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MATHEMATICAL MODELS OF CLASSICAL ELECTRODYNAMICS

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Abstract. Four mathematical models of classical electrodynamics based on vector fields, tensor spaces, geometric algebras and differential forms are represented in parallel and compared.

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1. Introduction

In December 1865, the great Scottish physicist James Maxwell presented to the Royal Society of London his original paper [11]. Initially Maxwell has formulated his theory in terms of twenty equations. Later, in 1873, he published his next important paper [12], where he reduced his system to twelve scalar equations. The further development showed that even eight scalar equations are sufficient! The main mathematical advantage of his system of equations was that he offered one electromagnetic model which "in principle" discovered all the important natural effects that are related to electromagnetic field theory. The main physical disadvantage of his model was that he thought that for the electromagnetic waves (predicted by him) is necessary of existence a special medium called "ether". Further experiments did not reveal the existence of such "artificial medium". He introduced this concept used analogy with acoustical waves that such a medium is necessary (like air, for example). Nevertheless his mathematical description of the electromagnetic waves was correctly. This was proved experimentally about twenty years after by the gifted German physicist Heinrich Hertz reported in [7].

The main difference between the acoustical waves and the electromagnetic waves is that the first ones are "scalar longitudinal waves", while the second ones are "vector transverse waves". The main Maxwell's inventions were to use the concept first introduced by the great English physicist Michael Faraday about "the existence of the electromagnetic field" even in the case of a single charge. The second charge is necessary just to prove this event. Actually the right system in Maxwell's equations was formulated after "avoidance the necessity of far space interaction". This idea was first introduced by Faraday.

The Maxwell's equations are based on four physical laws: 1) Faraday's law; 2) Ampere's law (modified by Maxwell in the range of the high frequency electromagnetic fields); 3) Gauss's law of the electric field; 4) Gauss's law of the magnetic field. Faraday's law shows that the time-variation of the magnetic field produces vortex electric field. Analogically corrected Ampere's law shows that the time-variation of the electric field produces magnetic field (that is always vortex)! Electric Gauss's law shows that the static electric field can be produced always in the presence of electric charge, while Magnetic Gauss's law shows that in the nature there is no magnetic charge and the static magnetic field can be only produced by the moving electrical charge (or current)! The Maxwell's equations show one very important property of the electromagnetic fields: the static DC electric and magnetic fields are independent, while the dynamic AC electric and magnetic fields are coupled!

2. Different Electromagnetic Models

2.1. The Concept of Vector Fields

The British electric engineer Oliver Heaviside, who was largely self-taught, was so eager to get grips with Maxwell's electromagnetic theory that he studied the treatise carefully until he was well enough versed to achieve his own way ahead with the theory. He published his main results in the field of Electromagnetic theory in the papers [5, 6].

His innovation greatly improved the readability of Maxwell's equations, he condensed the original 20 equations into four "new" equations: two of them (with "divergence operator" were in scalar form); another two of them (with "rotation operator" were in vector form). His main contribution was to write the Maxwell's equations in the following the most popular now form

$$\begin{aligned} \operatorname{div}\mathbf{D} &= \rho, \qquad & \operatorname{div}\mathbf{B} = 0 \\ \operatorname{rot}\mathbf{E} &= -\partial_t\mathbf{B}, \qquad & \operatorname{rot}\mathbf{H} = \mathbf{J} + \partial_t\mathbf{D} \end{aligned} \tag{1}$$

where ∂_t is a time-derivative. His innovation greatly improved the readability of Maxwell's equations, he condensed the original 20 equations only into 4 "new" equations. His main contribution was to write the Maxwell's equations in the most popular form (1) given above.

Here the vector $\mathbf{E}$ [V / m] is the electric field intensity. The vector $\mathbf{B}$ [Wb / m^2 = T] is the magnetic flux density, the vector $\mathbf{D}$ [C / m^2] is the electric displacement, the vector $\mathbf{H}$ [A / m] is the magnetic field intensity, ρ [C / m^3] is the electric charge density and the vector $\mathbf{J}$ [A / m^2] is the electric current density.

The first two equations of the system (1) are "divergence equations" – they relate closely the electromagnetic vectors with their sources. A source of the static electric field is only the electric charge, while a source of the static magnetic field is only the moving charge (i.e. the current). The second two equations are "rotation equations" – they show the character of both electromagnetic vectors. The magnetic field has always vortex character, while the electric field has such a character only in the case of time-varying magnetic field.

