Nineteenth International Conference on
Geometry, Integrability and Quantization
June 02–07, 2017, Varna, Bulgaria
Ivaïlo M. Mladenov and
Akira Yoshioka, Editors
Avangard Prima, Sofia 2018, pp 234–249
doi: 10.7546/giq-19-2018-234-249

STAR PRODUCT, STAR EXPONENTIAL AND APPLICATIONS

AKIRA YOSHIOKA

Department of Mathematics, Tokyo University of Science, 162-8601 Tokyo, Japan

Abstract. We introduce star products for certain function space containing polynomials, and then we obtain an associative algebra of functions. In this algebra we can consider exponential elements, which are called star exponentials. Using star exponentials we can define star functions in the star product algebra. We explain several examples.

MSC: 53D55
Keywords: Deformation quantization, star exponential, star product

1. Introduction: The Idea

The idea of star product is deeply related to the canonical commutation relation in Quantum mechanics, which is given by a pair of operators $\hat{p}, \hat{q}$ such that

$$[\hat{p}, \hat{q}] = \hat{p}\,\hat{q} - \hat{q}\,\hat{p} = \sqrt{-1}\,\hbar = \mathrm{i}\,\hbar$$

where $\hat{p} = \mathrm{i}\,\hbar\,\partial_q$ and $\hat{q}$ is a multiplication operator $q\times$ acting on the functions of q, and $\hbar$ is a constant equal to the Planck constant divided by 2π. The algebra generated by $\hat{p}$ and $\hat{q}$ is called the *Weyl algebra* which plays a fundamental role in quantum mechanics.

We have another way to produce the same algebra without using operators. The idea is to introduce an associative product into the space of functions of (q, p). The product is different from the usual multiplication of functions, but is given by a deformation of the usual multiplication in the following way. (cf. Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer [1], Moyal [9]).

For smooth functions f, g on $\mathbb{R}^2$, we have the canonical Poisson bracket

$$\{f, g\}(q, p) = \partial_p f \partial_q g - \partial_q f \partial_p g, \qquad (q, p) \in \mathbb{R}^2.$$

In deformation quantization, we often use the notation such as

$$\{f, g\} = f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) g = \partial_p f\, \partial_q g - \partial_q f\, \partial_p g.$$

The Moyal Product. The typical star product is the Moyal product given as follows. For smooth functions f, g we consider a product $f *_O g$ given by a formal power series of the biderivation $\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}$ such that

$$\begin{aligned} f *_O g &= f \exp \frac{\mathrm{i}\hbar}{2} \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\mathrm{i}\hbar}{2}\right)^k \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^k g \\ &= fg + \frac{\mathrm{i}\hbar}{2} f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) g + \frac{1}{2!} \left(\frac{\mathrm{i}\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^2 g \\ &\quad + \cdots + \frac{1}{k!} \left(\frac{\mathrm{i}\hbar}{2}\right)^k f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^k g + \cdots . \end{aligned}$$

The product is well-defined when f or g is a polynomial, and it is easy to see that the product is associative.

Now we calculate the commutator of the variables p and q. We see

$$\begin{aligned} p *_O q = p \exp \frac{\mathrm{i}\hbar}{2} \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) q &= p \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\mathrm{i}\hbar}{2}\right)^k \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right)^k q \\ &= pq + \frac{\mathrm{i}\hbar}{2} p \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}\right) q = pq + \frac{\mathrm{i}\hbar}{2} \cdot \end{aligned}$$

Similarly we see

$$q *_O p = pq - \frac{\mathrm{i}\hbar}{2} \cdot$$

Then the functions p and q satisfy the canonical commutation relation under the commutator of the product $*_O$

$$[p, q]_* = p *_O q - q *_O p = \mathrm{i}\hbar.$$

The product $*_O$ is associative on polynomials with canonical commutation relation, and then we obtain the Weyl algebra given by the ordinary polynomials with the product $*_O$. Using this Weyl algebra of the product $*_O$, we can obtain same results of quantum mechanics and some other extension.

In this note, we give a brief review on this subject mainly related our investigation.

2. Star Calculation of Eigenvalues

As an application of star product algebra, we calculate the eigenvalues of the harmonic oscillator by means of the star product $*_O$.

2.1. Eigenvalues of Harmonic Oscillator

Eigenvalues. The Schrödingier operator of the harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2}\left(\frac{\partial}{\partial q}\right)^2 + \frac{1}{2}q^2.$$

The eigenvalues are

$$E_n = \hbar(n + \frac{1}{2}), \qquad n = 0, 1, 2, \ldots .$$

and the eigenfunctions are given by the Hermite polynomials of q.

We calculate these values E_n by means of the star product $*_O$ and functions of p and q, parallel to the methods in quantum mechanics.

