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SEARCH FOR FUNDAMENTAL MODELS WITH AFFINE SYMMETRY: SOME RESULTS, SOME HYPOTHESES AND SOME ESSAY

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Abstract. We are dealing below with the tetrad models of gravitation and with certain modifications of this relatively old and partially forgotten subject. The problem is studied in a general n-dimensional manifold, not necessarily in the usual four-dimensional space-time. So strictly speaking, we are dealing with fields of frames (n-legs) rather than with literally understood tetrads (four-legs, Vierbeine). The main novelty of our models in comparison with traditional ones is that it is the total globally acting $\mathrm{GL}(n, \mathbb{R})$ that is used as the group of internal symmetries. In traditional approaches one uses the globally or locally acting Lorenz group $\mathrm{SO}(1, n-1)$. And obviously, our models, according to the Einstein–Hilbert paradigm are generally-covariant. We have shown that group spaces of Lie groups or certain their deformations are special vacuum-type homogeneous solutions of the model. Interaction with the bispinor field is briefly discussed and certain cosmological aspects of our models (cosmological expansions, escaping of galaxies) are mentioned. The model sheds new light onto this important problem. It may also have to do with the important problem of dark matter. In a sense, our model may be also interpreted in mechanical terms, as a theory of relativistic continuum with microstructure (relativistic micromorphic continuum), so in a sense, the traditional gap between field theory and mechanics becomes diffused. There are also certain features of our model which seem to be interesting from the point of view of multidimensional Kaluza–Klein universes (models generally covariant in a multidimensional sense, with fibration and fundamental group structures appearing as features of some special solutions). Spherically symmetric solutions of our models are in a sense formally similar to the t'Hooft–Polyakof monopole.

1. Introduction

For many years the tetrad models of gravitation do not seem a particularly interesting subject. There was some period of relatively strong interest initiated by Einstein himself who advocated the idea that teleparallelism, i.e., a field of linear frames (by duality equivalent to co-frames) in the space-time manifold is a promising candidate for describing gravitational field and perhaps also for creating some unified treatments. In any case the idea of expressing everything in terms of orthonormal frames looked interesting as a possibility of establishing some link between gravitation and specially-relativistic intuitions. Tetrad has also more degrees of freedom than the metric tensor and this motivated also some hopes for a unified field theory. And, a very important fact, tetrads are unavoidable when describing spinor fields in a manifold. There were also ideas that perhaps some tetrad models as admitting much stronger nonlinearity than that of Einstein equations may help to avoid singularities of generally-relativistic solutions [20, 21] (nowadays everybody just likes singularities and black holes, so today this motivation might look rather archaic although some new objections against black holes and singularities reappeared recently). Perhaps there was some influence of ideas underlying the Born–Infeld nonlinear electrodynamics. Einstein ideas of unified field theories failed however and the recent unification attempts are based rather on the gauge paradigm which turned out to be so successful in elementary particles theory and fundamental quantum interactions [16]. And finally, both the success of the standard general relativity, at least in macroscopic and cosmic scale (double pulsar radiation) and the essential mathematical difficulties within the framework of alternative treatments reduced the interest in modified approaches and unified treatments.

Nevertheless, it is still rather likely that on the microscopic and submicroscopic level the gauge approach and related formulations based on differential forms are promising. In any case they are mathematically very interesting because of their relationships with the modified versions of Born–Infeld nonlinearity celebrating now its come back to physics [5–7, 18, 24, 25]. And besides, the field-theoretic models with degrees of freedom described by differential forms have to do with the attractive idea of group spaces as non-excited vacua [13, 30].

Our approach below has to do with all these ideas and with certain concepts from the realm of relativistic mechanics of structured continua. Namely, from some point of view field of frames in the space-time manifold may be interpreted as the relativistic micromorphic medium consisting of infinitesimal affinely-rigid bodies [26,27]. Continua of this kind were discussed within a different context by Eringen and others [10, 19, 28]. Roughly speaking, one can interpret such approaches as something between field theory and continuum mechanics, as a kind of cosmic ether concept revisited.

2. Dynamical Models, Affine Invariance

The absolute framework of our field-theoretic model consists of a differential "space-time" manifold M and linear space V, both of the same real dimension n. The space V plays the role of the local model of M; usually one simply puts $V = \mathbb{R}^n$, however $\mathbb{R}^n$ is usually implicitly assumed with the plenty of structures felt as intrinsic, canonical, e.g., the Kronecker or perhaps normal-hyperbolic metric tensor (in relativistic applications). And here this would be just misleading; it is essential for us that V is a completely amorphous linear space. The above term "space-time" is not to be taken too literally. It may be as well some higher-dimensional Kaluza-type Universe ($n > 4$), some "elliptic" problems in three dimensions are also of interest. When for computational analytical purposes we identify V with $\mathbb{R}^n$, no metric or any other type of geometry is there explicitly assumed, unless otherwise stated. In particular, nothing like a fixed signature is fixed. On the contrary, our programme consists in "deriving" the signature, perhaps even the space-time dimensions from differential equations.

Fundamental objects in a differential manifold are linear frames in tangent spaces; all other geometric objects are their byproducts (with the proviso that spinors are so in a rather non-direct way). In other words, the most fundamental structure is the principal bundle of frames, or equivalently one of co-frames. All other bundles are its associated bundles.

So, it is natural to expect that the most fundamental physical field is the cross-section of the principal bundle of (co)frames. It is expected to describe gravitation, the most fundamental and universal interaction.

So, let $F^*(M,V)$ denote the manifold consisting of all linear isomorphisms $e_x : T_xM \to V$ at all possible points $x \in M$,

$$F^*(M,V) = \bigcup_{x\in M} LI(T_xM, V).$$

Obviously, when we put $V = \mathbb{R}^n$, these isomorphisms become simply linear co-frames $(\dots, e^A, \dots)$, $e^A \in T_x^*M$; then the bundle manifold is denoted simply by $F^*(M)$;

$$F^*M \subset \oplus_n T^*M = \bigcup_{x\in M} T_x^*M \times \cdots \times T_x^*M \quad (n \text{ factors, } n\text{-fold Whitney sum}).$$

$F^*(M,V)$ is in an obvious way the principal fibre bundle over M with $\mathrm{GL}(V)$ as the structural group

$$L \in \mathrm{GL}(V) : e \mapsto L \circ e.$$

When we put $V = \mathbb{R}^n$,

$$L \in \mathrm{GL}(n,\mathbb{R}) : e = (\dots, e^A, \dots) \mapsto Le := (\dots, L^A{}_B e^B, \dots). \tag{1}$$

The bundle of linear frames $F(M, V)$ consists of all isomorphisms $\tilde{e}_x : V \to T_xM$, $x \in M$. Obviously we have $\tilde{e}_x \circ e_x = \mathrm{id}_{T_xM}$, $e_x \circ \tilde{e}_x = \mathrm{id}_V$. When $V = \mathbb{R}^n$, then $\tilde{e} \in F(M, V)$ becomes $(\ldots, e^a, \ldots) \in F(M)$ and obviously $\langle e^A, e_B \rangle = \delta^A{}_B$ (duality).

The structure group $\mathrm{GL}(V)$ acts on $F(M, V)$ according to the dual rule

$$L \in \mathrm{GL}(V) : \tilde{e} \mapsto \tilde{e} \circ A^{-1}$$

so as to preserve duality. In analytical $V = \mathbb{R}^n$-description this reads

$$L \in \mathrm{GL}(n, \mathbb{R}) : \tilde{e} = (\ldots, e_A, \ldots) \mapsto L\tilde{e} := (\ldots, e_B L^{-1B}{}_A, \ldots).$$

Remark. Obviously the action of $\mathrm{GL}(V)$ (or $\mathrm{GL}(n, \mathbb{R})$) on (co)frames should not be confused with the action of $L_x \in \mathrm{GL}(T_xM) \subset T_xM \otimes T_x^*M$; the elements of $\mathrm{GL}(V)$ (or $\mathrm{GL}(n, \mathbb{R})$) cannot act on separate (co)vectors in T_x^*M or T_xM; only just on their n-tuples.

The bundle projections will be denoted by

$$\pi^* : F^*(M, V) \to M \quad \text{or} \quad \pi^* : F^*(M) \to M$$

and

$$\pi : F(M, V) \to M \quad \text{or} \quad \pi : F(M) \to M.$$

The canonical diffeomorphism of $F^*(M, V)$ onto $F(M, V)$ ($F^*(M)$ onto $F(M)$) will be denoted by J; obviously

$$\pi^* = \pi \circ J.$$

When local coordinates x^i are fixed in M, coframes and frames are parametrized by their components $e^A{}_i$, $e^i{}_A$ where

$$e^A{}_i e^i{}_B = \delta^A{}_B, \qquad e^i{}_A e^A{}_j = \delta^i{}_j.$$

This parametrization gives rise to local coordinates $(x^i, e^A{}_i)$, $(x^i, e^i{}_A)$ on the bundle manifold. It is advantage of models using only differential forms as degrees of freedom that they may be intrinsically differentiated without any auxiliary objects, just in the sense of exterior differentials; in particular no affine connection is necessary [30].

So, let e be some field of coframes, i.e. cross-section of the bundle $\pi^* : F^*(M, V) \to M$. The vector-valued differential form e may be differentiated resulting in the V-valued differential two-form $\mathrm{d}e$. So, if some basis in V is fixed ($\mathbb{R}^n$-identification), then we obtain the system of differential two-forms $\mathrm{d}e^A$, $A = 1, \ldots, n$, with the local representation $(\mathrm{d}e^A)_{ij} = e^A{}_{j,i} - e^A{}_{i,j}$. From this system we can construct a single mixed tensor

$$S^i{}_{jk} = \frac{1}{2} e^i{}_A (e^A{}_{j,k} - e^A{}_{k,j}) \tag{2}$$

i.e.,

$$S = -\frac{1}{2} e_A \otimes \mathrm{d}e^A.$$

This is a very well-defined object, namely the torsion of the teleparallelism affine connection induced by e. This connection is uniquely defined by the condition

$$\nabla e_A = 0, \qquad A = 1, \ldots, n.$$

Analytically its components are given by

$$\Gamma^i{}_{jk} = e^i{}_A e^A{}_{j,k}$$

so really (2) coincides with its torsion

$$S^i{}_{jk} = \Gamma^i{}_{[jk]} = \frac{1}{2}(\Gamma^i{}_{jk} - \Gamma^i{}_{kj}).$$

Γ and S are coordinate-independent first-order differential con-comitants of e. For any constant (x-independent) $L \in \mathrm{GL}(V)$ we have

$$S[Le] = S[e]$$

i.e., the prescription $e \rightarrow S[e]$ is globally $\mathrm{GL}(V)$-invariant (but not longer locally). This prescription is also natural (covariant) with respect to the diffeomorphism group

$$S[\varphi^* e] = \varphi^* S[e].$$

And this is very important, because we are going to construct $\mathrm{Diff}(M)$-invariant i.e., generally-covariant models. This is the Hilbert–Einstein paradigm according to which all fundamental theories should be generally-covariant, i.e., free of any implicit absolute objects.

Another important geometric interpretation of S is that in terms of the non-holonomy object of e. Indeed, one can easily show that

$$S = \frac{1}{2} C^K{}_{LM} e_K \otimes e^L \otimes e^M$$

where

$$[e_K, e_L] = C^M{}_{KL} e_M, \qquad \mathrm{d}e^K = \frac{1}{2} C^K{}_{LM} e^M \wedge e^L \tag{3}$$

and obviously $[X, Y]$ denotes the Lie bracket of vector fields. C is the non-holonomy object of e, it vanishes if and only if the field of frames e is holonomic.

Obviously the curvature tensor of Γ vanishes and the teleparallelism transport is path-independent, at least in topologically trivial domains.

Having defined the invariant tensorial derivative S of e we are prepared to construct generally covariant Lagrangian densities. No absolute geometric objects are allowed in M if the resulting model is to be $\mathrm{Diff}(M)$-invariant. On the other hand, in the internal space V some geometry may be fixed without violating general covariance in M. Moreover, during many years one assumed, incorrectly, that some kind of internal metric η in V has to be fixed if we are to be in position to construct Lagrangian, i.e., weight-one scalar density built algebraically of e and of its exterior differential. This was just the basic assumption of the original teleparallelism model. Let us now review the hierarchy of models starting from the smallest to the largest reasonable group of internal symmetries.

2.1. Pseudo-Euclidean Internal Space (V, η)

Here $\eta \in V^* \otimes V^*$ denotes a symmetric non-degenerate bilinear form (scalar product) on V. Any field of co-frames e on M gives rise to the field $h[e, \eta]$ of the symmetric non-degenerate metric tensor,

$$h[e, \eta]_x = e^*_x \cdot \eta$$

i.e., analytically,

$$h[e, \eta]_{ij} = \eta_{AB} e^A{}_i e^B{}_j.$$

The very important fact: $h[e, \eta]$ is automatically parallel with respect to the teleparallelism connection $\Gamma[e]$

$$\nabla h[e, \eta] = 0.$$

Obviously it is also parallel with respect to the Levi–Civita connection induced by $h[e, \eta]$. The structure $(M, \Gamma[e], h[e, \eta])$ is a (pseudo)Riemann–Cartan space and

$$\Gamma[e]^i{}_{jk} = e^i{}_A e^A{}_{j,k} = \{^{\,i}_{jk}\} + S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i$$

where the shift of tensor indices from their natural positions is meant in the sense of $h[e, \eta]$.

The metric field h is invariant under the group $\mathrm{O}(V, \eta) \in \mathrm{GL}(V)$ of η-orthogonal internal transformations,

$$h[Le, \eta] = h[e, \eta]$$

for any $L \in \mathrm{O}(V, \eta)$. Moreover, in virtue of the algebraic dependence of h on e (no derivatives), the above holds also for local η-orthonormal transformations, i.e., for any group-valued function $L : M \to \mathrm{O}(V, \eta)$ we have

$$h[L(x)e_x, \eta] = h[e_x, \eta]$$

where x runs over M. Such a local invariance does not hold for $S[e]$. And obviously, the prescription $e \to h[e, \eta]$ is generally-covariant

$$h[\varphi^* e, \eta] = \varphi^* h[e, \eta]$$

for any $\varphi \in \mathrm{Diff}(M)$. In the sequel $h[e, \eta]$ will be referred to as a Dirac–Einstein metric. It is just a byproduct of e, no independent field quantity. The coframe e and its dual $\tilde{e}$ are automatically $h[e, \eta]$-orthonormal. The only essential feature of η-geometry which matters here is the signature of η. In Einstein teleparallelism model and in more general tetrad theories, when $n = 4$, the normal-hyperbolic signature $(+---)$ or $(-+++)$ is used. In these theories $h[e,\eta]$ is interpreted as the spatio-temporal metric tensor. Nevertheless, it is e that is a primary field variable subject to the variational procedure. This is especially important when spinor fields are included into the treatment; the tetrad field is then indispensable as a fundamental field quantity. The metric $h[e,\eta]$ is flat if and only if e is holonomic, $S[e] = 0$. Spinor fields in matter Lagrangians are coupled directly to the co-frame e and this coupling is in an essential way irreducible to the coupling with $h[e,\eta]$. Having the metric $h[e,\eta]$ at disposal we can contract indices of other tensorial quantities like S and construct in this way scalar densities of weight one and scalars, in particular the ones quadratic in derivatives of e. Within the generally-covariant framework there are three independent quadratic scalars, namely, so-called Weitzenböck invariants

$$J_1 = h_{ia}h^{jb}h^{kc}S^{i}{}_{jk}S^{a}{}_{bc}$$

$$J_2 = h^{ij}S^{k}{}_{li}S^{l}{}_{kj}$$

$$J_3 = h^{ij}S^{a}{}_{ai}S^{b}{}_{bj}$$

for simplicity the labels e,η are omitted in h, S, when there is no confusion danger. Obviously, the upper-case h denotes the contravariant inverse of the lower-case h, $h^{ia}h_{aj} = \delta^{i}{}_{j}$. The simplest teleparallel models are based on Lagrangians quadratic in S (quadratic in derivatives), namely

$$L = c_1L_1 + c_2L_2 + c_3L_3 = (c_1J_1 + c_2J_2 + c_3J_3)\sqrt{|h|} \tag{4}$$

where, obviously, $|h|$ is the weight-two Weyl density, $|h| = |\det[h_{ab}]|$. Its square root and therefore the total L is a weight-one Weyl density, as any well-defined Lagrangian must be. The above model is generally-covariant (Diff(M)-invariant) and invariant under the global action of the internal group $\mathrm{O}(V,\eta) \subset \mathrm{GL}(V)$.

