Sixth International Conference on
Geometry, Integrability and Quantization
June 3–10, 2004, Varna, Bulgaria
Ivaïlo M. Mladenov and Allen C. Hirshfeld, Editors
SOFTEX, Sofia 2005, pp 78–125

BASIC ASPECTS OF SOLITON THEORY

VLADIMIR S. GERDJIKOV

Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
1784 Sofia, Bulgaria

Abstract. This is a review of the main ideas of the inverse scattering method (ISM) for solving nonlinear evolution equations (NLEE), known as soliton equations. As a basic tool we use the fundamental analytic solutions $\chi^{\pm}(x,\lambda)$ of the Lax operator $L(\lambda)$. Then the inverse scattering problem for $L(\lambda)$ reduces to a Riemann–Hilbert problem. Such construction has been applied to wide class of Lax operators, related to the simple Lie algebras. We construct the kernel of the resolvent of $L(\lambda)$ in terms of $\chi^{\pm}(x,\lambda)$ and derive the spectral decompositions of $L(\lambda)$. Thus we can solve the relevant classes of NLEE which include the NLS equation and its multi-component generalizations, the N-wave equations, etc. Applying the dressing method of Zakharov and Shabat we derive the N-soliton solutions of these equations.

Next we explain that the ISM is a natural generalization of the Fourier transform method. As appropriate generalizations of the usual exponential function we use the so-called "squared solutions" which are constructed again in terms of $\chi^{\pm}(x,\lambda)$ and the Cartan–Weyl basis of the relevant Lie algebra. One can prove the completeness relations for the "squared solutions" which in fact provide the spectral decompositions of the recursion operator Λ.

These decompositions can be used to derive all fundamental properties of the corresponding NLEE in terms of Λ: i) the explicit form of the class of integrable NLEE; ii) the generating functionals of integrals of motion; iii) the hierarchies of Hamiltonian structures. We outline the importance of the classical R-matrices for extracting he involutive integrals of motion.

1. Introduction

The modern development of the soliton theory in the last three decades of the 20th century has lead to a number of important applications and developments in several areas of contemporary physics and mathematics, see [41, 12, 6, 3, 2]. In this review I will outline the basic ideas of the **inverse scattering method** (ISM) on the example of the **nonlinear Schrödinger equation** (NLS) and its multicomponent generalizations.

The integrability of the well known (scalar) NLS equation

$$\mathrm{i}q_t + q_{xx} + 2\epsilon|q(x,t)|^2 q(x,t) = 0, \qquad \epsilon = \pm 1 \tag{1}$$

was discovered by Zakharov and Shabat in their pioneer paper [45] which strongly stimulated the search of other important integrable nonlinear evolution equations (NLEE). After the Korteweg–de Vries equation, this was the second NLEE integrable by the ISM. In the next few years the number of new integrable NLEE was growing quickly: the modified KdV equation [39], the N-wave equations [29, 31, 40], the vector NLS [34], the Toda chain [35], the principal chiral field equation [42, 43], etc.

The simplest nontrivial multicomponent generalizations of NLS is the vector NLS equation known also as the **Manakov model** [34]

$$\mathrm{i}\vec{q}_t + \vec{q}_{xx} + 2(q^\dagger \vec{q})\vec{q}(x,t) = 0 \tag{2}$$

where $\vec{q}$ is an n-component complex-valued vector

$$\vec{q}(x,t) = \begin{pmatrix} q_1(x,t) \\ \vdots \\ q_n(x,t) \end{pmatrix}$$

tending to zero fast enough for $x \to \pm\infty$. Another version of the Manakov model is

$$\mathrm{i}\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger \boldsymbol{\epsilon}\vec{q})\vec{q}(x,t) = 0 \tag{3}$$

where $\boldsymbol{\epsilon} = \mathrm{diag}(\epsilon_1, \dots, \epsilon_n)$ and $\epsilon_n = \pm 1$. Both the scalar and vector NLS equations (especially the one with $n = 2$) find wide applications in nonlinear optics, plasma physics, etc.

Equations (2) and (3) are particular cases of the matrix NLS equation which is obtained from the system

$$\begin{aligned} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\boldsymbol{q}\boldsymbol{r}\boldsymbol{q}(x,t) &= 0 \\ -\mathrm{i}\boldsymbol{r}_t + \boldsymbol{r}_{xx} + 2\boldsymbol{r}\boldsymbol{q}\boldsymbol{r}(x,t) &= 0. \end{aligned} \tag{4}$$

with appropriate reductions (involution). Here $\boldsymbol{q}(x,t)$ and $\boldsymbol{r}^T(x,t)$ are $n \times m$-matrix-valued functions of x and t with $n > 1$, $m > 1$ which are smooth enough and tend to zero fast enough for $x \to \pm\infty$. The best known involution compatible with the evolution of (4) is

$$\boldsymbol{r} = B - \boldsymbol{q}^\dagger B_+^{-1}, \qquad B_\pm = \mathrm{diag}(\eta_1^\pm, \dots, \eta_m^\pm), \qquad (\eta_s^\pm)^2 = 1 \tag{5}$$

and the corresponding MNLS equation is of the form

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\boldsymbol{q}B_-\boldsymbol{q}^\dagger B_+^{-1}\boldsymbol{q}(x,t) = 0. \tag{6}$$

For $n = m = 1$ and $r = \epsilon q^*$ the system (4) goes into the scalar NLS equation; for $m = 1$ and $n > 1$ and with appropriate choice of the involution (5) equation (4) can be transferred into the Manakov model or into equation (6).

The MNLS (6) is known to be closely related to the symmetric spaces [13]. All these versions of NLS are solvable by applying the ISM. Their **Lax representation** $[L, M] = 0$ is provided by a generalization of the **Zakharov–Shabat system**

$$L\psi \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + Q(x,t) - \lambda J \right) \psi(x,\lambda) = 0 \tag{7}$$

$$M\psi \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + V_0(x,t) + \lambda V_1(x,t) - 2\lambda^2 J \right) \psi(x,\lambda) = 0 \tag{8}$$

$$Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q}(x) \\ \boldsymbol{r}(x) & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{9}$$

where $Q(x,t)$ and J are $(n+m) \times (n+m)$ matrices with compatible block structure and $V_0(x,t)$, $V_1(x,t)$ are expressed in terms of Q and its x-derivative

$$V_1(x,t) = 2Q(x,t), \qquad V_0(x,t) = -[Q, \mathrm{ad}_J^{-1} Q] + 2\mathrm{i}\, \mathrm{ad}_J^{-1} Q_x. \tag{10}$$

Remark 1. *One can consider Lax operators which are more general than* (7). *For example, like in* [40, 11, 15] *one can choose $Q(x,t)$ and J to be elements of a simple Lie algebra $\mathfrak{g}$ such that $J \in \mathfrak{h}$ and $Q(x,t) = [J, \widetilde{Q}(x,t)]$. Such form of $Q(x,t)$ can always be achieved by a gauge transformation commuting with J. Such $Q(x,t)$ span the co-adjoint orbit of $\mathfrak{g}$ passing through J and can be viewed as the tangent plane to the homogeneous space $\mathcal{G}/\mathcal{J}$. Here $\mathcal{G}$ is the Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{J}$ is the subgroup of $\mathcal{G}$ commuting with J. Our choice of $Q(x,t)$ in equation* (7) *corresponds to the symmetric space* $\mathrm{SU}(n+m)/\mathrm{S}(\mathrm{U}(n) \otimes \mathrm{U}(m))$, *see also* [13].

Remark 2. *An effective tool to impose involutions is the reduction group introduced by Mikhailov* [36]. *The involution* (5) *(or $\mathbb{Z}_2$-reduction) can be written as*

$$BU^\dagger(x,t,\lambda^*)B^{-1} = U(x,t,\lambda) \tag{11}$$

where B is an automorphism of $\mathfrak{g}$ matrix such that $B^2 = \mathbb{1}$, $[J, B] = 0$ and

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \qquad B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}. \tag{12}$$

Reductions leading to new types of MNLS systems are demonstrated in [27, 21].

Section 2 is devoted to the direct scattering problem for the Lax operator L. Our analysis is based on the notion of fundamental analytic solution (FAS) which allows one to prove that the scattering problem for L is equivalent to a Riemann–Hilbert problem (RHP) for FAS. Two minimal sets of scattering data $\mathcal{T}_{1,2}$ are introduced and shown to determine uniquely both the scattering matrix $T(\lambda)$ and the corresponding potential $Q(x)$.

In Section 3 we approach the solution of the inverse scattering problem (ISP) through the RHP. The dressing Zakharov–Shabat method [43, 44] for the **symmetric spaces** $\mathrm{SU}(n+m)/\mathrm{S}(\mathrm{U}(n)\otimes \mathrm{U}(m))$ is outlined and used to construct explicitly singular solutions of the RHP. Thus we derive reflectionless potentials for L and soliton solutions for the relevant NLEE. We also define the resolvent of L through the FAS and prove the completeness relation for the Jost solutions of L.

Section 4 starts with the Wronskian relations as a tool to study the mapping between the potential $Q(x)$ and the scattering data of L generalizing the results of [1, 7, 8, 22]. Using them one is able to introduce the sets of 'squared solutions' $\{\boldsymbol{\Psi}\}$ and $\{\boldsymbol{\Phi}\}$ and prove that they are complete in the space of all allowed potentials. This makes more precise and explicit the results of [18]. We derive the expansions of $Q(x)$ and its variation $\mathrm{ad}_J^{-1}\,\delta Q(x)$ over $\{\boldsymbol{\Psi}\}$ and $\{\boldsymbol{\Phi}\}$ and demonstrate that the elements of the minimal sets of scattering data $\mathcal{T}_{1,2}$ and their variations appear as expansion coefficients. We also introduce the generating operators $\Lambda_\pm$ for which $\{\boldsymbol{\Psi}\}$ and $\{\boldsymbol{\Phi}\}$ are sets of eigen- and adjoint functions.

The tools developed in Section 4 are used in Section 5 to describe the fundamental properties of the NLEE. There we prove a theorem stating the equivalence of the NLEE to a corresponding set of linear evolution equations for the scattering data. Next we derive the hierarchy of the integrals of motion from the principle series in terms of Q and show that they play the role of Hamiltonians for the MNLS type equations. We display the hierarchy of Hamiltonian structures for these NLEE. In short we have demonstrated the complete analogy between the ISM and the usual Fourier transform thus generalizing the results of [1, 15, 25, 26, 30, 32].

Using the method of the classical R-matrix [12] we derive the Poisson brackets between the elements of the scattering matrix. As a consequence we prove that the integrals of motion from the principle series are in involution.

We end by a brief discussion in Section 6 on related methods and topics.

2. Direct and Inverse Scattering Problems for L

2.1. The Scattering Problem for L

Here we briefly outline the scattering problem for the system (7) for the class of potentials $Q(x,t)$ satisfying the following

Condition C1: $Q(x,t)$ are smooth enough and fall off to zero fast enough for $x \to \pm\infty$ for all t.

Condition C2: $Q(x,t)$ is such that L has a finite number of simple eigenvalues $\lambda_j^\pm \in \mathbb{C}_\pm$ for all t.

In this subsection t plays the role of an additional parameter; for the sake of brevity the t-dependence is not always shown. Condition C2 can not be formulated as a set of explicit conditions on $Q(x, t)$; its precise meaning will become clear below. The main tool here are the **Jost solutions** defined by their asymptotics at $x \to \pm\infty$

$$\lim_{x\to\infty} \psi(x, \lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x, \lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}. \tag{13}$$

Along with the Jost solutions we introduce

$$\xi(x, \lambda) = \psi(x, \lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} \qquad \varphi(x, \lambda) = \phi(x, \lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} \tag{14}$$

which satisfy the following linear integral equations

$$\xi(x, \lambda) = \mathbb{1} + \mathrm{i}\int_{\infty}^{x} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}\lambda J(x-y)}Q(y)\xi(y, \lambda)\mathrm{e}^{\mathrm{i}\lambda J(x-y)} \tag{15}$$

$$\varphi(x, \lambda) = \mathbb{1} + \mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}\lambda J(x-y)}Q(y)\varphi(y, \lambda)\mathrm{e}^{\mathrm{i}\lambda J(x-y)}. \tag{16}$$

These are Volterra type equations which, as is well known always have solutions providing one can ensure the convergence of the integrals in the right hand side. For λ real the exponential factors in (15) and (16) are just oscillating and the convergence is ensured by condition C1.

Obviously the Jost solutions as whole can not be extended for $\operatorname{Im}\lambda \neq 0$. However some of their columns can be extended for $\lambda \in \mathbb{C}_+$, others – for $\lambda \in \mathbb{C}_-$. Indeed, the equation (15) for the first column of $\xi(x, \lambda)$ contains only the exponential factor $\mathrm{e}^{\mathrm{i}\lambda(x-y)}$ which falls off for $\operatorname{Im}\lambda < 0$. More precisely we can write down the Jost solutions $\psi(x, \lambda)$ and $\phi(x, \lambda)$ in the following block-matrix form

$$\psi(x, \lambda) = (|\psi^-(x, \lambda)\rangle, |\psi^+(x, \lambda)\rangle) \qquad \phi(x, \lambda) = (|\phi^+(x, \lambda)\rangle, |\phi^-(x, \lambda)\rangle) \tag{17}$$

where the superscript $+$ and (resp. $-$) shows that the corresponding block-matrix allows analytic extension for $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$).

Solving the direct scattering problem means given the potential $Q(x)$ to find the scattering matrix $T(\lambda)$. By definition $T(\lambda)$ relates the two Jost solutions

$$\phi(x, \lambda) = \psi(x, \lambda)T(\lambda), \qquad T(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & -\boldsymbol{b}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & \boldsymbol{a}^-(\lambda) \end{pmatrix} \tag{18}$$

and has compatible block-matrix structure. In what follows we will need also the inverse of the scattering matrix

$$\psi(x, \lambda) = \phi(x, \lambda)\hat{T}(\lambda), \qquad \hat{T}(\lambda) \equiv \begin{pmatrix} \boldsymbol{c}^-(\lambda) & \boldsymbol{d}^-(\lambda) \\ -\boldsymbol{d}^+(\lambda) & \boldsymbol{c}^+(\lambda) \end{pmatrix} \tag{19}$$

where

$$\boldsymbol{c}^-(\lambda) = \hat{\boldsymbol{a}}^+(\lambda)(\mathbb{1} + \rho^-\rho^+)^{-1} = (\mathbb{1} + \tau^+\tau^-)^{-1}\hat{\boldsymbol{a}}^+(\lambda) \tag{20a}$$

$$\boldsymbol{d}^-(\lambda) = \hat{\boldsymbol{a}}^+(\lambda)\rho^-(\lambda)(\mathbb{1} + \rho^+\rho^-)^{-1} = (\mathbb{1} + \tau^+\tau^-)^{-1}\tau^+(\lambda)\hat{\boldsymbol{a}}^-(\lambda) \tag{20b}$$

$$\boldsymbol{c}^+(\lambda) = \hat{\boldsymbol{a}}^-(\lambda)(\mathbb{1} + \rho^+\rho^-)^{-1} = (\mathbb{1} + \tau^-\tau^+)^{-1}\hat{\boldsymbol{a}}^-(\lambda) \tag{20c}$$

$$\boldsymbol{d}^{+}(\lambda) = \hat{\boldsymbol{a}}^{-}(\lambda)\rho^{+}(\lambda)(\mathbb{1} + \rho^{-}\rho^{+})^{-1} = (\mathbb{1} + \tau^{-}\tau^{+})^{-1}\tau^{-}(\lambda)\hat{\boldsymbol{a}}^{+}(\lambda). \tag{20d}$$

The diagonal blocks of both $T(\lambda)$ and $\hat{T}(\lambda)$ allow analytic continuation off the real axis, namely $\boldsymbol{a}^{+}(\lambda)$, $\boldsymbol{c}^{+}(\lambda)$ are analytic functions of λ for $\lambda \in \mathbb{C}_{\pm}$, while $\boldsymbol{a}^{-}(\lambda)$, $\boldsymbol{c}^{-}(\lambda)$ are analytic functions of λ for $\lambda \in \mathbb{C}_{\pm}$.

By $\rho^{\pm}(\lambda)$ and $\tau^{\pm}(\lambda)$ above we have denoted the multicomponent generalizations of the reflection coefficients (for the scalar case, see [1, 7, 22, 32])

$$\rho^{\pm}(\lambda) = \boldsymbol{b}^{\pm}\hat{\boldsymbol{a}}^{\pm}(\lambda) = \hat{\boldsymbol{c}}^{\pm}\boldsymbol{d}^{\pm}(\lambda), \qquad \tau^{\pm}(\lambda) = \hat{\boldsymbol{a}}^{\pm}\boldsymbol{b}^{\mp}(\lambda) = \boldsymbol{d}^{\mp}\hat{\boldsymbol{c}}^{\pm}(\lambda). \tag{21}$$

We will need also the asymptotics for $\lambda \to \infty$

$$\begin{gathered}
\lim_{\lambda\to-\infty} \phi(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \lim_{\lambda\to\infty} \psi(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}, \qquad \lim_{\lambda\to\infty} T(\lambda) = \mathbb{1} \\
\lim_{\lambda\to\infty} \boldsymbol{a}^{+}(\lambda) = \lim_{\lambda\to\infty} \boldsymbol{c}^{-}(\lambda) = \mathbb{1}, \qquad \lim_{\lambda\to\infty} \boldsymbol{a}^{-}(\lambda) = \lim_{\lambda\to\infty} \boldsymbol{c}^{+}(\lambda) = \mathbb{1}.
\end{gathered} \tag{22}$$

The inverse to the Jost solutions $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ are solutions to

$$\mathrm{i}\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}x} - \hat{\psi}(x,\lambda)(Q(x) - \lambda J) = 0 \tag{23}$$

satisfying the conditions

$$\lim_{x\to\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\hat{\psi}(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\hat{\phi}(x,\lambda) = \mathbb{1}. \tag{24}$$

Now it is the collections of rows of $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ that possess analytic properties in λ

$$\hat{\psi}(x,\lambda) = \begin{pmatrix} \langle\hat{\psi}^{+}(x,\lambda)| \\ \langle\hat{\psi}^{-}(x,\lambda)| \end{pmatrix}, \qquad \hat{\phi}(x,\lambda) = \begin{pmatrix} \langle\hat{\phi}^{-}(x,\lambda)| \\ \langle\hat{\phi}^{+}(x,\lambda)| \end{pmatrix}. \tag{25}$$

Just like the Jost solutions, their inverse (25) are solutions to linear equations (23) with regular boundary conditions (24); therefore they can have no singularities in their regions of analyticity. The same holds true also for the scattering matrix $T(\lambda) = \hat{\psi}(x,\lambda)\phi(x,\lambda)$ and its inverse $\hat{T}(\lambda) = \hat{\phi}(x,\lambda)\psi(x,\lambda)$, i.e.

$$\boldsymbol{a}^{+}(\lambda) = \langle\hat{\psi}^{+}(x,\lambda)|\phi^{+}(x,\lambda)\rangle, \qquad \boldsymbol{a}^{-}(\lambda) = \langle\hat{\psi}^{-}(x,\lambda)|\phi^{-}(x,\lambda)\rangle \tag{26}$$

as well as

$$\boldsymbol{c}^{+}(\lambda) = \langle\hat{\phi}^{+}(x,\lambda)|\psi^{+}(x,\lambda)\rangle, \qquad \boldsymbol{c}^{-}(\lambda) = \langle\hat{\phi}^{-}(x,\lambda)|\psi^{-}(x,\lambda)\rangle \tag{27}$$

are analytic for $\lambda \in \mathbb{C}_{\pm}$ and have no singularities in their regions of analyticity. However they may become degenerate (i.e., their determinants may vanish) for some values $\lambda_j^{\pm} \in \mathbb{C}_{\pm}$ of λ. Below we analyze the structure of these degeneracies.