Star Product Calculation. The classical hamiltonian function is

$$H = \frac{1}{2}(p^2 + q^2).$$

We put functions such as

$$a = \frac{1}{\sqrt{2\,\hbar}}(p + \mathrm{i}q), \qquad a^\dagger = \frac{1}{\sqrt{2\,\hbar}}(p - \mathrm{i}q).$$

Then we calculate the product explicitly and obtain

$$a^\dagger *_O a = \frac{1}{2\,\hbar}\left(p *_O p + \mathrm{i}[p, q]_* + q *_O q\right) = \frac{1}{2\,\hbar}\left(p \cdot p + \mathrm{i} \cdot \mathrm{i}\hbar + q \cdot q\right).$$

Then we have

$$H = \hbar(N + \frac{1}{2}), \qquad N = a^\dagger *_O a.$$

The commutator with respect to the product $*_O$ is easily seen

$$[a, a^\dagger]_* = a *_O a^\dagger - a^\dagger *_O a = \frac{1}{2\,\hbar}2(-\mathrm{i})[p, q]_* = 1.$$

Now we consider a function

$$f_0 = \frac{1}{\pi\,\hbar}\exp(-2\,aa^\dagger) = \frac{1}{\pi\,\hbar}\exp(-\frac{1}{\hbar}(p^2 + q^2)).$$

By a direct calculation we see

$$a *_O f_0 = f_0 *_O a^\dagger = 0.$$

For every positive integer n we set a function

$$f_n = \frac{1}{n!}\underbrace{a^\dagger *_O \cdots *_O a^\dagger}_{n} *_O f_0 *_O \underbrace{a *_O \cdots *_O a}_{n}.$$

The relation $[a, a^\dagger]_* = 1$ induces $a *_O a^\dagger = a^\dagger *_O a + 1 = N + 1$ then

$$N *_O a^\dagger = (a^\dagger *_O a) *_O a^\dagger = a^\dagger *_O (a *_O a^\dagger) = a^\dagger *_O (N + 1).$$

Remark also that $a *_O f_0 = 0$ yields $N *_O f_0 = (a^\dagger *_O a) *_O f_0 = 0$. Then for example we calculate as

$$N *_O f_1 = N *_O (a^\dagger *_O f_0 *_O a) = a^\dagger *_O (N+1) *_O f_0 *_O a = f_1.$$

By a similar manner we easily see

$$N *_O f_k = f_k *_O N = k f_k.$$

Since $H = \hbar(N + \frac{1}{2})$ we have the solutions of the star eigenvalue problem

$$H *_O f_n = f_n *_O H = \hbar(n + \frac{1}{2}) f_n = E_n f_n, \qquad n = 0, 1, 2, \ldots$$

and thus we obtain the eigenvalues of the harmonic oscillator $\hat{H}$.

2.2. MIC-Kepler Problem (Kanazawa-Yoshioka [4])

Similarly the star product algebra also gives the exact eigenvalues and their multiplicities for the quantized Kepler problem and the more general system such as the MIC-Kepler problem, the Kepler problem under the influence of the Dirac magnetic monopole. For more details, see Appendix.

3. Star Products

Generalizing the ordering problem in physics, we give general star products as follows (cf. Omori-Maeda-Miyazaki-Yoshioka [10]).

3.1. Examples: Moyal, Normal, Anti-Normal Products

The Moyal product is a well-known example of star product.

As in the previous section, we define: for polynomials f, g of the variables

$$(u_1, \ldots, u_m, v_1, \ldots, v_m)$$

the Moyal product $f *_O g$ is given by the power series of the biderivation

$$\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) = \sum_j \left(\overleftarrow{\partial_{v_j}} \cdot \overrightarrow{\partial_{u_j}} - \overleftarrow{\partial_{u_j}} \cdot \overrightarrow{\partial_{v_j}}\right)$$

such that

$$f *_O g = f \exp \frac{i\hbar}{2} \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g$$
$$= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g + \frac{1}{2!} \left(\frac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^2 g$$
$$+ \cdots + \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g + \cdots .$$

Then we have

Theorem 1. The Moyal product is well-defined on polynomials, and associative.

Other typical star products are normal product $*_N$, anti-normal product $*_A$ given similarly by

$$f *_N g = f \exp i\hbar \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u}\right) g, \qquad f *_A g = f \exp -i\hbar \left(\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g.$$

These are also well-defined on polynomials and associative.
By direct calculation we see easily

Proposition 1. i) For these star products, the generators $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ satisfy the canonical commutation relations

$$[u_k, v_l]_{*_L} = -i\,\hbar \delta_{kl}, \qquad [u_k, u_l]_{*_L} = [v_k, v_l]_{*_L} = 0, \qquad k, l = 1, 2, \ldots, m$$

where $*_L$ stands for $*_O$, $*_N$, $*_A$.

ii) Then the algebras $(\mathbb{C}[u, v], *_L)$ $(L = O, N, A)$ are mutually isomorphic and isomorphic to the Weyl algebra.