Einstein model corresponds to the special case of the coefficients ratio $c_1 : c_2 : c_3 = 1 : 2 : (-4)$. The point is that the curvature scalar $R[h]$ of the metric tensor h may be expressed as follows

$$R[h] = J_1 + 2J_2 - 4J_3 + 4\nabla_i(S^{a}{}_{ab}h^{bi})$$

where the covariant derivative is meant in the Levi–Civita h-sense. Therefore, the Hilbert–Einstein Lagrangian $R\sqrt{|h|}$ differs from $(J_1 + 2J_2 - 4J_3)\sqrt{|h|}$ by the divergence term

$$4\nabla_i(S^{a}{}_{ab}h^{bi}\sqrt{|h|}) = 4\partial_i(S^{a}{}_{ab}h^{bi}\sqrt{|h|}).$$

This term absorbs all second derivatives, thus the tetrad Lagrangian of Einstein theory, $(J_1 + 2J_2 - 4J_3)\sqrt{|h|}$, is a well-defined scalar density of weight-one, explicitly free of second derivatives of dynamical variables e. Let us remind that when the usual metric description of general relativity is used, some second-derivatives term also may be subtracted, however, the remaining first-order Lagrangian has the unpleasant feature that it fails to be scalar density of weight one. Instead, its transformation rule is that of scalar density modulo some additional, rather unpleasant divergence term. This obscures the treatment and makes it artificially complicated.

Lagrangian (4) with $c_1 : c_2 : c_3 = 1 : 2 : (-4)$ is generally covariant, i.e., $\mathrm{Diff}(M)$-invariant modulo the correction term which is a "good" scalar density of the total divergence structure (without second derivatives). In other words, it is essentially $\mathrm{Diff}(M)$-invariant. Modifying the values of c_1, c_2, c_3 (more precisely their ratio) we can reduce drastically the infinite-dimensional local $\mathrm{O}(V, \eta)$-symmetry generated by ($\frac{1}{2}n(n-1)$ arbitrary functions, 6 in the physical case $n = 4$). So even in the absence of spinor fields, e involves some physical degrees of freedom, more general than those absorbed by $h[e, \eta]$. And there are some indications that such more general models, at least in some range of c_1, c_2, c_3 may be as compatible with experimental data as Einstein theory, being at the same time free of its non-desirable features. It is interesting that within the framework of first-order variational principles the metrical description of gravitational degrees of freedom does not admit any other dynamical model than the Hilbert–Einstein one modulo the cosmological term. In a sense kinematics determines dynamics. The tetrad formulation immediately opens a very huge possibility of dynamical models. By the way, it is difficult to decide a priori whether this arbitrariness is desirable or embarrassing.

The above quadratic models, leading to quasilinear equations, were suggested by Möler, Pellegrini and Plebański, Hehl and others [14, 19, 20]. General $\mathrm{O}(V, \eta)$-invariant models, especially those studied by the mentioned people, are usually written in the form

$$L(S, h) = c(S, h)\sqrt{|h|}$$

c denoting some scalar function built algebraically of (S, h) alone. For generic, non-exceptional forms of c, the resulting model is essentially $\mathrm{O}(V, \eta)$-invariant, i.e., there is no larger subgroup of $\mathrm{GL}(V)$ preserving it. The above representation is motivated by the quoted Weitzenöbck models quadratic in derivatives and by their qualitatively simple modifications. However, not always this representation is maximally convenient. According to the modified Born–Infeld paradigm, there is another pole of mathematical simplicity, motivated by the demand of essential non-linearity and in a sense opposite to quasilinear models quadratic in derivatives. This "Born–Infeld" class is based on Lagrangian densities of the form

$$L = \sqrt{|\det[L_{ij}]|} \tag{5}$$

where L_{ij}, the twice covariant "Lagrange tensor" is a polynomial of at most second degree in derivatives. The most general model of this type for essentially $\mathrm{O}(V, \eta)$-type class of action densities is given by

$$L_{ij} = T_{ij} + 4DS^a{}_{ib}S^c{}_{jd}h_{ac}h^{bd} + (EJ_1 + FJ_2 + GJ_3)h_{ij} + Nh_{ij} \tag{6}$$

where

$$T_{ij} = 4AS^k{}_{li}S^l{}_{kj} + 4BS^k{}_{ki}S^l{}_{lj} + 4CS^k{}_{lk}S^l{}_{ij} \tag{7}$$

A, B, C, D, E, F, G, N are constants and the 4-coefficients follow only from some non-essential conventions. It is seen that the above L_{ij} really is a second-degree polynomial of S; obviously the "cosmological" term controlled by N is zeroth-order in S.

One could try to use models with (5) multiplied by some scalar built of (S, h), but this of course would destroy the very nice Born–Infeld structure.

2.2. Weyl Geometry in the Internal Space

This is the first step of our programme of reducing and weakening the internal geometry in V. Now only the Weyl geometry, i.e., linear-conformal geometry is assumed. This means that η is fixed only up to the multiplicative factor; there exist standards of angles but not the ones of length. In order words, only some one-dimensional half-axis in $V^* \otimes V^*$ is fixed, it is not η itself but $\mathbb{R}^+\eta$ that describes geometry of V; the signature survives of course.

Now we are interested in models invariant under the Weyl group $\mathbb{R}^+\mathrm{O}(V, \eta) \subset \mathrm{GL}(V)$. Being also invariant under $\mathrm{O}(V, \eta)$ they form a subclass of (5), (6). The corresponding Lagrangian densities $L(S, h)$ must be homogeneous functions of degree zero in h,

$$L(S, h) = L(S, lh)$$

for any $l > 0$. In particular, Lagrangian tensors (6) are acceptable only if $N = 0$, i.e., the "cosmological" term is absent.

2.3. $\mathrm{GL}(V)$- and $\mathrm{SL}(V)$-Models

And now we decide V to be completely amorphous, i.e., no geometric object is fixed in V. A priori, when we once decide to leave the well-established ground of the local $\mathrm{O}(V, \eta)$-symmetry, the global $\mathrm{GL}(V)$-symmetry and the completely amorphous internal space V are the most natural assumptions.

Remark. With the purely e-degrees of freedom the local $\mathrm{GL}(V)$-symmetry would be meaningless. Indeed, local L-transformations (1) may produce every field of (co)frames from every other one. Therefore, a hypothetic local $\mathrm{GL}(V)$-local model of the e-field would be either trivial (every field would be a solution, tautologic field equations) or inconsistent (no solutions at all).

Lagrangian densities of $\mathrm{GL}(V)$-models must be algebraically built of S alone, $L(S)$, and now the most natural and simplest model is just the Born–Infeld Lagrangian (5) with the vanishing values of D, E, F, G, N, i.e.,

$$L = \sqrt{|\det[L_{ij}]|} = \sqrt{|\det[T_{ij}]|}. \tag{8}$$

The Lagrange tensor $L_{ij} = T_{ij}$ is given by (7), the only twice covariant tensor built of S alone in a $\mathrm{Diff}(M)$-invariant manner and quadratic in S. It is automatically a homogeneous second-degree polynomial of S. And now this generalized Born–Infeld structure is not a curiosity or "another pole of mathematical simplicity alternative with respect to models quadratic in derivatives". It is just the simplest model compatible with the assumed $\mathrm{GL}(V)$-symmetry. And roughly speaking, it is then "maximally linear" among the all by necessity nonlinear $\mathrm{GL}(V)$-models.

The above $\mathrm{GL}(V)$-Lagrangian is homogeneous of degree n in the tensor variable S, i.e., finally, in the tangent n-vector of the cross-section $e : M \to F^*M$. This means that such models are field-theoretical counterparts of the Finsler structure [23, 29]. One can show that every $\mathrm{Diff}(M)$- and $\mathrm{GL}(V)$-invariant Lagrangian $L(S)$ must be a homogeneous function of degree n of S,

$$S^k{}_{ij} \frac{\partial L}{\partial S^k{}_{ij}} = nL.$$

Let us observe that the symmetric part of (7)

$$T_{(ij)} = 4AS^k{}_{li}S^l{}_{kj} + 4BS^k{}_{ki}S^l{}_{lj}$$

may be considered as a candidate for the space-time metric tensor, which is globally $\mathrm{GL}(V)$-invariant and alternative with respect to the Dirac–Einstein prescription $h[e, \eta]$. This concerns particularly the Killing tensor

$$g[e]_{ij} = 4S^k{}_{li}S^l{}_{kj}. \tag{9}$$

(Strictly speaking some unit-fixing dimension factor A should be introduced).

Unlike the Dirac-Einstein metric, $g[e]$ in general is not covariantly constant under the teleparallelism connection. Its non-holonomic representation is

$$g[e] = g_{AB}e^A \otimes e^B, \qquad g_{AB} = C^K{}_{LA}C^L{}_{KB}, \tag{10}$$

where $C^K{}_{LM}$ are non-holonomy coefficients (3). When the contravariant vectors e_A form a Lie algebra, i.e., $C^K{}_{LM}$ are (structure) constants, then M becomes a Lie group space (and Lie group when some point $x_0 \in M$ is by convention chosen as the neutral element), g_{MN} become Lie-algebraic Killing–Cartan coefficients and $g[e]$ itself is the Killing metric field. Obviously $g[e]$ is the genuine non-singular metric if and only if $\tilde{e}$-spanned Lie algebra is semisimple. The metric $g[e]$ is then covariantly constant under the $\Gamma[e]$-teleparallelism and may be interpreted as the Dirac–Einstein metric with internal coefficients $\eta_{AB} = g_{AB}$.

The simplest Lagrangians (8) may be complicated by introducing some kind of "potentials", i.e. scalar multipliers f

$$L = f(s)\sqrt{|\det[T_{ij}]|}.$$

Here f are globally $\mathrm{GL}(V)$-invariant scalars built algebraically of S. All such "potentials" must be homogeneous of degree zero

$$S^k{}_{ij}\frac{\partial f}{S^k{}_{ij}} = 0$$

because, as mentioned, $L = f\sqrt{|T|}$, just as $\sqrt{|T|}$ itself, must be homogeneous of degree n. Let us quote a few examples:

$$g_{il}g^{jm}g^{kn}S^i{}_{jk}S^l{}_{mn}$$

$$g_{ij}S^a{}_{ai}S^b{}_{bj}$$

etc. Obviously, such models are much more complicated and loose the aesthetic and geometric value of (8).

Let us finish with $\mathrm{SL}(V)$-invariant models. Obviously, they are also amorphous in the internal space V, just as $\mathrm{GL}(V)$-models. In particular, they do not assume any volume standard in V, because $\mathrm{SL}(V)$ preserves simultaneously all volumes (all algebraic n-forms) in V. The corresponding Lagrangians are algebraic functions of S and the determinant $|e| = \det[e^A{}_i]$, $L = L(S, |e|)$. To use the scalar-density factorization we can write, e.g.,

$$L = a\left(S, \frac{|g|}{|e|^2}\right)|e| = b\left(S, \frac{|g|}{|e|^2}\right)\sqrt{|g|} = c\left(S, \frac{|g|}{|h|}\right)\sqrt{|h|}$$

etc. Here a, b, c are scalar functions built of the tensor S and of $\mathrm{SL}(V)$-invariant scalars like $|g||e|^{-2}$, $|g||h|^{-1}$, etc. One can also use "Born–Infeld" schemes like (7), (8), however with non-constant coefficients built algebraically of $\mathrm{SL}(V)$-invariant and generally-covariant scalars, just like, e.g., $|g||e|^{-2}$, $|g||h|^{-1}$.

3. Field Equations, Conservation Laws, Identities, Constraints

No doubt, the simplest and the most "Born–Infeld-like" models are those $\mathrm{GL}(V)$-invariant. The bare, amorphous internal space looks geometrically something best motivated.

Let us write the general structure of field equations, assuming for a while a more general model, when the Lagrange density is built algebraically of tensors S, h, i.e., it is a weight-one scalar density $L(S, h)$ covariant under $\mathrm{Diff}(M)$.

$$L\left(S[\varphi^* e], h[\eta, \varphi^* e]\right) = L(\varphi^* S[e], \varphi^* h[\eta, e]) = \varphi^* L(S[e], h[\eta, e])$$

for any $\varphi \in \mathrm{Diff}(M)$. This means that the action functional

$$I[e, \Omega] = \int_\Omega L[e]$$

is simply $\mathrm{Diff}(M)$-invariant, i.e.,

$$I[\varphi_* e, \varphi(\Omega)] = I[e, \Omega]$$

and Ω denotes an arbitrary n-dimensional domain in M with smooth $(n-1)$-dimensional boundary $\partial\Omega$.

Let us introduce some auxiliary weight-one tensor densities $H_k{}^{ij}$, Q^{ij},

$$H_k{}^{ij} := \frac{\partial L}{\partial S^k{}_{ij}} = -H_k{}^{ji}, \qquad Q^{ij} := \frac{\partial L}{\partial h_{ij}} = Q^{ji}$$

with appropriately defined differentiation convention of L with respect to skew-symmetric and symmetric tensors. These quantities are referred to respectively as the field momentum and the Dirac–Einstein stress. For models invariant under the Weyl group $\mathbb{R}^+\mathrm{O}(V,\eta)$ we have obviously

$$h_{ij}Q^{ij} = 0$$

because, as mentioned, L is then homogeneous of degree zero in h. And obviously for $\mathrm{GL}(V)$-models $Q^{ij} = 0$. The term "field momentum" is justified by the fact that

$$H_k{}^{ij} = e^A{}_k H_A{}^{ij} = e^A{}_k \frac{\partial L}{\partial e^A{}_{i,j}}.$$

When by analogy with electrodynamics, e^A are interpreted as a system of n "cov-ector potentials", then $F^A{}_{ji} := e^A{}_{i,j} - e^A{}_{j,i}$, i.e., a system of n "field strengths" is formally analogous to the $(\bar{E}, \bar{B})$-fields. And the quantities

$$H_A{}^{ij} := \frac{\partial L}{\partial e^A{}_{i,j}} = -H_A{}^{ji}$$

just the "field momenta" are analogous to the electromagnetic $(\bar{D}, \bar{E})$-fields. Obviously, the $\mathrm{Diff}(M)$-invariance implies that such an "n-electrodynamics" is not invariant under gradient transformations: $e^A \mapsto e^A + df^A$. L must depend algebraically on $e^A{}_i$, not only the $F^A{}_{ji} := e^A{}_{i,j} - e^A{}_{j,i}$ alone. In this sense this is a kind of "Mie n-electrodynamics". However, the gradient group is replaced by much more interesting $\mathrm{Diff}(M)$-group, also labelled by n arbitrary functions of n variables x^i. One can show that $\mathrm{Diff}(M)$-invariance implies in some kind of effective "gauge invariance" for Jacobi equations describing the dynamics of small perturbations to some fixed solutions.

Field equations following from the above Lagrangians may be concisely written in the following form

$$K_i{}^j := \nabla_k H_i{}^{jk} + 2S^m{}_{mk} H_i{}^{jk} - 2h_{ik}Q^{kj} = 0 \tag{11}$$

where, obviously, the covariant derivative ∇ is meant in the $\Gamma[e]$-teleparallelism sense. The weight-one tensor density K is a mixed-valence analogue of the Einstein tensor density from general relativity,

$$G_i{}^j := \left(R_i{}^j - \frac{1}{2}R^k{}_k\delta_i{}^j\right)\sqrt{|g|}.$$

In $\mathrm{GL}(V)$-models, when $Q^{ij} = 0$, our field equations simplify to

$$K_i{}^j := \nabla_k H_i{}^{jk} + 2S^m{}_{mk} H_i{}^{jk} = 0. \tag{12}$$

Another suggestive Mie-electrodynamic form of field equations is the following

$$H_A{}^{ij}{}_{,j} = -j^i{}_A$$

where $j^i{}_A$ are "self-interaction currents"

$$j^i{}_A = -\frac{\partial L}{\partial e^A{}_i}.$$

Obviously, field equations imply the "continuity equations"

$$j^k{}_{A,k} = 0. \tag{13}$$

H_A are skew symmetric contravariant tensor densities of weight-one, and j_A are contravariant vector densities of weight one. Therefore, the above divergence expressions, in spite of their being expressed through the usual partial derivatives, are well-defined quantities, respectively the vector and scalar densities of weight one. The self-currents j_A may be unified into a mixed weight-one tensor density

$$j^k{}_l = j^k{}_A e^A{}_l = -L\delta^k{}_l + 2H_i{}^{jk} S^i{}_{jl} = -L\delta^k{}_l + 2\frac{\partial L}{\partial S^i{}_{jk}} S^i{}_{jl}. \tag{14}$$

It may be expressed by the "energy-momentum complex"

$$t^k{}_l := e^A{}_{i,l} \frac{\partial L}{\partial e^A{}_{i,k}} - L\delta^k{}_l \tag{15}$$

namely

$$j^k{}_l = t^k{}_l + H_i{}^{kj} \Gamma_{tel}{}^i{}_{lj}. \tag{16}$$

These are general expressions valid for all generally-covariant models. If L is in addition $\mathrm{GL}(V)$-invariant (amorphous in the internal space), then

$$j^k{}_l = S^k{}_{ij} H_l{}^{ij}. \tag{17}$$

"Continuity equation" for j_A-currents implies the following equation

$$t^a{}_{b,a} = 0. \tag{18}$$

This is also valid for all $\mathrm{Diff}(M)$-invariant variational principles for the field of frames and may be interpreted as an improper conservation law following from the functions-labelled transformation group. Obviously, the complex t is not a tensor density; this is directly seen from the equation (15). Taken together (14), (15), (16), (18) mean that, in a sense, $j^a{}_b$ may be interpreted as the (non-conserved) energy-momentum tensor density of the field e.

