Seventh International Conference on
Geometry, Integrability and Quantization
June 2–10, 2005, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2005, pp 249–264

THE RELATIVISTIC HYPERBOLIC PARALLELOGRAM LAW

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Abstract. A gyrovector is a hyperbolic vector. Gyrovectors are equivalence classes of directed gyrosegments that add according to the gyroparallelogram law just as vectors are equivalence classes of directed segments that add according to the parallelogram law. In the "gyrolanguage" of this paper one attaches the prefix "gyro" to a classical term to mean the analogous term in hyperbolic geometry. The prefix stems from Thomas gyration, which is the mathematical abstraction of the relativistic effect known as Thomas precession. Gyrolanguage turns out to be the language one needs to articulate novel analogies that the classical and the modern in this paper share. The aim of this article is to employ recent developments in analytic hyperbolic geometry for the presentation of the relativistic hyperbolic parallelogram law, and the relativistic particle aberration.

1. Introduction

Einstein noted in 1905 that

> "Das Gesetz vom Parallelogramm der Geschwindigkeiten gilt also nach unserer Theorie nur in erster Annäherung."
>
> A. Einstein [1]

[Thus the law of velocity parallelogram is valid according to our theory only to a first approximation.] The important "velocity parallelogram" notion that appears in Einstein's 1905 original paper [1] as "Parallelogramm der Geschwindigkeiten" does not appear in its English translation [2]. It can be found, however, in other English translations as, for instance, the translation by H. Lorentz, H. Weyl and H. Minkowski [6, pp. 37–65; p. 50].

About a century later the geometry underlying Einstein's observation on the approximate validity of the velocity parallelogram was uncovered in [18, 23].

Einsteinian velocities are regulated by hyperbolic geometry and its gyrovector space algebraic structure just as Newtonian velocities are regulated by Euclidean geometry and its vector space algebraic structure. Accordingly, Einsteinian velocities obey the gyroparallelogram addition law of gyrovectors just as Newtonian velocities obey the parallelogram addition law of vectors. Gyrovectors or, equivalently, hyperbolic vectors, are introduced in [23].

The gyroparallelogram law (24) of gyrovector addition, shown in Fig. 3, is analogous to the parallelogram law of vector addition in Euclidean geometry, and is given by the coaddition of gyrovectors. Remarkably, in order to capture analogies between parallelograms and gyroparallelograms, we must employ both the gyrocommutative operation $\oplus$ and the commutative cooperation $\boxplus$ of gyrovector spaces. Gyrovectors are thus equivalence classes of directed gyrosegments in an Einstein gyrovector space that add according to the gyroparallelogram law, Fig. 2, just like vectors, which are equivalence classes of directed segments that add according to the parallelogram law.

Along the analogies, a remarkable disanalogy emerges as well. Newtonian velocity addition and the parallelogram addition law of Newtonian velocities coincide. In contrast, Einstein velocity addition and the gyroparallelogram addition law of Einsteinian velocities do not coincide. The reason is clear: Einstein velocity addition is in general noncommutative, while the gyroparallelogram addition law is commutative.

Definition 1 (Einstein Addition in the Ball). *Let* $\mathbb{V} = (\mathbb{V}, +, \cdot)$ *be a real inner product space* [7] *and let* $\mathbb{V}_s$ *be the s-ball of* $\mathbb{V}$,

$$\mathbb{V}_s = \{\mathbf{v} \in \mathbb{V};\ \|\mathbf{v}\| < s\} \tag{1}$$

where $s > 0$ *is an arbitrarily fixed constant. Einstein addition* $\oplus$ *is a binary operation in* $\mathbb{V}_s$ *given by the equation* [23]

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{2}$$

satisfying the ***gamma identity***

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right) \tag{3}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$, *where* $\gamma_{\mathbf{u}}$ *is the gamma factor*

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}} \tag{4}$$

in $\mathbb{V}_s$, *and where* $\cdot$ *and* $\|\ \|$ *are the inner product and norm that the ball* $\mathbb{V}_s$ *inherits from its space* $\mathbb{V}$.

In the special case when $\mathbb{V} = \mathbb{R}^3$ is the Euclidean 3-space, $\mathbb{V}_s$ reduces to the ball of $\mathbb{R}^3_s$ of all relativistically admissible velocities, and Einstein addition in $\mathbb{R}^3_c$ turns out to be the special relativistic Einstein addition law of relativistically admissible velocities, where c is the vacuum speed of light. We naturally use the notation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$.

When the vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{V}_s$ of $\mathbb{V}$ are parallel in $\mathbb{V}$, $\mathbf{u} \parallel \mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, Einstein addition reduces to a commutative and associative binary operation between parallel velocities

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{1}{s^2}\|\mathbf{u}\|\|\mathbf{v}\|}, \qquad \mathbf{u} \parallel \mathbf{v}. \tag{5}$$

Seemingly structureless, Einstein addition (2) is neither commutative nor associative. It is therefore important to realize that Einstein addition possesses a useful structure similar to, but richer than, that of a the common vector space operation. As such it gives rise to the Einstein gyrovector spaces [14, 15, 17, 18, 19, 20, 23] linked to Lie gyrovector spaces [4] and differential geometry [22].