2.2. The Fundamental Analytic Solutions

The next step is to construct the fundamental analytic solutions of (7). In our case this is done simply by combining the blocks of Jost solutions with the same analytic properties

$$\begin{aligned} \chi^+(x,\lambda) &\equiv (|\phi^+\rangle, |\psi^+\rangle)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^+(\lambda) = \psi(x,\lambda)\boldsymbol{T}^-(\lambda) \\ \chi^-(x,\lambda) &\equiv (|\psi^-\rangle, |\phi^-\rangle)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^-(\lambda) = \psi(x,\lambda)\boldsymbol{T}^+(\lambda) \end{aligned} \tag{28}$$

where the block-triangular functions $\boldsymbol{S}^\pm(\lambda)$ and $\boldsymbol{T}^\pm(\lambda)$ are given by

$$\begin{aligned} \boldsymbol{S}^+(\lambda) &= \begin{pmatrix} \mathbb{1} & \boldsymbol{d}^-(\lambda) \\ 0 & \boldsymbol{c}^+(\lambda) \end{pmatrix}, \qquad & \boldsymbol{T}^-(\lambda) &= \begin{pmatrix} \boldsymbol{a}^+(\lambda) & 0 \\ \boldsymbol{b}^+(\lambda) & \mathbb{1} \end{pmatrix} \\ \boldsymbol{S}^-(\lambda) &= \begin{pmatrix} \boldsymbol{c}^-(\lambda) & 0 \\ -\boldsymbol{d}^+(\lambda) & \mathbb{1} \end{pmatrix}, \qquad & \boldsymbol{T}^+(\lambda) &= \begin{pmatrix} \mathbb{1} & -\boldsymbol{b}^-(\lambda) \\ 0 & \boldsymbol{a}^-(\lambda) \end{pmatrix}. \end{aligned} \tag{29}$$

These triangular factors can be viewed also as generalized Gauss decompositions (see [28]) of $T(\lambda)$ and its inverse

$$\begin{aligned} T(\lambda) &= \boldsymbol{T}^-(\lambda)\hat{\boldsymbol{S}}^+(\lambda) = \boldsymbol{T}^+(\lambda)\hat{\boldsymbol{S}}^-(\lambda) \\ \hat{T}(\lambda) &= \boldsymbol{S}^+(\lambda)\hat{\boldsymbol{T}}^-(\lambda) = \boldsymbol{S}^-(\lambda)\hat{\boldsymbol{T}}^+(\lambda). \end{aligned} \tag{30}$$

The relations between $\boldsymbol{c}^\pm(\lambda)$, $\boldsymbol{d}^\pm(\lambda)$ and $\boldsymbol{a}^\pm(\lambda)$, $\boldsymbol{b}^\pm(\lambda)$ in equation (20) ensure that equations (30) become identities. From equations (28), (29) we derive

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda), \qquad G_0(\lambda) = \hat{D}^-(\lambda)(\mathbb{1} + K^-(\lambda)) \tag{31}$$

$$\chi^-(x,\lambda) = \chi^+(x,\lambda)\hat{G}_0(\lambda), \qquad \hat{G}_0(\lambda) = \hat{D}^+(\lambda)(\mathbb{1} - K^+(\lambda)) \tag{32}$$

valid for $\lambda \in \mathbb{R}$, where

$$D^-(\lambda) = \begin{pmatrix} \boldsymbol{c}^-(\lambda) & 0 \\ 0 & \boldsymbol{a}^-(\lambda) \end{pmatrix}, \qquad K^-(\lambda) = \begin{pmatrix} 0 & \boldsymbol{d}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & 0 \end{pmatrix} \tag{33}$$

$$D^+(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & 0 \\ 0 & \boldsymbol{c}^+(\lambda) \end{pmatrix}, \qquad K^+(\lambda) = \begin{pmatrix} 0 & \boldsymbol{b}^-(\lambda) \\ \boldsymbol{d}^+(\lambda) & 0 \end{pmatrix}. \tag{34}$$

Obviously the block-diagonal factors $D^+(\lambda)$ and $D^-(\lambda)$ are matrix-valued analytic functions for $\lambda \in \mathbb{C}_\pm$. Another well known fact about the FAS $\chi^\pm(x,\lambda)$ concerns their asymptotic behavior for $\lambda \to \pm\infty$, namely

$$\lim_{\lambda\to\infty} X^\pm(x,\lambda) = \mathbb{1} \tag{35}$$

where we have introduced:

$$X^\pm(x,\lambda) = \chi^\pm(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda J x}. \tag{36}$$

In the derivations that follow the analyticity properties of $X^\pm(x,\lambda)$ for $\lambda \in \mathbb{C}_\pm$ and equation (35) will play crucial role.

2.3. The Gel'fand–Levitan–Marchenko Equation

The first method to solve the ISM is based on the **Gel'fand–Levitan–Marchenko** (GLM) equation. For the Zakharov–Shabat system it is well known, for the block-matrix case – see the pedagogical exposition in [2]. The GLM equation is an integral equation for the transformation operator $\mathcal{K}$ which relate the Jost solution $\psi(x,\lambda)$ and the 'plane waves' $\exp(-\mathrm{i}\lambda J x)$ as follows

$$\psi(x,\lambda)\mathrm{e}^{\mathrm{i}J\lambda x} = \mathbb{1} + \int_x^\infty \mathrm{d}s\, \mathcal{K}(x,s)\mathrm{e}^{\mathrm{i}\lambda J(x-s)}. \tag{37}$$

Then the GLM equation is a Volterra-type integral equation of the form

$$\mathcal{K}(x,y) = \mathcal{F}(x+y) + \int_x^\infty \mathrm{d}s\, \mathcal{K}(x,s)\mathcal{F}(s+y) \tag{38}$$

where the kernel $\mathcal{F}(x)$ is expressed in terms of the scattering data of the operator L as follows

$$\begin{aligned} \mathcal{F}(x) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\,\lambda \begin{pmatrix} 0 & \rho^-(\lambda)\exp(-\mathrm{i}\lambda x) \\ \rho^+(\lambda)\exp(\mathrm{i}\lambda x) & 0 \end{pmatrix} \\ & + \mathrm{i}\sum_{j=1}^N \begin{pmatrix} 0 & \rho_j^- \exp(-\mathrm{i}\lambda_j^- x) \\ \rho_j^+ \exp(\mathrm{i}\lambda_j^+ x) & 0 \end{pmatrix} \end{aligned} \tag{39}$$

where $\rho^\pm(\lambda)$ are the reflection coefficients introduced in (21) above, $\lambda_j^\pm$ are the discrete eigenvalues of L and the constants $\rho_j^\pm$ characterize the norming of the Jost solutions $\psi^\pm(x,\lambda_j^\pm)$, see [2].

Let us now explain how the GLM equation allows one to solve the ISP for L. Indeed, given the scattering matrix $T(\lambda)$ we can construct the reflection coefficients $\rho^\pm(\lambda)$, the coefficients $\rho_j^\pm$ and the discrete eigenvalues $\lambda_j^\pm$, $j=1,\dots,N$. This provides us with the kernel $\mathcal{F}(x)$ of the GLM equation Since this is a Volterra type equation it always has a solution. If one is able to solve it and construct the transformation operator $\mathcal{K}(x,y)$ then the corresponding potential $Q(x)$ can be recovered through

$$Q(x) = \mathrm{i}J[J,\mathcal{K}(x,x)]. \tag{40}$$

Thus, given the scattering data we recover the corresponding potential $Q(x)$.

It is well known that in the reflectionless case $\rho^+ = \rho^- = 0$ the kernel $\mathcal{F}(x)$ becomes degenerated and the GLM equation can be solved exactly thus providing the reflectionless potentials for L, which in turn are directly related to the soliton solutions of the corresponding NLEE. We will derive them below using the method, known now as the **dressing method** [34, 42, 44].

2.4. Reductions of L

Typically one goes from the system (4) to the MNLS equations (1) with $\epsilon = \pm 1$ by imposing the condition $\boldsymbol{r} = \epsilon \boldsymbol{q}^\dagger$ which is known as reduction condition. The theory for constructing such reductions was proposed by Mikhailov in [36] and developed further in [19, 20, 33].

Here we analyze just $\mathbb{Z}_2$-reduction, which are most widely used in the literature

$$B(U^\dagger(x,t,\lambda^*))B^{-1} = U(x,t,\lambda), \quad U(x,t,\lambda) = Q(x,t) - \lambda J, \quad B^2 = \mathbb{1}. \tag{41}$$

Choosing B to be constant block-diagonal matrix

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix} \tag{42}$$

we find that equation (41) leads to

$$\boldsymbol{r} = B_- \boldsymbol{q}^\dagger B_+^{-1}, \qquad \boldsymbol{q} = B_+ \boldsymbol{r}^\dagger B_-^{-1}. \tag{43}$$

More precisely the reduction (41) means that

$$(\chi^+(x,t,\lambda^*))^\dagger = B^{-1}(\chi'^{,-}(x,t,\lambda))^{-1} B \tag{44}$$

where χ^+ and $\chi'^{,-}$ are conveniently chosen solutions of equation (7). If we identify $\chi^\pm$ as the FAS (28) analytic for $\lambda \in \mathbb{C}_\pm$ then

$$(\chi^+(x,t,\lambda^*))^\dagger = B^{-1}(\chi^-(x,t,\lambda)\hat{D}^-(\lambda))^{-1} B. \tag{45}$$

This result is derived easily by comparing the asymptotics of both sides of equation (45) for $x \to \pm\infty$. Skipping the detail we will list here the basic consequences of the reductions for the scattering data of L. For the scattering matrix we get

$$T^\dagger(t,\lambda^*) = B\hat{T}(t,\lambda)B^{-1} \tag{46}$$

or in 'components' (skipping the t-dependence)

$$\begin{aligned} (\boldsymbol{a}^\pm(\lambda^*))^\dagger &= B_\pm \boldsymbol{c}^\mp(\lambda)\hat{B}_\pm, \qquad (\boldsymbol{b}^\pm(\lambda^*))^\dagger = B_\pm \boldsymbol{d}^\mp(\lambda)\hat{B}_\mp \qquad \lambda \in \mathbb{R} \\ (\rho^-(\lambda))^\dagger &= B_- \rho^+(\lambda)\hat{B}_+, \qquad (\tau^-(\lambda))^\dagger = B_+ \tau^+(\lambda)\hat{B}_- \\ (D^+(\lambda^*))^\dagger &= B\hat{D}^-(\lambda)\hat{B}. \end{aligned} \tag{47}$$

For other interesting choices leading to new reductions of N-wave and MNLS-type equations, see [19, 20, 27, 21].

2.5. The Riemann–Hilbert Problem

The equations (31) and (32) can be written down as

$$X^+(x,\lambda) = X^-(x,\lambda)\hat{D}^-(\lambda)(\mathbb{1} + K^-(x,\lambda)), \qquad \lambda \in \mathbb{R} \tag{48}$$

or

$$X^-(x,\lambda) = X^+(x,\lambda)\hat{D}^+(\lambda)(\mathbb{1} - K^+(x,\lambda)), \qquad \lambda \in \mathbb{R} \tag{49}$$

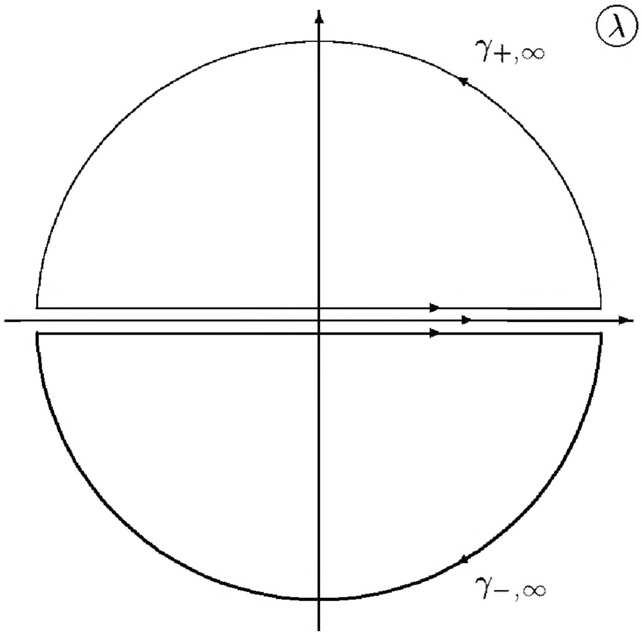

Figure 1. The contours $\gamma_{\pm} = \mathbb{R} \cup \gamma_{\pm\infty}$

where

$$K^{\pm}(x, \lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx} K^{\pm}(\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx}. \tag{50}$$

Equation (48) (resp. equation (49)) combined with (35) is known in the literature [14] as a Riemann–Hilbert problem (RHP) with canonical normalization. It is well known that RHP with canonical normalization has unique regular solution; the matrix-valued solutions $X_0^+(x, \lambda)$ and $X_0^-(x, \lambda)$ of (48), (35) is called regular if $\det X_0^{\pm}(x, \lambda)$ does not vanish for any $\lambda \in \mathbb{C}_{\pm}$.

Let us now apply the contour-integration method to derive the integral decompositions of $X^{\pm}(x, \lambda)$. To this end we consider the contour integral

$$\mathcal{J}_1(\lambda) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_+} \frac{\mathrm{d}\mu}{\mu - \lambda} X^+(x, \mu) - \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_-} \frac{\mathrm{d}\mu}{\mu - \lambda} X^-(x, \mu)\hat{D}^-(\mu) \tag{51}$$

where $\lambda \in \mathbb{C}_+$ and the contours $\gamma_{\pm}$ are shown on Fig. 1.

First we evaluate $\mathcal{J}_1(\lambda)$ by Cauchy theorem. The result is

$$\mathcal{J}_1(\lambda) = X^+(x, \lambda) + \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^-} \frac{X^-(x, \mu)\hat{D}^-(\mu)}{\mu - \lambda}. \tag{52}$$

We can also evaluate $\mathcal{J}_1(\lambda)$ by integrating along the contours. In integrating along the infinite semi-circles of $\gamma_{\pm,\infty}$ we use the asymptotic behavior of $X^{\pm}(x, \lambda)$ for $\lambda \to \infty$. The result is

$$\mathcal{J}_1(\lambda) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu - \lambda} X^-(x, \mu)\hat{D}^-(\mu)K^-(\mu) \tag{53}$$

where in evaluating the integrand we made use of equation (48). Equating the right hand sides of (52) and (53) we get the following integral decomposition for $X^+(x,\lambda)$

$$
\begin{aligned}
X^+(x,\lambda) = \mathbb{1} - \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^-} \frac{X^-(x,\mu)\hat{D}^-(\mu)}{\mu-\lambda} \\
+ \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda} X^-(x,\mu)\hat{D}^-(\mu)K^-(\mu). \quad (54)
\end{aligned}
$$

Quite analogously we derive the decomposition for $X^-(x,\lambda)$

$$
\begin{aligned}
X^-(x,\lambda) = \mathbb{1} - \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^+} \frac{X^+(x,\mu)\hat{D}^+(\mu)}{\mu-\lambda} \\
+ \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda} X^+(x,\mu)\hat{D}^+(\mu)K^+(\mu). \quad (55)
\end{aligned}
$$

Equations (54), (55) can be viewed as a set of singular integral equations which are equivalent to the RHP. For the MNLS these were first derived in [34].

2.6. The Minimal Set of Scattering Data

Obviously, given the potential $Q(x)$ one can solve the integral equations for the Jost solutions which determine them uniquely. The Jost solutions in turn determine uniquely the scattering matrix $T(\lambda)$ and its inverse $\hat{T}(\lambda)$. But $Q(x)$ contains at most $2mn$ independent complex-valued functions of x. Thus it is natural to expect that at most $2mn$ of the coefficients in $T(\lambda)$ for $\lambda \in \mathbb{R}$ will be independent; the rest must be functions of those.

The set of independent coefficients of $T(\lambda)$ are known as the minimal set of scattering data. As such we may use any of the following two sets $\mathcal{T}_i \equiv \mathcal{T}_{i,\mathrm{c}} \cup \mathcal{T}_{i,\mathrm{d}}$

$$
\begin{aligned}
\mathcal{T}_{1,\mathrm{c}} &\equiv \{\rho^+(\lambda), \rho^-(\lambda),\ \lambda \in \mathbb{R}\}, &\quad \mathcal{T}_{1,\mathrm{d}} &\equiv \left\{\rho_j^\pm, \lambda_j^\pm\right\}_{j=1}^{N} \\
\mathcal{T}_{2,\mathrm{c}} &\equiv \{\tau^+(\lambda), \tau^-(\lambda),\ \lambda \in \mathbb{R}\}, &\quad \mathcal{T}_{1,\mathrm{d}} &\equiv \left\{\tau_j^\pm, \lambda_j^\pm\right\}_{j=1}^{N}
\end{aligned} \quad (56)
$$

where the reflection coefficients $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ have been introduced in equation (20), $\lambda_j^\pm$ are (simple) discrete eigenvalues of L and $\rho_j^\pm$ and $\tau_j^\pm$ characterize the norming constants of the corresponding Jost solutions.

The reflection coefficients $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ are defined only on the real λ-axis, while the diagonal blocks $\boldsymbol{a}^\pm(\lambda)$ and $\boldsymbol{c}^\pm(\lambda)$ (or, equivalently, $D^\pm(\lambda)$) allow analytic extensions for $\lambda \in \mathbb{C}_\pm$. From the equations (20) there follows that

$$
\boldsymbol{a}^+(\lambda)\boldsymbol{c}^-(\lambda) = (\mathbb{1} + \rho^-\rho^+(\lambda))^{-1}, \qquad \boldsymbol{a}^-(\lambda)\boldsymbol{c}^+(\lambda) = (\mathbb{1} + \rho^+\rho^-(\lambda))^{-1} \quad (57)
$$

$$c^-(\lambda)a^+(\lambda) = (1\!\!1 + \tau^+\tau^-(\lambda))^{-1}, \qquad c^+(\lambda)a^-(\lambda) = (1\!\!1 + \tau^-\tau^+(\lambda))^{-1} \quad (58)$$

Given $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) we determine the right hand sides of (57) (resp. (58)) for $\lambda \in \mathbb{R}$. Combined with the facts about the limits

$$\begin{aligned} &\lim_{\lambda\to\infty} a^+(\lambda) = 1\!\!1, \qquad \lim_{\lambda\to\infty} c^-(\lambda) = 1\!\!1 \\ &\lim_{\lambda\to\infty} a^-(\lambda) = 1\!\!1, \qquad \lim_{\lambda\to\infty} c^+(\lambda) = 1\!\!1 \end{aligned} \quad (59)$$

each of the relations (57), (58) can be viewed as a RHP with canonical normalization. Such RHP can be solved explicitly in the one-component case (provided we know the locations of their zeroes) by using the Plemelj–Sokhotsky formulae [14]. These zeroes are in fact the discrete eigenvalues of L. One possibility to make use of these facts is to take log of the determinants of both sides of (57) getting

$$A^+(\lambda) + C^-(\lambda) = -\ln\det(1\!\!1 + \rho^-\rho^+(\lambda), \qquad \lambda \in \mathbb{R} \quad (60)$$

where

$$A^\pm(\lambda) = \ln\det a^\pm(\lambda), \qquad C^\pm(\lambda) = \ln\det c^\pm(\lambda). \quad (61)$$

Then Plemelj–Sokhotsky formulae allows us to recover $A^\pm(\lambda)$ and $C^\pm(\lambda)$

$$\mathcal{A}(\lambda) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda} \ln\det(1\!\!1 + \rho^-\rho^+(\mu)) + \sum_{j=1}^{N} \ln\frac{\lambda-\lambda_j^+}{\lambda-\lambda_j^-} \quad (62)$$

where $\mathcal{A}(\lambda) = A^+(\lambda)$ for $\lambda \in \mathbb{C}_+$ and $\mathcal{A}(\lambda) = -C^-(\lambda)$ for $\lambda \in \mathbb{C}_-$. In deriving (62) we have also assumed that $\lambda_j^\pm$ are simple zeroes of $A^\pm(\lambda)$ and $C^\pm(\lambda)$.

Remark 3. *The sets* (56) *were first derived for $m = n = 1$ in* [30], *see also* [22]. *Here we are using the gauge covariant formulation developed in* [25].

If we impose the reduction condition (47) then we get the following constraints on the sets $\mathcal{T}_{1,2}$

$$\rho^-(\lambda) = (B_-\rho^+(\lambda)B_+)^\dagger, \qquad \rho_j^- = (B_-\rho_j^+B_+)^\dagger, \qquad \lambda_j^- = (\lambda_j^+)^* \quad (63)$$

$$\tau^-(\lambda) = (B_+\tau^+(\lambda)B_-)^\dagger, \qquad \tau_j^- = (B_+\tau_j^+B_-)^\dagger, \qquad \lambda_j^- = (\lambda_j^+)^* \quad (64)$$

where $j = 1, \ldots, N$.

Remark 4. *For certain choices of the reduction conditions* (40), *such as $Q = -Q^\dagger$ the generalized Zakharov–Shabat system $L(\lambda)\psi = 0$ can be written down as an eigenvalue problem $\mathcal{L}\psi = \lambda\psi(x,\lambda)$ where $\mathcal{L}$ is a self-adjoint operator. The continuous spectrum of $\mathcal{L}$ fills up the whole real λ-axis thus 'leaving no space' for discrete eigenvalues. Such Lax operators have no discrete spectrum and the corresponding MNLS do not have soliton solutions.*

The full RHP's (57), (58) for n and $m > 1$ do not allow explicit solutions. However, from the general theory of RHP [14] one may conclude that (57), (58) allow unique solutions provided the number and types of the zeroes $\lambda_j^{\pm}$ are properly chosen.

Thus we can outline a procedure which allows one to reconstruct not only $T(\lambda)$ and $\hat{T}(\lambda)$ and the corresponding potential $Q(x)$ from each of the sets $\mathcal{T}_i$, $i = 1, 2$:

i) Given $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) solve the RHP (57) (resp. (58)) and construct $\boldsymbol{a}^{\pm}(\lambda)$ and $\boldsymbol{c}^{\pm}(\lambda)$ for $\lambda \in \mathbb{C}_{\pm}$.

ii) Given $\mathcal{T}_1$ we determine $\boldsymbol{b}^{\pm}(\lambda)$ and $\boldsymbol{d}^{\pm}(\lambda)$ as

$$\boldsymbol{b}^{\pm}(\lambda) = \rho^{\pm}(\lambda)\boldsymbol{a}^{\pm}(\lambda), \qquad \boldsymbol{d}^{\pm}(\lambda) = \boldsymbol{c}^{\pm}(\lambda)\rho^{\pm}(\lambda) \tag{65}$$

or if $\mathcal{T}_2$ is known then

$$\boldsymbol{b}^{\pm}(\lambda) = \boldsymbol{a}^{\pm}(\lambda)\tau^{\pm}(\lambda), \qquad \boldsymbol{d}^{\pm}(\lambda) = \tau^{\pm}(\lambda)\boldsymbol{c}^{\pm}(\lambda). \tag{66}$$

iii) The potential $Q(x)$ can be recovered from $\mathcal{T}_1$ by solving the GLM equation and using equation (39).

Another method for reconstructing $Q(x)$ from $\mathcal{T}_j$ uses the interpretation of the ISM as generalized Fourier transform, see [1, 15, 22, 25, 26, 30, 32] and Section 4 below.

3. The RHP and the Inverse Scattering Problem

There are two pioneer results that enhanced the development of the ISM. The first is the existence and explicit construction of the FAS for rather general class of Lax operators [4, 5, 26, 37, 38, 40]. The second is the discovery of the equivalence of the ISP for L to a RHP for the FAS [38, 42, 43, 44].