Actually the algebra isomorphism

$$I_O^N : (\mathbb{C}[u, v], *_O) \longrightarrow (\mathbb{C}[u, v], *_N)$$

is given explicitly by the power series of the differential operator such as

$$I_N^O(f) = \exp\left(-\frac{i\hbar}{2} \partial_u \partial_v\right)(f) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{i\hbar}{2}\right)^l (\partial_u \partial_v)^l (f).$$

And other isomorphisms are given in the similar form.

Remark 1. We remark here that these facts are well-known as ordering problem in physics.

3.2. General Star Product

Now by generalizing the biderivations in the previous products, we define a star product on complex domain.

Let n be even and let Λ be an arbitrary $n \times n$ complex matrix. We consider a biderivation on $\mathbb{C}^n$

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \cdots, w_n)$ is the coordinates of $\mathbb{C}^n$.

Now we define a star product by the power series of the above biderivation such that

Definition 1.

$$f *_{\Lambda} g = f \exp \frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) g.$$

Remark 2. i) The star product $*_{\Lambda}$ is a generalization of the previous products. Actually

- if we put $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then we have the Moyal product
- if $\Lambda = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, we have the normal product
- if $\Lambda = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ then the anti-normal product.

ii) If Λ is a symmetric matrix, the star product $*_{\Lambda}$ is commutative. Furthermore, if Λ is a zero matrix, then the star product is nothing but a usual commutative product.

Then similarly as before we see easily

Theorem 2. For an arbitrary Λ, the star product $*_{\Lambda}$ is well-defined on polynomials, and associative.

3.3. Star Product Representation of the Weyl Algebra

In this section, we fix the antisymmetric part of Λ in order to represent the Weyl algebra.

We assume the dimension is even, $n = 2m$. Let K be an arbitrary $2m \times 2m$ complex symmetric matrix. We put a complete matrix

$$\Lambda = J + K$$

where J is a fixed matrix such that

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since Λ is determined by the complex symmetric matrix K, we denote the star product by $*_K$ instead of $*_\Lambda$.
We consider polynomials of variables $(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$. By an easy calculation one obtains for an arbitrary K

Proposition 2. i) For a product $*_K$, the generators $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ satisfy the canonical commutation relations

$$[u_k, v_l]_* = -\mathrm{i}\hbar\delta_{kl}, \qquad [u_k, u_l]_* = [v_k, v_l]_* = 0, \qquad k, l = 1, 2, \ldots, m.$$

ii) Then the algebra $(\mathbb{C}[u, v], *_K)$ is isomorphic to the Weyl algebra, and the algebra is regarded as a polynomial representation of the Weyl algebra.

Equivalence. As in the case of typical star products, we have algebra isomorphisms as follows.

Proposition 3. For arbitrary star product algebras $(\mathbb{C}[u, v], *_{K_1})$ and $(\mathbb{C}[u, v], *_{K_2})$ we have an algebra isomorphism $I_{K_1}^{K_2} : (\mathbb{C}[u, v], *_{K_1}) \to (\mathbb{C}[u, v], *_{K_2})$ given by the power series of the differential operator $\partial_w(K_2 - K_1)\partial_w$ such that

$$I_{K_1}^{K_2}(f) = \exp\left(\frac{\mathrm{i}\hbar}{4}\partial_w(K_2 - K_1)\partial_w\right)(f)$$

where $\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

Remark 3. 1. By the previous proposition we see the algebras $(\mathbb{C}[u, v], *_K)$ are mutually isomorphic and isomorphic to the Weyl algebra. Hence we have a family of star product algebras $\{(\mathbb{C}[u, v], *_K)\}_K$ where each element is regarded as a polynomial representation of the Weyl algebra.

2. The above equivalences are also possible to make for star products $*_\Lambda$ for arbitrary Λ's with a common skew symmetric part.

By a direct calculation we have

Theorem 3. Then isomorphisms satisfy the following chain rule

1. $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = \mathrm{Id}$.
2. $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$.

3.4. Star Exponentials

Using polynomial expressions, we can consider exponential elements in the Weyl algebra.

Idea of Definition. Now we are considering general star product $*_\Lambda$.