In order to capture analogies with groups, we must introduce into gyrogroups $(G, \oplus)$ a second operation $\boxplus$, called **cooperation**. It is a **coaddition** that shares useful duality symmetries with its gyrogroup addition $\oplus$ [18, 23].

Definition 2 (The Gyrogroup Cooperation (Coaddition)). *Let $(G, \oplus)$ be a gyrogroup with gyrogroup operation (or, addition) $\oplus$. The gyrogroup **cooperation** (or, **coaddition**) $\boxplus$ is a second binary operation in G given by the equation*

$$a \boxplus b = a\oplus\mathrm{gyr}[a, \ominus b]b \tag{6}$$

for all $a, b \in G$.

Naturally, we use the notation $a \boxminus b = a \boxplus (-b)$. The gyrogroup cooperation is commutative if and only if the gyrogroup operation is gyrocommutative [23, Theorem 3.4, p. 50].

The gyrogroup cooperation $\boxplus$ is expressed in (6) in terms of the gyrogroup operation $\oplus$ and gyrator gyr. It can be shown that, similarly, the gyrogroup operation $\oplus$ can be expressed in terms of the gyrogroup cooperation $\boxplus$ and gyrator gyr by the identity [23, Theorem 2.10, p. 28],

$$a\oplus b = a \boxplus \mathrm{gyr}[a, b]b \tag{7}$$

for all a, b in a gyrogroup $(G, \oplus)$. Identities (6) and (7) exhibit one of the duality symmetries that the gyrogroup operation and cooperation share.

First gyrogroup theorems are presented in [18, 23], where it is shown in particular that any gyrogroup possesses a unique identity (left and right) and each element of any gyrogroup possesses a unique inverse (left and right). Furthermore, any gyrogroup obeys the left cancellation law

$$\ominus a \oplus (a \oplus b) = b \tag{8}$$

and the two right cancellation laws

$$(b \oplus a) \boxminus a = b \tag{9}$$

$$(b \boxplus a) \ominus a = b \tag{10}$$

[23, p. 33]. Identities (9) and (10) present a duality symmetry between a gyrogroup operation and cooperation.

Finally, it follows from the left cancellation law and the left gyroassociative law that gyrations in a gyrogroup are uniquely determined by the gyrogroup operation

$$\mathrm{gyr}[a, b]x = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus x)\}. \tag{11}$$

Identity (11) is therefore called the **gyrator identity**.

2. Gyroangles

Definition 3 (Unit Gyrovectors). *Let* $\ominus\mathbf{a}\oplus\mathbf{b}$ *be a nonzero gyrovector in an Einstein gyrovector space* $(\mathbb{V}_s, \oplus, \otimes)$*. Its gyrolength is* $\|\ominus\mathbf{a}\oplus\mathbf{b}\|$ *and its associated gyrovector*

$$\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|} \tag{12}$$

is called a unit gyrovector.

Definition 4 (The Gyrocosine Function And Gyroangles). *Let* $\ominus\mathbf{a}\oplus\mathbf{b}$ *and* $\ominus\mathbf{a}\oplus\mathbf{c}$ *be two nonzero rooted gyrovectors, rooted at a common point* $\mathbf{a}$ *in an Einstein gyrovector space* $(\mathbb{V}_s, \oplus, \otimes)$*. The gyrocosine of the measure of the gyroangle* α*,* $0 \le \alpha \le \pi$*, that the two rooted gyrovectors generate is given by the equation*

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|} \cdot \frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}. \tag{13}$$

Gyroangles are invariant under left gyrotranslations [23, Theorem 8.6]. *The gyroangle* α *in* (13) *is denoted by* $\alpha = \angle\mathbf{bac}$ *or, equivalently,* $\alpha = \angle\mathbf{cab}$*. Two gyroangles are congruent if they have the same measure.*

The gyrotrigonometry of the Einstein gyrovector plane, presented and studied in [23, Ex. (8), p. 328], is summarized in Fig. 1.

The operational interpretation of gyroangles in $\mathbb{R}^3_s$ is natural. The origin of the Einstein gyrovector space $(\mathbb{R}^3_s, \oplus, \otimes)$ is conformal. Hence, gyroangles and angles with vertex at the origin coincide. Accordingly, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are any three points of an Einstein gyrovector space $(\mathbb{R}^3_s, \oplus, \otimes)$, the measure of the gyroangle $\alpha_g = \angle_g\mathbf{bac}$ equals the measure of the angle $\alpha_a = \angle_a(\ominus\mathbf{a}\oplus\mathbf{b})\mathbf{0}(\ominus\mathbf{a}\oplus\mathbf{c})$ between the directions