Here we are considering rather special Lax operators for which the construction of the FAS does not require special efforts. Here we first prove the equivalence between RHP and the spectral problem L. Next we show how the FAS can be used to construct the resolvent of L and for the analysis of its spectral properties.

The use of the RHP rather than the GLM equations allowed for important effective extensions of the ISM; it allowed the treatment of generic Lax operators with rational dependence on λ [33, 42, 43], as well as of ones related to simple Lie algebras [15, 16, 19, 20, 27, 26]. In the next subsections we will make use of these developments to describe two types of soliton solutions for the MNLS.

3.1. Equivalence of RHP to ISP

The next important step is the possibility to reduce the solution of the ISP for the generalized Zakharov–Shabat system to a (local) RHP. Indeed the relation (31) can be rewritten as

$$X^+(x,t,\lambda) = X^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda \in \mathbb{R} \tag{67a}$$

$$G(x,t,\lambda) = \mathrm{e}^{-\mathrm{i}(\lambda J x - f(\lambda)t)} G_0(\lambda) \mathrm{e}^{\mathrm{i}(\lambda J x - f(\lambda)t)} \tag{67b}$$

$$G_0(\lambda) = \hat{S}^-(\lambda,t)S^+(\lambda,t)|_{t=0}. \tag{67c}$$

In other words the sewing function $G(x,t,\lambda)$ satisfies the equations

$$\mathrm{i}\frac{\mathrm{d}G}{\mathrm{d}x} - \lambda[J, G(x,t,\lambda)] = 0, \qquad \mathrm{i}\frac{\mathrm{d}G}{\mathrm{d}t} + [f(\lambda), G(x,t,\lambda)] = 0. \tag{68}$$

Here $f(\lambda) \in \mathfrak{h}$ determines the dispersion law of the NLEE.

Theorem 1 ([38]). *Let $X^+(x,t,\lambda)$ and $X^-(x,t,\lambda)$ be solutions to the RHP* (67) *with canonical normalization* (35) *allowing analytic extension in λ for $\lambda \in \mathbb{C}_\pm$, respectively. Then $\chi^\pm(x,t,\lambda) = X^\pm(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda J x}$ are fundamental analytic solutions of both operators L and M, i.e. satisfy equations* (7), (8) *with*

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda\left(J - X^\pm(x,t,\lambda)J\hat{X}^\pm(x,t,\lambda)\right). \tag{69}$$

Proof: Let us assume that $X^\pm(x,t,\lambda)$ are regular solutions to the RHP and let us introduce the function

$$g^\pm(x,t,\lambda) = \mathrm{i}\frac{\mathrm{d}X^\pm}{\mathrm{d}x}\hat{X}^\pm(x,t,\lambda) + \lambda X^\pm(x,t,\lambda)J\hat{X}^\pm(x,t,\lambda). \tag{70}$$

If $X^\pm(x,t,\lambda)$ are regular then neither $X^\pm(x,t,\lambda)$ nor their inverse $\hat{X}^\pm(x,t,\lambda)$ have singularities in their regions of analyticity $\lambda \in \mathbb{C}_\pm$. Then the functions $g^\pm(x,t,\lambda)$ also will be regular for all $\lambda \in \mathbb{C}_\pm$. Besides

$$\lim_{\lambda\to\infty} g^+(x,t,\lambda) = \lim_{\lambda\to\infty} g^-(x,t,\lambda) = \lambda J. \tag{71}$$

The crucial step in the proof [44] is based on the chain of relations

$$\begin{aligned}
g^+(x,t,\lambda) &\overset{(67)}{=} \mathrm{i}\frac{\mathrm{d}(X^-G)}{\mathrm{d}x}\hat{G}\hat{X}^-(x,t,\lambda) + \lambda X^- G J \hat{G}\hat{X}^-(x,t,\lambda) \\
&= \mathrm{i}\frac{\mathrm{d}X^-}{\mathrm{d}x}\hat{X}^-(x,t,\lambda) + X^-\left(\mathrm{i}\frac{\mathrm{d}G}{\mathrm{d}x}\hat{G} + \lambda G J \hat{G}\right)\hat{X}^-(x,t,\lambda) \\
&\overset{(68)}{=} \mathrm{i}\frac{\mathrm{d}X^-}{\mathrm{d}x}\hat{X}^-(x,t,\lambda) + X^-\left(\lambda[J,G]\hat{G} + \lambda G J \hat{G}\right)\hat{X}^-(x,t,\lambda) \\
&= \mathrm{i}\frac{\mathrm{d}X^-}{\mathrm{d}x}\hat{X}^-(x,t,\lambda) + \lambda X^- J \hat{X}^-(x,t,\lambda) \\
&\equiv g^-(x,t,\lambda), \qquad \lambda \in \mathbb{R}.
\end{aligned} \tag{72}$$

Thus we conclude that $g^+(x,t,\lambda) = g^-(x,t,\lambda)$ is a function analytic in the whole complex λ-plane except in the neighborhood of $\lambda \to \infty$ where $g^+(x,t,\lambda)$ tends to λJ, (71). Next from Liouville theorem we conclude that the difference

$g^+(x,t,\lambda) - \lambda J$ is a constant with respect to λ; if we denote this 'constant' by $-Q(x,t)$ we get

$$g^+(x,t,\lambda) - \lambda J = -Q(x,t). \tag{73}$$

Using the definition of $g^+(x,t,\lambda)$ (70) we find that $X^\pm(x,t,\lambda)$ satisfy

$$\mathrm{i}\frac{\mathrm{d}X^\pm}{\mathrm{d}x} + Q(x,t)X^\pm(x,t,\lambda) - \lambda[J, X^\pm(x,t,\lambda)] = 0 \tag{74}$$

i.e. that $\chi^\pm(x,t,\lambda)$ is a fundamental solution to L. The relation between $Q(x,t)$ and $X^\pm(x,t,\lambda)$ (69) is obtained by taking the limit of the left-hand sides of (73) for $\lambda \to \infty$ and using the asymptotic expansions of $X^\pm(x,\lambda)$ and $\hat{X}^\pm(x,\lambda)$

$$\begin{aligned} X^\pm(x,t,\lambda) &= 1\!\!1 + \sum_{s=1}^{\infty} \lambda^{-s} X_s^\pm(x,t) \\ \hat{X}^\pm(x,t,\lambda) &= 1\!\!1 + \sum_{s=1}^{\infty} \lambda^{-s} \hat{X}_s^\pm(x,t) \end{aligned} \tag{75}$$

we get

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda(J - X^\pm J\hat{X}^\pm(x,t,\lambda)] = [J, X_1^\pm(x,t)]. \tag{76}$$

Here we also used the fact that $X_1^\pm(x,t) = -\hat{X}_1^\pm(x,t)$, see equation (75).
Arguments along the same line applied to the functions $h^\pm(x,t,\lambda)$

$$h^\pm(x,t,\lambda) = \mathrm{i}\frac{\mathrm{d}X^\pm}{\mathrm{d}t}\hat{X}^\pm(x,t,\lambda) + 2\lambda^2 X^\pm(x,t,\lambda)J\hat{X}^\pm(x,t,\lambda) \tag{77}$$

can be used to prove that $\chi^\pm(x,t,\lambda)$ are fundamental solutions also of the operator M (8). Indeed, repeating the above arguments for $h^\pm(x,t,\lambda)$ we find that $h^+(x,t,\lambda) = h^-(x,t,\lambda)$ is a function analytic everywhere in $\mathbb{C}$ except at $\lambda \to \infty$ where it behaves like $2\lambda^2 J$. From Liouville theorem it follows that $h^\pm(x,t,\lambda) - 2\lambda^2 J$ is a linear function of λ equal to $-V_0(x,t) - \lambda V_1(x,t)$. Thus

$$\mathrm{i}\frac{\mathrm{d}X^\pm}{\mathrm{d}t} + (V_0(x,t) + \lambda V_1(x,t))X^\pm(x,t,\lambda) - [2\lambda^2 J, X^\pm(x,t,\lambda)] = 0. \tag{78}$$

To conclude the proof we have to account for possible zeroes and pole singularities of $X^\pm(x,t,\lambda)$ at the points $\lambda_j^\pm$. Below we derive the structure of these singularities and show that they do not influence the functions $g^\pm(x,t,\lambda)$ and $h^\pm(x,t,\lambda)$. □

3.2. The Dressing Zakharov–Shabat Method for the Symmetric Spaces $\mathrm{SU}(n+m)/\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(m))$

Let us outline how one can, starting from a given regular solutions $X_0^\pm(x,t,\lambda)$ of the RHP, construct new singular solutions $X^\pm(x,t,\lambda)$ having zeroes (singularities) at the prescribed points $\lambda_j^\pm \in \mathbb{C}_\pm$. The structure of these singularities are determined by the dressing factor $u_j(x,t,\lambda)$

$$\xi^{\pm}(x,t,\lambda) = u_j(x,t,\lambda)\xi_0^{\pm}(x,t,\lambda)w_{j,\pm}^{-1}(\lambda) \tag{79}$$

which in our case has a simple fraction-linear dependence on λ

$$u_j(x,t,\lambda) = 1\!\!1 + (c_j(\lambda)-1)P_j(x,t), \qquad c_j(\lambda) = \frac{\lambda-\lambda_j^+}{\lambda-\lambda_j^-}. \tag{80}$$

Here $P_j(x,t)$ is a projector $P_j^2 = P_j$, $w_{j,+}(\lambda) = 1\!\!1$ and $w_{j,-}(\lambda)$ will be defined below.

It is well known that, if $Q(x,t) - \lambda J$ takes values in the Lie algebra $\mathfrak{g}$ then $X^{\pm}(x,t,\lambda)$ must take values in the corresponding Lie group $\mathfrak{G}$. Therefore the dressing factor $u_j(x,t,\lambda)$ must also be element of the same group. But assuming $P_j(x,t)$ is a projector we can derive that

$$\det u(x,t,\lambda) = (c_j(\lambda))^{r_j}, \qquad r_j = \operatorname{rank} P_j(x,t). \tag{81}$$

Obviously the ansatz (80) is compatible with $\mathfrak{G} \simeq \mathrm{GL}(n+m)$, $\mathrm{SL}(n+m)$ and $\mathrm{U}(n+m)$, $\mathrm{SU}(n+m)$. The last two possibilities are realized provided proper reduction conditions like (63), (64) are imposed on $\lambda_j^{\pm}$ and $P_j(x,t) = BP_j^{\dagger}(x,t)B^{-1}$.

Usually $P_j(x)$ is chosen to be of rank 1. We will consider slightly more general case when $\operatorname{rank} P_j(x) = r_j \geq 1$; r_j is the multiplicity of the corresponding eigenvalues $\lambda_j^{\pm}$ and $r_j \leq \min(n,m)$. Then $P_j(x)$ can be written in the form

$$P_j(x) = |n_j\rangle \left(\langle m_j|n_j\rangle\right)^{-1} \langle m_j| \tag{82}$$

where the collections of r_j bra- (resp. ket-) eigenvectors $\langle m_j|$ (resp. $|n_j\rangle$) can be viewed also as rectangular $(n+m)\times r_j$ (resp. $r_j \times (n+m)$) matrices. Then $\langle m_j|n_j\rangle$ will be quadratic $r_j \times r_j$ matrix which we assume to be nondegenerate. If $r_j = 1$ equation (82) provides the standard expression for rank 1 projector. It is easy to check that the left hand side of (82) satisfies identically the relation $P_j^2 = P_j$.

From (79) there follows that the dressing factor $u(x,t,\lambda)$ satisfies the equation

$$\mathrm{i}\frac{\mathrm{d}u}{\mathrm{d}x} + Q(x,t)u(x,t,\lambda) - u(x,t,\lambda)Q_0(x,t) - \lambda[J, u(x,t,\lambda)] = 0 \tag{83}$$

where $Q_0(x,t)$ (resp. $Q(x,t)$) is the potential related to the regular $\chi_0^{\pm}(x,t,\lambda)$ (resp. singular $\chi^{\pm}(x,t,\lambda)$) solution of the RHP.

Next we insert the anzats (80) and request that it holds identically with respect to λ. To this end it is enough that equation (80) holds true for $\lambda = \lambda_j^+$, $\lambda \to \lambda_j^-$ and $\lambda \to \infty$. The first two conditions lead to the following equations for $\langle m_j|$ and $|n_j\rangle$

$$\mathrm{i}\frac{\mathrm{d}|n_j\rangle}{\mathrm{d}x} + U^{(0)}(x,t,\lambda_j^+)|n_j\rangle = 0, \qquad \mathrm{i}\frac{\mathrm{d}|n_j\rangle}{\mathrm{d}t} + V^{(0)}(x,t,\lambda_j^+)|n_j\rangle = 0 \tag{84}$$

$$\mathrm{i}\frac{\mathrm{d}\langle m_j|}{\mathrm{d}x} - \langle m_j|U^{(0)}(x,t,\lambda_j^-) = 0, \qquad \mathrm{i}\frac{\mathrm{d}\langle m_j|}{\mathrm{d}t} - \langle m_j|V^{(0)}(x,t,\lambda_j^-) = 0 \quad (85)$$

$$U^{(0)}(x,t,\lambda) = Q_0(x,t) - \lambda J, \qquad V^{(0)}(x,t,\lambda) = V(x,t,\lambda)|_{Q=Q_0}. \quad (86)$$

Here $V^{(0)}(x,t,\lambda)$ is obtained from $V(x,t,\lambda)$ (see (8), (10)) replacing $Q(x,t)$ by $Q_0(x,t)$. This construction is well defined also in the case when $\chi_0^{\pm}(x,\lambda)$ are singular solutions to the RHP, provided they are regular for $\lambda = \lambda_j^{\pm}$. In order to avoid technicalities in what follows we will treat only the case of one pair of discrete eigenvalues. From equations (84) there follows that

$$|n_j\rangle = \chi_{0j}^{+}(x,t)|n_j^0\rangle, \quad \langle m_j| = \langle m_j^0|\hat{\chi}_{0j}^{-}(x,t), \quad \chi_{0j}^{\pm}(x,t) = \chi_0^{\pm}(x,t,\lambda_j^{\pm}). \quad (87)$$

Note that since $\chi_0^{\pm}(x,t)$ are solutions of the regular RHP then $\chi_{0j}^{\pm}(x,t)$ exist and are nondegenerate. The equation (69) considered for $\lambda \to \infty$ gives the following relation between $Q_0(x,t)$, $Q(x,t)$ and $P(x,t)$

$$\begin{aligned} Q(x,t) &= Q_0(x,t) + \lim_{\lambda\to\infty} \lambda(J - u_j(x,t,\lambda)J\hat{u}_j(x,t,\lambda)) \\ &= Q_0(x,t) - (\lambda_j^+ - \lambda_j^-)[J, P_j(x,t)]. \end{aligned} \quad (88)$$

Thus, starting from a given regular solution of the RHP (and related solution $Q_0(x,t)$ to the NLEE) we can construct a singular solution to the RHP and a new solution $Q(x,t)$ of the NLEE depending on the $\lambda_j^{\pm}$ and on the eigenvectors of $P_j(x)$. If we start from the trivial solution $Q_0(x,t) = 0$ of the NLEE then we will get the one-soliton solution of the NLEE. Repeating the procedure N times we can get the N-soliton solution of the NLEE.

With the explicit formulae for $P_j(x)$ and using (79) we can establish the relationship between the scattering data of the regular RHP and the corresponding singular one. The dressing factor $u_j(x,t,\lambda)$ is determined by the collections of constant polarization vectors

$$|n_j^0\rangle = \begin{pmatrix} |n_{0,j}^1\rangle \\ |n_{0,j}^2\rangle \end{pmatrix}, \qquad \langle m_j^0| = \left(\langle m_{0,j}^1|, \langle m_{0,j}^2|\right) \quad (89)$$

which can not be quite arbitrary. They should be such that the constant $r_j \times r_j$ matrices $\langle m_{0,j}^1|n_{0,j}^1\rangle$ and $\langle m_{0,j}^2|n_{0,j}^2\rangle$ are nondegenerate.

Now we can evaluate the limits of the dressing factor $u(x,\lambda)$ for $x \to \pm\infty$ and then derive the interrelations between the scattering matrices $T_0(\lambda)$ and $T_1(\lambda)$ corresponding to the potentials $Q_0(x,t)$ and $Q_1(x,t)$. In what follows the elements of the scattering matrix $T_0(\lambda)$ will be denoted by the same letters as the ones of $T(\lambda)$ but with additional subscript 0.

Using the explicit formulae (87), (28), (29) we get

$$P_j^{\pm} = \lim_{x\to\pm\infty} P_j(x); \qquad P_j^{+} = \begin{pmatrix} P_{11,j}^{+} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_j^{-}(x) = \begin{pmatrix} 0 & 0 \\ 0 & P_{22,j}^{-} \end{pmatrix} \quad (90)$$

$$P^+_{11,j} = \boldsymbol{a}^+_0(\lambda^+_j)|n^1_{0,j}\rangle \left(\langle m^1_{0,j}|\boldsymbol{a}^+_0(\lambda^+_j)|n^1_{0,j}\rangle\right)^{-1} \langle m^1_{0,j}| \tag{91}$$
$$P^-_{22,j} = \boldsymbol{c}^+_0(\lambda^+_j)|n^2_{0,j}\rangle \left(\langle m^2_{0,j}|\boldsymbol{c}^+_0(\lambda^+_j)|n^2_{0,j}\rangle\right)^{-1} \langle m^2_{0,j}|$$

where obviously $\operatorname{rank} P^-_{11,j} = \operatorname{rank} P^-_{22,j} = r_j$. Therefore the limits of the dressing factor are given by

$$\lim_{x\to\infty} u(x,\lambda) = \begin{pmatrix} u^+_{11,j} & 0 \\ 0 & 1\!\!1 \end{pmatrix}, \qquad u^+_{11,j} = 1\!\!1 + (c_j(\lambda) - 1)P^+_{11,j} \tag{92}$$

$$\lim_{x\to-\infty} u(x,\lambda) = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & u^-_{22,j} \end{pmatrix}), \qquad u^-_{22,j} = 1\!\!1 + (c_j(\lambda) - 1)P^-_{22,j}. \tag{93}$$

Next we have to determine the matrices $w_{j,\pm}(\lambda)$ in the right hand side of equation (79) so that the asymptotics of the singular solution are compatible with (28), (29). This holds true if

$$w_{j,+}(\lambda) = 1\!\!1, \qquad w_{j,-}(\lambda) \equiv W(\lambda) = \begin{pmatrix} u^+_{11,j} & 0 \\ 0 & u^-_{22,j} \end{pmatrix}. \tag{94}$$

In addition we find the following relations between the scattering matrices

$$T(\lambda) = u^+_j T_0(\lambda)\hat{u}^-_j, \tag{95}$$

or 'in components'

$$\boldsymbol{a}^+(\lambda) = u^+_{11,j}\boldsymbol{a}^+_0(\lambda), \qquad \boldsymbol{b}^-(\lambda) = u^+_{11,j}\boldsymbol{b}^-_0(\lambda)\hat{u}^-_{22,j} \tag{96a}$$
$$\boldsymbol{a}^-(\lambda) = \boldsymbol{a}^-_0(\lambda)\hat{u}^-_{22,j}, \qquad \boldsymbol{b}^+(\lambda) = \boldsymbol{b}^+_0(\lambda) \tag{96b}$$
$$\boldsymbol{c}^+(\lambda) = u^-_{22,j}\boldsymbol{c}^+_0(\lambda), \qquad \boldsymbol{d}^+(\lambda) = u^-_{22,j}\boldsymbol{d}^+_0(\lambda)\hat{u}^+_{11,j} \tag{96c}$$
$$\boldsymbol{c}^-(\lambda) = \boldsymbol{c}^-_0(\lambda)\hat{u}^+_{11,j}, \qquad \boldsymbol{d}^-(\lambda) = \boldsymbol{d}^-_0(\lambda). \tag{96d}$$

The dressed solutions of the RHP are given by

$$\chi^+(x,\lambda) = u(x,\lambda)\chi^+_0(x,\lambda), \qquad \chi^-(x,\lambda) = u(x,\lambda)\chi^-_0(x,\lambda)\hat{W}(\lambda) \tag{97}$$
$$D^+(\lambda) = W(\lambda)D^+_0(\lambda), \qquad D^-(\lambda) = D^-_0(\lambda)\hat{W}(\lambda) \tag{98}$$
$$W(\lambda) = 1\!\!1 + (c_j(\lambda) - 1)W_j, \qquad W_j = \begin{pmatrix} P^+_{11,j} & 0 \\ 0 & P^-_{22,j} \end{pmatrix}. \tag{99}$$

The explicit construction of the dressed FAS allow us to reveal the structure of the FAS at $\lambda \simeq \lambda^\pm_j$. Here we first formulate the expansions of $\boldsymbol{a}^\pm(\lambda)$, $\boldsymbol{c}^\pm(\lambda)$ and their inverse in the vicinities of $\lambda^\pm_j$

$$\boldsymbol{a}^\pm(\lambda) = \boldsymbol{a}^\pm_j + (\lambda - \lambda^\pm_j)\dot{\boldsymbol{a}}^\pm(\lambda) + \mathcal{O}((\lambda - \lambda^\pm_j)^2) \tag{100a}$$
$$\boldsymbol{c}^\pm(\lambda) = \boldsymbol{c}^\pm_j + (\lambda - \lambda^\pm_j)\dot{\boldsymbol{c}}^\pm(\lambda) + \mathcal{O}((\lambda - \lambda^\pm_j)^2) \tag{100b}$$

$$\hat{\boldsymbol{a}}^{\pm}(\lambda) = \frac{\hat{\boldsymbol{a}}_j^{\pm}}{(\lambda - \lambda_j^{\pm})} + \hat{\boldsymbol{a}}^{\pm}(\lambda) + \mathcal{O}((\lambda - \lambda_j^{\pm})) \tag{100c}$$

$$\hat{\boldsymbol{c}}^{\pm}(\lambda) = \frac{\hat{\boldsymbol{c}}_j^{\pm}}{(\lambda - \lambda_j^{\pm})} + \hat{\boldsymbol{c}}^{\pm}(\lambda) + \mathcal{O}((\lambda - \lambda_j^{\pm})) \tag{100d}$$

where

$$\boldsymbol{a}_j^{\pm} = (\mathbb{1} - P_{11,j}^{+})\boldsymbol{a}_{0,j}^{+}, \qquad \hat{\boldsymbol{a}}_j^{\pm} = (\lambda_j^{+} - \lambda_j^{-})\hat{\boldsymbol{a}}_{0,j}^{+} P_{11,j}^{+} \tag{101a}$$

$$\boldsymbol{c}_j^{\pm} = \boldsymbol{c}_{0,j}^{+}(\mathbb{1} - P_{11,j}^{+}), \qquad \hat{\boldsymbol{c}}_j^{\pm} = (\lambda_j^{-} - \lambda_j^{+})P_{11,j}^{+}\hat{\boldsymbol{c}}_{0,j}^{+}. \tag{101b}$$

Let us outline the structure of the eigenspaces corresponding to the discrete eigenvalues $\lambda_j^{\pm}$. To avoid technicalities in doing this we will assume that: i) L has no other discrete eigenvalues and ii) that the regular potential $Q(x)$ is on finite support. Due to ii) one can prove that all Jost solutions and their inverse, as well as $T(\lambda)$ and $\hat{T}(\lambda)$ are meromorphic functions of λ and can be extended to the whole complex λ-plane. Considering equations (18), (19) at $\lambda = \lambda_j^{\pm}$ we derive the relations

$$|\phi_j^{\pm}(x)\rangle\hat{\boldsymbol{a}}_j^{\pm} = \pm|\psi_j^{\pm}(x)\rangle\rho_j^{\pm}, \qquad \rho_j^{\pm} = \boldsymbol{b}_j^{\pm}\hat{\boldsymbol{a}}_j^{\pm} \tag{102a}$$

$$|\psi_j^{\pm}(x)\rangle\hat{\boldsymbol{c}}_j^{\pm} = \pm|\phi_j^{\pm}(x)\rangle\tau_j^{\pm}, \qquad \tau_j^{\pm} = \boldsymbol{d}_j^{\mp}\hat{\boldsymbol{c}}_j^{\pm} \tag{102b}$$

where the index j means that we are taking the value of the corresponding function for $\lambda = \lambda_j^{+}$.

