Second International Conference on
Geometry, Integrability and Quantization
June 7–15, 2000, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001

THE STRUCTURE OF FORMAL SOLUTIONS TO NAVIER'S EQUILIBRIUM EQUATION

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Abstract. The local Lie structure of the orientation-reversing involutions on $\mathbb{R}^3$ is used to construct a family of orthogonally invariant operators that produce all formal solutions, up to biharmonic equivalence, of Navier's equation for elastic equilibrium. In this construction the value of Poisson's ratio associated with each solution is determined by the hyperbolic geometry of $sl_2(\mathbb{R})$. Empirically feasible values of the ratio are associated with 'spacelike' operators whereas values outside of this range are associated with 'timelike' operators.

1. Introduction

In the theory of linear elasticity, Navier's equation says that the displacement $\mathbf{U} = (u_1, u_2, u_3)$ of a point in a body subjected to surface forces and after acceleration has vanished must satisfy the equilibrium equation

$$\nabla^2 \mathbf{U} = \frac{1}{2\nu - 1} \nabla\nabla \cdot \mathbf{U}\,,$$

where ν is Poisson's ratio, a dimensionless constant of the material that expresses the ratio of transverse compression to longitudinal extension under deformation. In Poisson's original formulation this constant was presumed to have the value of $\frac{1}{4}$ for all materials satisfying the generalized Hooke's law hypothesis, thus prompting Poisson to characterize linearized elastic response as "directionless". Later predictions by Cauchy and others (based on thermodynamic properties of the strain-energy function — an excellent discussion can be found in [1], Chapter 8) that ν could vary over the interval $(0, \frac{1}{2})$ were verified by photoelastic stress measurement ([5], 250 ff). Typically, hyperbolic contours with asymptotic angle γ were generated by stressing the material beneath a plate of glass, whereby ν would be equal to the curvature at the vertex of the hyperbola tangent to the unit circle of reference. Since this curvature is equal to $\cot^2\gamma$ it is of phenomenological interest why γ should be bounded below by $\tan^{-1}\sqrt{2}$. However, when this measurement is recorded in terms of distance s in the Poincaré conformal disc model of the hyperbolic plane, we find $\cot^2\theta = \frac{1}{4}(1-\tanh s)$, a fact that does not appear to be noted anywhere in the literature. This "coincidence" suggests a description of $\mathbf{U}$ in terms of hyperbolic geometry.

Work dating back to Maxwell implies that the general solution to the equilibrium equation on a region Ω is spanned by terms of the form $\mathbf{U} = M\nabla\psi + H_\psi N\mathbf{x}$, where M and N are compatible endomorphisms, ψ is harmonic on Ω with Hessian H_ψ and $\mathbf{x}$ is the position vector.(1) Any $\mathbf{U}$ of this form will be called fundamental. We will construct representatives for the equivalence classes of fundamental solutions from involutions in $M_3(\mathbb{R})$.

2. Notation

Formally the equilibrium equation can be studied in any dimension (and on smooth manifolds via the Hodge theory). We will work with ordinary Cartesian notation throughout because our construction is most easily described with basic matrix calculations. Let $\mathbf{f}$ and $\mathbf{g}$ be smooth functions from a region $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Let $\mathrm{ad}_\mathbf{f}$ be the Lie derivation defined by

$$\mathrm{ad}_\mathbf{f}\,\mathbf{g} = J_\mathbf{f}\mathbf{g} - J_\mathbf{g}\mathbf{f}\,,$$

where $J_\mathbf{f}$ and $J_\mathbf{g}$ are the Jacobians of $\mathbf{f}$ and $\mathbf{g}$, respectively. Let $\Psi = \{\mathbf{g}\,;\ \mathbf{g} = \nabla\psi, \psi \text{ harmonic on } \Omega\}$. We say two solutions $\mathbf{U}$ and $\mathbf{U}'$ of the equilibrium equation are *biharmonically equivalent* on Ω provided $\mathbf{U} - \mathbf{U}' \in \Psi$. The following proposition is a straightforward exercise.