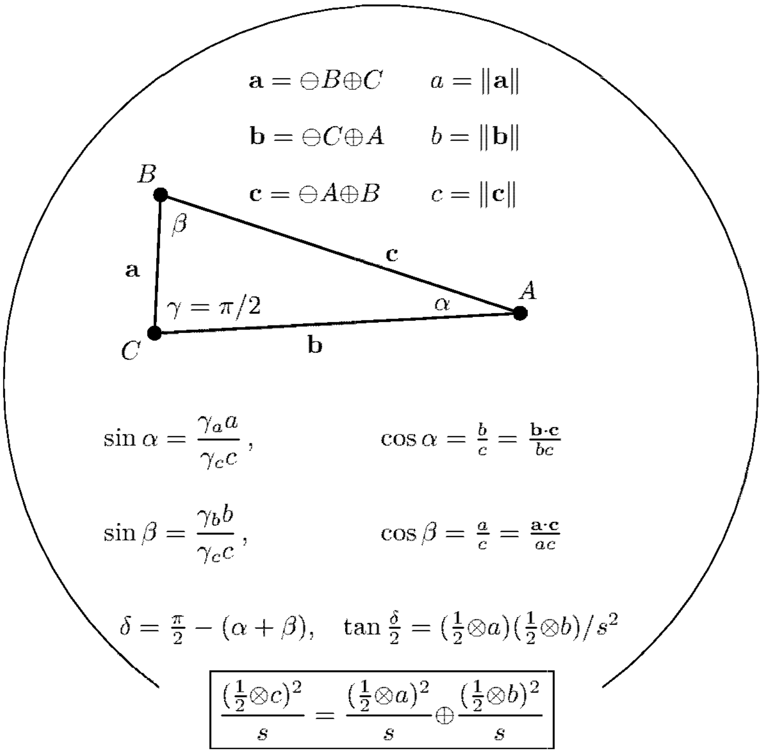

Figure 1. Gyrotrigonometry in Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. The gyrocosine and the gyrosine are elementary gyrotrigonometric functions. Their behaviour is identical with that of the elementary trigonometric functions modulo gamma factors. Thus, for instance, $\sin^2\alpha + \cos^2\alpha = 1$ and $\cot\alpha = \cos\alpha/\sin\alpha = \gamma_c b/(\gamma_a a)$ [23].

of motion of inertial frames $\Sigma_{\mathbf{b}}$ and $\Sigma_{\mathbf{c}}$ away from inertial frame $\Sigma_{\mathbf{a}}$, as seen by observers at rest relative to $\Sigma_{\mathbf{a}}$. This follows from the result that gyroangles are invariant under left gyrotranslations.

The gyrotriangle gyroangles determine uniquely the gyrotriangle side-gyrolengths. Using the standard notation for gyrotriangles as in Fig. 1 but in which gyroangle γ need not be $\pi/2$, we have

$$
\begin{aligned}
\gamma_{\mathbf{a}} &= \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma} \\
\gamma_{\mathbf{b}} &= \frac{\cos\beta + \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} \\
\gamma_{\mathbf{c}} &= \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}
\end{aligned}
\tag{14}
$$

and conversely

$$\begin{aligned}\cos\alpha &= \frac{-\gamma_{\mathbf{a}}+\gamma_{\mathbf{b}}\gamma_{\mathbf{c}}}{\gamma_{\mathbf{b}}\gamma_{\mathbf{c}}b_s c_s}\\ \cos\beta &= \frac{-\gamma_{\mathbf{b}}+\gamma_{\mathbf{a}}\gamma_{\mathbf{c}}}{\gamma_{\mathbf{a}}\gamma_{\mathbf{c}}a_s c_s}\\ \cos\gamma &= \frac{-\gamma_{\mathbf{c}}+\gamma_{\mathbf{a}}\gamma_{\mathbf{b}}}{\gamma_{\mathbf{a}}\gamma_{\mathbf{b}}a_s b_s}\end{aligned} \tag{15}$$

where $a_s = a/s$, etc. The special case of (14) when $\gamma = \pi/2$ is particularly interesting

$$\begin{aligned}\gamma_{\mathbf{a}} &= \frac{\cos\alpha}{\sin\beta}\\ \gamma_{\mathbf{b}} &= \frac{\cos\beta}{\sin\alpha}\\ \gamma_{\mathbf{c}} &= \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta\end{aligned} \tag{16}$$

implying

$$\gamma_{\mathbf{a}}\,\gamma_{\mathbf{b}} = \gamma_{\mathbf{c}} = \gamma_{\mathbf{a}\oplus\mathbf{b}} \tag{17}$$

for any right gyroangled gyrotriangle. As such, (17) may be viewed as a Pythagorean theorem in Einstein gyrovector spaces. Identities (14)–(15) follow from [23, Theorem 8.48, p. 280] with translation from Möbius to Einstein gyrovector spaces. The second equation in (17) follows from the gamma identity (3).

3. The Gyroparallelogram Law

A quadrilateral is a parallelogram if the lines containing opposite sides are parallel. Since the notion of parallelism between lines in vector spaces cannot be extended to gyrolines in gyrovector spaces, we note an equivalent definition of the parallelogram: A quadrilateral is a parallelogram if the midpoints of its two diagonals coincide. Accordingly, a gyroparallelogram is a gyroquadrilateral the two diagonals of which intersect at their gyromidpoints [16]. The formal definition of the gyroparallelogram in a gyrovector space thus follows.