But the eigenfunctions corresponding to the discrete eigenvalues must be square integrable. A necessary condition for this is the requirement that these eigenfunctions have no exponentially growing terms for both $x \to \infty$ and $x \to -\infty$. These limits for $\chi_j^{+}(x) \equiv \chi^{+}(x, \lambda_j^{+}) = (|\phi_j^{+}(x)\rangle, |\psi_j^{+}(x)\rangle)$ are equal to

$$\lim_{x\to\infty} \chi_j^{+}(x) = \mathrm{e}^{-\mathrm{i}\lambda_j^{+} Jx}\begin{pmatrix} \boldsymbol{a}_j^{+} & 0 \\ \boldsymbol{b}_j^{+} & \mathbb{1} \end{pmatrix}, \qquad \lim_{x\to-\infty} \chi_j^{+}(x) = \mathrm{e}^{-\mathrm{i}\lambda_j^{+} Jx}\begin{pmatrix} \mathbb{1} & \boldsymbol{d}_j^{-} \\ 0 & \boldsymbol{c}_j^{+} \end{pmatrix}. \tag{103}$$

Since generically $\boldsymbol{a}_j^{+} \neq 0$ and $\boldsymbol{c}_j^{+} \neq 0$ some of the columns of $\chi_j^{+}(x)$ will be exponentially growing and can not be interpreted as discrete eigenfunctions. However from equations (96) it is clear that both $\boldsymbol{a}_j^{+}$ and $\boldsymbol{c}_j^{+}$ are degenerate matrices of rank $n - r_j$ and $m - r_j$, respectively. Thus we find that r_j linear combinations of columns of $|\phi_j^{+}(x)\rangle$ and $|\psi_j^{+}(x)\rangle$ due to the relations

$$\boldsymbol{a}_j^{+}|n_{0,j}^{1}\rangle = 0, \qquad \boldsymbol{c}_j^{+}|n_{0,j}^{2}\rangle = 0 \tag{104}$$

decrease exponentially for both $x \to \infty$ and $x \to -\infty$. The eigenspace of L related to λ_j^{+} is spanned by r_j linearly independent discrete eigenfunctions which can be chosen among $|\phi_j^+(x)n_{0,j}^1\rangle$ or $|\psi_j^+(x)n_{0,j}^2\rangle$. These two sets of eigenfunctions satisfy linear relations which can be written compactly as

$$\chi^+(x,\lambda_j^+)|n_{0,j}\rangle \overset{(79)}{=} (\mathbb{1} - P_j(x))|n_j(x)\rangle = 0. \tag{105}$$

The r_j constant vectors $|n_{0,j}\rangle$ which determine the discrete eigenspace can be viewed as 'polarization' vectors of the corresponding soliton solution and parametrize its internal degrees of freedom.

The eigenspace corresponding to λ_j^- can be analyzed either independently along the same lines or by using the reduction properties of L, see subsection 2.4. Indeed, from (96b) and (96d) we get that only r_j linear combinations of the columns of $|\phi_j^-(x)\rangle$ (resp. $|\psi_j^-(x)\rangle$) do not have exponential growth for $x \to \infty$ (resp. $x \to -\infty$). The corresponding 'polarization' vectors $|n_{0,j}^+\rangle$ are related to $|n_{0,j}\rangle$ by

$$|n_{0,j}^+\rangle = D_{0,j}^+|n_{0,j}\rangle, \qquad D_{0,j}^+ = D_0^+(\lambda_j^+) \tag{106}$$

because

$$\boldsymbol{a}_j^-\boldsymbol{c}_{0,j}^+|n_{0,j}^2\rangle = 0, \qquad \boldsymbol{c}_j^-\boldsymbol{a}_{0,j}^+|n_{0,j}^1\rangle = 0. \tag{107}$$

The analog of equation (105) for $\chi_j^-(x)$ is the following relation

$$\chi_j^-(x)|n_{0,j}^+\rangle = 0, \qquad |n_{0,j}^+\rangle = D_0^+(\lambda_j^+)|n_{0,j}\rangle. \tag{108}$$

Remark 5. *Each of the eigenvalues $\lambda_j^\pm$ corresponds to r_j-dimensional eigensubspace with $1 \leq r_j \leq \min(n,m)$. In particular for the vector NLS (the Manakov model) we may have only $r_j = 1$. The reduction* (41) *relates the sets of polarization vectors by*

$$B^{-1}|m_{0,j}^\dagger\rangle = |n_{0,j}\rangle. \tag{109}$$

3.3. Reflectionless Potentials and Soliton Solutions

The simplest situation in which the dressing method can be applied is the one corresponding to vanishing potential and plane wave solution of (7)

$$Q_0(x) = 0, \qquad \chi^\pm(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx - 2\mathrm{i}\lambda^2 Jt}. \tag{110}$$

The corresponding scattering matrix of course is $T_0(\lambda) = \mathbb{1}$; therefore the corresponding reflection coefficients (21) are vanishing $\rho_0^\pm(\lambda) = 0$, $\tau_0^\pm(\lambda) = 0$ on the whole real λ axis.

Applying the dressing method to (110) we obtain for the simplest reflectionless potential

$$Q_1(x) = -(\lambda_j^+ - \lambda_j^-)[J, P_j(x,t)]. \tag{111}$$

Obviously, due to (96) the corresponding reflection coefficients $\rho_1^\pm(\lambda) = 0$, $\tau_1^\pm(\lambda) = 0$ are also vanishing on the whole real λ axis. Therefore $Q_1(x)$ is the simplest non-trivial reflectionless potential of L. The scattering matrix for this potential is block-diagonal, but is not trivial

$$T_{1s}(\lambda) = \begin{pmatrix} u^{+}_{11,j}(\lambda) & 0 \\ 0 & \hat{u}^{-}_{22,j}(\lambda) \end{pmatrix}. \tag{112}$$

The reflectionless potentials are closely related to the soliton solutions of the corresponding equation. Indeed, the derivation of the reflectionless potentials we considered the time t as an auxiliary parameter. In order to go from the reflectionless potential to the solution of the corresponding NLEE all we need to do is to impose the correct t-dependence on the 'polarization vectors' $|n_{0,j}\rangle$ and $\langle m_{0,j}|$. This can be determined from the dispersion law of the NLEE as follows

$$|n_{0,j}\rangle \to \exp(-2\mathrm{i}f(\lambda_j^+)t)|n_{0,j}\rangle, \qquad \langle m_{0,j}| \to \langle m_{0,j}|\exp(2\mathrm{i}f(\lambda_j^-)t). \tag{113}$$

Applying this procedure to Q_1 we get the one-soliton solution of the generic NLEE with dispersion law $f(\lambda)$ in the form

$$\begin{aligned} Q_{1s}(x,t) &= 2(\lambda_j^+ - \lambda_j^-) \\ &\times \begin{pmatrix} 0 & -|n_j^1(x,t)\rangle R_j^{-1}(x,t)\langle m_j^2(x,t)| \\ |n_j^2(x,t)\rangle R_j^{-1}(x,t)\langle m_j^1(x,t)| & 0 \end{pmatrix} \end{aligned} \tag{114}$$

$$\begin{aligned} |n_j^1(x,t)\rangle &= \mathrm{e}^{-\mathrm{i}(\lambda_j^+ x + 2f_{0,j}^+ t)}|n_{0,j}^1\rangle, \qquad |n_j^2(x,t)\rangle = \mathrm{e}^{\mathrm{i}(\lambda_j^+ x + 2f_{0,j}^+ t)}|n_{0,j}^2\rangle \\ R_j(x,t) &= \langle m_j^1(x,t)|n_j^1(x,t)\rangle + \langle m_j^2\ (x,t)|n_j^2(x,t)\rangle \\ \langle m_j^1(x,t)| &= \langle m_{j,0}^1|\mathrm{e}^{\mathrm{i}(\lambda_j^- x + 2f_{0,j}^-)t)}, \qquad \langle m_j^2(x,t)| = \langle m_{j,0}^2|\mathrm{e}^{-\mathrm{i}(\lambda_j^- x + 2f_{0,j}^-)t)} \end{aligned}$$

where $f_{0,j}^{\pm} = f_0(\lambda_j^{\pm})$ and $f(\lambda) = f_0(\lambda)J$ is the dispersion law of the NLEE. In order to obtain the one-soliton solution for the MNLS we have to replace $f_0(\lambda)J$ by $f_{0,\mathrm{NLS}} = -2\lambda^2$. If we put $r_j = 1$ and take into account the reduction (41) then $R_j(x,t)$ can be written down as

$$R_j(x,t) = 2R_{0,j}\cosh(2\nu_j x + 2\tilde{\mu}_j t + \xi_{0,j})$$

where

$$\begin{aligned} \nu_j &= \mathrm{i}(\lambda_j^- - \lambda_j^+)/2, \qquad \tilde{\mu}_j = \mathrm{i}(f(\lambda_j^-) - f(\lambda_j^+)) \\ R_{0,j} &= \sqrt{\langle m_{j,0}^1|n_{0,j}^1\rangle\langle m_{j,0}^2|n_{0,j}^2\rangle}, \qquad \xi_{0,j} = \frac{1}{2}\ln\frac{\langle m_{j,0}^1|n_{0,j}^1\rangle}{\langle m_{j,0}^2|n_{0,j}^2\rangle} \end{aligned}$$

thus reproducing the well know result for the one-soliton solution of the vector NLS and the rank-1 solutions of the MNLS [2, 34].

We can apply the dressing procedure again, starting with the potential $Q_{1s}(x,t)$ and the corresponding FAS $\chi_{1s}^{\pm}(x,\lambda)$. Doing this we have to use a dressing factor $u_{2s}(x,\lambda)$ like in (80) but with new locations $\lambda_k^{\pm}$ for the eigenvalues and a new choice for the projector $P_k(x)$ which may have different rank r_k and sets of vectors $|n_k(x)\rangle$, $\langle m_k(x)|$. Choosing $\lambda_k^\pm \neq \lambda_j^\pm$ the FAS $\chi_{1s}^\pm(x,\lambda)$ will be regular at the points $\lambda = \lambda_k^\pm$ which is all that is required for the procedure to be valid.

Repeating the procedure several times we get more and more complicated potentials which are still reflectionless.

Another way to derive these potentials and the related N-soliton solutions of the NLEE consists in using equations (54) and (55) with $K^+(x,\lambda) = K^-(x,\lambda) = 0$. In order to explain this better and following [34] we need to evaluate the residues of $X^\pm(x,\lambda)\hat{D}^\pm(\lambda)$ at $\lambda = \lambda_j^\pm$. Using equation (102) we obtain

$$\operatorname*{Res}_{\lambda=\lambda_j^+} X^+(x,\lambda)\hat{D}^+(\lambda) = X_j^+(x)\mathcal{K}_j^+(x) \tag{115}$$

$$\operatorname*{Res}_{\lambda=\lambda_j^-} X^-(x,\lambda)\hat{D}^-(\lambda) = -X_j^-(x)\mathcal{K}_j^-(x) \tag{116}$$

where $X_j^\pm(x) = X^\pm(x,\lambda_j^\pm)$ and

$$\mathcal{K}_j^+(x) = \begin{pmatrix} 0 & \rho_j^+(x) \\ \tau_j^+(x) & 0 \end{pmatrix}, \qquad \mathcal{K}_j^-(x) = \begin{pmatrix} 0 & \tau_j^-(x) \\ \rho_j^-(x) & 0 \end{pmatrix} \tag{117}$$
$$\rho_j^\pm(x) = \rho_j^\pm \mathrm{e}^{\pm 2\mathrm{i}\lambda_j^\pm x}, \qquad \tau_j^\pm(x) = \tau_j^\pm \mathrm{e}^{\mp 2\mathrm{i}\lambda_j^\pm x}.$$

Thus the equations (54), (55) take the form

$$X^+(x,\lambda) = \mathbb{1} - \sum_{j=1}^N \frac{1}{\lambda - \lambda_j^-} X_j^-(x)\mathcal{K}_j^-(x) \tag{118}$$

$$X^-(x,\lambda) = \mathbb{1} + \sum_{j=1}^N \frac{1}{\lambda - \lambda_j^+} X_j^+(x)\mathcal{K}_j^+(x). \tag{119}$$

Note that it is legitimate to put $\lambda = \lambda_k^+$ in equation (118) and $\lambda = \lambda_k^-$ in equation (119) thus getting an algebraic system of equations for $X_j^\pm(x)$. After solving for $X_j^\pm(x)$ we can insert them into equations (54), (55) thus recovering the FAS $X^\pm(x,\lambda)$ for all $\lambda \in \mathbb{C}_\pm$. The corresponding reflectionless potential is given by (see equation (76))

$$Q^{(N)}(x) = -\sum_{j=1}^N [J, X_j^-(x)\mathcal{K}_j^-(x)] = \sum_{j=1}^N [J, X_j^+(x)\mathcal{K}_j^+(x)]. \tag{120}$$

As we shall see in the next subsection the generalized Zakharov–Shabat system (7) with potential $Q^{(N)}(x)$ has $2N$ discrete eigenvalues located at $\lambda_j^\pm$ with r_j-dimensional eigenspaces. The corresponding N-soliton solution $Q_{\mathrm{Ns}}(x,t)$ is obtained from (120) by introducing the relevant t-dependence into the polarizations vectors $|n_{0,j}\rangle$ and $\langle m_{0,j}|$ or equivalently, in $\mathcal{K}_j^{\pm}(x,t)$ as follows

$$\rho_j^{\pm}(x,t) = \rho_j^{\pm}(x)\mathrm{e}^{\pm 2\mathrm{i} f_{0,j}^{\pm} t}, \qquad \tau_j^{\pm}(x,t) = \tau_j^{\pm}(x)\mathrm{e}^{\mp 2\mathrm{i} f_{0,j}^{\pm} t} \tag{121}$$

where $f_{0,j}^{\pm} = f_0(\lambda_j^{\pm})$ and $f(\lambda) = f_0(\lambda)J$ is the dispersion law of the NLEE.

3.4. Spectral Properties of L

Using the FAS one can construct the resolvent of (7)

$$\begin{aligned} R^{\pm}(x,y,\lambda) &= -\mathrm{i}\chi^{\pm}(x,\lambda)\Theta^{\pm}(x-y)\hat{\chi}^{\pm}(y,\lambda) \\ \Theta^{\pm}(z) &= \mathrm{diag}(\mp\theta(\mp z)\mathbb{1}, \pm\theta(\pm z)\mathbb{1}) \end{aligned} \tag{122}$$

where $\theta(z)$ is the standard step-function.

Let us consider $R^{\pm}(x,y,\lambda)$ as the kernel of an integral operator acting on vector-valued functions of x as follows

$$(\boldsymbol{R}_\lambda f)(x) = \int_{-\infty}^{\infty} \mathrm{d}\, y R^{\pm}(x,y,\lambda) f(y) \quad \text{for} \quad \lambda \in \mathbb{C}_{\pm}. \tag{123}$$

Theorem 2. *Let $Q(x)$ satisfy conditions* (C.1) *and* (C.2) *and let $\lambda_j^{\pm}$ be the zeroes of* $\det \boldsymbol{a}^{\pm}(\lambda)$. *Then*

1. *$R^{\pm}(x,y,\lambda)$ is an analytic function of λ for $\lambda \in \mathbb{C}_{\pm}$ having pole singularities at $\lambda_j^{\pm} \in \mathbb{C}_{\pm}$;*
2. *$R^{\pm}(x,y,\lambda)$ is a kernel of a bounded integral operator for* $\operatorname{Im}\lambda \neq 0$;
3. *$R^{\pm}(x,y,\lambda)$ is uniformly bounded function for $\lambda \in \mathbb{R}$ and provides a kernel of an unbounded integral operator;*
4. *$R^{\pm}(x,y,\lambda)$ satisfy the equation*

$$L(\lambda)R^{\pm}(x,y,\lambda) = \mathbb{1}\delta(x-y). \tag{124}$$

Idea of the proof:

1. is obvious from the fact that $\chi^{\pm}(x,\lambda)$ are the FAS of $L(\lambda)$. From the definition (28) there follows that $\det\chi^{\pm}(x,\lambda) = \det \boldsymbol{a}^{\pm}(\lambda)$, i.e. $\hat{\chi}^{\pm}(y,\lambda)$ and consequently, $R^{\pm}(x,y,\lambda)$ will develop pole singularities for all $\lambda_j^{\pm}$ for which $\det \boldsymbol{a}^{\pm}(\lambda) = 0$.
2. Assume that $\operatorname{Im}\lambda > 0$ and consider the asymptotic behavior of $R^{+}(x,y,\lambda)$ for $x, y \to \infty$. From equations (28), (29) we find that

$$R^{+}(x,y,\lambda) = \sum_{p=1}^{n} X^{+}(x,\lambda)\mathrm{e}^{-\mathrm{i}\lambda J(x-y)}\Theta^{+}(x-y)\hat{X}^{+}(y,\lambda) \tag{125}$$

Due to the fact that $\chi^{+}(x,\lambda)$ has block-triangular asymptotics for $x \to \infty$ and $\lambda \in \mathbb{C}_{+}$ and for the correct choice of $\Theta^{+}(x-y)$ (122) we check that the right hand side of (125) falls off exponentially for $x \to \infty$ and arbitrary choice of y. All other possibilities are treated analogously.

3. For $\lambda \in \mathbb{R}$ the arguments of item 2 can not be applied because the exponentials in the right hand side of (125) $\operatorname{Im}\lambda = 0$ only oscillate. Thus we conclude that $R^{\pm}(x,y,\lambda)$ for $\lambda \in \mathbb{R}$ is only a bounded function and thus the corresponding operator $R(\lambda)$ is an unbounded integral operator.
4. The proof of equation (124) follows from the fact that $L(\lambda)\chi^{+}(x,\lambda) = 0$ and
$$\frac{\mathrm{d}\Theta^{\pm}(x-y)}{\mathrm{d}x} = \mathbb{1}\delta(x-y). \tag{126}$$

The theorem is proved. □

From theorem 2, item 3 there follows that the continuous spectrum of L fills up the whole real λ-axis with multiplicity $n+m$. By definition the operator L may also have discrete eigenvalues at the points at which $R^{\pm}(x,y,\lambda)$ have pole singularities. From item 1 it follows that these are precisely the points $\lambda_j^{\pm}$.

Let us now analyze the structure of these singularities and evaluate the corresponding residues. To this end we insert equations (97), (99) into (122). The result is that $R^{\pm}(x,y,\lambda)$ have poles of first order in the neighborhood of $\lambda_j^{\pm}$ with residues

$$\operatorname*{Res}_{\lambda=\lambda_j^{+}} R^{+}(x,y,\lambda) = -\mathrm{i}(\lambda_j^{+}-\lambda_j^{-})(\mathbb{1}-P_j(x))\chi_{0,j}^{+}(x)\Theta^{+}(x-y)\hat{\chi}_{0,j}^{+}(y)P_j(y) \tag{127}$$

$$\operatorname*{Res}_{\lambda=\lambda_j^{-}} R^{-}(x,y,\lambda) = \mathrm{i}(\lambda_j^{+}-\lambda_j^{-})P_j(x)\chi_{0,j}^{-}(x)\Theta^{-}(x-y)\hat{\chi}_{0,j}^{-}(y)(\mathbb{1}-P_j(y)). \tag{128}$$

Formally in the right hand side of (127) there enter the discontinuous functions $\Theta^{\pm}(x-y)$. However, due to the special structure of the projectors $P_j(x)$ (see equations (82) and (102a), (102b) we obtain the following expressions for the residues of $R^{\pm}(x,y,\lambda)$

$$\operatorname*{Res}_{\lambda=\lambda_j^{\pm}} R^{\pm}(x,y,\lambda) = \pm\mathrm{i}|\psi_j^{\pm}(x)\rangle\rho_j^{\pm}\langle\psi_j^{\pm}(y)| \tag{129}$$

where $\rho_j^{\pm}$ are defined in equation (102) and there are no discontinuities present.