Let us introduce the following system of n^2 weight-one vector densities $H_A{}^B$

$$H_A{}^{Bi} = e^l{}_A e^B{}_k H_l{}^{ki}.$$

One can show that for $\mathrm{GL}(V)$-invariant models $L(S)$ equations of motion imply that

$$H_A{}^{Bi}{}_{,i} = 0.$$

This is the system of conservation laws following from the $\mathrm{GL}(V)$-invariance via the Noether theorem. Roughly speaking, the global quantities corresponding to the densities $H_A{}^B$ are Hamiltonian generators of $\mathrm{GL}(V)$ acting on the fields e via (1). Because of this, $H_A{}^B$ may be referred to as co-moving components of the affine spin of the system. For Lagrangians $L(S, h)$ invariant under $\mathrm{O}(V, \eta) \subset \mathrm{GL}(V)$ $H_A{}^B$ do not obey the conservation laws. However such laws hold then for their η-skew-symmetric parts

$$S_A{}^B := H_A{}^B - \eta_{AC}\eta^{BD}H_D{}^C \quad \text{and} \quad S_A{}^{Bi}{}_{,i} = 0.$$

The corresponding conserved global quantities may be interpreted as the co-moving components of the spin of e.

In models invariant under $\mathrm{SL}(V)$ the dilatation-free (shear) part of H is conserved,

$$\Delta_A{}^{Bi}{}_{,i} = 0, \qquad \Delta_A{}^B := H_A{}^B - \frac{1}{n}H_C{}^C\delta^A{}_B.$$

And for Lagrangians invariant under the Weyl group $\mathbb{R}^+\mathrm{O}(V, \eta)$ we have

$$\left(S_A{}^{Bi} + \frac{1}{n}H_C{}^{Ci}\delta_A{}^B\right)_{,i} = 0$$

i.e., separately

$$S_A{}^{Bi}{}_{,i} = 0, \quad H_C{}^{Ci}{}_{,i} = 0.$$

The continuity equations (13) for the self-interaction currents j_A and, in a sense equivalently, (18) have another geometric status than the above conservation laws. Namely, they have to do with differential identities following from the invariance of L under the infinite-dimensional functions-labelled transformation group $\mathrm{Diff}(M)$. They are not proper weak conservation laws following from finite-dimensional Lie groups like the internal symmetries $\mathrm{GL}(V)$, $\mathrm{O}(V, \eta)$, $\mathbb{R}^+\mathrm{O}(V, \eta)$ and $\mathrm{SL}(V)$. The total system of $\mathrm{Diff}(M)$-invariance differential identities reads

$$\frac{\partial L}{\partial e^A{}_{b,a}} = -\frac{\partial L}{\partial e^A{}_{a,b}}, \quad \text{i.e.,} \quad H_A{}^{ba} = -H_A{}^{ab} \tag{19}$$

$$t^k{}_{l,k} = e^A{}_{a,l}\mathcal{L}^a{}_A \tag{20}$$

$$j^i{}_A = -\frac{\partial L}{\partial e^A{}_i} = \left(t^i{}_j - \frac{\partial L}{\partial e^B{}_{k,i}}e^B{}_{j,k}\right)e^j{}_A \tag{21}$$

where $\mathcal{L}^a{}_A$ is the $({}^a{}_A)$-th Euler–Lagrange term,

$$\mathcal{L}^a{}_A = \frac{\partial L}{\partial e^A{}_a} - \left(\frac{\partial L}{\partial e^A{}_{a,b}}\right)_{,b}. \tag{22}$$

Equation (19) means that there was no our "free will" in constructing L as depending on derivatives $e^{A}{}_{i,j}$ through their skew-symmetric part

$$\frac{1}{2}F^{A}{}_{ij} = \frac{1}{2}\left(e^{A}{}_{j,i} - e^{A}{}_{i,j}\right)$$

thus finally, through $S^{k}{}_{ij}$. There is no other possibility for generally-covariant Lagrangians. Equation (21) is equivalent to (14). Equations (19), (20) and (21) imply that, just as in any generally-covariant theory, the energy-momentum complex is a curl modulo Euler–Lagrange terms,

$$t^{l}{}_{k} = \frac{D}{Dx^{a}}H_{k}{}^{la} - e^{A}{}_{k}\mathcal{L}^{l}{}_{A} \tag{23}$$

($\frac{D}{Dx^{a}}$-total derivatives). Thus, one obtains the following strong conservation laws

$$\frac{D}{Dx^{k}}(t^{k}{}_{l} + e^{A}{}_{l}\mathcal{L}^{k}{}_{A}) = 0 \tag{24}$$

and generalized "Bianchi identies"

$$\frac{D}{Dx^{k}}(e^{A}{}_{l}\mathcal{L}^{k}{}_{A}) - e^{A}{}_{a,l}\mathcal{L}^{a}{}_{A} = 0. \tag{25}$$

Strong "conservation laws" are ones satisfied independently on whether the field equations are fulfilled or not by e. Assuming that e satisfies the Euler–Lagrange equations one obtains the "weak conservation laws" which as usual in generally-covariant theories read

$$\frac{D}{Dx^{k}}t^{k}{}_{l} = 0. \tag{26}$$

In the theory of variational principles they are referred to as "improper" conservation laws, because on realistic (Euler–Lagrange compatible) field evolutions t becomes a curl. Any vector field u on M, i.e., any one-parameter subgroup of $\mathrm{Diff}(M)$ generates such an improper conservation laws. These improper laws imply the continuity equations (13) for "self interaction currents". Nevertheless, they are not weak conservation laws in the literal sense. It is so for any generally-covariant theory which involves vector fields as dynamical variables. In fact, the invariance of L under the one-parameter diffeomorphism group generated by the vector field u implies the identity

$$\frac{D}{Dx^{a}}\left(-t^{a}{}_{b}u^{b} - \frac{\partial L}{\partial e^{B}{}_{l,a}}e^{B}{}_{k}u^{k}{}_{,l}\right) = 0. \tag{27}$$

This is in principle an improper weak conservation laws. However, once derived, this equation may be slightly re-interpreted and instead of some fixed u we can substitute just one of the vector fields e_{A}. It turns out that the resulting laws are exactly the continuity equations (13). Roughly speaking, the corresponding conservation law follows from the invariance of L under the one-parameter group generated by

e_A; or more precisely – under the $\Gamma[e]$-parallel transport along the direction e_A in the first-jet bundle of FM.

When some "space-time" $1 + (n-1)$-decomposition of M (compatible with the signature of $g[e]$) is fixed, one can introduce global conserved quantities $K_A{}^B$, Q_A corresponding to divergence-free vector densities $H_A{}^{Bi}$, $j^i{}_A$. They are conserved, i.e., independent on the $(n-1)$-dimensional space-like section to which they are related. Interpretation of $j^a{}_b$ as a corrected (tensorially well-defined) energy-momentum density of e suggests us to interpret the charges Q_A as co-moving components of the total energy-momentum. This interpretation is supported by the fact that the conservation of Q_A is equivalent to the dynamical invariance of our model under parallel translations along e_A. If e_A is time-like in (M, g) and e_B with $B \neq A$ are space-like, then it is natural to interprete the formula for Q_A as a summation of rest energies of infinitesimal portions of the physical system described by e. Conservation of Q_A has to do with functionally-parametrized groups of symmetries, thus for smooth, non-singular solutions well-behaving at spatial infinity, Q_A will vanish.

The above scheme was thought on as candidate for alternative gravitation theory or some unifying treatment, (especially in higher dimension $n > 4$). At the same time it may be interpreted as a model of relativistic continuum with microstructure, a kind of micromorphic ether. If some solution g happens to be normal-hyperbolic, some e_B is time-like and all other e_A, $A \neq B$ are space like, one can interpret the integral curves of e_B as world-lines of some cosmic substratum (relativistic fluid) and the other tetrad legs e_A, $A \neq B$ are interpreted as describing internal degrees of freedom (a kind of Cosserat continuum, or rather micromorphic continuum). With such a philosophy the traditional gap between (continuum) mechanics and "true" field theory becomes diffused [10, 15, 19].

The questions arises how to couple other fields to the above model of e-gravity. For simplicity let us consider the complex scalar field Ψ. In Einstein theory the simplest and most natural scheme combines linearly the Hilbert gravitational Lagrangian $R[g]\sqrt{|g|}$ with the matter term, e.g., the Klein–Gordon one,

$$g^{ij}\bar{\Psi}_{,i}\Psi_{,j}\sqrt{|g|} - m^2\bar{\Psi}\Psi\sqrt{|g|}.$$

In our $\mathrm{GL}(V)$-invariant model the Einstein-like linear combination of the "gravitational" term like, e.g., $\sqrt{\det[g[e]_{ij}]}$ ($g[e]$ given by (9) with the "matter" term like, e.g.

$$g[e]^{ij}\bar{\Psi}_{,i}\Psi_{,j}\sqrt{|g[e]|} - m^2\bar{\Psi}\Psi\sqrt{|g[e]|} \tag{28}$$

would be rather artificial. The resulting field equations for e, Ψ would be coupled in a rather strange way through some second-order-derivatives terms. It seems much more natural to follow seriously the philosophy of generalized Born–Infeld non-linearities and postulate Lagrangians of the form (5) with the Lagrange tensor L_{ij} built in a quadratic way of derivatives of the system (e, Ψ), e.g.,

$$L_{ij} = (1 + \mu\bar{\Psi}\Psi)g[e]_{ij} + \lambda\bar{\Psi}_{,i}\Psi_{,j} \tag{29}$$

where λ, μ are constants. The parameter μ is responsible for the mass term of the field Ψ. One can also use more general form of L_{ij} involving all terms of T_{ij} from (7). Obviously for real values of μ, λ the lagrange tensor is hermitian,

$$L_{ij} = \bar{L}_{ji}$$

and the corresponding Lagrangian (15) is real, just as it should be. Let us observe by the way that (7) leads to Lagrangians (5) not only for real constants A, B, C but also for situations when A, B are real and C is purely imaginary, $\bar{C} = -C$.

We were dealing here with the simplest model of matter described by the complex scalar field. But for more complicated matter fields like spinor fields and multiplets of scalar and spinors a similar treatment is possible, based on generalized Born–Infeld models with Lagrange tensors depending in a second-order polynomial way on field derivatives.

Let us quote a few comments concerning the problem of Lagrangian constraints and the Dirac formalism [8,23]. It is well-known that in all realistic field theories variational principles are singular and the corresponding Legendre transformations lead to constraints in phase spaces. This is seen both on the level of infinite-dimensional symplectic phase spaces and in framework of finite-dimensional multisymplectic formalism. The only exceptional constraints-free models are those involving only scalar fields. They are however not only exceptional, but also non-physical, because except of hypothetic Higgs particles the fundamental level of matter is described by non-scalar fields (spinor fields and gauge fields).

Our $\mathrm{Diff}(M) \times \mathrm{GL}(V)$-invariant models for the field of frames are also based on singular variational principles. The very strong nonlinearity preserved us from performing explicitly the Dirac analysis of primary and secondary constraints, but some partial statements are possible. The same concerns metrical (V, η)-models invariant under $\mathrm{Diff}(M) \times \mathrm{O}(V, \eta)$. Let us assume that $(M, g[e])$ is normal-hyperbolic. More precisely, we restrict ourselves to such solutions e for which this is the case and use such coordinates

$$(\ldots, x^i, \ldots) = (x^0, x^1, \ldots, x^{n-1})$$

for which vectors field $\frac{\partial}{\partial x^0}$ is $g[e]$ time like, and the vectors fields $\frac{\partial}{\partial x^\mu}$, $\mu = 1, \ldots, n-1$ are space-like. Having this $1 + (n-1)$ formal "time + space" splitting we can perform the functional canonical formalism at least in principle. Let $\pi^i{}_A$ denote the $(n-1)$-dimensional densities (in the sense of "spatial" hypersurface

$x^0 =$ constant) of the functional canonical momenta conjugate to "covector potentials" $e^A{}_i$. Because our Lagrangians involve derivatives of $e^A{}_i$ only through the curl

$$f^A{}_{ij} = e^A{}_{j,i} - e^A{}_{i,j} = -f^A{}_{ji} \tag{30}$$

we obtain immediately the primary constraints

$$\pi^A{}_0 = 0, \qquad A = 0, \ldots, n-1. \tag{31}$$

There is complete analogy to the primary constraints in electrodynamics, where the field momenta conjugate to the "scalar potential" vanish. This is again "nonlinear Mie electrodynamic" feature of generally covariant n-leg ("tetrad") models. These constraints have vanishing functional Poisson brackets. They imply that $e^A{}_0$ are completely arbitrary purely gauge variables, just like the "scalar" potential A_0 in electrodynamics. One can fix the gauge for calculation purposes, e.g. by

$$e^A{}_0 = \delta^A{}_0 \tag{32}$$

or some kind of transversality Lorentz condition,

$$g[e]^{ij}\, e^A{}_{i,j} = 0 \tag{33}$$

something like $e^i{}_{A,i} = 0$ looks also reasonable. These are purely coordinate conditions. This gauge arbitrariness follows from the general covariance. It is interesting to compare the above results with the standard general relativity, when g_{ij} is a primary field variable. Denoting by π^{ij} the $(n-1)$-dimensional densities of canonical momenta conjugate to g_{ij} one obtains the primary constraints

$$\pi^{i0} = 0. \tag{34}$$

The resulting arbitrariness of g_{i0} may be fixed, e.g., by the Dirac condition

$$g_{i0} = \delta_{i0}. \tag{35}$$

Some harmonic "Lorentz conditions" are also possible. The above primary constraints are characteristic not only for our $\mathrm{Diff}(M) \times \mathrm{GL}(V)$-invariant models. They are common to all generally-covariant models for the field of frames, including those invariant under $\mathrm{Diff}(M) \times \mathrm{O}(V, \eta)$, i.e., based on the field equations (11). As mentioned, in a consequence of the very strong nonlinearity of our models it is very difficult to perform effectively the Dirac procedure [8, 9, 23] of the symplectic reduction. But it may be easily shown that secondary constraints are (at least partially) described by the following subsystem of the field equations (11)

$$K_i{}^0 = \nabla_j H_i{}^{0j} + 2S^k{}_{kj} H_i{}^{0j} - 2h_{ij} Q^{j0} = 0. \tag{36}$$

The reason is that the second "time" derivatives of our field variables, $e^A{}_{k,00}$, do not enter the above equations. Therefore, with respect to the "time" variable x^0 they are first order differential equations. Being independent on "accelerations" $e^A{}_{k,00}$ and involving only "velocities" $e^A{}_{k,\mu 0}$ "they are nonholonomic constraints" following form the structure of L. And this is just the standard representations of secondary constraints in the cotangent bundle T^*Q by so-called "Lagrangian constraints" in the tangent bundle TQ [23]. Let us remind something quite similar in the standard general relativity and electrodynamics. When one neglects matter and charges, then secondary constraints in Einstein theory and electrodynamic are represented respectively by the following subsystems of field equations

$$R_i{}^0 = 0 \tag{37}$$

or equivalently

$$G_i{}^0 = R_i{}^0 - \frac{1}{2} R^k{}_k \delta_i{}^0 = 0 \tag{38}$$

and

$$H^{0i}{}_{,i} = \mathrm{div}\bar{D} = 0. \tag{39}$$

4. Special Solutions, Group-Space Vacuums

Without rigorous solutions it is impossible to say anything convincing about physical utility of our model and it is interesting only because of its geometric curiosity. In consequence of the very strong nonlinearity the search for any explicit analytical solution is a very difficult task. And a priori it is even hard to decide whether the field equations are compatible at all, i.e., whether any solution does exist. It is always so in variational models invariant under infinite-dimensional transformations groups with elements labelled by arbitrary functions. It is very easy to formulate reasonably looking models with completely inconsistent field equations, e.g., with empty solutions. Dirac constraints may impose an effective veto against the field evolution. The point is that there are fictitious degrees of freedom which are eliminated by fixing some gauge, like e.g., (32) in $n-$ leg models or (35) in Einstein general relativity. But after imposing such gauge conditions we have only $n^2 - n = n(n-1)$ independent field variables $e^A{}_\mu$, $\mu = 1, \ldots, n-1$, at disposal and they have to fulfill the n^2 independent field equations (see (12) and (18)). So we obtain an over-determined system of differential equations. Secondary constraints make the situation even worse. There is always a good chance that an over-determined system of partial differential equations is inconsistent. As mentioned, in a consequence of very strong nonlinearity, it would be very difficult to decide the consistency problem on the basis of Dirac procedure [8,9,23]. The more so the search for rigorous solutions is important, because their existence proves explicitly that the model is non-empty. It is reasonable to expect that such "discoverable" solutions should be sought among geometrically distinguished fields of frames. In conventional Einstein relativity there exist such a priori evident solutions, namely those corresponding to the flat space-time. The same is true in metric teleparallel theories of gravitation based on quadratic Lagrangians (4). Quite independently of the assumed dynamical model (coefficients c_i) any holonomic field of frames is a solution. Indeed, if e is holonomic, then $S[e] = 0$ and the field equations are satisfied (they are linear in S). The corresponding pseudo-Riemannian manifold $(M, h[e, \eta])$ is flat. The fields e_A span an Abelian Lie algebra of vector fields, i.e., they generate a local commutative Lie group of transformations. The local action of this group on M is free and transitive. The existence of such solutions is a characteristic feature of the quadratic metric-teleparallel models; it seems to be a natural consequence of the restriction of the $\mathrm{GL}(V)$-symmetry to $\mathrm{O}(V, \eta)$.