Definition 5 (Gyroparallelograms). *Let* $\mathbf{a},\mathbf{b}$ *and* $\mathbf{b}'$ *be any three points in a gyrovector space* $(G,\oplus,\otimes)$. *The four points* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{b}'$, $\mathbf{a}'$ *in* G *are the vertices of the gyroparallelogram* $\mathbf{aba'b'}$, *Fig. 2, if* $\mathbf{a}'$ *satisfies the gyroparallelogram condition*

$$\mathbf{a}' = (\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}. \tag{18}$$

The gyroparallelogram is ***degenerate*** *if the three points* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{b}'$ *are gyrocollinear* [23, Def. 6.22].

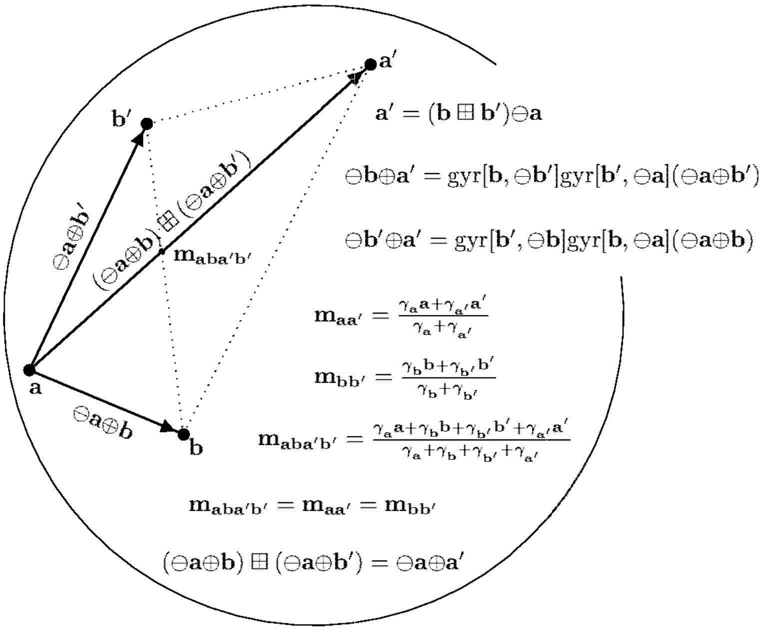

Figure 2. Einstein Gyroparallelogram, Def. 5, and the Relativistic Velocity Gyroparallelogram Addition Law, (24). Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{b}'$ be any three nongyrocollinear points in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, $\mathbb{V}_s$ being the s-ball of the real inner product space $(\mathbb{V}, +, \cdot)$, and let $\mathbf{a}'$ be given by the gyroparallelogram condition (18), $\mathbf{a}' = (\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}$. Then the four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{b}'$, $\mathbf{a}'$ are the vertices of the Einstein gyroparallelogram $\mathbf{aba'b'}$ and, by [23, Theorem 6.45], opposite sides are equal modulo gyrations. Shown are three expressions for the gyrocenter $\mathbf{m}_{\mathbf{aba'b'}} = \mathbf{m}_{\mathbf{aa'}} = \mathbf{m}_{\mathbf{bb'}}$ of the Einstein gyroparallelogram $\mathbf{aba'b'}$ in an Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. These can be obtained by relativistic CM (Center of Momentum) velocity considerations [19, 21, 23].

If the gyroparallelogram $\mathbf{aba'b'}$ *is non-degenerate, then the two vertices in each of the pairs* $(\mathbf{a}, \mathbf{a}')$ *and* $(\mathbf{b}, \mathbf{b}')$ *are said to be opposite to one another. The gyrosegments of adjacent vertices,* $\mathbf{ab}$*,* $\mathbf{ba}'$*,* $\mathbf{a'b'}$ *and* $\mathbf{b'a}$ *are the sides of the gyroparallelogram. The gyrosegments* $\mathbf{aa}'$ *and* $\mathbf{bb}'$ *that join opposite vertices in the non-degenerate gyroparallelogram* $\mathbf{aba'b'}$ *are the gyrodiagonals of the gyroparallelogram.*

The gyrogeometric significance of the gyroparallelogram condition (18) rests on its gyrocovariance with respect to rotations and left gyrotranslations. Indeed, $\mathbf{a}'$ of

(18) satisfies the identity [23, Eq. 3.64]

$$\mathbf{x}\oplus\mathbf{a}' = \mathbf{x}\oplus\{(\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}\} = \{(\mathbf{x}\oplus\mathbf{b}) \boxplus (\mathbf{x}\oplus\mathbf{b}')\}\ominus(\mathbf{x}\oplus\mathbf{a}) \tag{19}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{b}', \mathbf{x} \in G$, demonstrating that the point $\mathbf{a}'$ and its generating points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{b}'$ vary together under left gyrotranslations.