Now we can derive the completeness relation for the eigenfunctions of the Lax operator (7) by applying the contour integration method (see, e.g. [23, 1]) to the integral

$$\mathcal{J}(x,y) = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_+}\mathrm{d}\,\lambda R^{+}(x,y,\lambda) - \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_-}\mathrm{d}\,\lambda R^{-}(x,y,\lambda) \tag{130}$$

where the contours $\gamma_{\pm}$ are shown on the Fig. 1. Skipping the details we get

$$\begin{aligned}
\delta(x-y)J \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\,\lambda \left\{ |\phi^+(x,\lambda)\rangle \hat{\boldsymbol{a}}^+(\lambda)\langle \hat{\psi}^+(y,\lambda)| - |\phi^-(x,\lambda)\rangle \hat{\boldsymbol{a}}^-(\lambda)\langle \hat{\psi}^-(y,\lambda)| \right\} \\
&\quad - \mathrm{i}\sum_{j=1}^{N} \left(|\psi_j^+(x)\rangle \rho_j^+ \langle \psi_j^+(y)| - |\psi_j^-(x)\rangle \rho_j^- \langle \psi_j^-(y)| \right). \qquad (131)
\end{aligned}$$

The completeness relation (131) is a natural generalization of the one in [23] for the $\mathfrak{sl}(2)$ case. An important difference here is that now we have matrix-valued spectral functions $\boldsymbol{a}^{\pm}(\lambda)$ whose zeroes determine the location of the discrete eigenvalues.

Remark 6. *When both $n > 1$ and $m > 1$ there are two possible definitions of simple eigenvalues. One of them used in* [2] *is to define $\lambda_j^{\pm}$ as simple if* $\det \boldsymbol{a}^{\pm}(\lambda)$ *have simple zeroes for $\lambda = \lambda_j^{\pm}$. The other possible definition is: the eigenvalues $\lambda_j^{\pm}$ are simple if the resolvent of L has simple poles at $\lambda_j^{\pm}$. The singular solutions of the RHP* (80) *correspond to simple poles of the resolvent* (122) *although in the neighborhood of $\lambda \simeq \lambda_j^{\pm}$ we have that* $\det \boldsymbol{a}^{\pm}(\lambda)$ *behaves like $(\lambda - \lambda_j^{\pm})^{r_j}$.*

4. The Generalized Fourier Transforms for Nonregular J

The main result in this section consists in the fact that the analysis of [15] can be applied also to nonregular choices of J. In addition we briefly outline also how to take into account the presence of discrete spectrum of L. Skipping the details of the proof we formulate the completeness relation for the 'squared' solutions of L.

4.1. The Wronskian Relations

The analysis of the mapping $\mathcal{F}\colon \mathcal{M} \to \mathcal{T}$ between the class of allowed potentials $\mathcal{M}$ and the scattering data of L starts with the **Wronskian relations**, see [6, 7] for $\mathfrak{sl}(2)$-case and [18] for the block-matrix case (7). As we shall see, they would allow us

1. to formulate the idea that the ISM is a GFT;
2. to determine explicitly the proper generalizations of the usual exponents;
3. to introduce the skew-scalar product on $\mathcal{M}$ which provides it with a symplectic structure.

All these ideas will be worked out for the system (7). With (7) one can associate the systems

$$\mathrm{i}\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}x} - \hat{\psi}(x,t,\lambda)U(x,t,\lambda) = 0, \qquad U(x,\lambda) = Q(x) - \lambda J \qquad (132)$$

$$\mathrm{i}\frac{\mathrm{d}\delta\psi}{\mathrm{d}x} + \delta U(x,t,\lambda)\psi(x,t,\lambda) + U(x,t,\lambda)\delta\psi(x,t,\lambda) = 0 \qquad (133)$$

$$
\mathrm{i}\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}x} - \lambda J\psi(x,t,\lambda) + U(x,t,\lambda)\dot{\psi}(x,t,\lambda) = 0 \tag{134}
$$

where $\delta\psi$ corresponds to a given variation $\delta Q(x,t)$ of the potential, while by dot we denote the derivative with respect to the spectral parameter.

We start with the identity

$$
(\hat{\chi}J\chi(x,\lambda) - J)|_{x=-\infty}^{\infty} = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \hat{\chi}[Q(x),J]\chi(x,\lambda) \tag{135}
$$

where $\chi(x,\lambda)$ can be any fundamental solution of L. For convenience we choose them to be the FAS introduced above.

The left hand side of (135) can be calculated explicitly by using the asymptotics of $\chi^{\pm}(x,\lambda)$ for $x \to \pm\infty$, (28), (29). It would be expressed by the matrix elements of the scattering matrix $T(\lambda)$, i.e. by the scattering data of L as follows

$$
(\hat{\chi}^{+}J\chi^{+}(x,\lambda) - J)|_{x=-\infty}^{\infty} = -2\begin{pmatrix} 0 & \boldsymbol{d}^{-}(\lambda) \\ \boldsymbol{b}^{+}(\lambda) & 0 \end{pmatrix} \tag{136a}
$$

$$
(\hat{\chi}^{-}J\chi^{-}(x,\lambda) - J)|_{x=-\infty}^{\infty} = -2\begin{pmatrix} 0 & \boldsymbol{b}^{-}(\lambda) \\ \boldsymbol{d}^{+}(\lambda) & 0 \end{pmatrix}. \tag{136b}
$$

We will show that these Wronskian relations allow us to express the elements of each of the sets $\mathcal{T}_i$, $i = 1,2$ in equation (56) as integrals from the potential $Q(x)$ multiplied by some bilinear combination of eigenfunctions of L called 'squared solutions'. Let us now analyze the equations obtained from (135) after multiplying by the matrix E_{ab}, $(E_{ab})_{cd} = \delta_{ac}\delta_{bd}$ and taking the trace. Such operation will produce the a,b-matrix element of equation (135). In the right hand side of this equation we can use the invariance properties of the trace and rewrite it in the form:

$$
\mathrm{tr}((\hat{\chi}J\chi(x,\lambda) - J)E_{ab})|_{x=-\infty}^{\infty} = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}([Q(x),J]\boldsymbol{e}_{ab}(x,\lambda)) \tag{137}
$$

where

$$
e_{ab}(x,\lambda) = \chi E_{ab}\hat{\chi}(x,\lambda), \qquad \boldsymbol{e}_{ab}(x,\lambda) = P_{0J}(\chi E_{ab}\hat{\chi}(x,\lambda)) \tag{138}
$$

are the natural generalization of the 'squared solutions' introduced first for the $\mathfrak{sl}(2)$-case [32, 30]. By P_{0J} we have denoted the projector $P_{0J} = \mathrm{ad}_J^{-1}\mathrm{ad}_J$ on the block-off-diagonal part of the corresponding matrix-valued function.

The **squared solutions** obviously satisfy the equation

$$
\mathrm{i}\frac{\mathrm{d}\boldsymbol{e}_{ab}}{\mathrm{d}x} + [Q(x) - \lambda J, \boldsymbol{e}_{ab}(x,\lambda)] = 0. \tag{139}
$$

The indices a, b in (138)) are taking values in appropriate ranges; for convenience below i, h, k (resp. by l, r, s) will be taking values in the ranges

$$
1 \le i,h,k \le n, \qquad n+1 \le l,r,s \le n+m \tag{140}
$$

and by $i < r$ we will mean $1 \leq i \leq n$ and $n+1 \leq r \leq n+m$. We also introduce

$$\begin{aligned} \Psi_{ab}^{\pm}(x,\lambda) &= |\psi^{\pm}\rangle \epsilon_{ab} \langle \hat{\psi}^{\pm}|, & \Phi_{ab}^{\pm}(x,\lambda) &= |\phi^{\pm}\rangle \epsilon_{ab} \langle \hat{\phi}^{\pm}| \\ \Theta_{ab}^{\pm}(x,\lambda) &= |\phi^{\pm}\rangle \epsilon_{ab} \langle \hat{\psi}^{\pm}|, & \Xi_{ab}^{\pm}(x,\lambda) &= |\psi^{\pm}\rangle \epsilon_{ab} \langle \hat{\phi}^{\pm}| \end{aligned} \tag{141}$$

where by ϵ_{ab} we mean the relevant non-vanishing blocks of the matrices E_{ab}, e.g.

$$\begin{aligned} E_{ih} &= \begin{pmatrix} \epsilon_{ih} & 0 \\ 0 & 0 \end{pmatrix}, & E_{rs} &= \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{rs} \end{pmatrix} \\ E_{ir} &= \begin{pmatrix} 0 & \epsilon_{ir} \\ 0 & 0 \end{pmatrix}, & E_{ri} &= \begin{pmatrix} 0 & 0 \\ \epsilon_{ri} & 0 \end{pmatrix}. \end{aligned} \tag{142}$$

Using these definitions and equations (28), (29) we find that the 'squared solutions' can be expressed in terms of the Jost solutions as follows

$$\begin{aligned} e_{li}^{+}(x,\lambda) &= \hat{a}_{ih}^{+}(\lambda)\Psi_{lh}^{+}(x,\lambda), & e_{il}^{+}(x,\lambda) &= \hat{c}_{lr}^{+}(\lambda)\Phi_{ir}^{+}(x,\lambda) \\ e_{ih}^{+}(x,\lambda) &= \hat{a}_{hk}^{+}(\lambda)\Theta_{ik}^{+}(x,\lambda), & e_{lr}^{+}(x,\lambda) &= \hat{c}_{rs}^{+}(\lambda)\Xi_{ls}^{+}(x,\lambda) \end{aligned} \tag{143}$$

$$\begin{aligned} e_{il}^{-}(x,\lambda) &= \hat{a}_{lr}^{-}(\lambda)\Psi_{ir}^{-}(x,\lambda), & e_{li}^{-}(x,\lambda) &= \hat{c}_{ih}^{-}(\lambda)\Phi_{lh}^{-}(x,\lambda) \\ e_{ih}^{-}(x,\lambda) &= \hat{c}_{hk}^{-}(\lambda)\Theta_{ik}^{-}(x,\lambda), & e_{rs}^{-}(x,\lambda) &= \hat{a}_{sl}^{+}(\lambda)\Xi_{rl}^{-}(x,\lambda). \end{aligned} \tag{144}$$

Here we assume summation over the repeated indices in the relevant range (140).

We will also need the skew-scalar product

$$[\![X,Y]\!] = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \ \mathrm{tr}(X(x),[J,Y(x)]) \tag{145}$$

which is non-degenerate for $X(x), Y(x) \in \mathcal{M}$. Then we can write down the reflection coefficients $\rho_{ab}^{\pm}$ in the form

$$\rho_{lk}^{+}(\lambda) = -\mathrm{i}[\![Q(y), e_{il}^{+}(y,\lambda)]\!]\hat{a}_{ik}^{+} \tag{146}$$

and similar expressions for the other reflection coefficients. Thus we have a formula analogous to the standard Fourier transform in which $e_{il}^{+}(y,\lambda)$ can be viewed as the generalizations of the standard exponentials.

In order to work out the contributions from the discrete spectrum of L we will need the first two coefficients in the Taylor expansions of $\Psi_{ab}^{\pm}(x,\lambda)$ and $\Phi_{ab}^{\pm}(x,\lambda)$

$$\begin{aligned} \Psi_{ab}^{\pm}(x,\lambda) &= \Psi_{ab;j}^{\pm}(x) + (\lambda-\lambda_j^{\pm})\dot{\Psi}_{ab;j}^{\pm}(x) + \mathcal{O}((\lambda-\lambda_j^{\pm})^2) \\ \Phi_{ab}^{\pm}(x,\lambda) &= \Phi_{ab;j}^{\pm}(x) + (\lambda-\lambda_j^{\pm})\dot{\Phi}_{ab;j}^{\pm}(x) + \mathcal{O}((\lambda-\lambda_j^{\pm})^2) \end{aligned} \tag{147}$$

where

$$\Phi_{ab;j}^{\pm} = \Phi_{ab}^{+}(x,\lambda_j^{\pm}), \qquad \dot{\Phi}_{ab;j}^{\pm} = \left.\frac{\mathrm{d}\Phi_{ab}^{+}(x,\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_j^{\pm}}. \tag{148}$$

Indeed, from equation (141) there follows that these are regular functions for all $\lambda \in \mathbb{C}_\pm$. However the 'squared solutions' $e^{\pm}_{ab}(x,\lambda)$ have pole singularities of first order in the vicinities of $\lambda_j^\pm$ and therefore we will have

$$e^{\pm}_{ab}(x,\lambda) \simeq \frac{e^{\pm}_{ab;j}(x)}{(\lambda-\lambda_j^\pm)} + \dot{e}^{\pm}_{ab;j}(x) + \mathcal{O}(\lambda-\lambda_j^\pm) \tag{149}$$

where for $a = i$ and $b = l$ we have

$$e^{+}_{il;j}(x) = \hat{\boldsymbol{a}}^{+}_{ih;j}\Psi^{+}_{lh;j}(x), \qquad \dot{e}^{+}_{il;j}(x) = \hat{\dot{\boldsymbol{a}}}^{+}_{ih;j}\Psi^{+}_{lh;j}(x) + \hat{\boldsymbol{a}}^{+}_{ih;j}\dot{\Psi}^{+}_{lh;j}(x) \tag{150}$$

and similar expressions for $e^{\pm}_{ab;j}(x)$ and $\dot{e}^{\pm}_{ab;j}(x)$ with different choices for the indices a, b.

The second type of Wronskian relations, which we will consider relate the variation of the potential $\delta Q(x)$ to the corresponding variations of the scattering data. To this purpose we start with the identity

$$\hat{\chi}\delta\chi(x,\lambda)|_{x=-\infty}^{\infty} = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{i}\hat{\chi}\delta\chi)(x,\lambda). \tag{151}$$

To calculate the integrand in (151) we need to use the equation (133), satisfied by $\delta\chi(x,\lambda)$. Making use of equations (7) and (133) we get

$$\hat{\chi}\delta\chi(x,\lambda)|_{x=-\infty}^{\infty} = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \hat{\chi}\delta Q(x)\chi(x,\lambda). \tag{152}$$

We apply ideas similar to the ones above. Evaluating the left hand side of (152) with $\chi(x,\lambda) \equiv \chi^+(x,\lambda)$ and $\chi(x,\lambda) \equiv \chi^-(x,\lambda)$ we find

$$\hat{\chi}^+\delta\chi^+(x,\lambda)|_{x=-\infty}^{\infty} = \begin{pmatrix} \hat{\boldsymbol{a}}^+\delta\boldsymbol{a}^+(\lambda) & -\delta\tau^+(\lambda)\boldsymbol{c}^+(\lambda) \\ \delta\rho^+(\lambda)\boldsymbol{a}^+(\lambda) & -\hat{\boldsymbol{c}}^+\delta\boldsymbol{c}^+(\lambda) \end{pmatrix} \tag{153}$$

$$\hat{\chi}^-\delta\chi^-(x,\lambda)|_{x=-\infty}^{\infty} = \begin{pmatrix} \hat{\boldsymbol{c}}^-\delta\boldsymbol{c}^-(\lambda) & \delta\rho^-(\lambda)\boldsymbol{a}^-(\lambda) \\ -\delta\tau^-(\lambda)\boldsymbol{c}^-(\lambda) & -\hat{\boldsymbol{a}}^-(\lambda)\delta\boldsymbol{a}^-(\lambda) \end{pmatrix}. \tag{154}$$

Multiplying by E_{ab} and taking the trace we arrive at

$$\begin{aligned} \delta\rho^+_{lk}(\lambda) &= 2\mathrm{i}[\![e^+_{il}(y,\lambda), \mathrm{ad}_J^{-1}\,\delta Q(y)]\!]\hat{a}^+_{ik} \\ \delta\rho^-_{ik}(\lambda) &= -2\mathrm{i}[\![e^+_{li}(y,\lambda), \mathrm{ad}_J^{-1}\,\delta Q(y)]\!]\hat{a}^+_{lk}. \end{aligned} \tag{155}$$

These relations are basic in the analysis of the related NLEE and their Hamiltonian structures. Below we shall use them assuming that the variation of $Q(x)$ is due to its time evolution, and consider variations of the type

$$\delta Q(x,t) = Q_t\delta t + \mathcal{O}((\delta t)^2). \tag{156}$$

Keeping only the first order terms with respect to δt we find

$$\begin{aligned} \frac{\mathrm{d}\rho_{lk}^{+}(\lambda)}{\mathrm{d}t} &= 2\mathrm{i}[\![e_{il}^{+}(y,\lambda), \mathrm{ad}_J^{-1} Q_t(y)]\!]\hat{a}_{ik}^{+} \\ \frac{\mathrm{d}\rho_{ik}^{-}(\lambda)}{\mathrm{d}t} &= -2\mathrm{i}[\![e_{li}^{-}(y,\lambda), Q_t(y)]\!]\hat{a}_{lk}^{-}. \end{aligned} \tag{157}$$

4.2. Completeness of the 'Squared Solutions'

Let us introduce the sets of 'squared solutions'

$$\{\boldsymbol{\Psi}\} = \{\boldsymbol{\Psi}\}_{\mathrm{c}} \cup \{\boldsymbol{\Psi}\}_{\mathrm{d}}, \qquad \{\boldsymbol{\Phi}\} = \{\boldsymbol{\Phi}\}_{\mathrm{c}} \cup \{\boldsymbol{\Phi}\}_{\mathrm{d}} \tag{158}$$

$$\begin{aligned} \{\boldsymbol{\Psi}\}_{\mathrm{c}} &\equiv \left\{ \boldsymbol{\Psi}_{ri}^{+}(x,\lambda), \quad \boldsymbol{\Psi}_{ir}^{-}(x,\lambda), \quad i<r, \quad \lambda\in\mathbb{R} \right\} \\ \{\boldsymbol{\Psi}\}_{\mathrm{d}} &\equiv \left\{ \boldsymbol{\Psi}_{ri;j}^{+}(x), \quad \dot{\boldsymbol{\Psi}}_{ri;j}^{+}(x), \quad \boldsymbol{\Psi}_{ir;j}^{-}(x), \quad \dot{\boldsymbol{\Psi}}_{ir;j}^{-}(x) \right\}_{j=1}^{N} \end{aligned} \tag{159}$$

$$\begin{aligned} \{\boldsymbol{\Phi}\}_{\mathrm{c}} &\equiv \left\{ \boldsymbol{\Phi}_{ir}^{+}(x,\lambda), \quad \boldsymbol{\Phi}_{ri}^{-}(x,\lambda), \quad i<r, \quad \lambda\in\mathbb{R} \right\} \\ \{\boldsymbol{\Phi}\}_{\mathrm{d}} &\equiv \left\{ \boldsymbol{\Phi}_{ir;j}^{+}(x), \quad \dot{\boldsymbol{\Phi}}_{ir;j}^{+}(x), \quad \boldsymbol{\Phi}_{ri;j}^{-}(x), \quad \dot{\boldsymbol{\Phi}}_{ri;j}^{-}(x) \right\}_{j=1}^{N} \end{aligned} \tag{160}$$

where the subscripts 'c' and 'd' refer to the continuous and discrete spectrum of L. The 'squared solutions' in bold-face $\boldsymbol{\Psi}_{ri}^{+}, \ldots$ are obtained from $\Psi_{ri}^{+}, \ldots$ by applying the projector P_{0J}, i.e. $\boldsymbol{\Psi}_{ri}^{+}(x,\lambda) = P_{0J}\Psi_{ri}^{+}(x,\lambda)$, see equation (138).