For a polynomial H_* of the Weyl algebra, we want to define a star exponential $e_*^{t\frac{H_*}{i\hbar}}$. However, the expansion $\sum_n \frac{t^n}{n!}\left(\frac{H_*}{i\hbar}\right)^n$, where $\left(\frac{H_*}{i\hbar}\right)^n$ is an n-th power of $\frac{H_*}{i\hbar}$ with respect to the star product $*_\Lambda$, is not convergent in general. Then we define a star exponential by means of the differential equation.

Definition 2. *The star exponential* $e_*^{t\frac{H_*}{i\hbar}}$ *is given as a solution of the following differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}F_t = \frac{H_*}{\mathrm{i}\hbar} *_\Lambda F_t, \qquad F_0 = 1.$$

Examples. We are interested in the star exponentials of linear, and quadratic polynomials. For these, we can solve the differential equation explicitly. For simplicity, we consider $2m \times 2m$ complex matrices Λ with the skew symmetric part $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We write $\Lambda = J + K$ where K is a complex symmetric matrix. For these polynomials, for example we have the following explicit solutions for $*_\Lambda = *_K$.

Linear case. We denote by a linear polynomial by $l = \sum_{j=1}^{2m} a_j w_j$. We see

Proposition 4. For $l = \sum_j a_j w_j = \langle \mathbf{a}, \mathbf{w} \rangle$

$$\mathrm{e}_{*_\Lambda}^{t(l/\mathrm{i}\hbar)} = \mathrm{e}^{t^2 \mathbf{a}K\mathbf{a}/4\mathrm{i}\hbar}\mathrm{e}^{t(l/\mathrm{i}\hbar)}.$$

Quadratic Case.

Proposition 5. For a quadratic polynomial $Q_* = \langle \mathbf{w}A, \mathbf{w} \rangle_*$ where A is a $2m \times 2m$ complex symmetric matrix, we have

$$\mathrm{e}_{*_\Lambda}^{t(Q_*/\mathrm{i}\hbar)} = \frac{2^m}{\sqrt{\det(I - \kappa + \mathrm{e}^{-2t\alpha}(I+\kappa))}} \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}\langle \mathbf{w}\frac{1}{I-\kappa+\mathrm{e}^{-2t\alpha}(I+\kappa)}(I-\mathrm{e}^{-2t\alpha})J, \mathbf{w}\rangle}$$

where $\kappa = KJ$ and $\alpha = AJ$.

3.5. Star Functions

By the same way as in the ordinary exponential functions, we can obtain several non-commutative or commutative functions using star exponentials. For more examples, see OMMY [11]

There are many applications of star exponential. Here we show examples using star exponentials of linear polynomials.

In what follows, we consider the star product for the simplest case where $n \times n$ matrix is of the form

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0_{n-1} \end{pmatrix}, \qquad \rho \in \mathbb{C}.$$

Since Λ is symmetric, the star product is commutative and explicitly given by $f *_\Lambda g = f \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{u_1}}\overrightarrow{\partial_{u_1}}\right) g$. This means that the algebra is essentially reduced to space of functions of one variable u_1. Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) \mathrm{e}^{\frac{\tau}{2}\overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w).$$

A direct calculation gives that the star exponential of $\mathrm{i}\, tw$ with respect to $*_\tau$ is

Proposition 6.

$$\exp_{*_\tau} \mathrm{i}\, tw = \exp(\mathrm{i}\, tw - (\tau/4)t^2).$$

3.5.1. Star Hermite Function

Recall the identity

$$\exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!}$$

where $H_n(w)$ is an Hermite polynomial.

By the explicit formula $\exp_{*_\tau} \mathrm{i}\, tw = \exp(\mathrm{i}\, tw - (\tau/4)t^2)$, we see

$$\exp\left(\sqrt{2}\, tw - \frac{1}{2}t^2\right) = \exp_{*_\tau}(\sqrt{2}\, tw)_{\tau=-1}.$$

Since $\exp_{*_\tau}(\sqrt{2}tw) = \sum_{n=0}^{\infty}(\sqrt{2}w)^n_{*_\tau}\frac{t^n}{n!}$ we see the function $H_n(w)$ is the n-th power of w with respect to the star product $*_\tau$

$$H_n(w) = (\sqrt{2}w)^n_{*\tau}|_{\tau=-1}.$$

We define $*$-Hermite function by

$$H_n(w,\tau) = (\sqrt{2}\, w)^n_{*_\tau}, \qquad n = 0, 1, 2, \ldots$$

with respect to $*_\tau$ product. Remark that when we set $\tau = -1$ we have the usual Hermite functions. Then we have

$$\exp_{*_\tau}(\sqrt{2}\, tw) = \sum_{n=0}^{\infty} H_n(w,\tau)\, \frac{t^n}{n!}.$$

Identities. Trivial identity $\frac{\mathrm{d}}{\mathrm{d}t}\exp_{*_\tau}(\sqrt{2}\,tw) = \sqrt{2}w{*_\tau}\exp_{*_\tau}(\sqrt{2}\,tw)$ yields the identity