Of particular interest is the spacial case of (19) corresponding to $\mathbf{x} = \ominus\mathbf{a}$, giving rise to the identity

$$(\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{b}') = \ominus\mathbf{a}\oplus\{(\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}\} \tag{20}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{b}' \in G$. Identity (20) is a special kind of associative law enabling one to group together $\mathbf{b}$ and $\mathbf{b}'$ of the left hand side of (20). The need to employ this special kind of associative law will arise in the proof of the gyroparallelogram law in Theorem 2.

Theorem 1 (Gyroparallelogram Symmetries). *Every vertex of the gyroparallelogram $\mathbf{aba}'\mathbf{b}'$ satisfies the gyroparallelogram condition,* (18)*, that is,*

$$\begin{aligned} \mathbf{a} &= (\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}' \\ \mathbf{b} &= (\mathbf{a} \boxplus \mathbf{a}')\ominus\mathbf{b}' \\ \mathbf{b}' &= (\mathbf{a} \boxplus \mathbf{a}')\ominus\mathbf{b} \\ \mathbf{a}' &= (\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}. \end{aligned} \tag{21}$$

Furthermore, the two gyrodiagonals of the gyroparallelogram are concurrent, the concurrency point being the gyromidpoint of each of the two gyrodiagonals.

Proof: The last equation in (21) is valid by Definition 5 of the gyroparallelogram. By the right cancellation law (10) this equation is equivalent to the equation

$$\mathbf{a} \boxplus \mathbf{a}' = \mathbf{b} \boxplus \mathbf{b}'. \tag{22}$$

Since the coaddition $\boxplus$ is commutative in gyrovector spaces [23, Theorem 3.4], equation (22) is equivalent to each of the equations in (21) by the right cancellation law (10), thus verifying the first part of the theorem.

Equation (22) implies

$$\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{a}') = \tfrac{1}{2}\otimes(\mathbf{b} \boxplus \mathbf{b}'). \tag{23}$$

By [23, Def. 3.37] the left-hand side of (23) is the gyromidpoint of the gyrodiagonal $\mathbf{aa}'$ and the right-hand side of (23) is the gyromidpoint of the gyrodiagonal $\mathbf{bb}'$ [23, Theorem 6.33]. Hence, the gyromidpoints of the two gyrodiagonals $\mathbf{aa}'$ and $\mathbf{bb}'$ of the gyroparallelogram coincide, thus verifying the second part of the theorem. □

4. The Relativistic Velocity Gyroparallelogram Addition Law

Theorem 2 (The Gyroparallelogram Addition Law). *Let* $\mathbf{a}\mathbf{b}\mathbf{a}'\mathbf{b}'$ *be a gyroparallelogram in a gyrovector space* $(G, \oplus, \otimes)$*, Fig. 2. Then*

$$(\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{b}') = \ominus\mathbf{a}\oplus\mathbf{a}'. \tag{24}$$

Proof: By (20) and (18) we have

$$\begin{aligned}(\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{b}') &= \ominus\mathbf{a}\oplus\{(\mathbf{b} \boxplus \mathbf{b}')\ominus\mathbf{a}\} \\ &= \ominus\mathbf{a}\oplus\mathbf{a}'.\end{aligned} \tag{25}$$

□

The gyroparallelogram addition law (24) in an Einstein gyrovector space of relativistically admissible velocities $(\mathbb{R}^3_s, \oplus, \otimes)$, Fig. 2, is called the *relativistic velocity gyroparallelogram addition law.*

The relativistic velocity gyroparallelogram addition law plays in special relativity a role analogous to the role that the velocity parallelogram addition law plays in classical mechanics. In order to demonstrate this analogy we employ our study of the gyroparallelogram gyroangles in Sec. 5 to recover the standard angle of stellar aberration that results from the apparent shift in the position of stars due to the motion of the Earth as it orbits the Sun. The discovery of stellar aberration by Bradley around 1728 is described in [12].

5. The Gyroparallelogram Gyroangles

Let

$$\begin{aligned}\alpha &= \angle BAB' = \angle B'A'B \\ \beta &= \angle A'BA = \angle AB'A'\end{aligned} \tag{26}$$

be the two distinct gyroangles of the gyroparallelogram $ABA'B'$, Fig. 3. They are related to each other by the equations

$$\begin{aligned}\cos\alpha &= \frac{\gamma_a\gamma_b a_s b_s - (1+\gamma_a\gamma_b)\cos\beta}{1+\gamma_a\gamma_b - \gamma_a\gamma_b a_s b_s \cos\beta} \\ \cos\beta &= \frac{\gamma_a\gamma_b a_s b_s - (1+\gamma_a\gamma_b)\cos\alpha}{1+\gamma_a\gamma_b - \gamma_a\gamma_b a_s b_s \cos\alpha}\end{aligned} \tag{27}$$

as one can see by translating [23, Theorem 8.59, p. 297] from Möbius to Einstein gyrovector spaces by means of [23, Eqs. (6.309)–(6.310), p. 297].