Theorem 3 (see [18]). *The sets $\{\boldsymbol{\Psi}\}$ and $\{\boldsymbol{\Phi}\}$ form complete sets of functions in $\mathcal{M}_J$. The corresponding completeness relation has the form*

$$\begin{aligned} \delta(x-y)\Pi_{0J} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, (G^{+}(x,y,\lambda) - G^{-}(x,y,\lambda)) \\ - 2\mathrm{i}\sum_{j=1}^{N}(G_j^{+}(x,y) + G_j^{-}(x,y)) \end{aligned} \tag{161}$$

where

$$\Pi_{0J} = \sum_{i<r}(E_{ir}\otimes E_{ri} - E_{ri}\otimes E_{ir}) \tag{162}$$

$$\begin{aligned} G^{+}(x,y,\lambda) &= \sum_{i<r} \boldsymbol{e}_{ir}^{+}(x,\lambda)\otimes\boldsymbol{e}_{ri}^{+}(y,\lambda) \\ G^{-}(x,y,\lambda) &= \sum_{i<r} \boldsymbol{e}_{ri}^{-}(x,\lambda)\otimes\boldsymbol{e}_{ir}^{-}(y,\lambda) \end{aligned} \tag{163}$$

$$\begin{aligned} G_j^{+}(x,y) &= \sum_{i<r}(\boldsymbol{e}_{ir;j}^{+}(x)\otimes\dot{\boldsymbol{e}}_{ri;j}^{+}(y) + \dot{\boldsymbol{e}}_{ir;j}^{+}(x)\otimes\boldsymbol{e}_{ri;j}^{+}(y)) \\ G_j^{-}(x,y) &= \sum_{i<r}(\dot{\boldsymbol{e}}_{ri;j}^{-}(x)\otimes\boldsymbol{e}_{ir;j}^{-}(y) + \boldsymbol{e}_{ri;j}^{-}(x)\otimes\dot{\boldsymbol{e}}_{ir;j}^{-}(y)) \end{aligned} \tag{164}$$

Idea of the proof: Apply the contour integration method to a conveniently chosen Green function, see [18]. □

4.3. Expansions over the 'Squared Solutions'

Using the completeness relations one can expand any generic element $F(x)$ of the phase space $\mathcal{M}$ over each of the complete sets of 'squared solutions'. We remind that $F(x)$ is a generic element of $\mathcal{M}$ if it is a block-off-diagonal matrix-valued function, which falls off fast enough for $|x| \to \infty$. It can be written down in terms of its matrix elements $F_{\pm}(x)$ as

$$F(x) = \sum_{i<r} (F_{ir}(x) E_{ir} + F_{ri}(x) E_{ri}). \tag{165}$$

From (162) we get

$$-\frac{1}{2} \operatorname{tr}_1([J, F(x)] \otimes \mathbb{1}) \Pi_{0J} = \frac{1}{2} \operatorname{tr}_2 \Pi_{0J} (\mathbb{1} \otimes [J, F(x)]) = F(x) \tag{166}$$

where tr_1 (and tr_2) mean that we are taking the trace of the elements in the first (or the second) position of the tensor product. The result is

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} \left(\boldsymbol{e}_{ir}^{+}(x,\lambda) \gamma_{F;ir}^{+}(\lambda) - \boldsymbol{e}_{ri}^{-}(x,\lambda) \gamma_{F;ri}^{-}(\lambda) \right) \\ - 2\mathrm{i} \sum_{j=1}^{N} (Z_{F;j}^{+}(x) + Z_{F;j}^{-}(x)) \tag{167}$$

$$F(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} \left(\boldsymbol{e}_{ri}^{+}(x,\lambda) \gamma_{F;ri}^{+}(\lambda) - \boldsymbol{e}_{ir}^{-}(x,\lambda) \gamma_{F;ir}^{-}(\lambda) \right) \\ - 2\mathrm{i} \sum_{j=1}^{N} (\tilde{Z}_{F;j}^{+}(x) + \tilde{Z}_{F;j}^{-}(x)) \tag{168}$$

where

$$\gamma_{F;ab}^{\pm}(\lambda) = [\![\boldsymbol{e}_{ba}^{\pm}(y,\lambda), F(y)]\!] \tag{169}$$

$$\begin{aligned} Z_{F;j}^{+}(x) &= \operatorname*{Res}_{\lambda=\lambda_j^{+}} \sum_{i<r} \boldsymbol{e}_{ir}^{+}(x,\lambda) \gamma_{F;ir}^{+}(\lambda) \\ Z_{F;j}^{-}(x) &= \operatorname*{Res}_{\lambda=\lambda_j^{-}} \sum_{i<r} \boldsymbol{e}_{ri}^{-}(x,\lambda) \gamma_{F;ri}^{-}(\lambda) \end{aligned} \tag{170}$$

$$\begin{aligned} \tilde{Z}_{F;j}^{+}(x) &= \operatorname*{Res}_{\lambda=\lambda_j^{+}} \sum_{i<r} \boldsymbol{e}_{ri}^{+}(x,\lambda) \gamma_{F;ri}^{+}(\lambda) \\ \tilde{Z}_{F;j}^{-}(x) &= \operatorname*{Res}_{\lambda=\lambda_j^{-}} \sum_{i<r} \boldsymbol{e}_{ir}^{-}(x,\lambda) \gamma_{F;ir}^{-}(\lambda) \end{aligned} \tag{171}$$

The completeness relation (161) is directly related to the spectral decompositions of $\Lambda_{\pm}$, see, e.g. [17, 24] for $\mathfrak{g} \cong \mathfrak{sl}(n)$. It allows us to establish a one-to-one correspondence between the element $F(x) \in \mathcal{M}$ and its expansion coefficients. Indeed, from equations (169)–(171) we can easily prove the following

Proposition 1. *The function $F(x) \equiv 0$ if and only if all its expansion coefficients vanish, i.e.*

$$\gamma^{+}_{F;ir}(\lambda) = \gamma^{-}_{F;ri}(\lambda) = 0, \qquad i < r; \qquad Z^{+}_{F;j}(x) = Z^{-}_{F;j}(x) = 0 \tag{172a}$$

$$\gamma^{+}_{F;ri}(\lambda) = \gamma^{-}_{F;ir}(\lambda) = 0, \qquad i < r; \qquad \tilde{Z}^{+}_{F;j}(x) = \tilde{Z}^{-}_{F;j}(x) = 0 \tag{172b}$$

where $j = 1, \dots, N$.

Proof: To show that from $F(x) \equiv 0$ there follows (172a) we insert $F(x) \equiv 0$ into the right hand sides of the inversion formulae (169)–(171) getting (172). The fact that from (172a) there follows $F(x) \equiv 0$ is obtained by inserting (172a) into the right hand side of (169)–(171). The equivalence of $F(x) \equiv 0$ to (172b) is proved analogously. □

4.4. Expansions of $Q(x)$

Here we evaluate the expansion coefficients for $F(x) \equiv Q(x)$. As the reader have guessed already, their evaluation will be based on the Wronskian relations (136), (137) which we derived above. From them we have

$$[\![\boldsymbol{e}^{+}_{li}(x,\lambda), Q(x)]\!] = \mathrm{i}\boldsymbol{d}^{-}_{il}(\lambda), \qquad [\![\boldsymbol{e}^{-}_{il}(x,\lambda), Q(x)]\!] = \mathrm{i}\boldsymbol{d}^{+}_{li}(\lambda) \tag{173}$$

$$[\![\boldsymbol{e}^{+}_{il}(x,\lambda), Q(x)]\!] = \mathrm{i}\boldsymbol{b}^{+}_{li}(\lambda), \qquad [\![\boldsymbol{e}^{-}_{li}(x,\lambda), Q(x)]\!] = \mathrm{i}\boldsymbol{b}^{-}_{il}(\lambda). \tag{174}$$

Skipping the calculational details we get the following expansion of $Q(x)$ over the systems $\{\boldsymbol{\Phi}^{\pm}\}$ and $\{\boldsymbol{\Psi}^{\pm}\}$

$$Q(x) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} \left(\tau^{+}_{ir}(\lambda)\boldsymbol{\Phi}^{+}_{ir}(x,\lambda) - \tau^{-}_{ri}(\lambda)\boldsymbol{\Phi}^{-}_{ri}(x,\lambda)\right)$$
$$+ 2\sum_{k=1}^{N}\sum_{i<r}\left(\tau^{+}_{ir;j}\boldsymbol{\Phi}^{+}_{ir;j}(x) + \tau^{-}_{ri;j}\boldsymbol{\Phi}^{-}_{ri;j}(x)\right) \tag{175}$$

$$Q(x) = -\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} \left(\rho^{+}_{ri}(\lambda)\boldsymbol{\Psi}^{+}_{ri}(x,\lambda) - \rho^{-}_{ir}(\lambda)\boldsymbol{\Psi}^{-}_{ir}(x,\lambda)\right)$$
$$- 2\sum_{k=1}^{N}\sum_{i<r}\left(\rho^{+}_{ri;j}\boldsymbol{\Psi}^{+}_{ri;j}(x) + \rho^{-}_{ir;j}\boldsymbol{\Psi}^{-}_{ir;j}(x)\right). \tag{176}$$

4.5. Expansions of $\mathrm{ad}_J^{-1}\,\delta Q(x)$

Here we evaluate the expansion coefficients for $F(x) \equiv \mathrm{ad}_J^{-1}\,\delta Q(x)$. Their evaluation is based on the Wronskian relations (153), (154). Now we have

$$\begin{aligned}
[\![\boldsymbol{e}_{li}^{+}(x,\lambda), \mathrm{ad}_J^{-1}\,\delta Q(x)]\!] &= \frac{\mathrm{i}}{2}((\delta\tau^{+})\boldsymbol{c}^{+}(\lambda))_{il} \\
[\![\boldsymbol{e}_{il}^{-}(x,\lambda), \mathrm{ad}_J^{-1}\,\delta Q(x)]\!] &= -\frac{\mathrm{i}}{2}((\delta\tau^{-})\boldsymbol{c}^{-}(\lambda))_{li}
\end{aligned} \tag{177}$$

$$\begin{aligned}
[\![\boldsymbol{e}_{il}^{+}(x,\lambda), \mathrm{ad}_J^{-1}\,\delta Q(x)]\!] &= -\frac{\mathrm{i}}{2}((\delta\rho^{+})\boldsymbol{a}^{+}(\lambda))_{li} \\
[\![\boldsymbol{e}_{li}^{-}(x,\lambda), \mathrm{ad}_J^{-1}\,\delta Q(x)]\!] &= \frac{\mathrm{i}}{2}((\delta\rho^{-})\boldsymbol{a}^{-}(\lambda))_{il}.
\end{aligned} \tag{178}$$

Skipping the calculational details we get the following expansion of $\mathrm{ad}_J^{-1}\,\delta Q(x)$ over the systems $\{\boldsymbol{\Phi}^{\pm}\}$ and $\{\boldsymbol{\Psi}^{\pm}\}$

$$\begin{aligned}
\mathrm{ad}_J^{-1}\,\delta Q(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} \left(\delta\tau_{ir}^{+}(\lambda)\boldsymbol{\Phi}_{ir}^{+}(x,\lambda) + \delta\tau_{ri}^{-}(\lambda)\boldsymbol{\Phi}_{ri}^{-}(x,\lambda)\right) \\
+ \sum_{k=1}^{N}\sum_{i<r}\left(\delta' W_{ir;j}^{+}(x) - \delta' W_{ri;j}^{-}(x)\right)
\end{aligned} \tag{179}$$

$$\begin{aligned}
\mathrm{ad}_J^{-1}\,\delta Q(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} \left(\delta\rho_{ri}^{+}(\lambda)\boldsymbol{\Psi}_{ri}^{+}(x,\lambda) + \delta\rho_{ir}^{-}(\lambda)\boldsymbol{\Psi}_{ir}^{-}(x,\lambda)\right) \\
+ \sum_{k=1}^{N}\sum_{i<r}\left(\delta' \tilde{W}_{ir;j}^{+}(x) - \delta' \tilde{W}_{ri;j}^{-}(x)\right)
\end{aligned} \tag{180}$$

where

$$\delta' W_{ab;j}^{\pm}(x) = \delta\lambda_j^{\pm}\tau_{ab;j}^{\pm}\dot{\boldsymbol{\Phi}}_{ab;j}^{\pm}(x) + \delta\tau_{ab;j}^{\pm}\boldsymbol{\Phi}_{ab;j}^{\pm}(x) \tag{181}$$

$$\delta' \tilde{W}_{ab;j}^{\pm}(x) = \delta\lambda_j^{\pm}\rho_{ab;j}^{\pm}\dot{\boldsymbol{\Psi}}_{ab;j}^{\pm}(x) + \delta\rho_{ab;j}^{\pm}\boldsymbol{\Psi}_{ab;j}^{\pm}(x). \tag{182}$$

The expansions (175), (176) combined with proposition 1 is another way to establish the one-to-one correspondence between $Q(x)$ and each of the minimal sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$ (56). Likewise the expansions (179), (180) and proposition 1 establish the one-to-one correspondence between the variation of the potential $\delta Q(x)$ and the variations of the scattering data $\delta\mathcal{T}_1$ and $\delta\mathcal{T}_2$.

4.6. The Generating Operators

To complete the analogy between the standard Fourier transform and the expansions over the 'squared solutions' we need the analogs of the operator $D_0 =$

$-\mathrm{i}\,\mathrm{d}/\mathrm{d}x$. The operator D_0 is the one for which $\mathrm{e}^{\mathrm{i}\lambda x}$ is an eigenfunction: $D_0\mathrm{e}^{\mathrm{i}\lambda x} = \lambda\mathrm{e}^{\mathrm{i}\lambda x}$. Therefore it is natural to introduce the generating operators $\Lambda_\pm$ through

$$\begin{aligned} (\Lambda_+ - \lambda)\boldsymbol{\Psi}^+_{ri}(x,\lambda) &= 0, \qquad (\Lambda_+ - \lambda)\boldsymbol{\Psi}^-_{ir}(x,\lambda) = 0 \\ (\Lambda_- - \lambda)\boldsymbol{\Phi}^+_{ir}(x,\lambda) &= 0, \qquad (\Lambda_- - \lambda)\boldsymbol{\Phi}^-_{ri}(x,\lambda) = 0. \end{aligned} \tag{183}$$

Their derivation starts by introducing the splitting

$$e^\pm_{ab}(x,\lambda) = e^{\mathrm{d},\pm}_{ab}(x,\lambda) + \boldsymbol{e}^\pm_{ab}(x,\lambda), \qquad e^{\mathrm{d},\pm}_{ab}(x,\lambda) = (\mathbb{1} - P_{0J})e^\pm_{ab}(x,\lambda) \tag{184}$$

into the equation (139). Then equation (139) splits into

$$\mathrm{i}\frac{\mathrm{d}e^{\mathrm{d},\pm}_{ab}}{\mathrm{d}x} + [Q(x), \boldsymbol{e}^\pm_{ab}(x,\lambda)] = 0 \tag{185}$$

$$\mathrm{i}\frac{\mathrm{d}\boldsymbol{e}^\pm_{ab}}{\mathrm{d}x} + [Q(x), e^{\mathrm{d},\pm}_{ab}(x,\lambda)] = \lambda[J, \boldsymbol{e}^\pm_{ab}(x,\lambda)] \tag{186}$$

Equation (185) can be integrated formally with the result

$$e^{\mathrm{d},\pm}_{ab}(x,\lambda) = C^{\mathrm{d},\pm}_{ab;\epsilon}(\lambda) + \mathrm{i}\int_{\epsilon\infty}^{x} \mathrm{d}y\,[Q(y), \boldsymbol{e}^\pm_{ab}(y,\lambda)] \tag{187}$$

$$C^{\mathrm{d},\pm}_{ab;\epsilon}(\lambda) = \lim_{y\to\epsilon\infty} e^{\mathrm{d},\pm}_{ab}(y,\lambda), \qquad \epsilon = \pm 1. \tag{188}$$

Next insert (187) into (186) and act on both sides by ad_J^{-1}. This gives us

$$(\Lambda_\pm - \lambda)\boldsymbol{e}^\pm_{ab}(x,\lambda) = \mathrm{i}[C^{\mathrm{d},\pm}_{ab;\epsilon}(\lambda), \mathrm{ad}_J^{-1} Q(x)] \tag{189}$$

where the generating operators $\Lambda_\pm$ are given by

$$\Lambda_\pm X(x) \equiv \mathrm{ad}_J^{-1}\left(\mathrm{i}\frac{\mathrm{d}X}{\mathrm{d}x} + \mathrm{i}\left[Q(x), \int_{\pm\infty}^{x} \mathrm{d}y\,[Q(y), X(y)]\right]\right). \tag{190}$$

Thus, $\boldsymbol{e}^\pm_{ab}(x,\lambda)$ will be an eigenfunction of Λ_+ or Λ_- if only if $C^{\mathrm{d},\pm}_{ab}(y,\lambda) = 0$. Evaluating the limits of (188) for all combinations of indices a, b we find ($i < r$)

$$\begin{aligned} (\Lambda_+ - \lambda)\boldsymbol{\Psi}^+_{ri}(x,\lambda) &= 0, \qquad (\Lambda_+ - \lambda)\boldsymbol{\Psi}^-_{ir}(x,\lambda) = 0 \\ (\Lambda_+ - \lambda^+_j)\boldsymbol{\Psi}^+_{ri;j}(x) &= 0, \qquad (\Lambda_+ - \lambda^-_j)\boldsymbol{\Psi}^-_{ir;j}(x) = 0 \end{aligned} \tag{191}$$

$$\begin{aligned} (\Lambda_- - \lambda)\boldsymbol{\Phi}^+_{ir}(x,\lambda) &= 0, \qquad (\Lambda_- - \lambda)\boldsymbol{\Phi}^-_{ri}(x,\lambda) = 0 \\ (\Lambda_- - \lambda^+_j)\boldsymbol{\Phi}^+_{ir;j}(x) &= 0, \qquad (\Lambda_- - \lambda^-_j)\boldsymbol{\Phi}^-_{ri;j}(x) = 0. \end{aligned} \tag{192}$$

The rest of the squared solutions are not eigenfunctions of neither Λ_+ nor Λ_-

$$\begin{aligned} (\Lambda_+ - \lambda^+_j)\dot{\boldsymbol{\Psi}}^+_{ri;j}(x) &= \boldsymbol{\Psi}^+_{ri;j}(x), \qquad (\Lambda_+ - \lambda^-_j)\dot{\boldsymbol{\Psi}}^-_{ir;j}(x) = \boldsymbol{\Psi}^-_{ir;j}(x) \\ (\Lambda_- - \lambda^+_j)\dot{\boldsymbol{\Phi}}^+_{ir;j}(x) &= \boldsymbol{\Phi}^+_{ir;j}(x), \qquad (\Lambda_- - \lambda^-_j)\dot{\boldsymbol{\Phi}}^-_{ri;j}(x) = \boldsymbol{\Phi}^-_{ri;j}(x) \end{aligned} \tag{193}$$

i.e. $\dot{\mathbf{\Psi}}^{+}_{ri;j}(x)$ and $\dot{\mathbf{\Phi}}^{+}_{ir;j}(x)$ are adjoint eigenfunctions of Λ_+ and Λ_-. This means that $\lambda_j^{\pm}$, $j = 1, \dots, N$ are also the discrete eigenvalues of $\Lambda_\pm$ but the corresponding eigenspaces of $\Lambda_\pm$ have double the dimensions of the ones of L; now they are spanned by both $\mathbf{\Psi}^{\pm}_{ab;j}(x)$ and $\dot{\mathbf{\Psi}}^{\pm}_{ab;j}(x)$. Thus the sets $\{\Psi\}$ and $\{\Phi\}$ are the complete sets of eigen- and adjoint functions of Λ_+ and Λ_-.

5. Fundamental Properties of the MNLS-Type Equations

In this Section we describe the fundamental properties of the NLEE.

5.1. The Class of the MNLS-Type Equations

Let us insert variations of the form (156) into the expansions (179), (180). This results in

$$\operatorname{ad}_J^{-1}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\sum_{i<r}\left(\frac{\mathrm{d}\tau_{ir}^{+}}{\mathrm{d}t}\mathbf{\Phi}_{ir}^{+}(x,\lambda) + \frac{\mathrm{d}\tau_{ri}^{-}}{\mathrm{d}t}\mathbf{\Phi}_{ri}^{-}(x,\lambda)\right) + \sum_{k=1}^{N}\sum_{i<r}\left(W_{ir;j}^{\prime,+}(x,t) - W_{ri;j}^{\prime,-}(x,t)\right) \tag{194}$$

$$\operatorname{ad}_J^{-1}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\sum_{i<r}\left(\frac{\mathrm{d}\rho_{ri}^{+}}{\mathrm{d}t}\mathbf{\Psi}_{ri}^{+}(x,\lambda) + \frac{\mathrm{d}\rho_{ir}^{-}}{\mathrm{d}t}\mathbf{\Psi}_{ir}^{-}(x,\lambda)\right) + \sum_{k=1}^{N}\sum_{i<r}\left(\tilde{W}_{ir;j}^{\prime,+}(x,t) - \tilde{W}_{ri;j}^{\prime,-}(x,t)\right) \tag{195}$$

where

$$W_{ab;j}^{\prime,\pm}(x,t) = \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t}\tau_{ab;j}^{\pm}\dot{\mathbf{\Phi}}_{ab;j}^{\pm}(x) + \frac{\mathrm{d}\tau_{ab;j}^{\pm}}{\mathrm{d}t}\mathbf{\Phi}_{ab;j}^{\pm}(x) \tag{196}$$

$$\tilde{W}_{ab;j}^{\prime,\pm}(x) = \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t}\rho_{ab;j}^{\pm}\dot{\mathbf{\Psi}}_{ab;j}^{\pm}(x) + \frac{\mathrm{d}\rho_{ab;j}^{\pm}}{\mathrm{d}t}\mathbf{\Psi}_{ab;j}^{\pm}(x). \tag{197}$$

Next from (191), (192) there follows that

$$\begin{aligned} &(f_0(\Lambda_+) - f_0(\lambda))\mathbf{\Psi}_{ri}^{+}(x,\lambda) = 0, \qquad (f_0(\Lambda_+) - f_0(\lambda))\mathbf{\Psi}_{ir}^{-}(x,\lambda) = 0\\ &(f_0(\Lambda_+) - f_0(\lambda_j^{+}))\mathbf{\Psi}_{ri;j}^{+}(x) = 0, \qquad (f_0(\Lambda_+) - f_0(\lambda_j^{-}))\mathbf{\Psi}_{ir;j}^{-}(x) = 0\end{aligned} \tag{198}$$

$$\begin{aligned} (f_0(\Lambda_+) - f_0(\lambda_j^{+}))\dot{\mathbf{\Psi}}_{ri;j}^{+}(x) &= \dot{f}(\lambda_j^{+})\mathbf{\Psi}_{ri;j}^{+}(x)\\ (f_0(\Lambda_+) - f_0(\lambda_j^{-}))\dot{\mathbf{\Psi}}_{ir;j}^{-}(x) &= \dot{f}(\lambda_j^{-})\mathbf{\Psi}_{ir;j}^{-}(x)\end{aligned} \tag{199}$$

and similar relations between Λ_- and its eigenfunctions. Combining them with the expansions (175), (176) we get

$$\begin{aligned} f_0(\Lambda_-)Q(x) &= \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, f_0(\lambda)\sum_{i<r}\left(\tau_{ir}^+(\lambda)\boldsymbol{\Phi}_{ir}^+(x,\lambda) - \tau_{ri}^-(\lambda)\boldsymbol{\Phi}_{ri}^-(x,\lambda)\right) \\ &+ 2\sum_{k=1}^{N}\sum_{i<r}\left(f_0(\lambda_j^+)\tau_{ir;j}^+\boldsymbol{\Phi}_{ir;j}^+(x) + f_0(\lambda_j^-)\tau_{ri;j}^-\boldsymbol{\Phi}_{ri;j}^-(x)\right) \end{aligned} \quad (200)$$

$$\begin{aligned} f_0(\Lambda_+)Q(x) &= -\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{i<r} f_0(\lambda)\left(\rho_{ri}^+(\lambda)\boldsymbol{\Psi}_{ri}^+(x,\lambda) - \rho_{ir}^-(\lambda)\boldsymbol{\Psi}_{ir}^-(x,\lambda)\right) \\ &- 2\sum_{k=1}^{N}\sum_{i<r}\left(f_0(\lambda_j^+)\rho_{ri;j}^+\boldsymbol{\Psi}_{ri;j}^+(x) + f_0(\lambda_j^-)\rho_{ir;j}^-\boldsymbol{\Psi}_{ir;j}^-(x)\right). \end{aligned} \quad (201)$$

Now we can prove that the principal series of MNLS-type equations has the form

$$\mathrm{i}\,\mathrm{ad}_J^{-1}\frac{\mathrm{d}Q}{\mathrm{d}t} + f_0(\Lambda)Q(x,t) = 0 \quad (202)$$

where Λ can be either Λ_+ or Λ_- and $f_0(\lambda)$ determines the dispersion law of the corresponding NLEE.