Proposition 1. *If* $\mathbf{f}$ *is linear and* $\mathbf{g} \in \Psi$*, then* $\mathrm{ad}_\mathbf{f}\,\mathbf{g}$ *is divergence-free and*

$$\nabla^2\,\mathrm{ad}_\mathbf{f}\,\mathbf{g} = -2\nabla(\nabla\cdot J_\mathbf{f}\mathbf{g})\,.$$

If $\mathbf{f}$ is linear so that $J_\mathbf{f}$ is the endomorphism X we write ad_X to mean $\mathrm{ad}_\mathbf{f}$. Denote the identity endomorphism by $\mathbf{1}$ and let $V = \{X \in M_n(\mathbb{R})\,;\ X + X^T = c\mathbf{1},\ c \in \mathbb{R}\}$. Left action by members of V preserves Ψ, and V is conjugation invariant by $O(n)$. It follows from Proposition 1 that the function $\mathbf{U}$ obtained by replacing $J_\mathbf{f}$ in $\mathrm{ad}_\mathbf{f}\,\mathbf{g}$ with Y, where εX and Y are congruent mod V (i. e., $\varepsilon X - Y \in V$), solves the equilibrium equation with $\nu = \frac{1}{4}(3-\varepsilon)$. In

(1) The reader is referred to the extensive literature on biharmonic functions. Chapter 13 of [4] presents a collection of historically significant problems.

Section 3 we will use the geometry of orientation-reversing involutions on $\mathbb{R}^3$ to construct Y such that ε is determined by a parameter that represents distance in the hyperbolic plane.

From here on let us suppose that $n = 3$. Then

$$\operatorname{ad}_X \mathbf{g} = \frac{1}{3}(\operatorname{Tr} X)\nabla(2\psi - \mathbf{r}\cdot\nabla\psi) + \nabla \times (X_0\mathbf{r} \times \nabla\psi)\,,$$

where $X_0 = X - \dfrac{1}{3}(\operatorname{Tr} X)\mathbf{1}$ and $\mathbf{r} = (x_1, x_2, x_3)$. Unless otherwise indicated, lines and planes refer to subspaces of $\mathbb{R}^3$. For any plane Π let $M(\Pi)$ denote its stabilizer in $M_3(\mathbb{R})$. Choose a unit normal $\mathbf{n}$ for Π. If $X \in M(\Pi)$ we write $X = (\xi, \mathbf{x})$ where $\xi \in M_2(\mathbb{R})$ and $\mathbf{x} = X\mathbf{n}$. This will simplify notation by allowing us to identify $\mathbf{x}$ with $(\mathbf{0}, \mathbf{x})$, where $\mathbf{0}$ is the zero matrix. If M is an invertible member of $M(\Pi)$ that depends differentially on a parameter s then the Cartan image of M is defined by

$$C(M) = M_s M^{-1}(s)\,,$$

where $M_s = \dfrac{\partial M(s)}{\partial s}$. All endomorphisms in our construction will depend differentially on the hyperbolic parameter s that we define in Section 3.

We say $\mathbf{x}$ is an *affine* vector provided $\mathbf{x} = \mathbf{n} + \mathbf{y}$, for some $\mathbf{y} \in \Pi$. The affine group of Π is $AG(\Pi) = \{(\xi, \mathbf{x});\ \det\xi \neq 0, \mathbf{x} \text{ affine}\}$. Let $\mathfrak{g}_\Pi = \{(\xi, \mathbf{x});\ \operatorname{Tr}\xi = 0\}$, considered as a Lie algebra under the commutator product. Note that for any affine vector $\mathbf{x}_0$ the map

$$X \mapsto X(\mathbf{1} - \mathbf{x}_0)$$

is a Lie algebra homomorphism from $\mathfrak{g}_\Pi$ onto $sl_2(\mathbb{R})$ with kernel $\mathbb{R}^3$. By a **hyperbolic basis** for $sl_2(\mathbb{R})$ we mean an orthonormal triple (ξ, η, ξ^*) relative to the inner product $(\alpha, \beta) \mapsto \frac{1}{2}\operatorname{Tr}(\alpha\beta)$ such that $\frac{1}{2}[\eta, \xi] = \xi^*$, $\frac{1}{2}[\eta, \xi^*] = \xi$ and $\frac{1}{2}[\xi^*, \xi] = \eta$. In particular, the members anti-commute.