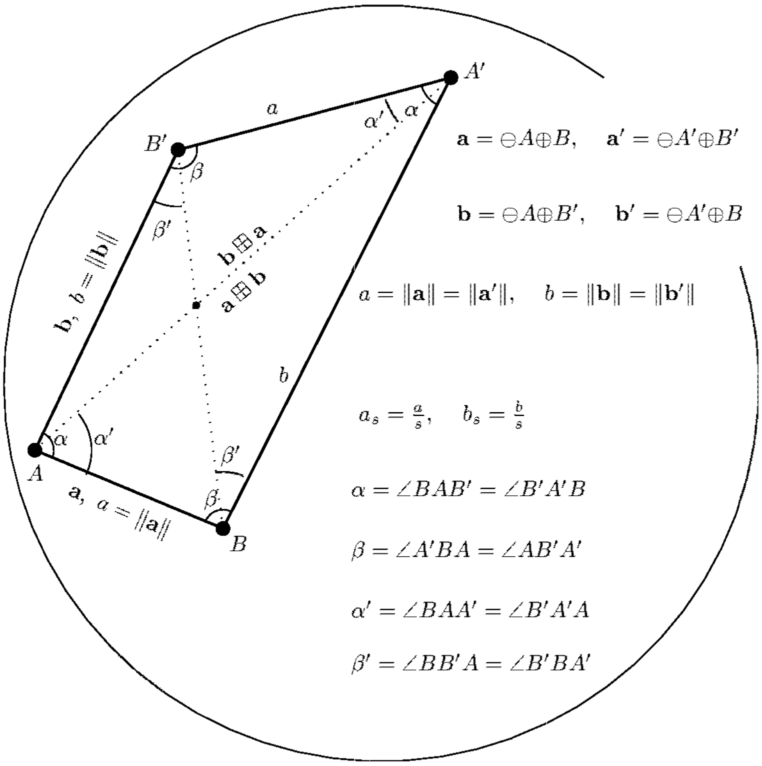

Figure 3. The Gyroangles of the Einstein Gyroparallelogram in Einstein gyrovector spaces $(\mathbb{V}_s, \oplus, \otimes)$. The two distinct gyroangles α and β of a gyroparallelogram in an Einstein gyrovector space are related to each other by (27)–(28).

It follows from (27) and the **gyrotrigonometric identity** $\sin^2\alpha+\cos^2\alpha = \sin^2\beta+\cos^2\beta = 1$, Fig. 1, that

$$\begin{aligned}\sin\alpha &= \frac{\gamma_a+\gamma_b}{1+\gamma_a\gamma_b-\gamma_a\gamma_b a_s b_s\cos\beta}\sin\beta \\ \sin\beta &= \frac{\gamma_a+\gamma_b}{1+\gamma_a\gamma_b-\gamma_a\gamma_b a_s b_s\cos\alpha}\sin\alpha.\end{aligned} \tag{28}$$

In the Newtonian limit, $s \to \infty$, γ_a and γ_b reduce to 1 while a_s and b_s vanish. Hence, identities (27)–(28) for the gyroparallelogram reduce to the identities $\cos\alpha = -\cos\beta$ and $\sin\alpha = \sin\beta$ for the parallelogram, implying $\beta = \pi - \alpha$ as expected.

According to the gyroparallelogram addition law, (24), the gyrodiagonal AA' of a gyroparallelogram $ABA'B'$, Fig. 3, satisfies the gyrovector equation

$$\ominus A \oplus A' = \mathbf{a} \boxplus \mathbf{b} \tag{29}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the gyrovectors

$$\begin{aligned} \mathbf{a} &= \ominus A \oplus B \\ \mathbf{b} &= \ominus A \oplus B' \end{aligned} \tag{30}$$

with magnitudes

$$\begin{aligned} a &= \|\mathbf{a}\| \\ b &= \|\mathbf{b}\|. \end{aligned} \tag{31}$$

Furthermore, by [23, Eqs. (3.156)–(3.157), p. 81] we have

$$\mathbf{a} \boxplus \mathbf{b} = \frac{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}}{\gamma_{\mathbf{a}}^2 + \gamma_{\mathbf{b}}^2 + \gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(1 + \mathbf{a}_s \cdot \mathbf{b}_s) - 1}(\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b}) \tag{32}$$

implying

$$\frac{1}{s}\|\mathbf{a} \boxplus \mathbf{b}\| = \frac{(\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}})\sqrt{(\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}})^2 - 2\{\gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(1 - \mathbf{a}_s \cdot \mathbf{b}_s) + 1\}}}{(\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}})^2 - \gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(1 - \mathbf{a}_s \cdot \mathbf{b}_s) - 1} \tag{33}$$

and

$$\gamma_{\mathbf{a} \boxplus \mathbf{b}} = \frac{\gamma_{\mathbf{a}}^2 + \gamma_{\mathbf{b}}^2 + \gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(1 + \mathbf{a}_s \cdot \mathbf{b}_s) - 1}{\gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(1 - \mathbf{a}_s \cdot \mathbf{b}_s) + 1} \tag{34}$$

where, as we see from Fig. 3

$$\mathbf{a}_s \cdot \mathbf{b}_s = a_s b_s \cos\alpha. \tag{35}$$

Here $\mathbf{a}_s = \mathbf{a}/s$, $a_s = a/s$, etc.