Theorem 4. *The NLEE* (202) *are equivalent to each of the following evolution equations for the scattering data of L*

$$\mathrm{i}\frac{\mathrm{d}\rho^{\pm}}{\mathrm{d}t} \mp 2f_0(\lambda)\rho^{\pm} = 0, \qquad \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t} = 0, \qquad \mathrm{i}\frac{\mathrm{d}\rho_{;j}^{\pm}}{\mathrm{d}t} \mp 2f_0(\lambda_j^{\pm})\rho_{;j}^{\pm} = 0 \quad (203)$$

$$\mathrm{i}\frac{d\tau^{\pm}}{\mathrm{d}t} \pm 2f_0(\lambda)\tau^{\pm} = 0, \qquad \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t} = 0, \qquad i\frac{\mathrm{d}\tau_{;j}^{\pm}}{\mathrm{d}t} \pm 2f_0(\lambda_j^{\pm})\tau_{;j}^{\pm} = 0. \quad (204)$$

Proof: Insert the expansions (195) and (201) into the left hand side of the NLEE (202) and use proposition 1. This immediately proves the equivalence of the NLEE to the linear equations (203). Analogously, from the expansions (194), (200) and proposition 1 one proves the equivalence between the NLEE and (204). □

5.2. Integrals of Motion–Principal Series

Dealing with the block Zakharov–Shabat system we can distinguish two types of NLEE and, consequently, two types of series of conservation laws that provide the Hamiltonians of the NLEE.

The MNLS equations have maximally degenerated dispersion law and belong to the principle series of NLEE. Their Hamiltonians have local densities, i.e. their densities depend on $Q(x,t)$ and its x-derivatives. At the same time they have maximal number of generating functionals of conservation laws: the whole blocks

$\boldsymbol{a}^{\pm}(\lambda)$ and $\boldsymbol{c}^{\pm}(\lambda)$. However not all of these functionals are in involution, see Subsection 5.4 below.

The NLEE characterized by generic dispersion laws have as a rule non-local Hamiltonian densities and a minimal possible number of generating functionals of integrals of motion. These are provided by the invariants (the eigenvalues) of $\boldsymbol{a}^{\pm}(\lambda)$ and $\boldsymbol{c}^{\pm}(\lambda)$. Special combinations of these as in equation (61) produce the Hamiltonians of the MNLS type equations. In this Subsection we will concentrate on those.

In Subsection 2.6 we showed how these functionals $A^{+}(\lambda)$ and $C^{+}(\lambda)$ can be expressed through the minimal sets of scattering data, see equation (62). Our aim here is to derive a recurrent procedure which would allow one to express their coefficients in the asymptotic expansions

$$A^{+}(\lambda) = \sum_{k=1}^{\infty} I_k \lambda^{-k}, \qquad C^{+}(\lambda) = \sum_{k=1}^{\infty} J_k \lambda^{-k} \tag{205}$$

in terms of $Q(x,t)$. To this end we make use of a Wronskian type relation

$$(\mathbb{1} - P_{0J})\hat{\chi}^{+}(x,\lambda)(\dot{\chi}^{+} + \mathrm{i}xJ\chi^{+})|_{x=-\infty}^{\infty} = \begin{pmatrix} \hat{\boldsymbol{a}}^{+}\dot{\boldsymbol{a}}^{+} & 0 \\ 0 & -\hat{\boldsymbol{c}}^{+}\dot{\boldsymbol{c}}^{+} \end{pmatrix} \tag{206}$$

valid for $\lambda \in \mathbb{C}_{+}$. Here and below by 'dot' we denote derivative with respect to λ. In what follows we will need also the standard formula

$$\operatorname{tr} \ln \boldsymbol{a}^{+}(\lambda) = \ln \det \boldsymbol{a}^{+}(\lambda) \tag{207}$$

and its consequence

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \operatorname{tr} \ln \boldsymbol{a}^{+}(\lambda) = \ln \det \hat{\boldsymbol{a}}^{+}(\lambda)\dot{\boldsymbol{a}}^{+}(\lambda). \tag{208}$$

Next we express $\chi^{+}(x,\lambda)$ in the form

$$\chi^{+}(x,\lambda) = (\mathbb{1} + W(x,\lambda))Z(x,\lambda)\mathrm{e}^{-\mathrm{i}\lambda Jx} \tag{209}$$

where

$$W(x,\lambda) = \begin{pmatrix} 0 & W^{(1)}(x,\lambda) \\ W^{(2)}(x,\lambda) & 0 \end{pmatrix}, \quad Z(x,\lambda) = \begin{pmatrix} Z^{(1)}(x,\lambda) & 0 \\ 0 & Z^{(2)}(x,\lambda) \end{pmatrix}. \tag{210}$$

Inserting (209) into (7) and separating the block-diagonal and block-off-diagonal parts we get for $W(x,\lambda)$ and $Z(x,\lambda)$ the following system

$$\mathrm{i}\frac{\mathrm{d}W}{\mathrm{d}x} + Q(x,t) - WQW(x,t,\lambda) = \lambda[J, W(x,t,\lambda)] \tag{211}$$

$$\mathrm{i}\frac{\mathrm{d}Z}{\mathrm{d}x}\hat{Z}(x,t,\lambda) + Q(x,t)W(x,t,\lambda) = 0. \tag{212}$$

Combining equations (209) and (206) we obtain

$$\hat{Z}(x,t,\lambda)\dot{Z}\Big|_{x=-\infty}^{\infty} = \begin{pmatrix} \hat{\boldsymbol{a}}^{+}(\lambda)\dot{\boldsymbol{a}}^{+}(\lambda) & 0 \\ 0 & -\hat{\boldsymbol{c}}^{+}(\lambda)\dot{\boldsymbol{c}}^{+}(\lambda) \end{pmatrix} \tag{213}$$

which leads, in view of (208) and (212) to

$$\hat{Z}(x,t,\lambda)\dot{Z}\Big|_{x=-\infty}^{\infty} = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \hat{Z}(x,t,\lambda)Q(x,t)\dot{W}(x,t,\lambda)Z(x,t,\lambda). \tag{214}$$

If we multiply both sides of (214) by $\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ take the trace and integrate over λ we get

$$\begin{aligned} A^{+}(\lambda) &\equiv \ln\det \boldsymbol{a}^{+}(\lambda) = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}\left(Q(x,t)W(x,t,\lambda)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\right) \\ C^{+}(\lambda) &\equiv \ln\det \boldsymbol{c}^{+}(\lambda) = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}\left(Q(x,t)W(x,t,\lambda)\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right). \end{aligned} \tag{215}$$

Using the asymptotic expansions of $A^{+}(\lambda)$, $C^{+}(\lambda)$ (see equation (205)) and $W(x,t,\lambda)$

$$W(x,t,\lambda) = \sum_{k=1}^{\infty} W_k(x,t)\lambda^{-k} \tag{216}$$

we arrive at the following expressions for the integrals of motion I_k and J_k

$$\begin{aligned} I_k &= \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}\left(Q(x,t)W_k(x,t)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\right) \\ J_k &= -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}\left(Q(x,t)W_k(x,t)\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right). \end{aligned} \tag{217}$$

The last step in these considerations consists in deriving recurrent relations for calculating $W_k(x,t)$ in terms of $Q(x,t)$ and its x-derivatives. This is done by inserting (216) into the equation (211) with the result

$$\begin{aligned} W_1(x,t) &= \frac{1}{4}[J, Q(x,t)] \\ W_{k+1}(x,t) &= \frac{1}{4}\left[J, \mathrm{i}\frac{\mathrm{d}W_k}{\mathrm{d}x} - \sum_{p+s=k} W_p(x,t)Q(x,t)W_s(x,t)\right]. \end{aligned} \tag{218}$$

In particular we have

$$W_2(x,t) = \frac{\mathrm{i}}{4}Q_x, \qquad W_3(x,t) = \frac{1}{16}\left[J, Q_{xx} + Q^3\right] \tag{219}$$

which leads to

$$\begin{aligned}
I_1 &= \frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}(qr), & J_1 &= -\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}(rq) \\
I_2 &= -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}(qr_x), & J_2 &= \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}(rq_x) \\
I_3 &= -\frac{\mathrm{i}}{8}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}(qr_{xx} + qrqr), & J_3 &= -\frac{\mathrm{i}}{8}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}(rq_{xx} + rqrq).
\end{aligned} \tag{220}$$

5.3. Hamiltonian Properties of the MNLS Equations

Let us briefly outline the Hamiltonian properties of the NLEE (202). Obviously the MNLS describes an infinite dimensional Hamiltonian system with Hamiltonian

$$H_{\mathrm{MNLS}} = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{tr}\left(QQ_{xx} + Q^4(x,t)\right) = 4\mathrm{i}(I_3 + J_3) \tag{221}$$

and Poisson brackets

$$\{\boldsymbol{q}_{ks}(y,t), \boldsymbol{r}_{ri}(x,t)\} = \mathrm{i}\delta_{ik}\delta_{rs}\delta(x-y) \tag{222}$$

or, equivalently, by the canonical symplectic form

$$\begin{aligned}
\Omega_0 &= \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}\,(\delta\boldsymbol{r}(x) \wedge \delta\boldsymbol{q}(x)) \\
&= \frac{1}{\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{tr}\left(\mathrm{ad}_J^{-1}\, \delta Q(x) \wedge [J, \mathrm{ad}_J^{-1}\, \delta Q(x)\right).
\end{aligned} \tag{223}$$

The second expression is preferable to us because it makes obvious the interpretation of $\delta Q(x,t)$ as local coordinate on the co-adjoint orbit passing through J. It is also expressed through the skew-scalar product by

$$\Omega_0 = \frac{1}{\mathrm{i}}[\![\mathrm{ad}_J^{-1}\, \delta Q \,\overset{\wedge}{,}\, \mathrm{ad}_J^{-1}\, \delta Q]\!]. \tag{224}$$

It can be evaluated in terms of the scattering data variations. To do this we insert the expansion of $\mathrm{ad}_J^{-1}\, \delta Q$ into Ω_0 and then use again the equations (177), (178). After some calculations we get

$$\Omega_0 = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda)\right) - \mathrm{i}\sum_{j=1}^{N}\left(\Omega_{0,j}^+ + \Omega_{0;j}^-\right) \tag{225}$$

$$\Omega_0^\pm(\lambda) = \frac{1}{2}\,\mathrm{tr}\,(\delta\tau^\pm \boldsymbol{c}^\pm \wedge \delta\rho^\pm \boldsymbol{a}^\pm(\lambda)), \qquad \Omega_{0,j}^\pm = \operatorname*{Res}_{\lambda=\lambda_j^\pm} \Omega_0^\pm(\lambda). \tag{226}$$

Here we skip the explicit expressions for $\Omega_{0,j}^\pm$ which are rather involved. From (225) it is not even obvious that Ω_0 is closed.

The Hamiltonian formulation of the MNLS equation with Ω_0 and H_0 is just one member of the hierarchy of Hamiltonian formulations of MNLS provided by

$$\Omega_k = \frac{1}{\mathrm{i}} [\![\mathrm{ad}_J^{-1} \delta Q \underset{'}{\wedge} \Lambda^k \, \mathrm{ad}_J^{-1} \delta Q]\!], \qquad \Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-) \tag{227}$$

$$H_k = 4\mathrm{i}(I_{k+3} + J_{k+3}). \tag{228}$$

We can also calculate Ω_k in terms of the scattering data variations. Doing this we will need also equations (198), (199). The answer is

$$\Omega_k = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, \lambda^k \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) - \mathrm{i} \sum_{j=1}^{N} \left(\Omega_{k,j}^+ + \Omega_{k;j}^- \right) \tag{229}$$

$$\Omega_{k,j}^{\pm} = \operatorname*{Res}_{\lambda=\lambda_j^{\pm}} \lambda^k \Omega_0^{\pm}(\lambda). \tag{230}$$

This allows one to prove that if we are able to cast Ω_0 in canonical form then all Ω_k will also be cast in canonical form and will be pair-wise equivalent.

5.4. The Classical R-Matrix and the NLEE of MNLS Type

One of the definitions of the classical R-matrix is based on the Lax representation for the corresponding NLEE. We will start from this definition, but before to state it will introduce the following notation

$$\left\{ U(x,\lambda) \underset{'}{\otimes} U(y,\mu) \right\} \tag{231}$$

which is an abbreviated record for the Poisson bracket between all matrix elements of $U(x,\lambda)$ and $U(y,\mu)$

$$\left\{ U(x,\lambda) \underset{'}{\otimes} U(y,\mu) \right\}_{ik,lm} = \left\{ U_{ik}(x,\lambda), U_{lm}(y,\mu) \right\}. \tag{232}$$

In particular, if $U(x,\lambda)$ is of the form

$$U(x,\lambda) = Q(x,t) - \lambda J, \qquad Q(x,t) = \sum_{i<r} (q_{ir} E_{ir} + p_{ri} E_{ri}) \tag{233}$$

and the matrix elements of $Q(x,t)$ satisfy (222), then

$$\left\{ U(x,\lambda) \underset{'}{\otimes} U(y,\mu) \right\} = \mathrm{i} \sum_{i<r} (E_{ir} \otimes E_{ri} - E_{ri} \otimes E_{ir}) \delta(x-y). \tag{234}$$

The classical R-matrix can be defined through the relation [12]

$$\left\{ U(x,\lambda) \underset{'}{\otimes} U(y,\mu) \right\} = \mathrm{i} [R(\lambda-\mu), U(x,\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U(y,\mu)] \delta(x-y). \tag{235}$$

which can be understood as a system of N^2 equation for the N^2 matrix elements of $R(\lambda - \mu)$. However, these relations must hold identically with respect to λ

and μ, i.e. (235) is an overdetermined system of algebraic equations for the matrix elements of R. It is far from obvious whether such $R(\lambda - \mu)$ exists, still less obvious is that it depends only on the difference $\lambda - \mu$. In other words far from any choice for $U(x, \lambda)$ and for the Poisson brackets between its matrix elements allow R-matrix description. Our system (235) allows an R-matrix given by

$$R(\lambda - \mu) = -\frac{\mathrm{i}}{2}\frac{P}{\lambda - \mu} \tag{236}$$

where P is a constant $N^2 \times N^2$ matrix

$$P = \sum_{a,b=1}^{N} E_{ab} \otimes E_{ba}. \tag{237}$$

The matrix P possesses the following special properties

$$P(X \otimes Y) = (Y \otimes X)P, \qquad P^2 \equiv 1\!\!1 \tag{238}$$

i.e. it interchanges the positions of the elements in the direct tensor product. By using these properties of P we are getting

$$[P, Q(x) \otimes 1\!\!1 + 1\!\!1 \otimes Q(x)] = 0 \tag{239}$$

i.e. the right hand side of (235) does not contain $Q(x, t)$. Besides

$$\begin{aligned}[P, \lambda J \otimes 1\!\!1 + \mu 1\!\!1 \otimes J] &= (\lambda - \mu)[P, J \otimes 1\!\!1] \\ &= -2(\lambda - \mu)\left(\sum_{i<r}(E_{ir} \otimes E_{ri} - E_{ri} \otimes E_{ir})\right)\end{aligned} \tag{240}$$

where we used the commutation relations between the matrices E_{ab}

$$[E_{ab}, E_{cd}] = E_{ad}\delta_{bc} - E_{cb}\delta_{da}. \tag{241}$$

The comparison between (239), (240) and (235) leads us to the result, that $R(\lambda - \mu)$ (236) indeed satisfies the definition (235).

Let us now show, that the classical R-matrix is a very effective tool for calculating the Poisson brackets between the matrix elements of $T(\lambda)$. It will be more convenient here to consider periodic boundary conditions on the interval $[-L, L]$, i.e. $Q(x - L) = Q(x + L)$ and to use the fundamental solution $T(x, y, \lambda)$ defined by

$$\mathrm{i}\frac{\mathrm{d}T(x, y, \lambda)}{\mathrm{d}x} + U(x, \lambda)T(x, y, \lambda) = 0, \qquad T(x, x, \lambda) = 1\!\!1. \tag{242}$$

Skipping the details we just formulate the following relation for the Poisson brackets between the matrix elements of $T(x, y, \lambda)$

$$\left\{T(x, y, \lambda) \underset{'}{\otimes} T(x, y, \mu)\right\} = [R(\lambda - \mu), T(x, y, \lambda) \otimes T(x, y, \mu)]\,. \tag{243}$$

The corresponding monodromy matrix $T_L(\lambda)$ describes the transition from $-L$ to L and $T_L(\lambda) = T(-L, L, \lambda)$. The Poisson brackets between the matrix elements of $T_L(\lambda)$ follow directly from equation (243) and are given by

$$\left\{ T_L(\lambda) \underset{,}{\otimes} T_L(\mu) \right\} = \left[R(\lambda - \mu), T_L(\lambda) \otimes T_L(\mu) \right]. \tag{244}$$

An elementary consequence of this result is the involutivity of the integrals of motion $I_{L,k}$ from the principal series which are from the expansions of

$$\ln \det \boldsymbol{a}_L^+(\lambda) = \sum_{k=1}^{\infty} I_{L,k} \lambda^{-k}, \qquad -\ln \det \boldsymbol{c}_L^-(\lambda) = \sum_{k=1}^{\infty} I_{L,k} \lambda^{-k} \tag{245}$$

$$\ln \det \boldsymbol{c}_L^+(\lambda) = \sum_{k=1}^{\infty} J_{L,k} \lambda^{-k}, \qquad -\ln \det \boldsymbol{a}_L^-(\lambda) = \sum_{k=1}^{\infty} J_{L,k} \lambda^{-k}. \tag{246}$$

An important property of the integrals $I_{L,k}$ and $J_{L,k}$ is their locality, i.e. their densities depend only on Q and its x-derivatives.

The simplest consequence of the relation (243) is the involutivity of $I_{L,k}$, $J_{L,k}$. Indeed, taking the trace of both sides of (243) shows that $\{\operatorname{tr} T_L(\lambda), \operatorname{tr} T_L(\mu)\} = 0$. We can also multiply both sides of (243) by $C \otimes C$ and then take the trace using equation (239); this proves

$$\{\operatorname{tr} T_L(\lambda) C, \operatorname{tr} T_L(\mu) C\} = 0. \tag{247}$$

In particular, for $C = \mathbb{1} + J$ and $C = \mathbb{1} - J$ we get the involutivity of

$$\begin{aligned} &\left\{ \operatorname{tr} \boldsymbol{a}_L^+(\lambda), \operatorname{tr} \boldsymbol{a}_L^+(\mu) \right\} = 0, \qquad \left\{ \operatorname{tr} \boldsymbol{a}_L^-(\lambda), \operatorname{tr} \boldsymbol{a}_L^-(\mu) \right\} = 0 \\ &\left\{ \operatorname{tr} \boldsymbol{c}_L^+(\lambda), \operatorname{tr} \boldsymbol{c}_L^+(\mu) \right\} = 0, \qquad \left\{ \operatorname{tr} \boldsymbol{c}_L^-(\lambda), \operatorname{tr} \boldsymbol{c}_L^-(\mu) \right\} = 0. \end{aligned} \tag{248}$$

Equation (243) was derived for the typical representation $V^{(1)}$ of $\mathfrak{G} \simeq \mathrm{SU}(n+m)$, but it holds true also for any other finite-dimensional representation of $\mathfrak{G}$. Let us denote by $V^{(k)} \simeq \wedge^k V^{(1)}$ the k-th fundamental representation of $\mathfrak{G}$; then the element $T_L(\lambda)$ will be represented in $V^{(k)}$ by $\wedge^k T_L(\lambda)$ – the k-th wedge power of $T_L(\lambda)$, see [28]. In particular, if we consider equation (243) in the representation $V^{(n)}$ and sandwich it between the highest and lowest weight vectors in $V^{(n)}$ we get

$$\{\det \boldsymbol{a}_L^+(\lambda), \det \boldsymbol{a}_L^+(\mu)\} = 0, \qquad \{\det \boldsymbol{c}_L^-(\lambda), \det \boldsymbol{c}_L^-(\mu)\} = 0. \tag{249}$$

Likewise considering (243) in the representation $V^{(m)}$ and sandwich it between the highest and lowest weight vectors in $V^{(m)}$ we get

$$\{\det \boldsymbol{a}_L^-(\lambda), \det \boldsymbol{a}_L^-(\mu)\} = 0, \qquad \{\det \boldsymbol{c}_L^+(\lambda), \det \boldsymbol{c}_L^+(\mu)\} = 0. \tag{250}$$

Since equations (249) and (250) hold true for all values of λ and μ we can insert into them the expansions (245) with the result

$$\{I_{L,k}, I_{L,p}\} = 0, \qquad \{J_{L,k}, J_{L,p}\} = 0, \qquad k, p = 1, 2, \dots. \tag{251}$$

Somewhat more general analysis along this lines allows one to see that only the eigenvalues of $\boldsymbol{a}_L^{\pm}(\lambda)$ and $\boldsymbol{c}_L^{\pm}(\lambda)$ produce integrals of motion in involution.

