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FLUX CONJECTURE ON SYMPLECTIC SUBMANIFOLDS

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Abstract. Let (M, ω) be a closed symplectic $2n$-dimensional manifold. According to the well-known result by Donaldson [5] there exist $2m$-dimensional symplectic submanifolds (V^{2m}, ω) of (M, ω), $1 \leq m \leq n-1$, with $(m-1)$-equivalent inclusions. In this paper, we have found a relation between the flux group and the kernel of the Lefschetz map. We have present also some properties of the flux groups for all symplectic $2m$-submanifolds (V^{2m}, ω) where $2 \leq m \leq n-1$.

1. Introduction

Let (M, ω) be a compact symplectic manifold and $\mathrm{Symp}_0(M)$ denote the identity component of the symplectomorphism group $\mathrm{Symp}(M)$ of (M, ω). Recall that the flux homomorphism

$$F_\omega : \pi_1(\mathrm{Symp}_0(M)) \to H^1(M, \mathbb{R})$$

can be defined as follows. For an element $\phi \in \pi_1(\mathrm{Symp}_0(M))$ and any homology class $\alpha \in H_1(M, \mathbb{R})$ set

$$(F_\omega(\phi), \alpha) = (\omega, \phi_t \alpha)$$

where $\phi_t \alpha$ denotes the trace of a loop α under the isotopy $\{\phi_t\}$ representing ϕ and $(\cdot, \cdot)$ is the natural pairing. It is well known that ϕ is represented by a Hamiltonian loop if and only if $F_\omega(\phi) = 0$. Define the flux group Γ_M of M by the image of the flux homomorphism, i.e.,

$$\Gamma_M = \mathrm{im}\{F_\omega : \pi_1(\mathrm{Symp}_0(M)) \to H^1(M, \mathbb{R})\} \subset H^1(M, \mathbb{R}).$$

The importance of this notion is due to the fact that the Hamiltonian diffeomorphism group $\mathrm{Ham}(M)$ is closed in $\mathrm{Symp}_0(M)$ if and only if Γ_M is a discrete subgroup of $H^1(M, \mathbb{R})$. The statement that Γ_M is discrete is known as the flux conjecture. Then we obtain a relation between the flux group Γ_M of M and

the kernel of the Lefschetz map of M, where the **Lefschetz map** Lef_M of M, $\mathrm{Lef}_M : H^1(M,\mathbb{R}) \xrightarrow{\cup[\omega]^{n-1}} H^{2n-1}(M,\mathbb{R})$, is defined by $\mathrm{Lef}_M(a) = a \cup \omega^{n-1}$.

Theorem 1.1. *Let (M,ω) be a compact symplectic manifold. If the Euler number $\chi(M)$ of M is not equal to zero, then the flux group Γ_M of M is included in the kernel of the Lefschetz map of M, i.e., $\Gamma_M \subset \ker(\mathrm{Lef}_M)$.*

A symplectic $2n$-manifold (M,ω) is called **Lefschetz** if the Lefschetz map is an isomorphism. In the Lefschetz case, Theorem 1.1 implies the following corollary.

Corollary 1.2. *Let (M,ω) be a compact symplectic Lefschetz manifold. If the Euler number $\chi(M)$ of M is not equal to zero, then the flux group Γ_M is trivial.*

Let (M,ω) be a compact symplectic manifold of dimension $2n$ with the cohomology class $[\omega]$ having a lift to an integral cohomology class h. Donaldson [5] proved the existence of some integer N_0 such that for any $N > N_0$ there exists a symplectic submanifold $V^{2(n-1)}$ of dimension $2(n-1)$ that realizes the Poincaré dual of Nh, that is,

$$\mathrm{PD}[V^{2(n-1)}] = Nh \in H^2(M).$$

Furthermore, such manifolds satisfy the Lefschetz theorem on hyperplane sections. This means that the inclusion $\iota : V^{2(n-1)} \to M$ is $(n-2)$-equivalent. By repeating this process, we get any even dimensional submanifold V^{2m}, $1 \leq m \leq n-1$, with $(m-1)$-equivalent inclusion $\iota : V^{2m} \to M$. These manifolds are called Donaldson submanifolds of M. Denote by $\mathrm{Lef}_{V^{2m}}$ the Lefschetz maps of Donaldson submanifolds V^{2m}, $2 \leq m \leq n-1$. Then we get relations between Lefschetz maps.