We now wish to calculate the gyroparallelogram gyroangle α' (β') generated by the gyroparallelogram gyrodiagonal AA' (BA'), Fig. 3, in terms of the gyroparallelogram gyroangle α (β) and its side gyrolengths $a = \|\mathbf{a}\|$ and $b = \|\mathbf{b}\|$. We therefore extend the gyrotriangle ABA' of the gyroparallelogram in Fig. 3 into a right gyroangled gyrotriangle ACA', as shown in Fig. 4; and apply the Einstein gyrotrigonometry, Fig. 1, to the resulting two right gyroangled gyrotriangles ACA' and BCA' in Fig. 4.

Applying Einstein gyrotrigonometry, Fig. 1, to the right gyroangled gyrotriangle BCA' in Fig. 4 we have

$$\begin{aligned} \cos(\pi - \beta) &= \frac{\|\ominus B \oplus C\|}{b} \\ \sin(\pi - \beta) &= \frac{\gamma_{\|\ominus A' \oplus C\|}\|\ominus A' \oplus C\|}{\gamma_b b}. \end{aligned} \tag{36}$$

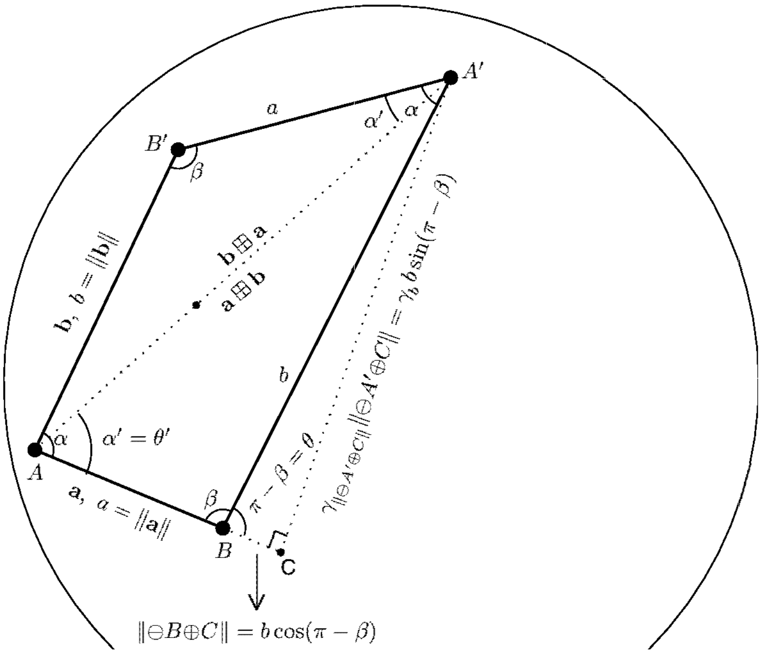

Figure 4. The Gyroangles of the Einstein Gyroparallelogram in Einstein gyrovector spaces $(\mathbb{V}_s, \oplus, \otimes)$. The two distinct gyroangles α and β of a gyroparallelogram in an Einstein gyrovector space are related to each other by (27)–(28). If B lies between A and C, as shown here, then $\|\ominus A\oplus C\| = \|\ominus A\oplus B\|\oplus\|\ominus B\oplus C\|$, and $\|\ominus B\oplus C\| = b\cos(\pi - \beta)$. If C lies between A and B then $\|\ominus A\oplus C\| = \|\ominus A\oplus B\|\ominus\|\ominus B\oplus C\|$, and $\|\ominus B\oplus C\| = b\cos\beta$.

Hence, noting that $\cos(\pi - \beta) = -\cos\beta$ and $\sin(\pi - \beta) = \sin\beta$ we have

$$\begin{aligned} \|\ominus B\oplus C\| &= -b\cos\beta \\ \gamma_{\|\ominus A'\oplus C\|}\|\ominus A'\oplus C\| &= \gamma_b b\sin\beta. \end{aligned} \tag{37}$$

It follows from the first equation in (37) that the gyrolength of gyrosegment AC, Fig. 4, is given by

$$\begin{aligned} \|\ominus A\oplus C\| &= \|\ominus A\oplus B\|\oplus\|\ominus B\oplus C\| \\ &= a\oplus(-b\cos\beta) \\ &= a\ominus b\cos\beta \\ &= \frac{a - b\cos\beta}{1 - a_s b_s\cos\beta} \end{aligned} \tag{38}$$

where we use the notation in (30)–(31), and employ Einstein addition for parallel velocities, (5). The first equation in (38) is the **gyrotriangle equality** studied in [23, Theorem 6.47, p. 146].

