

LAGUERRE'S FUNCTION OF DIRECTION IN A GENERALIZED WEYL HYPERSURFACE

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Abstract. In [1], the generalization of Laguerre's function of direction for a surface in ordinary space to a hypersurface of a Riemannian space is obtained. The Laguerre's function of direction for a hypersurface of a Weyl space has been derived in [2]. In this paper, the generalization of Laguerre's function of direction to a hypersurface of generalized Weyl space is made.

1. Introduction

An n -dimensional differentiable manifold W_n is said to be a Weyl space if it has a symmetric conformal metric tensor g_{ij} and a symmetric connection ∇ satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0, \quad (1.1)$$

where T_k are the components of a covariant vector field and ∇_k denotes the usual covariant derivative.

Let Γ_{jk}^i denote the coefficients of the connection ∇ . Then, from the compatibility condition given by (1.1) we get

$$\Gamma_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - \left(\delta_j^i T_k + \delta_k^i T_j - g^{li} g_{jk} T_l \right). \quad (1.2)$$

Under a renormalization of the fundamental tensor of the form $\tilde{g}_{ij} = \lambda^2 g_{ij}$ an object A admitting a transformation of the form $\tilde{A} = \lambda^p A$ is called a **satellite with weight** $\{p\}$ of the metric tensor g_{ij} .

The **prolonged covariant derivative** of the satellite A relative to ∇ , denoted by $\dot{\nabla}A$ is defined by [3]

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \quad (1.3)$$

An n -dimensional differentiable manifold having an asymmetric connection ∇^* and asymmetric conformal metric tensor g_{ij}^* preserved by ∇^* is called a generalized Weyl space [4]. Such a **generalized Weyl space** will be denoted by GW_n .

In local coordinates, we then have

$$\nabla_k^* g_{ij}^* - 2T_k^* g_{ij}^* = 0, \quad (1.4)$$

where T_k^* are the components of a covariant vector field called the complementary vector field of the generalized Weyl space.

The prolonged covariant derivative of the satellite A , with weight $\{p\}$, relative to ∇^* is defined as

$$\dot{\nabla}_k^* A = \nabla_k^* A - pT_k^* A, \quad (1.5)$$

where ∇_k^* denotes the usual covariant derivative.

Assume that g_{ij}^* is broken up into the sum of its symmetric and anti-symmetric parts $g_{(ij)}^*$ and $g_{[ij]}^*$, respectively, so that we have

$$g_{ij}^* = g_{(ij)}^* + g_{[ij]}^*. \quad (1.6)$$

Let us consider the generalized Weyl space GW_n having the same complementary vector field T as that of the Weyl space W_n having the symmetric part of g_{ij}^* as its metric tensor. The Weyl space W_n is called the associate space to the generalized Weyl space GW_n [5].

The coefficients L_{jk}^i of the connection ∇^* are obtained from the compatibility condition as [6]

$$L_{jk}^i = \Gamma_{jk}^i + \frac{1}{2} \left[\Omega_{kl}^h g_{(jh)}^* + \Omega_{jl}^h g_{(hk)}^* + \Omega_{jk}^h g_{(hl)}^* \right] g^{*(li)} \quad (1.7)$$

or, putting

$$Q_{jk}^i = \frac{1}{2} \left[\Omega_{kl}^h g_{(jh)}^* + \Omega_{jl}^h g_{(hk)}^* + \Omega_{jk}^h g_{(hl)}^* \right] g^{*(li)} \quad (1.8)$$

we have

$$L_{jk}^i = \Gamma_{jk}^i + Q_{jk}^i, \quad (1.9)$$

where $\Omega_{jk}^i = L_{jk}^i - L_{kj}^i$ are the components of the **torsion tensor** of the connection ∇^* .

2. Frenet Formulas in a Generalized Weyl Space

Let \mathbf{t} be the tangent vector field, normalized by the condition $g_{(ij)}^* t^i t^j = 1$, to the curve $C: x^i = x^i(s)$ in the associate Weyl space W_n of the generalized Weyl space GW_n and let s be the arclength of C measured from a fixed point on C .