An **involution** refers to any endomorphism with principal invariants $(\iota_1, \iota_2, \iota_3) = (1, 1, -1)$. Endomorphisms with invariants $(1, -1, 1)$ will be called **dual**. The following proposition, which allows us to focus on these linear maps, can be inferred from the orbit structure of $SL(3, \mathbb{R})$ under conjugation by $O(3)$.

Proposition 2. *If $X \in SL(3, \mathbb{R})$ then X is congruent mod V to an involution or to a dual (possibly both).*

We show that involutions and their duals are related to spacelike and timelike elements, respectively, in hyperbolic space.

3. Geometry of Involutions and their Duals

Let X be an involution, let Π_X be the plane fixed pointwise by X and let $\mathbf{x}_-$ span the line with eigenvalue -1. Let Π be any plane containing $\mathbf{x}_-$. Then Π is stable under X. Conversely, Π contains $\mathbf{x}_-$ for any involution X that stabilizes it. We write (Π, X) provided this relation holds. For any plane Π with unit normal $\mathbf{n}$, three parameters (s, t, ϕ) will suffice to describe the collection of all X such that (Π, X). We define these parameters as follows. Let $\mathbf{x}_+$ span $\Pi \cap \Pi_X$, let θ be the angle between $\mathbf{x}_+$ and $\mathbf{x}_-$, and set $s = \tanh^{-1}(\cos\theta)$. That is, θ is the angle-of-parallelism corresponding to distance s in the hyperbolic plane.

To define the parameters t and ϕ, we note there exists $X^* \in \mathfrak{g}_\Pi \cap AG(\Pi)$, unique up to inverse, such that (Π, X^*X) and the restriction of X^*X to Π is a reflection. Thus $\det X^* = 1$ and $XX^*X = (X^*)^{-1}$. It follows that X and X^* share a fixed line spanned by $X^*X + XX^*$ (with affine vector $\mathbf{x}_0$) and that $\langle X^*X + XX^* \mid (\Pi, X)\rangle$ is a plane containing $\mathbf{n}$. Let $\mathbf{e}_1$ be a unit vector orthogonal to this plane and let ϕ be the angle between $\mathbf{x}_0$ and $\mathbf{n}$, where $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $\mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1$. Then $\mathbf{x}_0 = \mathbf{n} + \tan\phi \mathbf{e}_2$.

For any $M \in M(\Pi)$ let $d(M) = (\mathbf{1} - M)\mathbf{n}$. If $M \in AG(\Pi)$ we call $d(M)$ the **derivation** of M *into* Π. In particular, if X is an involution note that $d(X) \in \langle \mathbf{x}_- \rangle$. More generally, if $Y \in \mathfrak{g}_\Pi$ satisfies $(1 - X)Y \in \langle \mathbf{x}_- \rangle$ we say that Y induces the **normal derivation** d_Y on X. An important case will be the normal derivation induced by the exponential shift $(\mathrm{e}^{-s}Y)_s$ which we denote by $d_{\vec{Y}}$.

X^* is the *dual* of the involution X with respect to Π. Let $H = H_\Pi(X) = X^*X$, the **Cartan kernel** of X with respect to Π. The triple (X, H, X^*) is the *frame* determined by X. Let $(\xi_\phi, \eta_\phi, \xi^*_\phi)$ be the image of the frame determined by X under the Lie homomorphism induced by $\mathbf{x}_0$.