Applying Einstein gyrotrigonometry, Fig. 1, to the right gyroangled gyrotriangle ACA' in Fig. 4 we have

$$\cos\alpha' = \frac{\|\ominus A\oplus C\|}{\|\mathbf{a} \boxplus \mathbf{b}\|}$$
$$\sin\alpha' = \frac{\gamma_{\ominus A'\oplus C}\|\ominus A'\oplus C\|}{\gamma_{\mathbf{a}\boxplus\mathbf{b}}\|\mathbf{a} \boxplus \mathbf{b}\|} \tag{39}$$

so that

$$\cot\alpha' = \frac{\gamma_{\mathbf{a}\boxplus\mathbf{b}}\|\ominus A\oplus C\|}{\gamma_{\ominus A'\oplus C}\|\ominus A'\oplus C\|}. \tag{40}$$

Substituting into (40)

i) $\gamma_{\mathbf{a}\boxplus\mathbf{b}}$ from (34);
ii) $\|\ominus A\oplus C\|$ from (38);
iii) $\gamma_{\|\ominus A'\oplus C\|}\|\ominus A'\oplus C\|$ from (37); and
iv) expressing $\cos\beta$ and $\sin\beta$ in terms of $\cos\alpha$ and $\sin\alpha$ by (27)–(28),

we obtain the following remarkably simple and elegant expression

$$\cot\alpha' = \frac{\gamma_a\, a + \gamma_b\, b\cos\alpha}{\gamma_b\, b\sin\alpha}. \tag{41}$$

Similarly, by symmetry considerations

$$\cot(\alpha - \alpha') = \frac{\gamma_b\, b + \gamma_a\, a\cos\alpha}{\gamma_a\, a\sin\alpha}. \tag{42}$$

Furthermore, similarly to (41)–(42) we have

$$\cot\beta' = \frac{\gamma_b\, b + \gamma_a\, a\cos\beta}{\gamma_a\, a\sin\beta} \tag{43}$$

and

$$\cot(\beta - \beta') = \frac{\gamma_a\, a + \gamma_b\, b\cos\beta}{\gamma_b\, b\sin\beta}. \tag{44}$$

To express α' in terms of β we substitute $\cos\alpha$ and $\sin\alpha$ from (27)–(28) into (41) obtaining

$$\cot\alpha' = \gamma_a\,\frac{a - b\cos\beta}{b\sin\beta}. \tag{45}$$

Using the notation

$$\begin{aligned} \theta' &= \alpha' \\ \theta &= \pi - \beta \end{aligned} \tag{46}$$

suggested in Fig. 4, identity (45) takes the form

$$\cot\theta' = \gamma_a \frac{a_s + b_s\cos\theta}{b_s\sin\theta} = \gamma_a \frac{\frac{a}{b} + \cos\theta}{\sin\theta} \tag{47}$$

called the relativistic particle **aberration formula**.

Formula (47) is identical, modulo notation, to the standard relativistic particle aberration formula which is commonly obtained by applying the Lorentz transformation [5, Eq. (5.4), p. 13]. Here, in contrast, we establish the standard relativistic particle aberration formula by means of the relativistic velocity gyroparallelogram addition law.

When $\beta = \pi/2$, the gyroangle θ' in (47) reduces to θ'_0 given by

$$\cot\theta'_0 = \gamma_a \frac{a}{b}. \tag{48}$$

The gyroangle $\theta - \theta'$ is called the aberration gyroangle of the gyroparallelogram $ABA'B'$ in Fig. 4, giving rise to the relativistic particle **aberration angle**.

In the special case when $\beta = \pi/2$ the aberration gyroangle reduces to $\pi/2 - \theta'_0$, given by

$$\tan\left(\frac{\pi}{2} - \theta'_0\right) = \cot\theta'_0 = \gamma_a \frac{a}{b}. \tag{49}$$

In the special case when b approaches s, (47) reduces to the equation

$$\cot\theta' = \gamma_a \frac{a_s + \cos\theta}{\sin\theta} \tag{50}$$

called the *relativistic photon (or light, or stellar) aberration formula.* It implies the following equivalent formulas

$$\begin{aligned} \sin\theta' &= \frac{1}{\gamma_a}\frac{\sin\theta}{1 + a_s\cos\theta} \\ \cos\theta' &= \frac{a_s + \cos\theta}{1 + a_s\cos\theta} \end{aligned} \tag{51}$$

and

$$\tan\frac{\theta'}{2} = \sqrt{\frac{1 - a_s}{1 + a_s}}\tan\frac{\theta}{2} \tag{52}$$

noting the trigonometric identity

$$\tan\frac{\theta'}{2} = \frac{\sin\theta'}{1 + \cos\theta'}. \tag{53}$$

Interestingly, i) the relativistic particle aberration formula (47) is identical with Rindler's particle aberration formula, [10, p. 86], and ii) the relativistic photon aberration formula (50) is identical with the well-known relativistic electromagnetic wave aberration formula observed in *stellar aberration* [2, pp. 146 – 149], [3, pp. 132 – 133], [5, pp. 12 – 14], [8, pp. 84 – 87], [9, pp. 57 – 58], [10, pp. 81 – 82], [11, p. 71], [13, p. 146]. This identity indicates that the gyroparallelogram law plays in relativistic mechanics the same role that the parallelogram law plays in classical mechanics.

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