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau)+\sqrt{2}wH_n(w,\tau)=H_{n+1}(w,\tau), \qquad n=0,1,2,\ldots$$

for every $\tau\in\mathbb{C}$. The exponential law

$$\exp_{*_\tau}(\sqrt{2}sw_*)*\exp_{*_\tau}(\sqrt{2}tw_*)=\exp_{*_\tau}(\sqrt{2}(s+t)w_*)$$

yields the identity

$$\sum_{k+l=n}\frac{n!}{k!l!}H_k(w,\tau)*_\tau H_l(w,\tau)=H_n(w,\tau).$$

3.5.2. Star Theta Function

We can express the Jacobi's theta functions by using star exponentials.

Recall the formula

$$\exp_{*_\tau}\mathrm{i}tw=\exp\mathrm{i}\,tw-(\tau/4)t^2).$$

Hence for $\mathrm{Re}\,\tau>0$, the star exponential $\exp_{*_\tau} n\mathrm{i}\,w=\exp(n\mathrm{i}\,w-(\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then we can consider summations for τ such that $\mathrm{Re}\,\tau>0$

$$\sum_{n=-\infty}^{\infty}\exp_{*_\tau}2n\mathrm{i}\,w=\sum_{n=-\infty}^{\infty}\exp\left(2n\mathrm{i}\,w-\tau n^2\right)=\sum_{n=-\infty}^{\infty}q^{n^2}\mathrm{e}^{2ni\,w}, \qquad q=\mathrm{e}^{-\tau}$$

This is convergent and gives Jacobi's theta function $\theta_3(w,\tau)$. Then we have expressions of theta functions as

$$\theta_{1*_\tau}(w)=\frac{1}{\mathrm{i}}\sum_{n=-\infty}^{\infty}(-1)^n\exp_{*_\tau}(2n{+}1)\mathrm{i}\,w, \quad \theta_{2*_\tau}(w)=\sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n{+}1)\mathrm{i}\,w$$

$$\theta_{3*_\tau}(w)=\sum_{n=-\infty}^{\infty}\exp_{*_\tau}2n\mathrm{i}\,w, \qquad \theta_{4*_\tau}(w)=\sum_{n=-\infty}^{\infty}(-1)^n\exp_{*_\tau}2n\mathrm{i}\,w.$$

Remark that $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k=1,2,3,4$ respectively. We have trivial identities because of the exponential law

$$\begin{aligned}\exp_{*_\tau}2\mathrm{i}\,w*_\tau\theta_{k*_\tau}(w)&=\theta_{k*_\tau}(w), && k=2,3\\ \exp_{*_\tau}2\mathrm{i}\,w*_\tau\theta_{k*_\tau}(w)&=-\theta_{k*_\tau}(w), && k=1,4.\end{aligned}$$

Then using $\exp_{*_\tau}2\mathrm{i}\,w=\mathrm{e}^{-\tau}\mathrm{e}^{2\mathrm{i}\,w}$ and the product formula directly we have

$$\begin{aligned}\mathrm{e}^{2\mathrm{i}\,w-\tau}\theta_{k*_\tau}(w+\mathrm{i}\,\tau)&=\theta_{k*_\tau}(w), && k=2,3\\ \mathrm{e}^{2\mathrm{i}\,w-\tau}\theta_{k*_\tau}(w+\mathrm{i}\,\tau)&=-\theta_{k*_\tau}(w), && k=1,4.\end{aligned}$$

3.5.3. $*$-Delta Functions

Since the $*_\tau$-exponential $\exp_{*_\tau}(itw) = \exp(itw - \frac{\tau}{4}t^2)$ is rapidly decreasing with respect to t when $\mathrm{Re}\,\tau > 0$. Then the integral of $*_\tau$-exponential

$$\int_{-\infty}^{\infty} \exp_{*_\tau}(\mathrm{i}t(w-a)_*)\,\mathrm{d}t = \int_{-\infty}^{\infty} \exp(\mathrm{i}t(w-a) - \frac{\tau}{4}t^2)\mathrm{d}t$$

converges for any $a \in \mathbb{C}$. We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_*(\mathrm{i}t(w-a)_*)\mathrm{d}t$$

which has a meaning at τ with $\mathrm{Re}\,\tau > 0$. It is easy to see for any element $p_*(w) \in (\mathbb{C}[w], *_\tau)$

$$p_*(w) *_\tau \delta_*(w-a) = p(a)\delta_*(w-a),\ w *_\tau \delta_*(w) = 0.$$

Using the Fourier transform we have

Proposition 7.

$$\theta_{1*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + \frac{\pi}{2} + n\pi)$$

$$\theta_{2*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + n\pi)$$

$$\theta_{3*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w + n\pi)$$

$$\theta_{4*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w + \frac{\pi}{2} + n\pi).$$

Now, we consider the τ satisfying the condition $\mathrm{Re}\,\tau > 0$. Then we calculate the integral and obtain $\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right)$. Then we have

$$\begin{aligned}
\theta_3(w,\tau) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w+n\pi) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w+n\pi)^2\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \theta_{3*}(\frac{2\pi w}{\mathrm{i}\tau}, \frac{\pi^2}{\tau}).
\end{aligned}$$

We also have similar identities for other $*$-theta functions by the similar way.