The prolonged derivatives of \mathbf{t} along C , relative to ∇ and ∇^* denoted, respectively, by $\frac{\delta t}{\delta s}$ and $\frac{\delta^* t}{\delta s}$ are given by

$$\begin{aligned} \frac{\delta t^i}{\delta s} &= t^j \dot{\nabla}_j t^i, & \frac{\delta^* t^i}{\delta s} &= t^j \dot{\nabla}_j^* t^i, \\ t^h &= \frac{dx^h}{ds}. \end{aligned} \quad (2.1)$$

Frenet formulae for W_n can be written as [3],

$$\begin{aligned} \frac{\delta t_r^i}{\delta s} &= \kappa_{r+1} t_{r+1}^i - \kappa_r t_{r-1}^i, \\ t_0^i &= t^i, \quad \kappa_0 = \kappa_n = 0, \\ r &= 0, 1, \dots, n-1, \end{aligned} \quad (2.2)$$

where κ_r is the r -th **curvature** of the curve C .

Similarly, the Frenet formulae for the space GW_n can be written in the form

$$\frac{\delta^* t_r^i}{\delta s} = \kappa_{r+1}^* t_{r+1}^i - \kappa_r^* t_{r-1}^i, \quad \kappa_0^* = \kappa_n^* = 0 \quad (2.3)$$

where κ_r^* is the r -th curvature of the curve C relative to GW_n .

If v^i is the contravariant components of any vector \mathbf{v} in GW_n , by using (1.9) and (2.1), we get

$$\frac{\delta^* v^i}{\delta s} = \frac{\delta v^i}{\delta s} + Q_{jk}^i v^j t^k. \quad (2.4)$$

Replacing v^i in (2.4) by $t_0^i, t_1^i, t_2^i, \dots, t_{n-1}^i$ and using (2.1) we obtain respectively

$$\begin{aligned} \frac{\delta^* t_0^i}{\delta s} &= \kappa_1 t_1^i + Q_{jk}^i t^j t^k \\ \frac{\delta^* t_1^i}{\delta s} &= (\kappa_2 t_2^i - \kappa_1 t_1^i) + Q_{jk}^i t_1^j t^k \\ \frac{\delta^* t_2^i}{\delta s} &= (\kappa_3 t_3^i - \kappa_2 t_2^i) + Q_{jk}^i t_2^j t^k \\ &\dots\dots\dots \\ \frac{\delta^* t_{n-1}^i}{\delta s} &= -\kappa_{n-1} t_{n-2}^i + Q_{jk}^i t_{n-1}^j t^k. \end{aligned} \tag{2.5}$$

These formulae may be replaced by the single equation

$$\frac{\delta^* t_r^i}{\delta s} = (\kappa_{r+1} t_{r+1}^i - \kappa_r t_{r-1}^i) + Q_{jk}^i t_r^j t^k. \tag{2.6}$$

Let us find the relationship between the curvatures κ_n and κ_n^* of the curve C relative to W_n and GW_n .

Since the vectors t_0, t_1, \dots, t_{n-1} are mutually orthogonal

$$g_{(ij)}^* t_p^i t_q^j = \delta_q^p \quad i, j = 1, 2, \dots, n; \quad p, q = 0, 1, \dots, n-1. \tag{2.7}$$

Multiplying (2.2) by $g_{(ij)}^* t_{r-1}^j$ and summing over i and j we find

$$\kappa_r = -g_{(ij)}^* \left(\frac{\delta t_r^i}{\delta s} \right) t_{r-1}^j. \tag{2.8}$$

Using (2.3), (2.6) and (2.8) we obtain

$$\kappa_r^* = \kappa_r - Q_{hjk} t_r^h t_{r-1}^j t^k, \tag{2.9}$$

where $g_{(ij)}^* Q_{hk}^i = Q_{hjk}$.

3. Laguerre's Function of Direction in a Generalized Weyl Hypersurface

Let GW_n be a hypersurface with coordinates $u^i (i = 1, 2, \dots, n)$ in a generalized Weyl space GW_{n+1} with coordinates $x^a (a = 1, 2, \dots, n, n+1)$.