Theorem 1.3 ([4]). *Let (M^{2n},ω) be a compact symplectic manifold and (V^{2m},ω) be a Donaldson submanifold of (M,ω) for each m, $2 \leq m \leq n-1$. Then*

$$\ker(\mathrm{Lef}_M) = \ker(\mathrm{Lef}_{V^{2m}}).$$

Using this fact, we get the following Corollary when (M,ω) is a Lefschetz manifold.

Corollary 1.4. *Let (M,ω) be a compact symplectic Lefschetz manifold. Then the flux groups $\Gamma_{V^{2m}}$ of Donaldson submanifolds V^{2m} are discrete for all dimensions $2m$, $2 \leq m \leq n-1$.*

In the case when (M,ω) is closed, Lalonde, McDuff and Polterovich proved that the flux group Γ_M of M is discrete if the first Betti number $\beta_1(M)$ is equal to one. Then we can show that the flux groups $\Gamma_{V^{2m}}$ of Donaldson submanifolds V^{2m}, $2 \leq m \leq n-1$, are discrete under the same assumption as above, i.e., $\beta_1(M) = 1$.

Theorem 1.5. *Let (M,ω) be a closed symplectic manifold with the first Betti number equal to one. Then the flux conjecture holds for all Donaldson submanifolds V^{2m}, $2 \leq m \leq n-1$.*

A compact symplectic manifold (M,ω) is said to be **symplectically aspherical** if $\omega|_{\pi_2(M)} = 0$. In this case, we obtain that all Donaldson submanifolds V^{2m} of M are also symplectically aspherical.

Theorem 1.6. *Let (M,ω) be a compact, symplectically aspherical manifold. Then all Donaldson submanifolds (V^{2m},ω) of (M,ω), $2 \leq m \leq n-1$, are also symplectically aspherical.*

In the symplectically aspherical case, the flux group Γ_M is trivial if the center of the fundamental group of M is trivial [4]. Using this fact and the above theorem, we get the following result for Donaldson submanifolds V^{2m} of M.

Corollary 1.7. *Let (M,ω) be a compact, symplectically aspherical manifold. If the fundamental group of M has a trivial center, then all flux groups of Donaldson submanifolds (V^{2m},ω), $2 \leq m \leq n-1$, are trivial.*

For any class ϕ in $\pi_1(\mathrm{Symp}_0(M))$, we can construct a symplectic fibration over S^2 with fiber (M,ω) (see Section 4 for details). Then, the fibration over S^2 gives rise to the Wang exact sequences

$$\cdots \to H_{i+2}(P_\phi) \to H_i(M) \xrightarrow{\partial_\phi} H_{i+1}(M) \xrightarrow{\iota_*} H_{i+1}(P_\phi) \to \cdots$$

and

$$\cdots \to H^i(P_\phi) \xrightarrow{\iota^*} H^i(M) \xrightarrow{\partial_\phi^*} H^{i-1}(M) \to H^{i+1}(P_\phi) \to \cdots$$

where $\partial_\phi A = [\phi_t(A)]$, $\langle \partial_\phi^* a, A\rangle = \langle a, \partial_\phi A\rangle$ for $A \in H_i(M)$ and $a \in H^{i+1}(M)$. We say that a symplectic fibration is Hamiltonian if the corresponding loop of symplectomorphisms is homopotic to a Hamiltonian loop. Using the property $F_\omega(\phi) = 0$ if and only if $\phi \in \pi_1(\mathrm{Ham}(M))$, we can prove the following