Let us observe that holonomic fields e admit adapted charts in which $e^A{}_i$ are constant. If we consider slightly perturbed fields ${}'e^A{}_i = e^A{}_i + u^A{}_i$, then, in the first order of approximation infinitesimal diffeomorphisms of M result in the following transformations rule for perturbations u^A

$$u^A{}_i \to u^A{}_i + f^A{}_{,i} \tag{40}$$

where f^A are scalars. This resembles the gradient gauge rule for vector fields. The general covariance of rigorous equations leads to the gradient invariance of their Jacobi equations. This possibility of deriving the Abelian gauge invariance from the general covariance (i.e. from the $\mathrm{Diff}(M)$-invariance) is interesting in itself and may lead to certain reflections and hypotheses.

Such solutions do not exist in $\mathrm{GL}(V)$-invariant models, because it is obvious from the very beginning that e must be non-holonomic; otherwise $g[e]$ certainly could not be non-singular.

Let us introduce an important concept which will be essential in our search for the solution:

A field of frames $e : M \to FM$ is said to be Killing-nonsingular (K-nonsingular) if its Killing tensor $g[e]$ is non-degenerate.

Obviously, our search for solutions must be restricted to the variety of K-nonsingular fields, at least when Lagrange tensor coincides with $g[e]$.

Holonomic fields of frames satisfy $S = 0$, i.e., $C^K{}_{LM} = 0$. The simplest natural generalization of such fields consists in putting $C^K{}_{LM} = \mathrm{const}$. The torsion tensor S is then covariantly constant under the e-parallelism

$$\nabla S = 0. \tag{41}$$

We say that such fields are closed.

If e is closed, then its contravariant "legs" e_A span a Lie algebra in the Lie-bracket-sense. The corresponding local Lie group of transformations acts freely and transitively in open domains of M. The tensor $g[e]$ is then covariantly constant with respect to the teleparallelism connection $\Gamma[e]$

$$\nabla g[e] = 0.$$

This means that $(M, g[e], \Gamma[e])$ is a Riemann–Cartan space.

The nonholonomic components of $g[e]$, $g_{AB} = g(e_A, e_B) = g_{ji}e^i{}_A e^j{}_B$ are constant; they coincidence with coefficients of the natural Killing form of the Lie algebra $\bigoplus_{A=1}^{n} \mathbb{R}e_A$,

$$g_{AB} = C^K{}_{AL}C^L{}_{BK}. \tag{42}$$

If e is K-nonsingular and closed, then $L = \bigoplus_{A=1}^{n} \mathbb{R}e_A$ is a semisimple Lie algebra. In this way we obtain a local semisimple Lie group of transformations acting freely and transitively in M. If we fix some "origin" $x_0 \in M$, then M becomes a local semisimple Lie group with x_0 as the identity element. Linear combinations of the vector fields e_A, $A = 1, \ldots, n$ (with constant coefficients) become generators of left regular translations; obviously, they are right-invariant vector fields on the resulting Lie group. The left-invariant vector fields corresponding to e_A will be denoted by $e^*{}_A$. Their linear shell (over reals) generates the group of right translations. We have the following system of basic commutators (Lie brackets)

$$\begin{aligned} [e_K, e_L] &= C^M{}_{KL}\, e_M \\ [e^*{}_K, e^*{}_L] &= -C^M{}_{KL}\, e^*_M \\ [e_K, e^*{}_L] &= 0. \end{aligned} \tag{43}$$

Obviously, the fields $e^*{}_A$ depend not only on the original fields e_B, $B = 1, \ldots, n$, but also-on the choice of the "origin" $x_0 \in M$ (neutral element). The metric tensor $g[e]$ admits an at least $2n$-dimensional group of motions, because e_A and $e^*{}_A$ are Killing vectors (infinitesimal isometries)

$$\pounds_{e_A} g[e] = \pounds_{e^*_A} g[e] = 0. \tag{44}$$

Closed fields of frames provide the simplest Lie-algebraic generalization of holonomic ones and at the same time they do not exclude the required nonsingularity of the Killing tensor $g[e]$. Thus, they seem to be a candidate for geometrically privileged solutions of $\mathrm{GL}(V)$-invariant dynamical models. One can easily show, the conjecture is true.

Theorem 1. *Any closed K-nonsingular field of linear frames is a solution of $\mathrm{GL}(V)$-invariant field equations* (12). *Indeed, for any $\mathrm{GL}(V)$-invariant model the quantity H is an algebraic function of the teleparallelism torsion S. Thus, the parallel invariance of S implies that $\nabla_l H_k{}^{ij} = 0$, in particular $\nabla_j H_k{}^{ij} = 0$. At the same time $S^m{}_{mj} = 0$, because for any semisimple Lie algebra the structural* *constants are traceless,* $C^{L}{}_{LK} = 0$*. Therefore, the both terms of* (12) *do vanish, i.e., closed* K*-nonsingular frames satisfy our field equations.*

In this way, the invariance under $\mathrm{GL}(V)$ seems to be responsible for the existence of solutions equivalent to local semisimple Lie groups of transformations acting freely and transitively on M. Abelian and semisimple Lie groups are opposite special cases within the family of all Lie groups. In this sense affinely-invariant models and metric-teleparallel models are "complementary". Therefore, closed-parallelism solutions of $\mathrm{GL}(V)$-invariant models seem to be conceptual counterparts of holonomic flat-space solutions in metric-teleparallel theories. Unfortunately, in four-dimensional space-time there are no solution of this type, because there are no four-dimensional semisimple Lie algebras. Thus, if we insist on Lie algebraic solutions as something fundamental, then we must accept Kaluza's philosophy of multi-dimensional space times. The "usual" four-dimensional space-time would be merely some aspect of "Kaluza's world", e.g., a quotient manifold or a submanifold-"membrane". We could also try to consider some kind of a complexified four-dimensional space time, because in eight real dimensions there exist semisimple Lie algebras, e.g., $\mathfrak{su}(3)$, $\mathfrak{sl}(3, \mathbb{R})$. However, it is also possible to retain intuitive special solutions of group-theoretical origin without introducing the mentioned complications (increase of dimension, Kaluza's universe, etc.). It turns out that dimensions "semisimple plus one" are also acceptable. Obviously, this covers the physical dimension four, because there are two simple three-dimensional Lie algebras, $\mathfrak{so}(3) = \mathfrak{su}(2)$, $\mathfrak{so}(1, 2) = \mathfrak{sl}(2, \mathbb{R})$. We shall now describe briefly those group-theoretical solutions adapted to dimensions "semisimple plus one".

The following notational convention will be used: coordinate and tensorial indices in an n-dimensional manifold run from 0 to $(n-1)$ and are denoted by Latin letters; Greek indices ("spatial" ones) run from 1 to $(n - 1)$. Nonholonomic indices are denoted, as usual, by capital symbols, with the same convention concerning Latin and Greek types. The reader is apologized for this not very popular, although sometimes used convention.

Let us begin with an auxiliary field of frames

$$(\dots, E_A, \dots) = (E_0, \dots, E_\Lambda, \dots), \qquad \Lambda = 1, \dots, n-1 \tag{45}$$

with the following properties:

- E_Λ are invariant under E_0, i.e.,

$$[E_0, E_\Lambda] = 0. \tag{46}$$

- E_Λ span an $(n - 1)$-dimensional semisimple Lie algebra, i.e.,

$$[E_\Lambda, E_\Sigma] = G^{\Delta}{}_{\Lambda\Sigma} E_\Delta \tag{47}$$

where $G^{\Delta}{}_{\Lambda\Sigma}$ are constant and the Killing matrix built of

$$g[e]_{\Lambda\Sigma} = G^{\Delta}{}_{\Pi\Lambda} G^{\Pi}{}_{\Delta\Sigma} \tag{48}$$

is non-degenerate.

The n-leg $(\ldots, E_A, \ldots) = (E_0, \ldots, E_\Lambda, \ldots)$ is a basis of an n-dimensional Lie algebra. Obviously, being the direct product of the one-dimensional centre spanned by E_0 and of the $(n-1)$-dimensional semisimple algebra spanned by $(\ldots, E_\Lambda, \ldots)$ it is not semisimple. Thus, it is certainly inapplicable as a candidate for the solution of affinely-invariant equations (12); the corresponding Killing tensor would be singular. However, we can easily construct from E some modified fields of frames which are free of these disadvantages and turn out to be solutions of (12). Let $(e_0, \ldots, e_\Lambda, \ldots)$ be a cross-section of FM given by

$$\begin{aligned} e_0 &:= E_0 \\ e_\Lambda &:= E_\Sigma \lambda^{\Sigma}{}_{\Lambda} \end{aligned} \tag{49}$$

where $\lambda : M \rightarrow \mathrm{GL}(n-1, \mathbb{R})$ is a matrix-valued function on M constant on all $(n-1)$-dimensional integral surfaces of the distribution spanned by the vectors $(e_0, \ldots, e_\Lambda, \ldots)$, or, equivalently, by $(\ldots, E_\Lambda, \ldots)$. Obviously, (49) implies that this distribution is integrable. If λ is not constant all over M, then neither the $\mathbb{R}$-linear span of $(\ldots, e_A, \ldots)$, nor that of $(\ldots, E_\Lambda, \ldots)$ are Lie algebras; instead of this we have

$$[e_A, e_B] = C^{K}{}_{AB} e_K \tag{50}$$

where the coefficients $C^{K}{}_{AB}$ are non-constant functions on M.

Nevertheless, they are constant along all integral surfaces of the distribution spanned by the system of "spatial" vectors $(\ldots, e_\Lambda, \ldots)$. They can be easily expressed through the structural constants G and deformation matrices λ,

$$\begin{aligned} C^{0}{}_{0\Lambda} &= 0 \\ C^{\Sigma}{}_{0\Lambda} &= (\lambda^{-1}\dot{\lambda})^{\Sigma}{}_{\Lambda} \\ C^{\Sigma}{}_{\Lambda\Pi} &= G^{\Omega}{}_{\Delta\Gamma} \lambda^{\Delta}{}_{\Lambda} \lambda^{\Gamma}{}_{\Pi} (\lambda^{-1})^{\Sigma}{}_{\Omega} \end{aligned} \tag{51}$$

where

$$\dot{\lambda} = E_0 \lambda = e_0 \lambda. \tag{52}$$

The last quantity is also constant on all integral manifolds of the $(\ldots, E_\Lambda, \ldots)$-distribution. Coefficients of the Killing object are given by

$$\begin{aligned} g[e]_{00} &= C^{\Sigma}{}_{0\Lambda} C^{\Lambda}{}_{0\Sigma} = \mathrm{Tr}((\lambda^{-1}\dot{\lambda})^2) \\ g[e]_{\Lambda\Sigma} &= C^{\Delta}{}_{\Lambda\Pi} C^{\Pi}{}_{\Sigma\Delta} = g[E]_{\Pi\Delta} \lambda^{\Pi}{}_{\Lambda} \lambda^{\Delta}{}_{\Sigma} \\ g[e]_{0\Lambda} &= C^{\Delta}{}_{0\Pi} C^{\Pi}{}_{\Lambda\Delta}. \end{aligned} \tag{53}$$

$g[e]_{00}$ is constant along integral surfaces of the "spatial" subframe $(\ldots, e_\Lambda, \ldots)$. "Spatial" coefficients $g[e]_{\Lambda\Sigma}$ depend only on the "spatial" components $G^\Omega{}_{\Pi\Delta}$ of the total nonholonomic object $G^A{}_{BC}$. They are built of them in the sense of the $(n-1)$-dimensional Killing formula. Moreover, they are λ-transforms of the Killing form for the $(n-1)$-dimensional Lie algebra spanned by the original fields E_Λ. Therefore,

$$g[e] = g[e]_{\Lambda\Sigma} e^\Lambda \otimes e^\Sigma = g[E]_{\Lambda\Sigma} E^\Lambda \otimes E^\Sigma. \tag{54}$$

In the other words, the "spatial triad" ($(n-1)$-leg) $(\ldots, e_\Lambda, \ldots)$ "breathes" in the course of "time" (group parameter)of e_0, nevertheless, the corresponding "spatial metric" does not feel this breathing and equals the Lie-algebraic Killing expression built of the field $(\ldots, e_\Lambda, \ldots)$. In adapted coordinates $(\ldots, x^i, \ldots) = (x^0, \ldots, x^\mu, \ldots) = (t, \ldots, x^\mu, \ldots)$ the $(n-1)$-dimensional integral surfaces of the distribution spanned by $(\ldots, E_\Sigma, \ldots)$ are given by $t = \mathrm{const}$, the matrix λ depends only on t and substituting (49) to the field equations (12) we obtain some over-determined system of ordinary differential equations for the functions $\lambda^\Sigma{}_\Delta(t)$. Obviously, the auxiliary field of frames E has the form

$$E_0 = \frac{\partial}{\partial t}, \qquad E_\Sigma = E^\mu{}_\Sigma(x^\nu)\frac{\partial}{\partial x^\mu}. \tag{55}$$

Equation (54) implies that the metric fields constructed according to the above prescription are "stationary" in the sense of the "time" variable $t = x^0$. They are also "static" in the sense that E_0 and e_0 are orthogonal to E_Σ and e_Σ if and only if the matrix $(\dot{\lambda}\lambda^{-1}) = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\lambda^{-1}$ is $g[E]$-symmetric,

$$g[E]_{\Lambda\Delta} g[E]^{\Sigma\Pi} (\dot{\lambda}\lambda^{-1})^\Delta{}_\Pi = (\dot{\lambda}\lambda^{-1})^\Sigma{}_\Lambda \tag{56}$$

i.e., λ is purely deformative. As yet we did not manage to check the existence of general solutions of this type. In some cosmological problems it may be important to answer the question as to the existence of the interference term $g[e]_{0\Lambda}$, when e_0 and e_Σ are not $g[e]$-orthogonal. It may be shown that the orthogonality occurs when the "affine velocity" [26,27] $(\dot{\lambda}\lambda^{-1})$ is symmetric with respect to the metric $g[E]$ in non-holonomic representation, i.e., when

$$g[E]_{\Lambda\Delta} g[E]^{\Sigma\Pi} (\dot{\lambda}\lambda^{-1})^\Delta{}_\Pi = (\dot{\lambda}\lambda^{-1})^\Sigma{}_\Lambda \tag{57}$$

(obviously the summation extended only over the "spatial" range of capital Greek indices). Any part of $\dot{\lambda}\lambda^{-1}$ skew-symmetric with respect to $[g[E]_{\Gamma\Omega}]$ contributes to the "cosmic rotation" with non-vanishing $g[e]_{0\Lambda}$, i.e., with non-orthogonal (in the $g[e]$-sense) e_0, e_Λ.

