξ is an $(n-1)$-dimensional surface with normal frame field $\{N, \xi\}$. We denote $\xi^\perp = -\sin\varphi\,\xi_1 + \cos\varphi\,\xi_2$ and $\omega = -\sin\varphi\,\eta_1 + \cos\varphi\,\eta_2$. Using (20) we get

$$\nabla'_x N = -\nu\,\omega(x)\,\xi^\perp, \qquad \nabla'_x\xi = -\overline{q}\,\omega(x)\,\xi^\perp, \qquad x \in \mathfrak{X}M_\xi^{n-1}$$

where $\overline{q} = p\,\sin\varphi + q\,\cos\varphi + \sin\varphi\,\mathrm{d}\varphi(\xi_1) - \cos\varphi\,\mathrm{d}\varphi(\xi_2)$. Hence, according to Lemma 1, locally M_ξ^{n-1} is a developable surface of codimension two with canonical normal frame field $\{N, \xi\}$. Moreover, since $\nu \neq 0$, the surfaces M_ξ^{n-1} are non-trivial ($M_\xi^{n-1} \neq \mathbb{E}^{n-1}$).

Now let us consider the case $\nu_1 \neq \nu_2$. Let Δ_ξ be an involutive distribution of M^n, where $\xi = \cos\varphi\,\xi_1 + \sin\varphi\,\xi_2$. Its integral submanifolds M_ξ^{n-1} are $(n-1)$-dimensional surfaces in $\mathbb{E}^{n+1}$ with normal frame field $\{N, \xi\}$. Once again we denote $\xi^\perp = -\sin\varphi\,\xi_1 + \cos\varphi\,\xi_2$ and $\omega = -\sin\varphi\,\eta_1 + \cos\varphi\,\eta_2$. Then, $\Delta_\xi = \Delta_0 + \mathrm{span}\{\xi^\perp\}$. Let $\{e_1, \dots, e_{n-2}\}$ be a local orthonormal basis of Δ_0. Using (15) we obtain

$$\begin{aligned} \nabla'_{\xi_1}\xi_1 &= \mathrm{d}\ln\nu_1(e_\alpha)\,e_\alpha + \frac{\mathrm{d}\nu_1(\xi_2)}{\nu_1 - \nu_2}\,\xi_2 + \nu_1\,N \\ \nabla'_{\xi_2}\xi_2 &= \mathrm{d}\ln\nu_2(e_\alpha)\,e_\alpha - \frac{\mathrm{d}\nu_2(\xi_1)}{\nu_1 - \nu_2}\,\xi_1 + \nu_2\,N \\ \nabla'_{\xi_1}\xi_2 &= \frac{\nu_2 - \nu_1}{\nu_2}\,\gamma(e_\alpha)\,e_\alpha - \frac{\mathrm{d}\nu_1(\xi_2)}{\nu_1 - \nu_2}\,\xi_1 \\ \nabla'_{\xi_2}\xi_1 &= \frac{\nu_2 - \nu_1}{\nu_1}\,\gamma(e_\alpha)\,e_\alpha + \frac{\mathrm{d}\nu_2(\xi_1)}{\nu_1 - \nu_2}\,\xi_2. \end{aligned}$$

Further, we calculate

$$\begin{aligned}\nabla'_{\xi^\perp}\xi = {} & (\mathrm{d}\varphi(e_\alpha) - \gamma(e_\alpha))\, e_\alpha - (\sin\varphi\, \mathrm{d}\varphi(\xi_1) - \cos\varphi\, \mathrm{d}\varphi(\xi_2))\, \xi^\perp \\ & - \left(\sin\varphi\, \frac{\mathrm{d}\nu_1(\xi_2)}{\nu_1 - \nu_2} - \cos\varphi\, \frac{\mathrm{d}\nu_2(\xi_1)}{\nu_1 - \nu_2}\right) \xi^\perp + (\nu_2 - \nu_1) \sin\varphi \cos\varphi\, N.\end{aligned} \tag{21}$$

On the other hand, the equation (15) imply

$$\nabla'_{e_\alpha}\xi = (\mathrm{d}\varphi(e_\alpha) - \gamma(e_\alpha))\; \xi^\perp, \quad \alpha = 1, \dots, n-2. \tag{22}$$

We denote

$$\begin{gathered} q = \sin\varphi\, \mathrm{d}\varphi(\xi_1) - \cos\varphi\, \mathrm{d}\varphi(\xi_2) + \sin\varphi\, \frac{\mathrm{d}\nu_1(\xi_2)}{\nu_1 - \nu_2} - \cos\varphi\, \frac{\mathrm{d}\nu_2(\xi_1)}{\nu_1 - \nu_2} \\ \varphi_\alpha = \mathrm{d}\varphi(e_\alpha), \qquad \gamma_\alpha = \gamma(e_\alpha), \qquad \alpha = 1, \dots, n-2. \end{gathered}$$

Using the unique decomposition of an arbitrary vector field $x \in \Delta_\xi$ in the form

$$x = g(x, e_\alpha)\, e_\alpha + \omega(x)\, \xi^\perp$$

we obtain from (21) and (22)

$$\begin{aligned}\nabla'_x\xi = {} & (\varphi_\alpha - \gamma_\alpha)\, \omega(x)\, e_\alpha + g(x, e_\alpha)(\varphi_\alpha - \gamma_\alpha)\, \xi^\perp \\ & - q\, \omega(x)\, \xi^\perp + (\nu_2 - \nu_1) \sin\varphi \cos\varphi\, \omega(x)\, N.\end{aligned} \tag{23}$$

From the equality (1) we get

$$\nabla'_x N = -(\nu_1 \sin^2\varphi + \nu_2 \cos^2\varphi)\, \omega(x)\, \xi^\perp - (\nu_2 - \nu_1) \sin\varphi \cos\varphi\, \omega(x)\, \xi. \tag{24}$$

Let us denote $\overline{B} = (\gamma_\alpha - \varphi_\alpha)\, e_\alpha$ ($\overline{B} \in \Delta_0$) and let B be a unit vector field, such that $\overline{B} = bB,\ b = |\overline{B}|$. If β is the unit one-form, corresponding to B, then

$$b\beta(x) = bg(x, B) = (\gamma_\alpha - \varphi_\alpha) g(x, e_\alpha).$$

Hence, setting $p = \nu_1 \sin^2\varphi + \nu_2 \cos^2\varphi,\ \mu = (\nu_2 - \nu_1) \sin\varphi \cos\varphi$ and using (23) and (24), we get

$$\begin{aligned}\nabla'_x N &= -p\, \omega(x)\, \xi^\perp - \mu\omega(x)\, \xi \\ \nabla'_x\xi &= -b\, \omega(x)\, B - b\, \beta(x)\, \xi^\perp - q\, \omega(x)\, \xi^\perp + \mu\omega(x)\, N\end{aligned}$$

where $p \neq 0$. In general, when $b \neq 0$, Lemma 4 implies that locally M^{n-1}_ξ is a semi-developable surface of codimension two with main normal vector field N. In particular, when $b = 0$, according to Lemma 1 locally M^{n-1}_ξ is a developable surface of codimension two.

Consequently, locally M^n is a one-parameter system of developable or semi-developable surfaces of codimension two. □

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