Corollary 1.8. *Let (M,ω) satisfy one of the following conditions:*

i) *(M,ω) is a compact, symplectic Lefschetz manifold,*

ii) *(M,ω) is a compact, symplectically aspherical manifold with fundamental group having trivial center.*

Then every symplectic fibration $(M,\omega) \to P_\phi \to S^2$ induced by the element $\phi \in \pi_1(\mathrm{Symp}_0(M))$ is Hamiltonian. Furthermore, in the case ii)*, every symplectic fibration $(V^{2m},\omega) \to P_{\phi_{V^{2m}}} \to S^2$ induced by $\phi_{V^{2m}} \in \pi_1(\mathrm{Symp}_0(V^{2m}))$ is also Hamiltonian for all Donaldson submanifolds V^{2m}, $2 \leq m \leq m-1$.*

2. Relations Between the Flux Group and the Lefschetz Map

Let (M, ω) be a compact symplectic manifold. Recall that the Lefschetz map

$$\mathrm{Lef}_M : H^1(M, \mathbb{R}) \overset{\cup[\omega]^{n-1}}{\longrightarrow} H^{2n-1}(M, \mathbb{R})$$

defined as $\mathrm{Lef}_M[a] = [a] \cup [\omega]^{n-1}$ for a class $[a] \in H^1(M, \mathbb{R})$ we can define an evaluation map

$$\mathrm{ev} : \pi_1(\mathrm{Symp}_0(M)) \to \pi_1(M)$$

by $\mathrm{ev}(\phi) = \phi(x_0)$, for a base point $x_0 \in M$. By $\widetilde{\mathrm{ev}}$ we denote a homomorphism from $\pi_1(\mathrm{Symp}_0(M))$ to $H_1(M) = H_1(M, \mathbb{Z})/\mathrm{torsion}$ which is given by the composition of ev and the **Hurewicz homomorphism** $h : \pi_1(M) \to H_1(M)$, i.e.,

$$\widetilde{\mathrm{ev}} : \pi_1(\mathrm{Symp}_0(M)) \overset{\mathrm{ev}}{\longrightarrow} \pi_1(M) \overset{h}{\longrightarrow} H_1(M).$$

Lemma 2.1 ([9]). *The following diagram is commutative up to a positive constant*

$$\begin{array}{ccccc} \pi_1(\mathrm{Symp}_0(M)) & \xrightarrow{\widetilde{\mathrm{ev}}} & H_1(M) & \xrightarrow{PD} & H^{2n-1}(M) \\ \| & & & & \downarrow \\ \pi_1(\mathrm{Symp}_0(M)) & \xrightarrow{F_\omega} & H^1(M, \mathbb{R}) & \xrightarrow{\cup[\omega]^{n-1}} & H^{2n-1}(M, \mathbb{R}) \end{array}$$

where PD is the Poincaré duality.

Let M^M be the space of continuous maps from M to M. Then $\pi_1(M^M)$ goes to $\pi_1(M)$ under the evaluation map $\mathrm{ev} : \pi_1(M^M) \to \pi_1(M)$. Denote by $G(M)$ the image of $\pi_1(M^M)$ under the evaluation map. This $G(M)$ is known as **Gottlieb group** of M.

Theorem 2.2 ([7]). *Suppose X has the same homotopy type as a compact connected polyhedron. If the Euler number $\chi(M)$ is not equal to zero, then the Gottlieb group $G(M)$ is trivial.*

Proof of Theorem 1.1: For any $\phi = \{\phi_t\}$ in $\pi_1(\mathrm{Symp}_0(M))$, $\phi_t \in \mathrm{Symp}_0(M)$ is in M^M, for all $t \in [0, 1]$, since $\mathrm{Symp}_0(M) \subset M^M$. Then the image of $\pi_1(\mathrm{Symp}_0(M))$ under the evaluation map $\mathrm{ev}(\phi) = \mathrm{ev}(\{\phi_t(x_0)\})$ is included in the Gottlieb group $G(M)$. But $G(M)$ is trivial by Theorem 2.2. This means that $\widetilde{\mathrm{ev}} : \pi_1(\mathrm{Symp}_0(M)) \to H_1(M)$ has trivial image. Then, from the commutativity of the diagram in Lemma 2.1