4. Appendix: MIC-Kepler Problem

Background. McIntosh and Cisneros [7] studied the dynamical system describing the motion of a charged particle under the influence of Dirac's monopole field besides the Coulomb's potential. Iwai-Uwano [2] give the Hamiltonian description for the MIC-Kepler problem and Mladenov-Tsanov [8] have quantized geometrically this system.

Point. Iwai-Uwano showed that the classical system of MIC-Kepler problem is obtained by the S^1-reduction, or Marsden-Weinstein reduction method for symplectic manifolds. Star product is using classical system with deformed product. Then by using star product calculation which consists of classical system with star product, we can expect to deal with the quantized system of MIC-Kepler problem by means of Marsden-Weinstein reduction method in natural way. We discuss this in this subsection.

MIC-Kepler problem. Now the MIC-Kepler problem is given in the following way. We consider a closed two form on $\dot{\mathbb{R}}^3 = \mathbb{R}^3 - \{\mathbf{0}\}$ such that

$$\Omega = \left(q_1 \, \mathrm{d}q_2 \wedge \mathrm{d}q_3 \; + \; q_2 \, \mathrm{d}q_3 \wedge \mathrm{d}q_1 \; + \; q_3 \, \mathrm{d}q_1 \wedge \mathrm{d}q_2 \right)/r^3$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$. We consider the cotangent bundle $T^*\dot{\mathbb{R}}^3$ and a symplectic form

$$\sigma_\mu = \mathrm{d}p_1 \wedge \mathrm{d}q_1 \; + \; \mathrm{d}p_2 \wedge \mathrm{d}q_2 \; + \; \mathrm{d}p_3 \wedge \mathrm{d}q_3 \; + \Omega_\mu$$

where $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, q_2, q_3, p_1, p_2, p_3) \in T^*\dot{\mathbb{R}}^3$ and the *two*-form $\Omega_\mu \equiv \mu \, \Omega$ stands for Dirac's monopole field of strength $\mu \in \mathbb{R}$. Then the MIC-Kepler problem is given as the triple $(T^*\dot{\mathbb{R}}^3, \, \sigma_\mu \, , \, H_\mu \,)$ where H_μ is the Hamiltonian function such that

$$H_\mu \, (\boldsymbol{q}, \, \boldsymbol{p}) = \frac{1}{2} \left(p_1^2 + p_2^2 + p_3^2 \right) + \frac{\mu^2}{2r^2} - \frac{k}{r}$$

and k is a positive constant. When $\mu = 0$ the system is just the Kepler problem.

S^1**-action.** The MIC-Kepler problem is obtained by the S^1-reduction from the conformal Kepler problem on $T^*\dot{\mathbb{R}}^4$ (Iwai-Uwano [2]).

We denote the points by $y \in \mathbb{R}^4$ and $(y, \eta) \in T^*\mathbb{R}^4$. We identify the point of $T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4)$ by

$$T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4) \mapsto (z_1, z_2, \zeta_1, \zeta_2) \in T^*\mathbb{C}^2 = \mathbb{C}^4$$

where

$$z_1 = y_1 + \mathrm{i}y_2, \qquad z_2 = y_3 + \mathrm{i}y_4, \qquad \zeta_1 = \eta_1 + \mathrm{i}\eta_2, \qquad \zeta_2 = \eta_3 + \mathrm{i}\eta_4.$$

The canonical one form θ on $T^*\mathbb{R}^4$ is written as

$$\theta(z, \zeta) = \mathrm{Re} \, (\bar{\zeta} \cdot \mathrm{d}z).$$

Now we define an S^1 action on $\dot{\mathbb{R}}^4 = \mathbb{R}^4 - \{\mathbf{0}\} = \dot{\mathbb{C}}^2$ by $z \mapsto \mathrm{e}^{\mathrm{i}t} z$ which induces the action on the cotangent bundle $T^*\dot{\mathbb{R}}^4$

$$\varphi_t : (z, \zeta) \mapsto (\mathrm{e}^{\mathrm{i}t} z, \mathrm{e}^{\mathrm{i}t} \zeta).$$

The induced action φ_t preserves the canonical one form θ and then is an exact symplectic action.