Everything that may be relatively easily shown is that for the extreme case of $g[E]$-symmetry, namely for the purely dilatational λ-matrix, $\lambda^\Sigma{}_\Delta = \lambda\delta^\Sigma{}_\Delta$ any field of the type (49) is a solution; again a kind of universal homogeneous solutions (non-excited vacuums).

Theorem 2. *For any dilatation matrix* $\lambda^{\Sigma}{}_{\Delta}(t) = \lambda(t)\delta^{\Sigma}{}_{\Delta}$ *with the factor* λ *quite arbitrary but without critical points (*$\dot{\lambda}(t) \neq 0$ *for any* t*),* (49) *is a solution of* (12). *Such solutions will be called "breathing-closed" solutions.*

We have then the following expression for the Killing metric $g[e]$

$$\begin{aligned} g[e] &= (n-1)(\dot{\lambda}/\lambda)^2 e^0 \otimes e^0 + g[e]_{\Lambda\Sigma} e^{\Lambda} \otimes e^{\Sigma} \\ &= (n-1)(\dot{\lambda}/\lambda)^2 E^0 \otimes E^0 + g[E]_{\Lambda\Sigma} E^{\Lambda} \otimes E^{\Sigma} \end{aligned} \tag{58}$$

where the last term

$$\underset{(n-1)}{g}[E] = g[e]_{\Lambda\Sigma} e^{\Lambda} \otimes e^{\Sigma} = g[E]_{\Lambda\Sigma} E^{\Lambda} \otimes E^{\Sigma}$$

describes the $(n-1)$-dimensional metric geometry of integral surfaces of the distribution spanned by E_{Σ}, $\Sigma = 1, \ldots, (n-1)$. (As usual E^A, e^A and E_A, e_A are elements of the mutually dual fields of co-frames and frames, respectively). Let us notice that if vector fields E_{Σ}, $\Sigma = 1, \ldots, (n-1)$ span a compact Lie algebra, then $\underset{(n-1)}{g}[E]$ is *negatively* definite and the Killing tensor $g[e]$ is automatically normal-hyperbolic, i.e., its signature equals $(+ - \cdots -)$. In the physical case $n = 4$ we have at disposal two simple Lie algebras, $\mathfrak{so}(3) = \mathfrak{su}(2)$ and $\mathfrak{so}(1,2) = \mathfrak{sl}(2,\mathbb{R})$. As $\mathfrak{so}(3)$ is compact, the corresponding $g[e]$ is normal-hyperbolic, e_0 is time-like, and E_{Σ}, e_{Σ}, $\Sigma = 1, 2, 3$ are space-like. Maximal integral surfaces of the e_{Σ}-distribution are spatial sections, e_0 is a reference frame ("ether") and in this way the above metaphoric terms "time" and "space" acquire a literal relativistic meaning.

The Killing signature of $\mathfrak{so}(1,2) = \mathfrak{sl}(2,\mathbb{R})$ is $(+ + -)$, thus the total 4-dimensional $g[e]$ again is normal-hyperbolic with the signature $(+ + + -)$, but the vector e_0 is now space-like. However, from the global point of view such a model is useless because the time-like dimension corresponds to the compact subgroup of planar rotations in $\mathrm{SO}(1,2)$. Pseudo-Riemannian manifolds with closed time-like curves are (as yet) unacceptable as realistic models of the physical space-time. Obviously, from this point of view the universal covering group $\overline{\mathrm{SO}(1,2)}$ is a good manifold because there are no closed time-like curves, however it is not clear for us whether some other pathologies do not occur in the space-time model $\mathbb{R} \times \overline{\mathrm{SO}(1,2)}$, e.g., ones with the domains of dependence or other casual problems.

The factor λ is arbitrary and has the status of a coordinate-like gauge variable. Its existence follows from the general covariance. If t is the group parameter of the vector field $E_0 = e_0$ (as matter of fact the coordinate x^0 in M), then, obviously, the most natural choice of λ is given by

$$\lambda = A \exp(\pm t/\sqrt{n-1}) \tag{59}$$

where A is an arbitrary constant. Obviously, for any constant c we can take as well

$$\lambda = A\exp(\pm ct/\sqrt{n-1}) \tag{60}$$

and the "velocity of light" $|c|$ appears as an "integration constant". The Killing-type metric tensor is then given by

$$\begin{aligned} g[e] &= (n-1)\left(\frac{\mathrm{d}\ln\lambda}{\mathrm{d}t}\right)^2 \mathrm{d}t\otimes\mathrm{d}t + \underset{(n-1)}{g}[E]_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu \\ &= c^2\mathrm{d}t\otimes\mathrm{d}t + \underset{(n-1)}{g}[E]_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu \end{aligned} \tag{61}$$

where the previously introduced coordinates (t, x^μ) are used and $\underset{(n-1)}{g}[E]_{\mu\nu}$ is, roughly speaking, the Killing metric tensor on the $(n-1)$-dimensional spatial surfaces $t = \mathrm{const}$,

$$\underset{(n-1)}{g}[E]_{\mu\nu} = 4S[E]^\alpha{}_{\mu\beta}S[E]^\beta{}_{\nu\alpha} = 4\underset{(n-1)}{S}[E]^\alpha{}_{\mu\beta}\underset{(n-1)}{S}[E]^\beta{}_{\nu\alpha} \tag{62}$$

and

$$\underset{(n-1)}{S}[E]^\alpha{}_{\mu\nu} = \frac{1}{2}E^\alpha{}_\Lambda(E^\Lambda{}_{\mu,\nu} - E^\Lambda{}_{\nu,\mu}) = G^\Lambda{}_{\Sigma\Pi}E^\alpha{}_\Lambda E^\Sigma{}_\mu E^\Pi{}_\nu. \tag{63}$$

Our metric $g[e]$ admits at least $(2n-1)$ independent Killing vectors. The first $(n+1)$-tuple is given by

$$X_0 = E_0, \qquad X_\Sigma = E_\Sigma.$$

Obviously, their Lie brackets have the form

$$[X_0, X_\Sigma] = 0, \qquad [X_\Lambda, X_\Sigma] = G^\Pi{}_{\Lambda\Sigma}X_\Pi.$$

In this way locally our manifold becomes locally a group space of the non-semi-simple Lie group $\mathbb{R}\times\mathrm{G}$ (possibly $\mathrm{U}(1)\times\mathrm{G}$ if we are not afraid of causal anomalies), where G is a Lie group locally fixed uniquely by the structure constants $G^\Pi{}_{\Lambda\Sigma}$. Fixing as previously by convention some "unit" element $x^0 \in M$ we turn this manifold locally into a Lie group $\mathbb{R}\times\mathrm{G}$. Then we can construct the invariant fields X^*_Σ acting from the opposite side than X_Σ. Their Lie bracket relations have the following form

$$[X_0, X^*_\Sigma] = 0, \qquad [X^*_\Lambda, X^*_\Sigma] = -G^\Pi{}_{\Lambda\Sigma}X^*_\Pi, \qquad [X_\Lambda, X^*_\Sigma] = 0.$$

Together one obtains $2n-1$ Killing vector linearly independent over CONSTANT reals, because due to the assumed semisimplicity of G, X^*_Σ are in this sense independent of X_Σ. Independence of X_0 on X_Λ, X^*_Σ is trivial because X_0 is transversal to G-surfaces (X_Σ, X^*_Σ are tangent to G).

5. Cosmology, Spinors, Dark Matter

It is interesting that although the field of frames e exponentially "expands" or "contracts" in the course of time, the metric tensor $g[e]$ constructed of such an e is static and stationary. All quantities built of e in a $\mathrm{GL}(V)$-invariant way are "time"-independent. In particular the $(n-1)$-dimensional "spatial" metric $\underset{(n-1)}{g}[e]$ is "constant". Therefore, if the above G is compact, in particular, if it is equivalent to $\mathrm{SU}(2) \cong \mathbb{S}^3(0, R)$, we obtain the static Einstein world as a fundamental "vacuum" solution. It is difficult to escape here certain cosmological reflections. As it is well known, spinor models are rather reluctant to affine internal symmetries. The well established spinor field theories in curved manifolds are based on the Dirac-Einstein metric $h[e, \eta]$ in V. And it does not seem possible (at least as yet) to formulate viable models without the use of tetrads as V-valued differential form on M. The term "tetrad" is used because we have in mind applications in the real four-dimensional space time, but of course any dimension n of the type "semisimple + one" is possible (then "n-leg" instead "tetrad"). Let us consider the "breathing" solution e with λ given by (60) and the corresponding Killing metric $g[e]$, (54,58). And we inject some fermionic matter described by the spinor field Ψ into the empty space-time M endowed with the geometry given by e. We assume this matter to be so rarified that the feedback of e-geometry through Ψ may be neglected, e is just the above solution of equations of the tetrad field alone and Ψ satisfies "Dirac equation" in (M, e) – the e-structured manifold M. This Dirac equation is based on the Dirac–Einstein metric $h[e, \eta]$, where the most natural assumption concerning $\eta \in V^* \otimes V^*$ is as follows

$$[\eta_{AB}] = \begin{bmatrix} [c^2] & \\ & [\kappa_{\Lambda\Sigma}] \end{bmatrix} \tag{64}$$

where three-dimensional "spatial" part is given by

$$\kappa_{\Lambda\Sigma} = g[E]_{\Lambda\Sigma} \tag{65}$$

i.e., by (48). The corresponding Dirac–Einstein metric is then given by

$$h[e, \eta] = \eta_{AB} E^A \otimes E^B \tag{66}$$

therefore,

$$h[e, \eta] = c^2 E^0 \otimes E^0 + A^{-2} \exp(-2\alpha t) \kappa_{\Lambda\Sigma} E^\Lambda \otimes E^\Sigma, \qquad \alpha = c/\sqrt{n-1} \tag{67}$$

$$h[e, \eta] = c^2 \mathrm{d}t \otimes \mathrm{d}t + A^{-2} \exp(-2\alpha t) \kappa_{\mu\nu}(x^\lambda) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{68}$$

Strictly speaking, the above form of η is "aesthetically" convincing, but not necessary. We could use just as well the standard form

$$[\eta_{AB}] = \mathrm{diag}(1, -1, -1, -1).$$

This is equivalent to transforming e like in (1), with the use of some fixed L. The one point is essential, namely, that the "spatial" three-dimensional part

$$\underset{3}{h}[e,\eta]_{\mu\nu} = A^{-2}\exp(-2\alpha t)\kappa_{\mu\nu}(x^{\lambda})\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu} \tag{69}$$

is time-dependent. It is expanding for $\alpha < 0$ and contacting for $\alpha > 0$.

It is just this metric which is felt by spinor fields. And of course the fundamental heavy matter, i.e., leptons and quarks, is described by bispinor fields. The conclusion is that matter injected into empty Universe with geometry described by the above "breathing" tetrads will witness about the cosmological expansion or contraction, e.g., about escaping of galaxies. Obviously, this reasoning is based on the over-simplified model of the test matter influenced by the spatio-temporal geometry, but non-affecting this geometry. This is of course a very crude, rough approximation, Nevertheless, the arguments look convincing. Therefore, if "true" spatio-temporal metric relations are meant in the sense of the fundamental Killing tensor $g[e]$, the cosmological expansion is a kind of "illusion". The "time" parameter t runs the range $[-\infty,\infty]$, and the creation or annihilation of the Universe is an "illusion", too.

For a given auxiliary field of frames defined by (45), we can also construct a little different field of "breathing" frames than the above e, namely the field $\varepsilon = (\ldots,\varepsilon_A,\ldots)$ given by

$$\varepsilon_A = \lambda E_A, \qquad A = 0,1,\ldots,n-1$$

with the same as previously conditions for λ

$$E_{\Sigma}\lambda = 0, \qquad \Sigma = 1,\ldots,n-1$$

i.e., λ is a function of t only. One can show that such fields ε are also solutions for $\mathrm{Diff}(M)\times\mathrm{GL}(V)$ invariant models. As a matter of fact locally they are identical with the previous solutions e and differ only in the choice of the group parameter of e_o. There are however some interesting global differences, just ones concerning the cosmic scenario. The Killing tensors of e, ε are identical,

$$g[e] = g[\varepsilon]$$

however, for the new Dirac–Einstein tensor $h[\varepsilon,\eta]$, we have

$$h[\varepsilon,\eta] = c^2\mathrm{d}T\otimes\mathrm{d}T + A^{-2}\alpha^2T^2\kappa_{\mu\nu}(x^{\lambda})\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu} \tag{70}$$

where now the cosmic time T is related to the group parameter t of E_0 as follows

$$T = \pm\frac{1}{\alpha}\exp(-\alpha t). \tag{71}$$

If $\alpha < 0$ and we put $T = \pm\frac{1}{\alpha}\exp(-\alpha t)$, then $T\in[0,\infty]$ when $t\in[-\infty,\infty]$ and the three-dimensional (similarly for the general $(n-1)$-dimensional) Universe with the three-metric

$$\underset{3}{h}[e,\eta]_{\mu\nu} = A^{-2}\alpha^2 T^2 \kappa_{\mu\nu}(x^\lambda)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{72}$$

expands uniformly in the cosmic time; the spatial distances are linearly increasing. And there is a "creation" at $T = 0$. To admit the total real range $[-\infty, \infty]$ for T we must sew together the two solutions corresponding to the positive and negative values of α with the same $|\alpha|$. These two solutions describe respectively the contraction and expansion eras joined together at the singular point $T = 0$ (annihilation and creation simultaneously).

The natural question arises as to the cosmic scenarios with the oscillatory behaviour of the three-dimensional ($(n-1)$-dimensional) spatial metric. They are possible within a slightly different framework, namely, when some complex scalar field is admitted in addition to the tetrad, and the Lagrange tensor is given by (29). Another possibility is based on the complexification idea and the use of complex tetrads (complex n-legs) [11, 12]. The oscillatory regime for the spatial metric becomes then possible.

Another natural question is the one concerning the existence of relationship between $\mathrm{GL}(V)$-models and the standard General Relativity, in any case a well-established theory. Let us define the spatio-temporal metric tensor as a multiple of the standard Killing tensor by some constant A, cf. (9), necessary just for the dimensional reasons,

$$g[e]_{ij} = 4AS^a{}_{bj}S^b{}_{aj}. \tag{73}$$

It is a known fact [13] that due to its very structure this tensor satisfies identically the equation

$$\mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}g_{ij} = \frac{2-n}{8A}g_{ij} \tag{74}$$

i.e., Einstein equations with a (negative!) cosmological constant $\Lambda = \frac{2-n}{8A}$. In the physical case $n = 4$,

$$\Lambda = -\frac{1}{4A}. \tag{75}$$

Therefore some relationship with Einstein theory exists at least in some neighborhood of semisimple group-space-solutions.

There are some other important points concerning the status of the global $\mathrm{GL}(V)$-symmetry for tetrads from the spinor point of view. As mentioned, spinors "do not like" affine symmetry. Their theory is based on some fixed metric $\eta = V^* \otimes V^*$ and, in standard approaches, it is locally invariant under $\mathrm{O}(V, g)$, or, more precisely, under its covering group.

If we do not use the sophisticated fibre-bundle language, bispinor field in a four-dimensional space-time manifold is described as a mapping $\Psi : M \to P$ where P is a complex four-dimensional linear space. Analytically, it is a quadruplet of complex-valuated functions Ψ^r, however to avoid some artefacts and mistakes it is convenient to distinguish between P and the arithmetic space $\mathbb{C}^4$, just as the internal space V of the gravitational co-tetrads was not identified with $\mathbb{R}^4$. The real linear space of sesquilinear hermitian forms on P will be denoted by $H(P)$, we use the convention of antilinearity in the first arguments. The evaluation of $\Gamma \in H(P)$ on vectors u, v will be denoted by $\Gamma(u, v)$; analytically

$$\Gamma(u, v) = \overline{\Gamma(v, u)} = \Gamma_{\bar{r}s}\bar{u}^{\bar{r}}v^s \tag{76}$$

where the bar symbol over the vector components denotes the complex conjugation. The matrix $[\Gamma_{\bar{r}s}]$ is hermitian. $\mathrm{Dim}H(P) = 16$; this dimension is meant over reals.