$$\begin{array}{ccccc} \pi_1(\mathrm{Symp}_0(M)) & \xrightarrow{\widetilde{\mathrm{ev}}} & H_1(M) & \xrightarrow{PD} & H^{2n-1}(M) \\ \| & & & & \downarrow \\ \pi_1(\mathrm{Symp}_0(M)) & \xrightarrow{F_\omega} & H^1(M, \mathbb{R}) & \xrightarrow{\cup[\omega]^{n-1}} & H^{2n-1}(M, \mathbb{R}) \end{array}$$

any class a in the flux group $\Gamma_M = F_\omega(\pi_1(\mathrm{Symp}_0(M)))$ goes to the trivial element in $H^{2n-1}(M,\mathbb{R})$ by the Lefschetz map, $H^1(M,\mathbb{R}) \overset{\cup[\omega]^{n-1}}{\longrightarrow} H^{2n-1}(M,\mathbb{R})$. □

Now, we can prove Corollary 1.2.

Proof of Corollary 1.2: By Theorem 1.1, the flux group Γ_M is included in the kernel of the Lefschetz map Lef_M. That is, $\Gamma_M \subset \ker(\mathrm{Lef}_M)$. On the other hand, we know that the Lefschetz map Lef_M is an isomorphism since (M,ω) is a Lefschetz manifold. Therefore, the kernel of Lef_M is trivial and this completes the proof. □

3. On Symplectic Submanifolds

Let (M,ω) be a compact symplectic manifold. Donaldson proved the existence of symplectic manifolds of codimension two which can be realized by the Poincaré dual of Nh, for sufficiently large N, where h is the integral lift of ω.

Theorem 3.1 ([5]). *Let (M^{2n},ω) be a compact symplectic manifold and $h \in H^2(M)$ be an integral lift of $[\omega]$. Then for a large enough N the Poincaré dual of Nh, in $H_{2n-2}(M)$, can be realized by a symplectic submanifold V^{2n-2} of M^{2n}. Moreover, we can choose V^{2n-2} such that the inclusion $\iota : V^{2n-2} \to M^{2n}$ is an $(n-2)$-equivalence, i.e., the homomorphism $\iota_\sharp : \pi_k(V^{2n-2}) \to \pi_k(M)$ is an isomorphism for $k \leq n-2$ and an epimorphism for $k = n-1$.*

By repeating this process, we get for any even dimensional submanifold V^{2m}, $1 \leq m \leq n-1$, $(m-1)$-equivalent inclusions $\iota : V^{2m} \to M$. These manifolds are called **Donaldson submanifolds** of M. The following example shows that $\mathbb{C}P^m$, $0 \leq m \leq n-1$, are Donaldson submanifolds of $\mathbb{C}P^n$.

Example. Let $\mathbb{C}^n$ be the complex n-dimensional vector space with the standard symplectic form $\omega = \frac{i}{2}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$. Consider the Hamiltonian action of the circle S^1 on $\mathbb{C}^n$ given by $e^{i\theta}(z_1,\ldots,z_n) = (e^{i\theta}z_1,\ldots,e^{i\theta}z_n)$ with a moment map $H : \mathbb{C}^n \to \mathbb{R}$ given by $H(z_1,\ldots,z_n) = |z_1|^2+\cdots+|z_n|^2$. Then $1 \in \mathbb{R}$ is a regular value of H and $H^{-1}(1) = S^{2n-1}$ is a smooth submanifold of $\mathbb{C}^n$ preserved by the circle action. Moreover, the quotient $H^{-1}(1)/S^1 = \mathbb{C}P^{n-1}$ is just the symplectic reduction. The inverse image $D(1) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n;\, H(z_1,\ldots,z_n) \leq 1\}$ is decomposed into the open unit disc $D^0(1) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n;\, H(z_1,\ldots,z_n) < 1\}$ and the unit sphere $S^{2n-1} = \{(z_1,\ldots,z_n) \in \mathbb{C}^n;\, H(z_1,\ldots,z_n) = 1\}$, i.e., $D(1) = D^0(1) \cup S^{2n-1}$. By symplectic cutting, we have $\mathbb{C}P^n = D^0(1) \cup S^{2n-1}/S^1 \supset \mathbb{C}P^{n-1}$. Considering this process, we have also $\mathbb{C}P^n \supset \mathbb{C}P^{n-1} \supset \cdots \supset \mathbb{C}P^1 \supset \{\mathrm{pt.}\}$.