Proposition 3. *The triple $(\xi_\phi, \eta_\phi, \xi^*_\phi)$ is a hyperbolic basis for $sl_2(\mathbb{R})$. Further, with respect to s, $\eta_\phi = C(X) = C(X^*)$.*

Proof: Let

$$\omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With respect to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$, $X = (\xi, \mathbf{x})$ where $\xi = (\cosh s \cos t)\omega_1 + (\cosh s \sin t)\omega_2 + (\sinh s)\omega_3$, for some $t \in [0, 2\pi)$, and $\mathbf{x} = \mathbf{x}_0 + \tan\phi[(\sinh s - \cosh s \sin t)\mathbf{e}_1 + (\cosh s \cos t)\mathbf{e}_2]$. Further, $X^* = (\xi^*, \mathbf{x}^*)$ where the dualizations are obtained by applying the permutation $\sinh s \leftrightarrow \cosh s$. It follows that $\xi^*_\phi = X_s$ and $\xi_\phi = X^*_s$. The assertions in the proposition now follow by direct calculations. $\square$

Remark. *Since the Cartan map is a derivation of $AG(\Pi)$ into $sl_2(\mathbb{R})$ with respect to the action of conjugation we have $C(X^*X) = C(X^*) + X^*C(X)(X^*)^{-1}$. However, $X^*\eta_\phi(X^*)^{-1} = -\eta_\phi$ and so $C(H)$ is the zero map.*

The proposition shows that every frame is associated with a copy of $sl_2(\mathbb{R})$ in $\mathfrak{g}_\Pi$ and that these copies are parameterized by ϕ. Let $\mathfrak{g}_\phi$ denote the copy spanned by $(\xi_\phi, \eta_\phi, \xi^*_\phi)$, and let $\varphi\colon \mathfrak{g}_\Pi \to \mathfrak{g}_\phi$ denote the homomorphism induced by $\mathbf{x}_0$.

4. A Ruled 3-Surface

The frames (X, H, X^*) are partitioned by the Cartan kernels H. Each equivalence class of frames corresponds to a pair of curves $X(s)$ and $X^*(s)$ in $AG(\Pi)$ whose images in $\mathfrak{g}_\phi$ are comprised of poles (spacelike unit vectors) and points (timelike unit vectors), respectively, of hyperbolic $\mathbb{R}^3$. The one-dimensional space $\mathfrak{L}_H = \{\varphi(X + X^*);\ (X, H, X^*) \text{ is a frame}\}$ is a generator of the corresponding light cone.

Definition. *If $d_{\vec{\mathcal{X}}}(H) = (\operatorname{Tr}\mathcal{X})_s d(H)$ then $\mathcal{X}$ is an s-character of H of weight $\chi = \operatorname{Tr}\mathcal{X}$. We say $\mathcal{X}$ and $\mathcal{X}^*$ are co-characters provided $\mathcal{X}$ and $\mathcal{X}^*$ are s-characters of non-zero weight such that*

$$d_{\vec{\mathcal{X}}+\vec{\mathcal{X}}^*}(H) = \chi_s\chi^*_s d(H)\,.$$

A pair of co-characters $\mathcal{X}, \mathcal{X}^*$ are associated with the frame (X, H, X^*) provided the normal derivation on H induced by $X\mathcal{X} + X^*\mathcal{X}^*$ is $\chi\chi^* d(H)$. In this case, set $\hat{X} = \hat{X}(\chi) = X + \mathcal{X}$ and $\hat{X}^* = X^* + \mathcal{X}^*$.

Lemma. *For any frame (X, H, X^*) there are unique associated co-characters $\mathcal{X}$ and $\mathcal{X}^*$ such that $C(X\hat{X})$ and $C(X^*\hat{X}^*)$ are in $\langle\mathbf{n}\rangle$. In particular, $\hat{X}$ and $\hat{X}^*$ are invertible.*