Taking the limit $L \to \infty$ we are able to transfer these results also for the case of potentials with zero boundary conditions. Indeed, let us multiply (243) by $E(y,\lambda) \otimes E(y,\mu)$ on the right and by $E^{-1}(x,\lambda) \otimes E^{-1}(x,\mu)$ on the left, where $E(x,\lambda) = \exp(-\mathrm{i}\lambda Jx)$ and take the limit for $x \to \infty$, $y \to -\infty$. Since

$$\lim_{x\to\pm\infty} \frac{\mathrm{e}^{\mathrm{i}x(\lambda-\mu)}}{\lambda-\mu} = \pm\mathrm{i}\pi\delta(\lambda-\mu) \tag{252}$$

we get

$$\left\{T(\lambda) \underset{'}{\otimes} T(\mu)\right\} = R_+(\lambda-\mu)T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu)R_-(\lambda-\mu)$$

$$R_\pm(\lambda-\mu)$$
$$= -\frac{1}{2(\lambda-\mu)}\left(\sum_{ik} E_{ik}\otimes E_{ki} + \sum_{rs} E_{rs}\otimes E_{sr}\right) \pm \mathrm{i}\pi\delta(\lambda-\mu)\Pi_{0J} \tag{253}$$

where Π_{0J} is defined by equation (162). Analogously we prove that

i) the integrals $I_k = \lim_{L\to\infty} I_{L,k}$ and $J_p = \lim_{L\to\infty} J_{L,p}$ are in involution, i.e.

$$\{I_k, I_p\} = \{I_k, J_p\} = \{J_k, J_p\} = 0$$

for all positive values of k and p;

ii) only the eigenvalues of $\boldsymbol{a}^{\pm}(\lambda)$ and $\boldsymbol{c}^{\pm}(\lambda)$ produce integrals of motion in involution.

5.5. Generic NLEE

Now we apply the expansions above to the analysis of the generic NLEE related to L. Each of these NLEE is determined by its dispersion law

$$f(\lambda) = \sum_{k=1}^{r} \lambda^k B_k = \begin{pmatrix} f_1(\lambda) & 0 \\ 0 & -f_2(\lambda) \end{pmatrix}, \qquad B_k = \begin{pmatrix} B_{k,1} & 0 \\ 0 & -B_{k,2} \end{pmatrix}. \tag{254}$$

The analysis of these NLEE is based on the expansions $[B_k, \mathrm{ad}_J^{-1} Q(x,t)]$ over the systems $\{\boldsymbol{\Psi}\}$ and $\{\boldsymbol{\Phi}\}$. The corresponding expansion coefficients are obtained by multiplying equation (136) on the right by B_k and taking the trace. Then apply Λ^k to both sides of the expansions. This proves a theorem generalizing theorem 4.

Theorem 5. *The generic NLEE with polynomial dispersion law $f(\lambda)$ (254) are of the form*

$$\mathrm{i}\,\mathrm{ad}_J^{-1}\frac{\mathrm{d}Q}{\mathrm{d}t} + \sum_{k=1}^{r} \Lambda^k[B_k, \mathrm{ad}_J^{-1} Q(x,t)] = 0 \tag{255}$$

and are equivalent to each of the following evolution equations for the scattering data of L

$$\begin{aligned}
&\mathrm{i}\frac{\mathrm{d}\rho^+}{\mathrm{d}t} - f_2(\lambda)\rho^+(\lambda,t) - \rho^+(\lambda,t)f_1(\lambda) = 0 \\
&\mathrm{i}\frac{\mathrm{d}\rho^-}{\mathrm{d}t} + f_1(\lambda)\rho^-(\lambda,t) + \rho^-(\lambda,t)f_2(\lambda) = 0 \\
&\mathrm{i}\frac{\mathrm{d}\rho_j^+}{\mathrm{d}t} - f_{2,j}^+\rho_j^+(t) - \rho_j^+(t)f_{1,j}^+ = 0, \qquad f_{1,2;j}^{\pm} = f_{1,2}(\lambda_j^{\pm}) \\
&\mathrm{i}\frac{\mathrm{d}\rho_j^-}{\mathrm{d}t} + f_{1,j}^-\rho_j^-(t) - \rho_j^-(t)f_{2,j}^- = 0, \qquad \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t} = 0
\end{aligned} \tag{256}$$

$$\begin{aligned}
&\mathrm{i}\frac{\mathrm{d}\tau^+}{\mathrm{d}t} + f_1(\lambda)\tau^+(\lambda,t) + \tau^+(\lambda,t)f_2(\lambda) = 0 \\
&\mathrm{i}\frac{\mathrm{d}\tau^-}{\mathrm{d}t} - f_2(\lambda)\tau^-(\lambda,t) - \tau^-(\lambda,t)f_1(\lambda) = 0 \\
&\mathrm{i}\frac{\mathrm{d}\tau_j^+}{\mathrm{d}t} + f_{1,j}^+\tau_j^+(t) + \tau_j^+(t)f_{2,j}^+ = 0, \qquad f_{1,2;j}^{\pm} = f_{1,2}(\lambda_j^{\pm}) \\
&\mathrm{i}\frac{\mathrm{d}\tau_j^-}{\mathrm{d}t} - f_{2,j}^-\tau_j^-(t) - \tau_j^-(t)f_{1,j}^- = 0, \qquad \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t} = 0.
\end{aligned} \tag{257}$$

As a consequence of theorem 5 we get that $S^{\pm}(\lambda)$ and $T^{\pm}(\lambda)$ satisfy linear set of equations

$$\mathrm{i}\frac{\mathrm{d}S^{\pm}}{\mathrm{d}t} + [f(\lambda), S^{\pm}(\lambda,t)] = 0, \qquad \mathrm{i}\frac{\mathrm{d}T^{\pm}}{\mathrm{d}t} + [f(\lambda), T^{\pm}(\lambda,t)] = 0 \tag{258}$$

which can be easily solved. In addition the block-diagonal matrices $D^{\pm}(\lambda)$ are not integrals of motion but rather satisfy

$$\mathrm{i}\frac{\mathrm{d}D^{\pm}}{\mathrm{d}t} + [f(\lambda), D^{\pm}(\lambda,t)] = 0. \tag{259}$$

From equation (259) there follows that for generic $f(\lambda)$ only the invariants of $D^{\pm}(\lambda,t)$ (or, equivalently, only the invariants of $\boldsymbol{a}^{\pm}(\lambda,t)$ and $\boldsymbol{c}^{\pm}(\lambda,t)$) provide series of integrals of motion in involution. As a consequence the generic NLEE (255) possess soliton solutions whose velocities may depend on time. Examples of such NLEE and the properties of their soliton solutions called boomerons and trappons have been analyzed by Calogero and Degasperis [7, 8, 9, 10].

6. Discussion

In order to understand better the idea that the expansions over the 'squared solutions' are generalized Fourier transforms we will outline the limit to small potentials, i.e. $Q(x) \ll 1$, for which the Born approximation is adequate. We will see that in this limit all these expansions turn out to be just the usual Fourier transforms.

It is known [1] that the corresponding Zakharov–Shabat system may have discrete eigenvalues only provided $Q(x)$ is large enough, while in the Born approximation it does not possess discrete spectrum. The eigenfunctions of the continuous spectrum are well approximated by the 'plane waves'

$$\chi^+(x,\lambda) \simeq \chi^-(x,\lambda) \simeq \mathrm{e}^{-\mathrm{i}\lambda Jx} \tag{260a}$$

and the scattering data are provided by the Born approximation

$$\begin{gathered}\rho^+_{ir}(\lambda) = \mathrm{i}\int_{-\infty}^{\infty} \boldsymbol{q}_{ir}(x)\mathrm{e}^{2\mathrm{i}\lambda x}\,\mathrm{d}x, \qquad \rho^-_{ri}(\lambda) = \mathrm{i}\int_{-\infty}^{\infty} \boldsymbol{r}_{ri}(x)\mathrm{e}^{-2\mathrm{i}\lambda x}\,\mathrm{d}x \\ a^+(\lambda) \simeq 1, \qquad a^-(\lambda) \simeq 1.\end{gathered} \tag{260b}$$

As a result the "squared solutions" are also approximated by 'plane waves'

$$\boldsymbol{\Psi}^+_{ri}(x,\lambda) \simeq \boldsymbol{\Phi}^-_{ri}(x,\lambda) \simeq E_{ir}\mathrm{e}^{2\mathrm{i}\lambda x}, \qquad \boldsymbol{\Psi}^-_{ir}(x,\lambda) \simeq \boldsymbol{\Phi}^+_{ri}(x,\lambda) \simeq E_{ir}\mathrm{e}^{-2\mathrm{i}\lambda x}.$$

The completeness relation (161) acquires the form

$$\delta(x-y)\Pi_{0J} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}x \sum_{i<r}\left(\mathrm{e}^{-2\mathrm{i}\lambda(x-y)}E_{ir}\otimes E_{ri} - \mathrm{e}^{2\mathrm{i}\lambda(x-y)}E_{ri}\otimes E_{ir}\right)$$

of the usual Fourier transform for matrix-valued functions. The recursion operators $\Lambda_\pm$ in this limit go into the purely differentiation operator

$$\Lambda^{\mathrm{as}}_{\pm}\cdot \simeq \frac{\mathrm{i}}{4}\left[\sigma_3, \frac{\mathrm{d}}{\mathrm{d}x}\cdot\right] \tag{261}$$

whose spectrum is purely continuous.

On the other hand the expansions over the "squared solutions" can be understood as expansions over the eigenfunctions of Λ_+ and Λ_-. Note, that while $\mathrm{i}\mathrm{d}/\mathrm{d}x$ does not have discrete spectrum, the operators $\Lambda_\pm$ may have discrete eigenvalues whenever L has. Finally, if we restrict ourselves to the special class of reflectionless potentials for which $\rho^\pm(\lambda) \equiv 0$ for all $\lambda \in \mathbb{R}$, we obtain finite dimensional subspace of $\mathcal{M}$. Then the operators $\Lambda_\pm$ operate nontrivially only in this subspace.

7. Conclusions

We showed that the interpretation of the ISM as a generalized Fourier transform holds true for the generalized Zakharov–Shabat systems related to the symmetric spaces $\mathrm{SU}(n+m)/\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(n))$. Using the standard dressing method we outlined the structure of the singularities of the fundamental analytic solutions and constructed the soliton solutions with multidimensional eigenspaces (the projectors $P_j(x)$ may have rank $r_j > 1$).

The expansions over the 'squared solutions' are natural tool to derive the fundamental properties not only of the MNLS type equations, but also of the NLEE with generic dispersion laws. Some of these equations, besides the intriguing properties as dynamical systems allowing for boomerons, trappons, etc., may also have interesting physical applications.

Another interesting area for further investigations is to study and classify the reductions of these NLEE. For results along this line for the MNLS equations see the reports [27] and [21]. Reductions of other types of NLEE have been considered in [16, 19, 20, 33].

One can also treat generalized Zakharov–Shabat systems related to other symmetric spaces. This would require substantial changes in the dressing factors. The expansions over the 'squared solutions' can be closely related to the graded Lie algebras, and to the reduction group and provide an effective tool to derive and analyze new soliton equations. For more details and further reading see [11, 16].

Acknowledgements

This work has been supported by the National Science Fund of the Bulgarian Ministry of Education and Science, Contract No. F–1410.

References

[1] Ablowitz M., Kaup D., Newell A. and Segur H., *The Inverse Scattering Transform – Fourier Analysis for Nonlinear Problems*, Stud. Appl. Math. **53** (1974) 249–315.

[2] Ablowitz M., Prinari B. and Trubatch A., *Discrete and Continuous Nonlinear Schrödinger Systems*, London Mathematical Society, Lecture Notes Series **302**, Cambridge University Press, 2004.

[3] Ablowitz M. and Seegur H., *Solitons and the Inverse Scattering Transform*, SIAM Studies in Applied Mathematics, Philadelphia, SIAM, 1981.

[4] Beals R. and Coifman R., *Scattering and Inverse Scattering for First Order Systems*, Commun. Pure & Appl. Math. **37** (1984) 39–90.
— *Inverse Scattering and Evolution Equations*, Commun. Pure & Appl. Math. **38** (1985) 29–42.
Caudrey P. *The Inverse Problem for the Third Order Equation* $u_{xxx}+q(x)u_x+r(x)u=-\mathrm{i}\zeta^3u$, Phys. Lett. A **79A** (1980) 264–268;
— *The Inverse Problem for a General* $n\times n$ *Spectral Equation*, Physica **D6** (1982) 51–66.

[5] Beals R. and Sattinger D. *On the Complete Integrability of Completely Integrable Systems*, Commun. Math. Phys. **138** (1991) 409–436.

[6] Calogero F. and Degasperis A., *Spectral Transform and Solitons*, Vol. I, North Holland, Amsterdam, 1982.

[7] Calogero F. and Degasperis A., *Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform I,II*, Nuovo Cim. B **32** (1976) 201–242; **39** (1976) 1–54.

[8] Calogero F. and Degasperis A., *Coupled Nonlinear Evolution Equations Solvable via the Inverse Spectral Transform and Solitons that Come Back: the Boomeron*, Lett. Nuovo Cimento **16** (1976) 425–433.

[9] Calogero F. and Degasperis A., *New Integrable Equations of Nonlinear Schrödinger Type*, Stud. Appl. Math. (to appear).

[10] Degasperis A., *Solitons, Boomerons, Trappons*, In: Nonlinear Evolution Equations Solvable by the Spectral Transform, F. Calogero (Ed), Pitman, London 1978, pp 97–126.

[11] Drinfel'd V. and Sokolov V., *Lie Algebras and Equations of Korteweg–de Vries Type*, Sov. J. Math. **30** (1985) 1975–2036.

[12] Faddeev L. and Takhtadjan L. *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[13] Fordy A. and Kulish P., *Nonlinear Schrodinger Equations and Simple Lie Algebras*, Commun. Math. Phys. **89** (1983) 427–443.

[14] Gakhov F., *Boundary Value Problems*, Translated from Russian, I. Sneddon (Ed), Pergamon Press, Oxford, 1966.

[15] Gerdjikov V., *Generalized Fourier Transforms for the Soliton Equations. Gauge Covariant Formulation*, Inverse Problems **2** (1986) 51–74.
— *Generating Operators for the Nonlinear Evolution Equations of Soliton Type Related to the Semisimple Lie Algebras* (in Russian), Doctor of Sciences Thesis, JINR, Dubna, USSR, 1987.

[16] Gerdjikov V., *Algebraic and Analytic Aspects of Soliton Type Equations*, Contemporary Mathematics **301** (2002) 35–68; **nlin.SI/0206014**.

[17] Gerdjikov V., *On the Spectral Theory of the Integro-Differential Operator Λ, Generating Nonlinear Evolution Equations*, Lett. Math. Phys. **6** (1982) 315–324.

[18] Gerdjikov V., *Complete Integrability, Gauge Equivalence and Lax Representations of the Inhomogeneous Nonlinear Evolution Equations*, Theor. Math. Phys. **92** (1992) 374–386.

[19] Gerdjikov V., Grahovski G. and Kostov N., *Reductions of N-wave Interactions Related to Low-Rank Simple Lie Algebras*. I: *$\mathbb{Z}_2$-Reductions*, J. Phys. A: Math. & Gen. **34** (2001) 9425–9461.

[20] Gerdjikov V., Grahovski G., Ivanov R. and Kostov N., *N-wave Interactions Related to Simple Lie Algebras. $\mathbb{Z}_2$-Reductions and Soliton Solutions*, Inverse Problems **17** (2001) 999–1015.

[21] Gerdjikov V., Grahovski G. and Kostov N., *On the Multi-Component NLS-type Models and Their Gauge Equivalent*, Report at the International Conference "Contemporary Aspects of Astronomy, Theoretical and Gravitational Physics", Sofia, May 20–22, 2004.

[22] Gerdjikov V. and Khristov E., *On the Evolution Equations Solvable with the Inverse Scattering Problem*. I. *The Spectral Theory*; II. *Hamiltonian Structures and Bäcklund Transformations*, Bulg. J. Phys. **7** (1980) 28–41; **7** (1980) 119–133.

[23] Gerdjikov V. and Kulish P. *A Method for the Derivation of the Bäcklund Transformation in the Formalism of the Inverse Scattering Method*, (in Russian), Theor. Math. Phys. **39** (1979) 69–74.

[24] Gerdjikov V. and Kulish P., *The Generating Operator for the* $n \times n$ *Linear System*, Physica D **3** (1981) 549–564.

[25] Gerdjikov V. and Yanovski A., *Gauge Covariant Formulation of the Generating Operator*. II. *Systems on Homogeneous Spaces*, Phys. Lett. A **110** (1985) 53–58;
– *Gauge Covariant Formulation of the Generating Operator* I, Commun. Math. Phys. **103** (1986) 549–568.

[26] Gerdjikov V. and Yanovsky A., *Completeness of the Eigenfunctions for the Caudrey–Beals–Coifman System*, J. Math. Phys. **35** (1994) 3687–3725.

[27] Grahovski G., Gerdjikov V. and Kostov N., *On the Multi-Component NLS-type Equations on Symmetric Spaces: Reductions and Soliton Solutions*, this volume pp 203–217.

[28] Helgasson S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.

[29] Kaup D., *The Three Wave Interaction – a Non-Dispersive Phenomenon*, Stud. Appl. Math. **55** (1976) 9–44.

[30] Kaup D., *Closure of the Squared Zakharov–Shabat Eigenstates*, J. Math. Annal. Appl. **54** (1976) 849–864.

[31] Kaup D., Reiman A. and Bers A., *Space-Time Evolution of Nonlinear Three-Wave Interactions*. I. *Interaction in a Homogeneous Medium*, Rev. Mod. Phys. **51** (1979) 275–310.

[32] Kaup D. and Newell A., *Soliton Equations, Singular Dispersion Relations and Moving Eigenvalues*, Adv. Math. **31** (1979) 67–100.

[33] Lombardo S. and Mikhailov A., *Reductions of Integrable Equations. Dihedral Group*, J. Phys. A: Math. Gen. **37** (2004) 7727–7742; (E-print: **nlin.SI/0404013**);
— *Reduction Groups and Automorphic Lie Algebras*, E-print: **math-ph/0407048**.

[34] Manakov S., *On the Theory of Two-Dimensional Stationary Self-Focusing Electromagnetic Waves*, Zh. Exp. Teor. Fiz. **65** (1973) 505–516; Manakov S., Sov. Phys. JETP **38** (1974) 248–253.

[35] Manakov S. *Complete Integrability and Stochastization of Discrete Dynamical Systems*, Sov. Phys. JETP **40** (1975) 703–716 (Zh. Exp. Teor. Fiz. **67** (1974) 543–555). Flashka H. *Toda Lattice II*, Prog. Theor. Phys. **51** (1974) 703–716.

[36] Mikhailov A. *The Reduction Problem and the Inverse Scattering Problem*, Physica D **3** (1981) 73–117.

[37] Shabat A., *The Inverse Scattering Problem for a System of Differential Equations*, Functional Annal. & Appl. **9** (1975) 75–78.

[38] Shabat A., *The Inverse Scattering Problem* (in Russian), Diff. Equations **15** (1979) 1824–1834.

[39] Wadati M., *The Modified Korteweg-de Vries Equation*, J. Phys. Soc. Jpn. **34** (1972) 1289–1296.

[40] Zakharov V. and Manakov S., *The Theory of Resonant Interaction of Wave Packets in Nonlinear Media* (in Russian), Zh. Exp. Teor. Fiz. **69** (1975) 1654-1673 .
— *Exact Theory of Resonance Interactions of Wave Packets in Nonlinear Media*, Preprint 74-41 INF, Novosibirsk, 1975.

[41] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons: The Inverse Scattering Method*, Plenum, Consultants Bureau, N.Y., 1984.

[42] Zakharov V. and Mikhailov A., *On the Integrability of Classical Spinor Models in Two-Dimensional Space–Time*, Commun. Math. Phys. **74** (1980) 21–40.

[43] Zakharov V. and Mikhailov A., *Relativistically Invariant Two-Dimensional Models of Field Theory Which are Integrable by Means of the Inverse Scattering Problem Method*, Zh. Exp. Teor. Fiz. **74** (1978) 1953–1973.

[44] Zakharov V. and Shabat A., *A Scheme for Integrating Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform, I*, Funct. Annal. Appl. **8** (1974) 43–53;
— *A Scheme for Integrating Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform, II*, Funct. Anal. Appl. **13** (1979) 13–23.

[45] Zakharov V. and Shabat A., *Exact Theory of Two-dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media*, Zh. Exp. Teor. Fiz. **61** (1971) 118–134.