Recall that a symplectic manifold is called Lefschetz if the Lefschetz map Lef_M is an isomorphism. Then it follows from Lemma 2.1 that the flux groups are discrete for Lefschetz manifolds. Indeed, in the diagram in Lemma 2.1, the image of the up arrows is discrete in $H^{2n-1}(M,\mathbb{R})$. Then the flux group is discrete since $\cup[\omega]^{n-1}$ is an injective map. Thus, we can show that the flux conjecture holds for Donaldson submanifold V^{2m} if the Lefschetz map $\mathrm{Lef}_{V^{2m}}$ is injective.

Proof of Corollary 1.4: By Theorem 1.3, the kernel of $\mathrm{Lef}_{V^{2m}}$ is equal to the kernel of Lef_M for each m, $2 \leq m \leq n-1$. Since M is Lefschetz, the kernel of Lef_M is trivial and therefore the kernels of $\mathrm{Lef}_{V^{2m}}$ are also trivial. This means that $\mathrm{Lef}_{V^{2m}}$ are injective maps for all dimensions $2m$, $2 \leq m \leq n-1$. Then, by the above statement, all flux groups $\Gamma_{V^{2m}}$ are discrete. □

In the case when (M,ω) is closed, the following theorem is proved by Lalonde, McDuff and Polterovich.

Theorem 3.2 ([10]). *Let (M,ω) be a closed symplectic manifold with the first Betti number $\beta_1(M)$ equal to one. Then the flux conjecture holds for M.*

Donaldson submanifolds (V^{2m},ω), $2 \leq m \leq n-1$, are closed manifolds when (M,ω) is closed. Then we can obtain the same result as Theorem 3.2 for Donaldson submanifolds.

Lemma 3.3. *Let (M,ω) be a closed symplectic manifold and (V^{2m},ω) be Donaldson submanifolds for M, $2 \leq m \leq n-1$. Then the first Betti number of V^{2m} is equal to the first Betti number of M, i.e.,*

$$\beta_1(V^{2m}) = \beta_1(M)$$

for any m, $2 \leq m \leq n-1$.

Proof: For a given closed symplectic manifold (M,ω), we can consider Donaldson submanifolds (V^{2m},ω) which are closed and have $(m-1)$-equivalent inclusions $\iota : V^{2m} \to M$. That is, $\iota_\sharp : \pi_i(V^{2m}) \to \pi_i(M)$ is an isomorphism for $i \leq (m-1)$ and an epimorphism for $i = m$. Then, by Whitehead theorem, $\iota_* : H_1(V^{2m}) \to H_1(M)$ is an isomorphism for every m, $2 \leq m \leq n-1$. Thus we get that $\beta_1(V^{2m}) = \beta_1(M)$. □

Proof of Theorem 1.5: From the just proven Lemma 3.3, $\beta_1(V^{2m}) = \beta_1(M)$ for any m, $2 \leq m \leq n-1$. Applying Theorem 3.2 to our case, we obtain the result we want. □

Now we consider flux groups of symplectically aspherical manifolds. Recall that a symplectic manifold (M,ω) is called symplectically aspherical if the symplectic

form ω of M vanishes on spherical images, i.e., $\int \omega : H_2(M) \to \mathbb{R}$ vanishes on $H_2^S(M)$. This means that

$$\int_{S^2} f^*\omega = 0$$

for every map $f : S^2 \to M$. Then we can prove that every Donaldson submanifold is also symplectically aspherical.