Proof: The proof is computationally intensive. We outline the main points. By Proposition 3, $C(X\hat{X}) = \eta_\phi + XC(\hat{X})X$ so we must have $C(\hat{X}) - C(X) = \alpha\mathbf{n}$, $\alpha \in \mathbb{R}$. This condition implies that $\hat{X}\mathbf{n} = (u, v, w)$ where u, v and w satisfy the linear system

$$\begin{aligned} u + \mathrm{i}(w - v) = {}& \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{2})}(u_s + \mathrm{i}v_s) + [(\sinh s \sin t - \cos t - \cosh s) \\ & + \mathrm{i}(\sinh s \cos t + \sin t)]w_s\,. \end{aligned}$$

Applying $\sinh s \leftrightarrow \cosh s$ provides the conditions on $\hat{X}^*\mathbf{n}$. The condition that $\mathcal{X}$ and $\mathcal{X}^*$ be co-characters implies $w = \chi$ and $w^* = \chi^*$ (whereby $\alpha = [\ln(1 + \chi)]_s$ and $\alpha^* = [\ln(1 + \chi^*)]_s$). The system may then be solved to find

$\mathcal{X} = (\xi + \xi^*, \chi\mathbf{x})$ and $\mathcal{X}^* = (\xi + \xi^*, \chi^*\mathbf{x}^*)$, corresponding respectively to $X = (\xi, \mathbf{x})$ and $X^* = (\xi^*, \mathbf{x}^*)$. But then χ and χ^* must satisfy the system

$$\chi + \chi^* = \chi\chi^*$$
$$\chi_s + \chi_s^* = \chi_s\chi_s^* \, ,$$

which has the unique solution $\chi = 1 + \tanh s, \chi^* = 1 + \coth s$. $\square$

Corollary to the proof. $\hat{X}$ is congruent to $(2+\tanh s)X$ and $\hat{X}^*$ is congruent to $(2 + \coth s)X^* \mod V$.

Remark. *From the proof we also infer that* $X + X^* = \frac{1}{\chi}\mathcal{X} + \frac{1}{\chi^*}\mathcal{X}^*$*. Thus, if* $\mathfrak{R}_\phi$ *is a ruled 3-surface in* $\mathfrak{g}_\Pi$ *comprised of the lines* $L_X = \{r\mathcal{X}+(1-r)\mathcal{X}^* \, ; \, r \in \mathbb{R}\}$*, then* $\mathfrak{R}_\phi$ *is mapped by* φ *to the light cone and the pre-image of the generator* $\mathfrak{L}_H$ *is the collection of all* L_X *such that* (X, H, X^*) *is a frame.*

5. Main Theorem and an Application to the Planar Strain Energy

We can now state our main theorem, which follows immediately from Propositions 1, 2 and the above Lemma. For any involution X and its dual relative to Π let $\mathrm{ad}_{(\Pi,X)}\,\mathbf{g} = \hat{X}\mathbf{g} - J_{\mathbf{g}}X\mathbf{r}$ and let $\mathrm{ad}_{(\Pi,X^*)}\,\mathbf{g} = \hat{X}^*\mathbf{g} - J_{\mathbf{g}}X^*\mathbf{r}$.

Theorem. *Any fundamental solution of Navier's equilibrium equation on* Ω *with* $\nu \in (0, \frac{1}{2})$*, (resp.* $\left|\nu^* - \frac{1}{4}\right| > \frac{1}{4}$*) is biharmonically equivalent to* $\mathbf{U} = \mathrm{ad}_{(\Pi,X)}\,\mathbf{g}$ *(resp.* $\mathbf{U}^* = \mathrm{ad}_{(\Pi,X^*)}\,\mathbf{g}$*) for some* X, X^*, Π *and* $\mathbf{g} \in \Psi$*. In particular,* $\nu = \frac{1}{4}(1 - \tanh s)$ *and* $\nu^* = \frac{1}{4}(1 - \coth s)$*.*

Corollary. *The solutions* $\mathbf{U}$ *with* $s = 0$ *correspond to Poisson's "directionless" case* $\nu = \frac{1}{4}$*.*

Remarks:

1. Since $4\nu\nu^* = \nu + \nu^*$ it follows that ν and ν^* are related by the linear fractional transformation which fixes 0 and $\frac{1}{2}$ and which interchanges $\frac{1}{4}$ and ∞. That our construction approaches the values 0 and $\frac{1}{2}$ asymptotically perhaps relates to the interpretation of these ratios: $\nu = \frac{1}{2}$ corresponds to the constant divergence situation which implies that all directions are characteristic whereas linear isotropic elasticity requires no characteristic directions (see [2]); $\nu = 0$ implies that the stress and strain tensors are related by a scalar matrix, effectively, that the equilibrium equation is independent of the divergence of the displacement.