The induced vector field $v(z, \zeta)$ on $T^*\dot{\mathbb{R}}^4$ of the action is

$$v(z, \zeta) = (\mathrm{i}z, \mathrm{i}\,\zeta)$$

and

$$\psi(z, \zeta) = \iota_v \theta(z, \zeta) = \mathrm{Im}\ \zeta \cdot \bar{z} = (\zeta \cdot \bar{z} - \bar{\zeta} \cdot z)/2\,\mathrm{i}$$

is a moment map ψ of the action.

S^1**-Reduction.** Following the Marsden-Weinstein reduction theory, we consider a level set of the moment map $\psi^{-1}(\mu)$ for $\mu \in \mathbb{R}$. Then the S^1-bundle $\pi_\mu : \psi^{-1}(\mu) \to \psi^{-1}(\mu)/S^1$ has the symplectic structure ω_μ such that $\iota_\mu^* \mathrm{d}\theta = \pi_\mu^* \omega_\mu$, hence we have a reduced symplectic manifold $(\psi^{-1}(\mu)/S^1, \omega_\mu)$, where

$$\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$$

is the inclusion map. Then one can show

Proposition 8. The reduced phase space is diffeomorphic to the symplectic manifold of the MIC-Kepler problem

$$(\psi^{-1}(\mu)/S^1, \sigma_\mu) \simeq (T^*\dot{\mathbb{R}}^3, \sigma_\mu).$$

Conformal Kepler Problem on $T^*\dot{\mathbb{R}}^4$. Now we consider a harmonic oscillator on $T^*\dot{\mathbb{R}}^4$

$$H_0(z, \zeta) = \frac{1}{2}\,|\zeta|^2 + \frac{1}{2}\,\omega^2\,|z|^2.$$

Iwai-Uwano [2] introduces the conformal Kepler problem with the Hamiltonian

$$H_{CF}(z, \zeta) = \frac{1}{4|z|^2}\,(H_0(z, \zeta) - 4k) - \frac{1}{8}\omega^2 = \frac{1}{8|z|^2}|\zeta|^2 - \frac{k}{|z|^2}.$$

The MIC-Kepler problem is the reduced hamiltonian system of the conformal Kepler problem, i.e.,

$$\pi_\mu^* H_\mu = \iota_\mu^* H_{CF}.$$

The conformal Kepler problem is related to the harmonic oscillator on $T^*\dot{\mathbb{R}}^4$ as

$$4|z|^2 \left(H_{CF}(z, \zeta) + \frac{1}{8}\omega^2 \right) = H_0(z, \zeta) - 4k.$$

Hence the energy surfaces in $T^*\dot{\mathbb{R}}^4$ coincide, i.e.,

$$H_{CF} = -\frac{1}{8}\omega^2 \Longleftrightarrow H_0 = -4k.$$

Star Product Calculation of the Eigenvalues. On 8-dimensional phase space $T^*\dot{\mathbb{R}}^4$, we have the canonical Poisson bracket and then by the same way as the previous section, we have the star product $*_O$.

We consider functions

$$\begin{cases} b_1(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, z_1 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\,\zeta_1\right), \qquad b_1(z,\zeta)^\dagger = \overline{b_1(z,\zeta)} \\ b_2(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, z_2 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\,\zeta_2\right), \qquad b_2(z,\zeta)^\dagger = \overline{b_2(z,\zeta)} \\ b_3(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_1 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\,\bar{\zeta}_1\right), \qquad b_3(z,\zeta)^\dagger = \overline{b_3(z,\zeta)} \\ b_4(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_2 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\,\bar{\zeta}_2\right), \qquad b_4(z,\zeta)^\dagger = \overline{b_4(z,\zeta)}. \end{cases}$$

We see that the commutators of these functions are

$$[b_j, b_k]_* = [b_j^\dagger, b_k^\dagger]_* = 0, \qquad [b_j, b_k^\dagger]_* = \delta_{jk}, \qquad j,k = 1,2,3,4.$$

We set

$$N = b_1^\dagger *_O b_1 + b_2^\dagger *_O b_2 + b_3^\dagger *_O b_3 + b_4^\dagger *_O b_4.$$

Then we see

$$H_0 = \hbar\omega\,(N+2)$$

and the moment map $\psi(z,\zeta)$ is written in terms of $b_j,\ b_j^\dagger$ as

$$\psi(z,\zeta) = \frac{\hbar}{2}\,(-b_1^\dagger *_O b_1 - b_2^\dagger *_O b_2 + b_3^\dagger *_O b_3 + b_4^\dagger *_O b_4).$$