Proof of Theorem 1.6: For any map $f : S^2 \to V^{2m}$, $2 \leq m \leq n-1$, we need to show that $\int_{S^2} f^*\omega = 0$. For a given f consider a map $f_M : S^2 \to M$ which is a composition of $f : S^2 \to V^{2m}$ with the inclusion $\iota : V^{2m} \to M$. That is, $f_M = \iota \circ f : S^2 \xrightarrow{f} V^{2m} \xrightarrow{\iota} M$. Then,

$$\int_{S^2} f^*\omega = \int_{f(S^2)} \omega = \int_{f(S^2)} \iota^*\omega = \int_{\iota \circ f(S^2)} \omega = \int_{f_M(S^2)} \omega.$$

But $\int_{f_M(S^2)} \omega = \int_{S^2} {f_M}^*\omega = 0$ since (M, ω) is symplectically aspherical. □

Theorem 3.4 (see [4]). *Let (M, ω) be compact and symplectically aspherical. If $\pi_1(M)$ has trivial center, then the flux group of M is trivial.*

Thus we can prove the triviality of the flux groups of Donaldson submanifolds of a symplectically aspherical manifold as stated in Corollary 1.7.

Proof of Corollary 1.7: Theorem 3.1 implies that the inclusions $\iota : V^{2m} \to M$ are $(m-1)$-equivalent. Thus $\iota_\sharp : \pi_1(V^{2m}) \to \pi_1(M)$ is an isomorphism for all m, $2 \leq m \leq n-1$. Then, by hypothesis, every center of fundamental group $Z(\pi_1(V^{2m}))$ is trivial and Donaldson submanifolds are symplectically aspherical by Theorem 1.6. Finally, after applying Theorem 3.4 to our case, the proof is completed. □

4. Relations with Symplectic Fibrations

There is a correspondence between loops in the group of symplectic diffeomorphisms and symplectic fibrations over S^2 with fiber (M, ω). By definition, a **symplectic fibration** is a such fibration for which the changes of trivialization preserve the given symplectic form ω on the fibers. In other words, the structure group of the fibration is $\mathrm{Symp}(M)$. The correspondence is given by assigning to each symplectic loop ϕ in $\mathrm{Symp}_0(M)$ the fibration

$$(M, \omega) \to P_\phi \to S^2$$

obtained by gluing a copy of $D_2^+ \times M$ with $D_2^- \times M$ along their boundaries in the following way

$$\phi : (2\pi t, x) \mapsto (-2\pi t, \phi_t(x)).$$

Here D_2 is the closed disc of radius one in the plane and

$$P_\phi = D_2^+ \times M \cup_\phi D_2^- \times M.$$

We always assume that the base S^2 is oriented with an orientation induced by D_2^+. Note that this correspondence can be reversed: given a symplectic fibration over the oriented 2-sphere with fiber M, one can reconstruct the homotopy class of ϕ. We can notice also that any given symplectic fibration P_ϕ is Hamiltonian if $F_\omega(\phi) = 0$, since $F_\omega(\phi) = 0$ if and only if $\phi \in \pi_1(\mathrm{Ham}(M))$.

Proof of Corollary 1.8: $F_\omega(\phi) = 0$ if and only if $\phi \in \pi_1(\mathrm{Ham}(M))$. Thus, every symplectic fibration $(M, \omega) \to P_\phi \to S^2$ induced by $\phi \in \pi_1(\mathrm{Symp}_0(M))$ is Hamiltonian if the flux group $\Gamma_M = \mathrm{im}\{F_\omega : \pi_1(\mathrm{Symp}_0(M)) \to H^1(M, \mathbb{R})\}$ is trivial. By Corollary 1.2, the flux group is trivial in the case i) and the flux group is trivial when (M, ω) satisfies the condition ii) by Theorem 3.4. Furthermore, in case ii) all flux groups $\Gamma_{V^{2m}}$ of Donaldson submanifolds are trivial by Corollary 1.7. □

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