In spinor theory P is endowed with some geometry given by a fixed form $G \in H(P)$ of the neutral signature $(+\ +\ -\ -)$. This form establishes some antilinear isomorphism $P \ni u \mapsto \tilde{u} \in P^*$ known as Dirac conjugation; analytically

$$\tilde{u}_r = \bar{u}^{\bar{s}}G_{\bar{s}r}. \tag{77}$$

In the linear space $\mathrm{L}(P)$ of linear endomorphisms of P we distinguish the subspace $H(P, G) \subset \mathrm{L}(P)$ of G-hermitian mappings, i.e., such ones that

$$G(Au, v) = G(u, Av) \tag{78}$$

for any $u, v \in P$. Obviously, $H(P, G)$ is also real-16-dimensional, and the spaces $H(P, G)$, $H(P)$ are canonically isomorphic. Analytically $\Gamma \in H(P, G)$ and its isomorphic image $\tilde{\Gamma} \in H(P)$ are related to each other as follows

$$\tilde{\Gamma}_{\bar{r}s} = G_{\bar{r}z}\Gamma^z{}_s, \qquad \Gamma^r{}_s = G^{r\bar{z}}\tilde{\Gamma}_{\bar{z}s} \tag{79}$$

where, obviously,

$$G^{r\bar{z}}G_{\bar{z}s} = \delta^r{}_s, \qquad G_{\bar{r}z}G^{z\bar{s}} = \delta_{\bar{r}}{}^{\bar{s}}.$$

The scalar product G distinguishes in $\mathrm{GL}(P)$ the real subgroup $\mathrm{U}(P, G)$ of G-pseudounitary matrices, i.e., ones preserving G,

$$G(Au, Av) = G(u, v) \tag{80}$$

for any $u, v \in P$. It is (non-canonically) isomorphic with the standard inclusion $\mathrm{U}(2, 2) \subset \mathrm{GL}(4, \mathbb{C})$.

The semisimple subgroup $\mathrm{SU}(P, G) \subset \mathrm{U}(P, G)$ consists of mappings with the unit determinant. It is well-known that $\mathrm{SU}(2, 2) \subset \mathrm{U}(2, 2)$ is the covering group of the conformal group $\mathrm{CO}(1, 3)$ with the Lorentz signature. The Lie algebra $\mathfrak{u}(P, G) \subset \mathrm{L}(P)$ consists of G-antihermitian mappings (such ones that $G(Au, v) = -G(u, Av)$), therefore, briefly

$$\mathfrak{u}(P, G) = \mathrm{i}H(P, G). \tag{81}$$

Lie algebra of $\mathrm{SU}(P, G)$, i.e., $\mathfrak{su}(P, G)$ isomorphic with $\mathfrak{su}(2, 2)$ consists of traceless elements of $\mathfrak{u}(P, G)$. There is neither place here (nor any profit from) to get into the general details of Clifford analysis; instead we shall make use of the peculiarities of $\dim M = 4$. So from now on V is endowed with some normal-hyperbolic metric $\eta = V^* \otimes V^*$ of signature $(+\ -\ -\ -)$. Some Clifford injection will be assumed $\gamma : V \hookrightarrow H(P, G)$ such that the following holds

$$\{\gamma(u), \gamma(v)\} = \gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2\eta(u, v)\,\mathrm{Id}_P \tag{82}$$

Id_P denoting the identity mapping of P onto itself. Analytically, fixing some η-orthonormal basis ε_A in V and denoting $\gamma_A = \gamma(\varepsilon_A)$, we have for the resulting "Dirac matrices"

$$\{\gamma_A, \gamma_B\} = \gamma_A\gamma_B + \gamma_B\gamma_A = 2\eta_{AB}\,\mathrm{Id}_P\,. \tag{83}$$

In terms of Hermitian forms

$$\tilde{\gamma}_A{}^{r\bar{z}}\tilde{\gamma}_{B\bar{z}s} + \tilde{\gamma}_B{}^{r\bar{z}}\tilde{\gamma}_{A\bar{z}s} = 2\eta_{AB}\delta^r{}_s.$$

Raising the capital indices with the help of η, we obtain, e.g.,

$$\{\gamma^A, \gamma^B\} = 2\eta^{AB}\,\mathrm{Id}_P\,. \tag{84}$$

Using the tetrad and its dual co-tetrad, i.e., analytically $e^i{}_A$, $e^A{}_i$, one introduces the Dirac "world matrices"

$$\gamma_i = e^A{}_i\gamma_A, \qquad \gamma^i = e^i{}_A\gamma^A. \tag{85}$$

They satisfy obviously the anticommutation rules

$$\begin{aligned} \{\gamma_i, \gamma_j\} &= \gamma_i\gamma_j + \gamma_j\gamma_i = 2h[e, \eta]_{ij}\,\mathrm{Id}_P \\ \{\gamma^i, \gamma^j\} &= \gamma^i\gamma^j + \gamma^j\gamma^i = 2h[e, \eta]^{ij}\,\mathrm{Id}_P\,. \end{aligned} \tag{86}$$

The shift of indices here is meant in the sense of $h[e, \eta]$.

Remark. The analytical term "Dirac matrices" is used for historical reasons; γ_A, γ^B are linear mapping from P into P, to be distinguished from hermitian forms $\tilde{\gamma}_A$, $\tilde{\gamma}^B$. We avoid however the term "Dirac operators", because it has another meaning in literature.

The Clifford injection γ is quite arbitrary except it must satisfy the above conditions. Various possible choices are known in literature as various "representations" of γ^A-s.

When some injection $\gamma : V \hookrightarrow H(P, G)$ is fixed then within the group $\mathrm{U}(P, G)$ there is a distinguished subgroup $\mathrm{U}[\gamma] \subset \mathrm{U}(P, G)$ such that for any its element $U \in \mathrm{U}[\gamma]$ the following holds

$$U\gamma(v)U^{-1} = \gamma(L(U)v), \qquad L(U) \in \mathrm{O}(V, \eta) \tag{87}$$

for any $v \in V$. Analytically

$$U\gamma_A U^{-1} = \gamma_B L(U)^B{}_A \tag{88}$$

and if $v = v^A \gamma_A$, then

$$(L(u)v)^A = L(U)^A{}_B v^B. \tag{89}$$

Obviously, the elements of $\mathrm{U}[\gamma]$ proportional to Id_P give rise to $L = \mathrm{Id}_V$, so strictly speaking it is only the restriction of the assignment $U \mapsto L(U)$ to $\mathrm{SU}(P, G)$ that is essential. This assignment is a homomorphism (group epimorphism)

$$L(U_1 U_2) = L(U_1) L(U_2), \qquad L(\mathrm{Id}_P) = \mathrm{Id}_V\,.$$

This is the covering of $\mathrm{O}(V, \eta)$, in particular the connected subgroup of $\mathrm{SU}(P, G)$ is under the above projection the universal covering of the restricted Lorentz group $\mathrm{SO}^{\uparrow}(V, \eta)$. Therefore, this connected component is isomorphic with $\mathrm{SO}(1, 3)$. Using sophisticated terms of Clifford-algebraic theory: one deals here with some particular realization of groups $\mathrm{Pin}(4, \eta)$ and $\mathrm{Spin}(4, \eta)$.

One can easily show that

$$L(U)^A{}_B = \frac{1}{4}\,\mathrm{Tr}(\gamma^A U \gamma_B U^{-1}) \tag{90}$$

for any $U \in \mathrm{U}[\gamma]$; the shift of capital indices meant as usual in the η-sense.

In analytical language, when V, P are identified respectively with $\mathbb{R}^4$, $\mathbb{C}^4$, the above U-mappings are identified with complex matrices assigned to elements of $\mathrm{SL}(2, \mathbb{C})$. The particular choice of the assignment $\mathrm{SL}(2, \mathbb{C}) \ni A \mapsto U(A) \in \mathrm{SU}(P, G)$ is also non-essential; it is a matter of representation only. For example, the Infeld–Weyl–Van der Waerden and Dirac representations are respectively given by

$$U(A) = \begin{bmatrix} A & 0 \\ 0 & A^{-1+} \end{bmatrix}, \qquad U(A) = \frac{1}{2}\begin{bmatrix} A + A^{-1+} & A - A^{-1+} \\ A - A^{-1+} & A + A^{-1+} \end{bmatrix}. \tag{91}$$

They correspond respectively to the following analytical representations of the scalar product G

$$[G_{\bar{r}s}] = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad [G_{\bar{r}s}] = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{92}$$

where I denotes the 2×2 identity matrix.

Let us introduce for a moment an auxiliary concept, namely, the $\mathrm{SO}(V, \eta)$-ruled connection form on M. It is a differential form Γ with values in $\mathfrak{so}(V, \eta)$, the Lie algebra of $\mathrm{SO}(V, g)$

$$M \ni x \mapsto \Gamma_x \in L(T_x M, \mathfrak{so}(V, \eta)).$$

Analytically it is represented by a system of differential one-forms $\Gamma^A{}_{Bi}$, where after the η-shift of indices

$$\Gamma_{ABi} + \Gamma_{BAi} = 0. \tag{93}$$

We consider local transformations, i.e., x-dependent mappings U

$$M \ni x \mapsto U(x) \in \mathrm{U}[\gamma].$$

The quantities Ψ, e, Γ transform then pointwisely as follows

$$\begin{aligned} [\Psi^r] &\mapsto [U^r{}_s \Psi^s] \\ [e^K{}_i] &\mapsto [L(U)^K{}_M e^M{}_i] \\ [\Gamma^K{}_{Ai}] &\mapsto [L(U)^K{}_B \Gamma^B{}_{Ci} L(U)^{-1C}{}_A - \frac{\partial L(U)^K{}_B}{\partial x^i} L(U)^{-1B}{}_A]. \end{aligned} \tag{94}$$

Obviously, $h[e, \eta]$ is invariant under such transformations. The quantities $e^A{}_i$, $\Gamma^A{}_{Bi}$ give rise to the affine connection

$$\Gamma^i{}_{jk} = e^i{}_A \Gamma^A{}_{Bk} e^B{}_j + e^i{}_A e^A{}_{j,k} = e^i{}_A \Gamma^A{}_{Bk} e^B{}_j + \Gamma[e]^i_{\mathrm{tel}jk} \tag{95}$$

the last term denoting the teleparallelism connection built of e. The Levi–Civita affine connection built of $h[e, \eta]$, $\left\{ {i \atop j\ k} \right\}$, is related to $\Gamma^i{}_{jk}$ as follows

$$\Gamma^i{}_{jk} = \left\{ {i \atop j\ k} \right\} + S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i \tag{96}$$

$$S^i{}_{jk} = \Gamma^i{}_{[jk]} = \frac{1}{2}(\Gamma^i{}_{jk} - \Gamma^i{}_{kj}).$$

Obviously, the connection (96), is metrical,

$$\nabla h[e, \eta] = 0. \tag{97}$$

If in the gravitational sector we do not use any object except the tetrad e itself, then the only natural possibility is the Levi–Civita connection built of $h[e, \eta]$, i.e., $S = 0$ (do not confuse this S with that built of the teleparalelism connection). Obviously, this implicitly imposes certain constraints on the coefficients $\Gamma^A{}_{Bk}$. In any case the cooefficients $\Gamma^A{}_{Bk}$ may be obtained from (95) by simple substituting of $\left\{ {i \atop j\ k} \right\}$ instead $\Gamma^i{}_{jk}$. The next step is to introduce the bispinor connection ω, a differential one-form with values in $\mathfrak{u}[\gamma] \subset \mathrm{L}(P)$ – the Lie algebra of $\mathrm{U}[\gamma] \subset \mathrm{SU}(P, \gamma)$. It is given by

$$\omega_k = \frac{1}{2} \Gamma_{ABk} \Sigma^{AB} \tag{98}$$

where, as usual, the shift of indices is meant in the η-sense, and Σ^{AB} are the basic generators of the action of $\mathrm{U}[\gamma]$ in P,

$$\Sigma^{AB} := \frac{1}{4}(\gamma^A \gamma^B - \gamma^B \gamma^A) = \frac{1}{4}[\gamma^A, \gamma^B]. \tag{99}$$

The inverse formula reads

$$\Gamma^A{}_{Bk} = \frac{1}{2} \operatorname{Tr}(\gamma^A \omega_k \gamma_B). \tag{100}$$

Local transformations (94) affect the connection form in the usual way

$$\omega_k \mapsto U\omega_k U^{-1} - \frac{\partial U}{\partial x^k} U^{-1}. \tag{101}$$

Obviously, the action of the group $\mathrm{U}[\gamma]$ is equivalent to the Dirac representation $D^{1/2,0} \oplus D^{0,1/2}$ of $\mathrm{SL}(2, \mathbb{C})$. The finite actions of $\mathrm{U}[\gamma]$ are given by

$$\mathrm{U}(\varepsilon) = \exp\left(\frac{1}{2}\varepsilon_{AB}\Sigma^{AB}\right) \tag{102}$$

where $\varepsilon_{AB} = -\varepsilon_{BA}$ are canonical coordinates.
The covariant derivatives of bispinor fields are given by

$$D_k\Psi = \partial_k\Psi + \omega_k\Psi. \tag{103}$$

Dirac Lagrangian for the bispinor field is given by [8]

$$L_{\mathrm{mat}}(\Psi; e, \omega) = \frac{\mathrm{i}}{2} e^k{}_A \gamma^{Ar}{}_s \left(\tilde{\Psi}_r D_k \Psi^s - D_k \tilde{\Psi}_r \Psi^s\right) \sqrt{|h|} - m \tilde{\Psi}_r \Psi^r \sqrt{|h|} \tag{104}$$

$$= \frac{\mathrm{i}}{2} e^k{}_A \tilde{\gamma}^A{}_{\bar{r}s} \left(\bar{\Psi}_{\bar{r}} D_k \Psi^s - D_k \bar{\Psi}_{\bar{r}} \Psi^s\right) \sqrt{|h|} - m G_{\bar{r}s} \bar{\Psi}_{\bar{r}} \Psi^s \sqrt{|h|}$$

where m denotes the mass of the particle. The subscript "mat" in L_{mat} refers to matter, of course. The above Lagrangian for Ψ, controlled by (e, ω) is locally invariant under $\mathrm{U}(P, \gamma)$. The total Lagrangian for the matter-gravitation system is a combination of (104) and of an appropriate gravitational term built of (e, Γ), or, equivalently, of (e, ω). In Einstein theory it is only e that is used as a gravitational potential, and $\Gamma^i{}_{jk} = \left\{{}_j{}^i{}_k\right\}$ is then the Levi–Civita connection built of $h[e, \eta]$.

And the gravitational Lagrangian is proportional to $R[h[e, \eta]]\sqrt{|h[e, \eta]|}$, where $R[h[e, \eta]]$ is the curvature scalar built of $h[e, \eta]$; thus one deals then with the Hilbert Lagrangian. In some alternative treatments like Einstein–Cartan theory or more general gauge approaches e and Γ, or equivalently e and ω are *a priori* to some extent independent and the torsion term is admitted. Γ is then the Einstein–Cartan theory and $\Gamma^A{}_{Bi}$ are subject only to the-η-antisymmetry condition, $\Gamma_{ABi} + \Gamma_{BAi} = 0$. Such alternative Lagrangians $L_{\mathrm{gr}}[e, \Gamma]$ may be linear in the curvature tensor $R[h[e, \eta]]^i{}_{jkl}$ (Hilbert prescription), or, in more general approaches, some terms quadratic in curvature and torsion of Γ are admitted. In any case, in all the mentioned models of gravitation, including the conventional Einstein theory, L_{gr} is invariant under local, i.e., x-dependent internal Lorentz rotations, $M \ni x \mapsto \mathrm{L}(x) \in \mathrm{SO}(V, \eta)$. And so is the total Lagrangian, i.e., the linear combination (coefficients depending on the gravitational constant, or on the system of gravitational constants in alternative gauge approaches). And now we just go back to our $\mathrm{GL}(V)$-models for the field of frames. Obviously, they are globally invariant under $\mathrm{GL}(V)$. The local invariance is impossible because any two fields of frames may be related to each other by an appropriate local (x-dependent) $\mathrm{GL}(V)$-transformation. Therefore, a hypothetic local $\mathrm{GL}(V)$-model would be either trivial (field equations identically satisfied by any e) or empty (no solutions at all).