We put for $j = 1,2,3,4$

$$f_{j,0}(z,\zeta) = \frac{1}{\pi\hbar}\mathrm{e}^{-2\,b_j^\dagger\, b_j}, \quad f_{j,k}(z,\zeta) = \frac{1}{k!}\,(b_j^\dagger)_*^k *_O f_{j,0} *_O (b_j)_*^k.$$

We consider

$$f_{\overrightarrow{n}} = f_{1,n_1} *_O f_{2,n_2} *_O f_{3,n_3} *_O f_{4,n_4}, \qquad \overrightarrow{n} = (n_1, n_2, n_3, n_4).$$

Parallel to Iwai-Uwano [3], we can calculate the eigenvalues of the MIC-Kepler problem as follows.

Similarly as before we easily see

$$H_0 *_O f_{\overrightarrow{n}} = \hbar\omega(N+2) *_O f_{\overrightarrow{n}} = \hbar\omega(n_1+n_2+n_3+n_4+2)\, f_{\overrightarrow{n}}$$

and

$$\psi *_O f_{\overrightarrow{n}} = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)\, f_{\overrightarrow{n}}.$$

Hence the energy levels are

$$H_{CF} = -\frac{1}{8}\omega^2 \Longleftrightarrow H_0 = -4k \qquad \text{and} \qquad \psi = \mu$$

is read as

$$-4k = \hbar\omega(n_1+n_2+n_3+n_4+2)$$

and

$$\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4).$$

Thus the quantized energy level of H_{CF} is $-\frac{1}{8}\omega^2 = -\frac{2k^2}{\hbar^2(n_1+n_2+n_3+n_4+2)^2}$ and the strength of magnetic monopole is quantized as $\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)$. Thus we have

Theorem 4. The eigenvalues of the MIC-Kepler problem with the strength of magnetic monopole $\hbar\,\frac{m}{2}$ are

$$E_n = -\frac{2k^2}{\hbar^2(n+2)^2}, \qquad n \geq |m|, \quad \text{and} \quad n \pm m \equiv 0 \mod 2.$$

The multiplicity of the eigenvalue E_n is

$$\frac{(n+m+2)(n-m+2)}{4}.$$

Acknowledgements

The author expresses his acknowledgements to Professor Ivaïlo Mladenov for his encouragement. This work was supported by JSPS KAKENHI Grant Number JP15K04856.

References

[1] Bayen F., Flato M., Fronsdal C., Lichnerowicz A. and Sternheimer D., *Deformation Theory and Quantization I & II*, Ann. Phys. **111** (1978) 61–151.

[2] Iwai T. and Uwano Y., *The Four-Dimensional Conformal Kepler Problem Reduces to the Three-Dimensional Kepler Problem with a Centrifugal Potential and Dirac's Monopole Field. Classical Theory*, J. Math. Phys. **27** (1986) 1523–1529.

[3] Iwai T. and Uwano Y., *The Quantised MIC-Kepler Problem and Its Symmetry Group for Negative Energies*, J. Phys. A: Math. & Gen. **21** (1988) 4083–4104.

[4] Kanazawa T. and Yoshioka A., *Star Product and Its Application to the MIC-Kepler Problem*, J. Geom. Symmetry Phys. **25** (2012) 57–75.

[5] Kanazawa T., *A Specific Illustration of Section Derived from $*$-Unitary Evolution Function*, In: New Developments in Geometric Mechanics, T. Iwai (Ed), RIMS, Kyoto 2012, pp 172–191.

[6] Kanazawa T., *Green's Function, Wavefunction and Wigner Function of The MIC-Kepler Problem*, J. Geom. Symmetry Phys. **30** (2013) 63–73.

[7] McIntosh H. and Cisneros A., *Degeneracy in the Presence of a Magnetic Monopole*, J. Math. Phys. **11** (1970) 896–916.

[8] Mladenov I. and Tsanov V., *Geometric Quantisation of the MIC-Kepler Problem*, J. Phys. A: Math. & Gen. **20** (1987) 5865–5871.

[9] Moyal E. and Bartlett S., *Quantum Mechanics as a Statistical Theory*, Math. Proc. Cambr. Phil. Soc. **45** (1949) 99–124.

[10] Omori H., Maeda Y., Miyazaki N. and Yoshioka A., *Star Exponential Functions as Two-Valued Elements*, In: The Breadth of Symplectic and Poisson Geometry, Progress in Math. **232** (2005) 483–492.

[11] Omori H., Maeda Y., Miyazaki N. and Yoshioka A., *Deformation of Expressions for Elements of an Algebra*, In: Symplectic, Poisson, and Noncommutative Geometry, MSRI publications **62** (2014) 171–209, arXiv:1307.0267.