Suppose that the metrics of GW_n and GW_{n+1} are elliptic and that they are given respectively, by $g_{ij}^* du^i du^j$ and $g_{ab}^* dx^a dx^b$ which are connected by the relations

$$g_{ij}^* = g_{ab}^* x_i^a x_j^b, \quad i, j = 1, 2, \dots, n; \quad a, b = 1, 2, \dots, n+1$$

from which it follows that

$$g_{(ij)}^* = g_{(ab)}^* x_i^a x_j^b, \quad g_{[ij]}^* = g_{[ab]}^* x_i^a x_j^b,$$

where x_i^a denotes the covariant derivative of x^a with respect to u^i .

Let n^a be the contravariant components of the vector field in GW_{n+1} normal to GW_n and let it be normalized by the condition $g_{ab}^* n^a n^b = 1$. Then, we have

$$g_{(ab)}^* n^a n^b = 1. \quad (3.1)$$

The moving frame $\{x_a^i, n_a\}$ on GW_n reciprocal to the moving frame $\{x_i^a, n^a\}$ is defined by [7]

$$n_a n^a = 1, \quad n_a x_i^a = 0, \quad n^a x_a^i = 0, \quad x_i^a x_a^j = \delta_i^j. \quad (3.2)$$

On the other hand, differentiating covariantly x_i^a with respect to u^k , we get

$$\dot{\nabla}_k^* x_i^a = \nabla_k^* x_i^a = A_{ik} n^a + B_{ik}^j x_j^a$$

which yields, with the help of (3.1) and (3.2)

$$A_{ik} = g_{(ab)}^* (\dot{\nabla}_k^* x_i^a) n^b, \quad B_{ik}^j = x_a^j (\dot{\nabla}_k^* x_i^a).$$

The **normal curvature** and the **geodesic torsion** of the curve C in GW_n are respectively,

$$\rho_n^* = A_{(ij)} t^i t^j \quad (3.3)$$

$$\tau_g^* = A_{(ij)} t^i t_1^j. \quad (3.4)$$

If the generalized prolonged derivative of (3.3) in the direction of C is taken and if the fact that the weight of ρ_n^* is $\{-1\}$ is used, we find that

$$\begin{aligned} \frac{\delta^* \rho_n^*}{\delta s} &= t^h \dot{\nabla}_h^* \rho_n^* \\ &= t^h (\nabla_h^* \rho_n^* + T_h \rho_n^*) \\ &= t^h [\nabla_h^* (A_{(ij)} t^i t^j)] + t^h T_h \rho_n^* \\ &= t^h (\nabla_h^* A_{(ij)}) t^i t^j + A_{(ij)} t^h (\nabla_h^* t^i) t^j + A_{(ij)} t^h (\nabla_h^* t^j) t^i + t^h T_h \rho_n^* \end{aligned}$$

and hence

$$\frac{\delta^* \rho_n^*}{\delta s} = t^h (\nabla_h^* A_{(ij)}) t^i t^j + 2A_{(ij)} t^h (\nabla_h^* t^j) t^i + t^h T_h \rho_n^*. \quad (3.5)$$

By virtue of (2.1), (2.3) and (3.4) the equation (3.5) reduces to

$$\frac{\dot{\delta}^* \rho_n^*}{\delta s} = t^h (\nabla_h^* A_{(ij)}) t^i t^j + 2\tau_g^* \kappa_1^* + t^h T_h A_{(ij)} t^i t^j,$$

or, putting

$$\mathcal{L} = \frac{\dot{\delta}^* \rho_n^*}{\delta s} - 2\tau_g^* \kappa_1^*,$$

we obtain

$$\mathcal{L} = t^h (\nabla_h^* A_{(ij)}) t^i t^j + t^h T_h A_{(ij)} t^i t^j \quad (3.6)$$

which is the generalized Laguerre's function of direction to a hypersurface in a generalized Weyl space. If, in particular, $T_h = 0$, i. e. if the space is Riemannian, then we obtain the expression for Laguerre's direction function of a Riemannian hypersurface.

Definition. *A curve in a hypersurface will be called a **Laguerre line** if and only if the Laguerre function of direction along the curve vanishes identically.*

The differential equation of Laguerre lines on a generalized Weyl hypersurface is, by (3.6)

$$\mathcal{L} = \left[(\nabla_h^* A_{(ij)}) t^i t^j + T_h A_{(ij)} t^i t^j \right] t^h = 0.$$

References

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