Now, the above-described "cosmological" aspects of our global $\mathrm{GL}(V)$-models suggest what follows: to combine the globally $\mathrm{GL}(V)$-invariant "gravitational" model for the tetrad field e with the locally $\mathrm{SO}(V, \eta)$-invariant Lagrangian (104) for the spinorial matter. Global invariance of our "gravitational" Lagrangians for tetrads means that there exist in e more degrees of freedom than those merely contributing to $h[e, \eta]$. They are supposed to represent some hypothetic physics. Let us mention, there was an interesting idea due to von Borzeszkowski [1–4] according to which the tetrad degrees of freedom ruled by the global Lorentz group $\mathrm{O}(V, \eta)$ might be responsible for the dark matter. In any case, from the point of view of Einstein General Relativity this is really a kind of "matter".

So, again we are faced with some "cosmological" aspects of the globally-invariant tetrad models.

6. Kaluza–Klein Aspect

We told that perhaps some $\mathrm{Diff}(M) \times \mathrm{GL}(V)$-invariant models could be interesting in dimensions $n > 4$ within the framework of something like generalized Universes of the Kaluza–Klein type. As a matter of fact, models with smaller invariance groups like $\mathrm{Diff}(M) \times \mathrm{O}(V, \eta)$ a priori are also an admissible possibility, although we feel more convinced to the internal $\mathrm{GL}(V)$-symmetry.

In such modified Kaluza–Klein-like treatments the n-dimensional co-frame will play a similar role as the n-dimensional metric tensor in usual Kaluza–Klein models. This might be a way out of certain non-pleasant features of Kaluza's scheme and fibre-bundle-based formulation of gauge theories. Namely, in those treatments there are certain absolute geometric objects in n-dimensional Universes, and the resulting dynamics, although generally-covariant in the four-dimensional sense, fails to be generally-covariant on the n-dimensional level. This is an evident drawback from the point of view of the general covariance paradigm which we owe to Einstein and Hilbert. Successes of non-Abelian gauge theories of electroweak and strong interactions seem to prove the real existence of multidimensional Universe, thus, it is natural to expect that the Hilbert–Einstein paradigm should work also in n-dimensional Kaluza-like world, not only in the four-dimensional space-time of our every-day experience. We intuitively feel that the really fundamental theories in their basic formulation should be as amorphous as possible. What appears to us an absolute object should be in fact a special background solution of fundamental equations, hidden beyond the effective dynamics of small vibrations.

In traditional Kaluza-like treatments one assumes the existence of at least one absolute structure in n-dimensions, namely, the foliation defining the very bundle structure; usual four-dimensional space-time is a quotient manifold with respect to that foliation. In fibre-bundle formulations of gauge theories there exists an additional entity frozen into n-dimensional Universe, namely, the very structure of a Lie group acting along fibres. As dynamical variables one assumes there the metric on the base manifold (four-dimensional space-time) and the connection form on the bundle universe. But if one believes in the real existence of multidimensional universe, then this scheme seems very artificial. It is a very natural temptation to search for hypothetic models where also the structure of a principal fibre bundle i.e., the very foliation and the structure group, are not absolute but follow from some special solutions. In Lagrangians for such models there must be nothing but dynamical variables. And, in my opinion, one of very natural ways would be just a non-metrical modification of Kaluza'a scheme. The background universe would be a completely amorphous n-dimensional differential manifold without any fixed geometry introduced by hand, except of course the very differential structure. It seems that the most natural candidate for a unified description of physical fields would be the field of co-frames in a multidimensional universe, i.e., the n-tuple of covector fields. This object is rich enough to model both the foliation and the structural group of a principal fibre bundle over the resulting quotient space. In fact, a quadruple of one-forms e^A, $A = 0, 1, 2, 3$ may define a global foliation (if the corresponding Pfaff problem is integrable). The remaining $(n-4)$-tuple of forms $e^{\mathbf{R}}$, $\mathbf{R} = 4, 5, \ldots, (n-1)$ has a sufficient freedom to define a connection, i.e., infinitesimal transversality on the resulting bundle structure. If we take the dual contravariant frame $(e_A, e_{\mathbf{R}})$ ($A = 0, 1, 2, 3$, $\mathbf{R} = 4, 5, \ldots, (n-1)$ again), then the $(n-4)$-tuple of vectors $e^{\mathbf{R}}$ may generate a free and transitive action of an $(n-4)$-dimensional Lie group along $(n-4)$-dimensional fibres in such a way that the covectors $e_{\mathbf{R}}$, $\mathbf{R} = 4, 5, \ldots, (n-1)$ will be components of some connection form on the resulting bundle. The mentioned Lie group will be a structural group. Obviously, the main point is, how to construct reasonable Lagrangians for co-frames, admitting as solutions principal fibrations and connections. Such hypothetic models could provide a nontrivial unification scheme of gravitation and other fundamental interactions where a priori all covector potentials appear on the same "democratic" footing, and their specification as multiplets of some gauge groups and carriers of some specific interactions appears on the level of solutions, not in the basic formulation of degrees of freedom. In particular, the specific gauge groups are then also implied by differential equations, without being introduced by hand from outside. Thus, structure constants of gauge groups and metric tensors in internal spaces would become in some sense a dynamical concept.

We have seen above that something like Lie-group structures may be obtained dynamically, as particular "vacuum" solutions of $\mathrm{GL}(V)$-invariant equations for frames. Incidentally, solutions of this type exist also in some other, e.g., $\mathrm{O}(V,\eta)$-invariant models, however, with fixed η field equations impose some conditions on the admissible structure constants. Unlike this, in $\mathrm{GL}(V)$-models any semi-simple Lie algebra (or its central extension) is a solution. Therefore, one can reasonably suspect that this scheme of dynamical group-structure-generation makes higher-dimensional $\mathrm{GL}(V)$-models a promising framework for constructing Kaluza-like models without any absolute objects in n dimensions. It is difficult to decide a priori whether this conjecture is true. Let us only quote a few formulas to express precisely the question. To visualize in an explicit form the field of frames (n-legs) described above, let us take the trivial manifold model $M = X \times G$, X being a four-dimensional space-time, and G being an $(n-4)$-dimensional Lie group. Coordinates on X, G will be denoted by x^i, $y^{\mathbf{r}}$, respectively, $i = 0,1,2,3$, $\mathbf{r} = 4,\ldots,n-1$. For simplicity (notation economy) the same symbols will be used for their pull-backs to M. The field of co-frames e described above is given by

$$e^A = e^A{}_a(x)\,\mathrm{d}x^a, \qquad e^{\mathbf{R}} = \theta^{\mathbf{R}}{}_{\mathbf{r}}(y)\,\mathrm{d}y^{\mathbf{r}} + U(y)^{\mathbf{R}}{}_{\mathbf{W}}A^{\mathbf{W}}{}_a(x)\,\mathrm{d}x^a$$

where the system of one-forms $\theta^{\mathbf{R}} = \theta^{\mathbf{R}}{}_{\mathbf{r}}(y)\,\mathrm{d}y^{\mathbf{r}}$ represents the canonical one-form (canonical co-frames) of G and $[U(y)^{\mathbf{R}}{}_{\mathbf{W}}]$ is the matrix of the co-adjoint representation of G. The one-forms $\theta^{\mathbf{R}}{}_{\mathbf{r}}$ are related to the structure constants $C^{\mathbf{R}}{}_{\mathbf{WZ}}$ of G as follows

$$\mathrm{d}\theta^{\mathbf{R}} = \frac{1}{2}C^{\mathbf{R}}{}_{\mathbf{WZ}}\theta^{\mathbf{Z}} \wedge \theta^{\mathbf{W}}.$$

The indices A, a above run over the range $(0,1,2,3)$ and so does the summation over them when the Einstein convention is used. Similarly, the range of indices $\mathbf{R}$, $\mathbf{W}$, $\mathbf{Z}$, $\mathbf{r}$ is $4,5,\ldots,n-1$. This concerns both the free indices and the Einstein convention. Now let $e^a{}_A$, $\theta^{\mathbf{r}}{}_{\mathbf{Z}}$ be dually defined by

$$\theta^{\mathbf{R}}{}_{\mathbf{r}}\theta^{\mathbf{r}}{}_{\mathbf{Z}} = \delta^{\mathbf{R}}{}_{\mathbf{Z}}, \qquad e^A{}_a e^a{}_B = \delta^A{}_B$$

with the same as previously convention concerning the range and summation. So, roughly speaking, we are dealing with the mutually dual frames and co-frames in G, and X. One can easily show that the contravariant dual $\tilde{e}$ of e is given by

$$e_A = e^a{}_A(x)\frac{\partial}{\partial x^a} - \theta^{\mathbf{r}}{}_{\mathbf{R}}(y)U^{\mathbf{R}}{}_{\mathbf{Z}}(y)A^{\mathbf{Z}}{}_a(x)e^a{}_A(x)\frac{\partial}{\partial y^{\mathbf{r}}}$$

$$e_{\mathbf{R}} = \theta^{\mathbf{r}}{}_{\mathbf{R}}(y)\frac{\partial}{\partial y^{\mathbf{r}}}.$$

Obviously, $\theta_{\mathbf{R}}$ may be identified with the basic Lie-algebraic fields on G,

$$[\theta_{\mathbf{R}}, \theta_{\mathbf{Z}}] = C^{\mathbf{W}}{}_{\mathbf{RZ}}\theta_{\mathbf{W}}.$$

Philosophy of the above field of (co)frames e on M is as follows: $e^{A}{}_{a}(x)$, $e^{a}{}_{A}(x)$ represent the gravitational (co)tetrad, $A^{\mathbf{R}}{}_{a}$ are gauge fields, G is the Lie group which rules the gauge potentials $A^{\mathbf{R}}$. The system of differential forms $e^{\mathbf{R}}$, $\mathbf{R} = 4, \ldots, n-1$ on M represents the connection form corresponding to the gauge field $A^{\mathbf{R}}{}_{a}$. The vector fields $\theta_{\mathbf{R}}$ are fundamental vector fields of the bundle structure, and the very fibres of this bundle are maximal integral manifolds of the distribution spanned by $\theta_{\mathbf{R}}$ (maximal submanifolds with tangent vectors anihilated by e^{A}, $A = 0, 1, 2, 3$).

Obviously, this yet a wishful thinking. The most important questions to be answered when thinking in this way are the following ones:

1. Do exist reasonable Lagrangians for the (co) n-leg field e in a manifold M (assumed a priori quite arbitrary, without the splitting $X \times G$) such that the corresponding Euler–Lagrange equations possess solutions which in some coordinates have the above form, rigorously or in a reasonable approximation?
2. If so, do exist among these Lagrangians such ones that the fields $A^{\mathbf{R}}$ satisfy rigorously or in a reasonable approximation (e.g., a weak-approximation) equations known from gauge theories (generalized Yang–Mills equations)?
3. Does the cotetrad part e^{A} satisfy equations reasonably corresponding with the gravitation theory?

There is yet no answer to these questions. The formerly discussed group-space-solutions seem to suggest that $\mathrm{GL}(V)$-invariant models are the best candidates to offer an affirmative answer.

7. Search for Spherical Solutions

What we have done as yet is merely a preliminary step. We have given some heuristic arguments for hypothesis that the generally covariant and affinely invariant Lagrangian dynamics of the "tetrad" field could be useful as a geometric model of some fundamental interaction. Then the general mathematical formalism was given, and finally we have presented two simplest kinds of vacuum solutions (closed and breathing closed fields). Thus, certainly, our field equations are non empty. However, this is not yet physics. To be acceptable as an alternative gravitation theory, our equations must possess solutions with non trivial g_{00}-components and with a reasonable Newtonian and Schwarzschild asymptotics. Thus, from now on, we focus our attention on spherically symmetric fields.

Let us put $M = \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ and denote the natural coordinates by x^i, $i = 0, 1, 2, 3$. The coordinate x^0, denoted also by t, is a "time-like" variable, whereas x^μ, $\mu = 1, 2, 3$ will be "spatial" coordinates (in the sequel the Greek indices always run over the range $1, 2, 3$ whilst the Latin ones the range $0, 1, 2, 3$). This means that we restrict ourselves to such tetrad fields e that the vector fields $\frac{\partial}{\partial x^0}$, $\frac{\partial}{\partial x^\mu}$ are respectively time-like and space-like with respect to the Killing metric tensor $g[e]$. We shall also use the spherical coordinates r, θ, φ in $\mathbb{R}^3$ and the versor components $n^\mu = \frac{1}{r}x^\mu$.

Isotropic tetrads will be sought in the following form, quite analogous to the famous t'Hooft–Polyakov monopole [16]

"temporal" leg,

$$e_0 = K(r,t)\frac{\partial}{\partial t} + J(r,t)x^\mu \frac{\partial}{\partial x^\mu} = K(r,t)\frac{\partial}{\partial t} + J(r,t)\frac{\partial}{\partial r} \tag{105}$$

"spatial" legs, $\Lambda = 1, 2, 3$

$$\begin{aligned} e_\Lambda &= I(r,t)x_\Lambda \frac{\partial}{\partial t} + [F(r,t)\delta^\mu{}_\Lambda + G(r,t)x^\mu x_\Lambda + H(r,t)\varepsilon_{\Lambda\nu}{}^\mu x^\nu]\frac{\partial}{\partial x^\mu} \\ &= rI(r,t)n_\Lambda \frac{\partial}{\partial t} + (F(r,t) + r^2 G(r,t))n_\Lambda \frac{\partial}{\partial r} \\ &\quad - \frac{1}{r}F(r,t)\varepsilon_{\Lambda\mu}{}^\nu n^\mu + H(r,t)D_\Lambda \end{aligned} \tag{106}$$

where

$$D_\Lambda = \varepsilon_{\Lambda\alpha}{}^\beta x^\alpha \frac{\partial}{\partial x^\beta}$$

the shifting of all indices is meant in the sense Kronecker deltas, and F, G, H, I, J, K are certain shape functions depending only on the variables (t, r). The raising and lowering of indices at δ, x, ε is understood in those formulas in the trivial $\mathbb{R}^3$-Kronecker sense; it is used only for "cosmetic" purposes, e.g., to avoid "graphical" conflicts with the summation convention. The above formulas describe the most general tetrad field covariant with respect to the group $\mathrm{SO}(3)$ acting as natural diffeomorphism group of $M = \mathbb{R} \times \mathbb{R}^3$. The term 'covariant" is understood in the sense that the components $e^i{}_A$ satisfy the following conditions

$$\begin{aligned} e^\mu{}_\Lambda(t, Rx) &= R^\mu{}_\nu \Phi^\nu{}_\Sigma(t,x)(R^{-1})^\Sigma{}_\Lambda \\ e^0{}_\Lambda(t, Rx) &= e^0{}_\Sigma(t,x)(R^{-1})^\Sigma{}_\Lambda \\ e^\mu{}_0(t, Rx) &= R^\mu{}_\nu e^\nu{}_0(t,x) \\ e^0{}_0(t, Rx) &= e^0{}_0(t,x) \end{aligned} \tag{107}$$

for any $R \in \mathrm{SO}(3)$ and $t \in \mathbb{R}$, $x \in \mathbb{R}^3$.

The Killing metric tensor is then also spherically symmetric

$$\begin{aligned}
g_{00} &= g_{00}(t, r) \\
g_{0\mu} &= g_{\mu 0} = g_0(t, r)x_\mu = rg_0(t, r)n_\mu \\
g_{\mu\nu} &= g[0](t, r)\delta_{\mu\nu} + g[2](t, r)x_\mu x_\nu \\
&= g[0](t, r)\delta_{\mu\nu} + r^2 g[2](t, r)n_\mu n_\nu
\end{aligned} \tag{108}$$

where g_{00}, g_0, $g[0]$, $g[2]$ are certains functions of (t, r) built in a rational way of the above functions F, G, H, I, J, K and their first order derivatives.