2. Let $\epsilon = \frac{1}{\chi^* - \chi}$. Then the symmetric identity

$$\left(\frac{1}{\chi} + \epsilon_s\right) \mathcal{X} + \left(\frac{1}{\chi^*} - \epsilon_s\right) \mathcal{X}^* = \left(\frac{1}{\chi_s} + \epsilon\right) \mathcal{X}_s + \left(\frac{1}{\chi_s^*} - \epsilon\right) \mathcal{X}_s^*$$

is satisfied by the co-characters $\mathcal{X}$ and $\mathcal{X}^*$.

Finally, our construction has the following consequence regarding the strain-energy function

$$W(\mathbf{U}) = \frac{\nu}{1-2\nu}(\nabla \cdot \mathbf{U})^2 + \frac{1}{4}\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2$$

where the Lamé constant μ has been normalized to 1. The case of plane strain (which applies to prismatic solids with forces orthogonal to the longitudinal axis) is obtained by taking $\phi = 0$, and $\mathbf{g} = \bar{f}$ for some complex analytic function $f(z)$ on Ω. Write $Z_1 \cdot Z_2 = \mathrm{Re}(Z_1 \overline{Z}_2)$ for any two complex quantities Z_1, Z_2 and let R be the hyperbolic rotation $x \mapsto x\cosh s + y \sinh s$, $y \mapsto x \sinh s + y \cosh s$. Then

$$\begin{aligned} W(\mathbf{U}) = 2\Big[&(\mathrm{e}^{2s} + 1)(f' \cdot \mathrm{e}^{-\mathrm{i}t})^2 + |R_s z|^2 |f''|^2\Big] \\ &+ (2\,\mathrm{e}^s - \mathrm{e}^{-s})\Big[(R_s z) \cdot (\mathrm{i}\bar{f}' f'') + (2\,\mathrm{e}^s - \mathrm{e}^{-s})|f'|^2\Big]\,, \\ W(\mathbf{U}^*) = 2\Big[&(\mathrm{e}^{2s} - 1)(f' \cdot \mathrm{e}^{-\mathrm{i}t})^2 + |R'_s z|^2 |f''|^2\Big] \\ &+ (2\,\mathrm{e}^s + \mathrm{e}^{-s})\Big[(R'_s z) \cdot (\mathrm{i}\bar{f}' f'') + (2\,\mathrm{e}^s + \mathrm{e}^{-s})|f'|^2\Big], \end{aligned}$$

whereby

$$W(\mathbf{U}) - W(\mathbf{U}^*) = (f' \cdot \mathrm{e}^{-\mathrm{i}t})^2 - 4|f'|^2 + (z - 3\mathrm{i}\bar{z}) \cdot (\mathrm{i}\bar{f}' f'')\,.$$

That is, the difference between the strain potential of a 'spacelike' displacement and that of its dual 'timelike' solution is independent of ν.

References

[1] Malvern L. E., *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, London 1969.

[2] Olver P. J., *Applications of Lie Groups to Differential Equations*, 2nd ed., Springer-Verlag, New York, Berlin, Heidelberg 1993.

[3] Sarli J. and Torner J., *Representations of $\mathbb{C}$, Biharmonic Vector Fields, and the Equilibrium Equation of Planar Elasticity*, J. Elasticity **32** (1993) 223–241.

[4] Soutas-Little R. W., *Elasticity*, Dover, Mineola, New York 1999.

[5] Timoshenko S. and Goodier J. N., *Theory of Elasticity*, McGraw-Hill, New York 1951.