In other words

$$\begin{aligned}
g_{00}(t, Rx) &= g_{00}(t, x) \\
g_{0\mu}(t, Rx) &= g_{0\nu}(t, x)(R^{-1})^\nu{}_\mu \\
g_{\mu\nu}(t, Rx) &= g_{\alpha\beta}(t, x)(R^{-1})^\alpha{}_\mu(R^{-1})^\beta{}_\nu.
\end{aligned} \tag{109}$$

Substituting the above form of e to our field equations (11), (12) we obtain a system of 6 partial differential equations for 6 functions F, G, H, I, J of two variables (t, r). These have the form

$$\begin{aligned}
K_0{}^0 = 0, \qquad K_0 = 0, \qquad K^0 = 0 \\
K[0] = 0, \quad K[1] = 0, \quad K[2] = 0
\end{aligned} \tag{110}$$

where above K-s are shape functions of the tensor $K_i{}^j$ which, obviously has also the isotropic structure

$$\begin{aligned}
K^0{}_0 &= K^0{}_0(t, r) \\
K_0{}^\mu &= K_0(t, r)x^\mu = rK_0(t, r)n^\mu \\
K_\mu{}^0 &= K^0(t, r)x_\mu = rK^0(t, r)n_\mu \\
K_\mu{}^\nu &= K[0](t, r)\delta_\mu{}^\nu + K[1](t, r)\varepsilon_\mu{}^{\nu\kappa}x_\kappa + K[2](t, r)x_\mu x^\nu \\
&= K[0]\delta_\mu{}^\nu + rK[1]\varepsilon_\mu{}^{\nu\kappa}n_\kappa + r^2 K[2]n_\mu n^\nu.
\end{aligned} \tag{111}$$

Let us notice that coordinates x^i are not uniquely fixed by the demand that e should have the isotropic structure (105), (106). Indeed, any smooth change of coordinates on the (t, r)-plane is admissible

$$(t, r) \to (\overline{t}, \overline{r}) = (a(t, r), b(t, r)). \tag{112}$$

Such transformations do not affect either the isotropic form of e (105), (106) the field equation (110); this is a consequence of general covariance. The above transformation formula involves two arbitrary functions a, b of two variables (r, t). Thus, the system of shape functions $(F, \ldots, K)$ is redundant, because in principle two of them can be given any a priori prescribed form.

In Einstein theory field equations together with a proper choice of coordinates in the (t, r)-plane enable one to eliminate the time variable; this elimination reduces the equations for spherically symmetric fields to ORDINARY differential equations for the shape functions. This implies in particular that the Schwarzschild solution (gravitational field of point mass) is static. In our affinely-invariant theory of tetrad field it does not seem possible to get rid of the time variable by the above (a, b)-change of coordinates. Nevertheless, on the basis of analogy with breathing-closed solutions we can show that there exists a natural class of isotropic solutions described by ordinary differential equations. First of all, let us observe that breathing closed field e, as mentioned, can be alternatively described by the formulas

$$e_0 = \lambda(t)E_0, \qquad e_\Sigma = \lambda(t)E_\Sigma. \tag{113}$$

This modification does not include the formulas for the Killing tensor $g[e]$. Thus, it is natural to postulate the following form of isotropic solutions

$$(F, G, H, I, J, K)(t, r) = \exp(\alpha t)(f(r), g(r), h(r), i(r), j(r), k(r)) \tag{114}$$

where α is constant. This conjecture is correct. If we substitute it to equations of motion, then the time variable drops out of these equations and we obtain a system of six second-order ordinary differential equations for six functions (f, g, h, i, j, k) of the radial variable r. The system does not impose any restrictions on the parameter α. Roughly speaking, again the "velocity of light" is an integration constant.

Ordinary differential equations satisfied by $(f, \ldots, k)$ are strongly nonlinear. Their left-hand sides are very complicated rational expressions of the shape functions and their first-order derivatives; the second derivatives enter in a linear way.

The over-all exponential time dependence of the tetrad implies that coefficients of the teleparallelism connection and its torsion do not depend on time. Therefore, also the Killing metric $g[e]$ is time-independent. The pseudo-Riemannian manifold $(\mathbb{R} \times \mathbb{R}^3, g[e])$ is stationary, although in general non-static ($g_{0\mu}$ need not vanish).

The time-exponential coordinate conditions (114) impose certain additional restrictions on the coordinate system, nevertheless, there exists still some rather large gauge freedom. Namely, the exponential factorization is preserved by the following deformations of coordinates

i) radial variable deformations

$$(t, r) \mapsto (\overline{t}, \overline{r}) = (t, \omega(r)) \tag{115}$$

ii) r-dependent time translations

$$(t, r) \mapsto (\overline{t}, \overline{r}) = (t + \varepsilon(r), r) \tag{116}$$

where ω and ε are in principle arbitrary functions of r.

These transformations preserve the exponentially-factorized shape of e and result respectively in the following transformations of $(f, \ldots, k)$

$$
\begin{aligned}
\overline{f}(\omega(r)) &= f(r)\frac{\omega(r)}{r} \\
\overline{g}(\omega(r)) &= g(r)\frac{r^2}{\omega(r)}\frac{\mathrm{d}\ln\omega(r)}{\mathrm{d}r} + \frac{f(r)}{\omega(r)}\left(\frac{\mathrm{d}\ln\omega(r)}{\mathrm{d}r} - \frac{1}{r}\right) \\
\overline{h}(\omega(r)) &= h(r) \\
\overline{i}(\omega(r)) &= i(r)\frac{r}{\omega(r)} \\
\overline{j}(\omega(r)) &= j(r)r\frac{\mathrm{d}\ln\omega(r)}{\mathrm{d}r} \\
\overline{k}(\omega(r)) &= k(r)
\end{aligned}
\tag{117}
$$

and

$$
\begin{aligned}
\overline{f} &= \exp(-\alpha\varepsilon)f \\
\overline{g} &= \exp(-\alpha\varepsilon)g \\
\overline{h} &= \exp(-\alpha\varepsilon)h \\
\overline{i} &= \exp(-\alpha\varepsilon)\left(i + \frac{1}{r}(f + gr^2)\frac{\mathrm{d}\varepsilon}{\mathrm{d}r}\right) \\
\overline{j} &= \exp(-\alpha\varepsilon)j \\
\overline{k} &= \exp(-\alpha\varepsilon)\left(k + rj\frac{\mathrm{d}\varepsilon}{\mathrm{d}r}\right).
\end{aligned}
$$

These transformation rules involve two arbitrary functions, thus, to some extent it is possible to deform two of the six functions $(f, \ldots, k)$ to any a priori fixed shape. This remains from the general covariance when one restricts oneself to spherically-symmetric fields.

Thus, when some gauge is fixed, we are dealing with a system of six second-order ordinary differential equations imposed on four shape functions. As mentioned above, these equations are extremely complicated and, when written down explicitly, completely obscure. It is rather hard to expect rigorous solutions in analytical form. We suppose that more realistic and physically interesting is the following problem: to estimate the asymptotic behaviour of the shape functions $(f, \ldots, k)$ and the Killing tensor $g[e]$ about the origin $r = 0$. This is necessary when we intend to compare our model with well-established consequences of Einstein theory of gravitation and with Newton theory.

The strong nonlinearity of our field equations and the Born–Infeld structure of Lagrangian L enable one to conjecture that perhaps there exist solutions finite at

$r = 0$. It is interesting whether there exist black-holes and horizon-effects with hypothetic solutions finite at $r = 0$. As yet, we are unable to answer such questions.

It is, at least apparently, much more easy to discuss the correspondence with Newton potential and with Schwarzschild metric in the weak-field approximation. We have seen that there exist explicitly known isotropic solutions, namely, "breathing-closed" solutions corresponding to the $\mathfrak{su}(2)$-Lie algebra. The manifold M becomes then locally identical with $\mathbb{R} \times \mathrm{SU}(2) \simeq \mathbb{R} \times \mathbb{S}^3$, or with $\mathbb{R} \times \mathrm{SO}(3)$. Let us parameterize $\mathrm{SO}(3)$ with the help of the rotation vector $\overline{\rho} = (\rho^1, \rho^2, \rho^3)$ (canonical coordinates of the first kind on $\mathrm{SO}(3)$). This parameterization identifies $\mathrm{SO}(3)$ with the closed ball $\varrho \leq \pi$ in $\mathbb{R}^3$ with the proviso that antipodal points on the surface $\rho = \pi$ are identified. Similarly, $\mathrm{SU}(2)$ becomes the ball $\rho \leq 2\pi$ with the proviso that the whole surface $\rho = 2\pi$ is identified with $-I$, where I denotes the 2×2 identity matrix. The shape functions corresponding to the $\mathrm{SO}(3)$ "breathing-closed" solutions are given by

$$f_0 = \frac{\rho}{2} \operatorname{ctg} \frac{\rho}{2}, \; g_0 = \frac{1}{\rho^2}\left(1 - \frac{\rho}{2} \operatorname{ctg} \frac{\rho}{2}\right), \; h_0 = \pm\frac{1}{2}, \; i_0 = j_0 = 0, \; k_0 = 1. \tag{118}$$

This parameterization is inconvenient because it leads to expressions in which trigonometric functions are badly mixed with algebraic ones. Thus, it is better to use the vector of finite rotation $\overline{r}$ as a parameterization of $\mathrm{SO}(3)$. It is related to canonical coordinates through the formulas

$$\frac{\overline{r}}{r} = \frac{\overline{\rho}}{\rho}, \qquad r = \operatorname{tg} \frac{\rho}{2}. \tag{119}$$

This parameterization identifies $\mathrm{SO}(3)$ with the projective space $\mathbb{RP}^3$; rotations by $\pi/2$ are represented by points at infinity. When we deal with $\mathrm{SU}(2) \simeq \mathbb{S}^3(0, 1)$, it is so as if two copies of $\mathbb{RP}^3$ were "glued" so as to become a single connected manifold. The above parameterization eliminates all trigonometric functions. The $\mathrm{SO}(3)$ "breathing-closed" solution is given by

$$f_0 = g_0 = h_0 = \frac{1}{2}, \qquad i_0 = j_0 = 0, \qquad k_0 = 1. \tag{120}$$

The corresponding Killing tensor has the form

$$\mathrm{d}s^2 = 3\alpha^2 \mathrm{d}t^2 - \frac{8}{(1 + r^2)^2} \mathrm{d}r^2 - \frac{8r^2}{1 + r^2} (\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\varphi^2) \tag{121}$$

where (r, θ, φ) are spherical coordinates in $\mathbb{R}^3$.

We shall now consider small spherically-symmetric perturbations of the above breathing-closed solutions. Therefore, we put

$$f = \frac{1}{2} + \varphi, \quad g = \frac{1}{2} + \gamma, \quad h = \frac{1}{2} + \chi, \quad i = \mu, \quad j = \nu, \quad k = 1 + \kappa \tag{122}$$

where φ, γ, χ, μ, $\nu 4$, κ are small corrections depending only on the variable r. Substituting these expressions to (110), (111) and neglecting higher-order terms, we obtain a system of six linear ordinary differential equations imposed on six functions $(\varphi, \ldots, \kappa)$. The general covariance enables us to eliminate two of these functions. Let us consider infinitesimal transformations (115), (116), i.e., we put $\omega = 1 + \eta$ and assume that η and ε are "small". Linearizing transformation rules, we obtain the following formulas

$$\begin{aligned} &\overline{\varphi} = \varphi + \frac{1}{2r}\eta, \qquad \overline{\gamma} = \gamma - \frac{1+2r^2}{2r^3}\eta + \frac{1+r^2}{2r^2}\frac{\mathrm{d}\eta}{\mathrm{d}r} \\ &\overline{\chi} = \chi, \qquad \overline{\mu} = \mu, \qquad \overline{\nu} = \nu, \qquad \overline{\kappa} = \kappa \end{aligned} \tag{123}$$

and

$$\begin{aligned} &\overline{\varphi} = \varphi - \frac{1}{2}\alpha\varepsilon, \qquad \overline{\gamma} = \gamma - \frac{1}{2}\alpha\varepsilon, \qquad \overline{\chi} = \chi - \frac{1}{2}\alpha\varepsilon \\ &\overline{\mu} = \mu + \frac{1+r^2}{2r}\frac{\mathrm{d}\varepsilon}{\mathrm{d}r}, \qquad \overline{\nu} = \nu, \qquad \overline{\kappa} = \kappa - \alpha\varepsilon. \end{aligned} \tag{124}$$

It is interesting that among all infinitesimal shape functions $(\varphi, \ldots, \kappa)$, ν is the only purely physical quantity invariant under coordinate gauge transformations. The most convenient choice is $\mu = 0$, $\gamma = 0$, because second derivatives of these functions do not enter linear equations for small corrections. One can show that in linear approximation

$$g_{00} = 3\alpha^2 + \frac{2\alpha(r^2-3)}{1+r^2}\nu - 2\alpha r\frac{\mathrm{d}\nu}{\mathrm{d}r}. \tag{125}$$

Therefore, in this approximation the gravitational potential g_{00} is controlled by the shape function ν alone, i.e., by the "spatial" component of e_0, just by the only gauge-independent shape function. And this is really correct if g_{00} is to represent gravitational scalar potential ("scalar" in the three-dimensional sense of course). If we tried to interpret the total symmetric part of the tensor T_{ij} as a metric tensor, then we would have to multiply the above expression for g_{00} by $(A + 3B)$.

Calculations leading to linear equations for small corrections $(\varphi, \ldots, \kappa)$ are very strenuous. The final equations are also rather complicated, thus, we do not quote them here. The explicit form of equations depends on the choice of Lagrangian L. Our calculations were based on the "Born–Infeld" model (8), (7). The resulting linear equations have non-constant coefficients, thus, to obtain any explicit result we have to use the Frobenius method of power series. We are especially interested in the asymptotic behaviour of g_{00} about the origin $r = 0$, thus, all shape functions $(\varphi, \ldots, \kappa)$ should be expressed in terms of power series of the variable r. Luckily, it may be shown that $r = 0$ is a proper (regular) singular point of our system of equations. The asymptotics of solutions at the origin is determined by the characteristic equation of the system. It has two roots: $p = 0$, $p = -3$. Because of their integer difference there are some problems with the Frobenius method. There are however other, more serious difficulties. Let us report them briefly:

1. The system is over-determined (six equations and essentially four gauge-free variables). So, it is not clear whether there exist non-trivial solutions for correction terms, i.e., such ones that at least one of the variables $(\varphi, \ldots, \kappa)$ is non-vanishing. There exist non-vanishing solutions for the lowest-power terms, but it is quite possible that the higher-order equations (higher powers) impose back some restrictions due to which the previously found expressions must vanish. The complicated structure of equations prevented us from obtaining any convincing answer.
2. The assumption that $(\varphi, \ldots, \kappa)$ are "small" may be in a neighbourhood of $r = 0$ incompatible with some terms involving the negative powers of r.
3. The most important problem is the following one. We linearize generally-covariant equations about the background $\mathbb{R} \times \mathrm{SU}(2)$-solution with non-trivial Killing vectors. But it is well-known that just in such situations the linearization procedure may (although need not) fail. Quite often it happens so that the background solutions with continuous symmetry groups are "cusps" in the variety of all solutions. They are so-to-speak non-manifold points in the general solution; the tangent space is not well-defined there and because of this the linearization procedure fails there. And if so, some peculiar, purely constructive, non-perturbative methods may be applicable. As yet we were not successful in finding a proper procedure. In a consequence of the very complicated structure of our strongly nonlinear equations, it is quite possible that only some very sophisticated computer algebra may be effective.

8. Some Mechanical Comments

We have mentioned about the relationship between the above field-theoretic models and affine mechanical systems [26, 27]. More precisely, the expression for $T_{(ij)}$ (the symmetric part of (7)) and the corresponding Lagrangian (8) may be interpreted as a field-theoretic counterpart of the affinely-invariant model of the kinetic energy of affinely rigid body. If configurations of affinely-invariant body in n dimensions are represented by linear frames $(e_1, \ldots, e_A, \ldots, e_n)$ or their duals $(e^1, \ldots, e^A, \ldots, e^n)$ and the affine velocity in laboratory and co-moving representations is given by

$$\Omega^i{}_j = \frac{\mathrm{d}e^i{}_A}{\mathrm{d}t} e^A{}_j, \qquad \widehat{\Omega}^A{}_B = e^A{}_i \frac{\mathrm{d}e^i{}_B}{\mathrm{d}t}$$

then affinely-invariant kinetic energies are expressed as follows

$$T = \frac{A}{2}\operatorname{Tr}(\Omega^2) + \frac{B}{2}(\operatorname{Tr}\Omega^2) = \frac{A}{2}\operatorname{Tr}(\widehat{\Omega}^2) + \frac{B}{2}(\operatorname{Tr}\widehat{\Omega}^2).$$

It is easily seen (and one can formulate this in rigorous mathematical terms) that the above T is exactly the mechanical counterpart of (7), (8).

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