Another three supplementary vector equations are presented in the following form

$$\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{J} = \sigma\mathbf{E}$$

where ε [F / m] is the permittivity of the medium, μ [H / m] is its permeability, σ [S / m] is the conductivity. The German physicist Boltzmann, better known for his contributions to thermodynamics, proposed his own version of the Electromagnetic laws. However, later he became a proponent of the Maxwell's theory [1]. At the end of 19-th century the Dutch physicist Hendrik Lorentz published own version of the Maxwell's equations. At this time the experiments found that the real sources of the conductive currents are the moving electrons. He explained his "microscopic electromagnetic field theory" in [9]. Lorentz succeeded to prove that the main equations of his electron theory lead to the original "macroscopic Maxwell's theory". However, his main contributions were that he discovered an important symmetry of the Maxwell's basic equations: they are invariant with respect to the change of one inertial system with another one. This very important property of Maxwell's theory was called "*covariance*"! We will use only a part of his theory where he proved that the Maxwell's equations are covariant with respect of scalar potentials. We consider here the case of a charge moving with a constant velocity "v" along the "x-axis" of one Cartesian coordinate system of equations – the Lorentz's transforms are presented by the equations

$$x \to (x - vt)/\sqrt{1 - v^2/c^2}, \qquad y \to y, \qquad z \to z. \tag{2}$$

This important property called "covariance" was used later in Special Theory of Relativity of Einstein. Here $c = 3 \times 10^8$ m / s is the velocity of light in free space.

2.2. Tensor Notations

In the beginning of 20th century two Italian mathematicians Ricci and Levi-Civita have published a paper [13] where they introduced a new powerful method of differential calculus known today as tensor analysis. In year 1905 the great German physicist (of Jew's origin) Albert Einstein published his historical paper [3] in which the basic ideas of his famous Special Theory of Relativity were explained. Originally this theory was proposed on the language of vector analysis.

However, it was his colleague Hermann Minkowski, who brought the Einstein's theory onto tensor analysis in 4-D "space-time". This new notations have two big advantages: 1) shorter than vector notations; 2) so-called "covariance' (or automatic fulfillment of Lorentz's transforms (2)). He introduced new 4-D tensors for the electromagnetic field: $\{J^\mu\}$ is a four-tensor 1st rank of the electromagnetic sources and $\{F^{\mu\nu}\}$ is a 2nd rank tensor of the electromagnetic fields – the last one combines both vectors $\{(\mathbf{E}, \mathbf{B})\}$ in one anti-symmetric tensor with 6 different components as shown in the equations below

$$[x] = [x, y, z, (jct)]^T, \qquad [\partial] = [\partial_x, \partial_y, \partial_z, \partial_{(jct)}]^T$$

$$[J] = [Z_0 J_x, Z_0 J_y, Z_0 J_z, j\rho/\varepsilon_0]^T$$

$$[F] = \begin{bmatrix} 0 & cB_z & -cB_y & -jE_x \\ -cB_z & 0 & cB_x & -jE_y \\ cB_y & -cB_x & 0 & -jE_z \\ jE_x & jE_y & jE_z & 0 \end{bmatrix}$$

where $j = \mathrm{i} = \sqrt{-1}$ is the imaginary unit, $Z_0 = \sqrt{\mu_0/\varepsilon_0} \approx 377\,\Omega$ is the free space impedance and $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m is the free space permittivity. The last equations could be described in the following more concise *tensor form*

$$\partial_\nu F^{\mu\nu} = J^\mu, \qquad \partial_\rho F_{\sigma\tau} + \partial_\sigma F_{\tau\rho} + \partial_\tau F_{\rho\sigma} = 0 \tag{3}$$

where the four-vector of the covariant first derivative is $\partial_\nu = (\partial_x, \partial_y, \partial_z, \partial_{jct})$, while $x^\mu = (x, y, z, jct)$ are the four-coordinates in 4-D space-time. In first equation the Einstein's rule (summation over the repeated index ν) is applied. In modern quantum electrodynamics the language of tensors is very fruitful to write the basic equations of the photon and electron.

2.3. Geometric Algebra Model

The first papers dedicated on this new mathematical approach were published by the British mathematician William Clifford. The geometric algebra was developed further by the American physicist D. Hestenes. He published a better version of this new mathematical language in [8]. The geometric algebra is a vector space approach in which the roles of second-rank skew-symmetric tensors are replaced by the bi-vectors, a single entity formed by the direct multiplication of two orthogonal vectors. This is in strict contrast to the quaternions, developed by Hamilton [4], where a product effectively reverts to a vector. As there is no continually refer to some assumed spatial frame, such as $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, geometric algebra provides a versatile coordinate free approach.

This means that rather than effectively being labels for ordered components, symbols such ∇ actually stand for the vectors themselves. Nevertheless, we may still express these vectors as $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ and $\nabla = \partial_x\hat{\mathbf{x}} + \partial_y\hat{\mathbf{y}} + \partial_z\hat{\mathbf{z}}$, or in terms of whatever other basis may find convenient. This is a *coordinate-free approach*. The geometric algebra formalism of multi-vectors allows for a graded hierarchy of entities called n-vectors (or multi-vectors), where 0-vector is a scalar, one-vector is the familiar sort of vector (a polar vector), two-vector is a bi-vector (or an axial vector), and three-vector is a three-vector (or a pseudoscalar). It is a key feature that different grades of n-vector may be added as well as multiplied, resulting in what generally a multi-vector of mixed grade. In order to distinguish them from the usual 3-D vectors, we will write general n-vectors and multi-vectors in bold, e.g. $\mathbf{u}$, $\mathbf{v}$, and so on. By the *basic rules of geometric multiplication*, the product $\mathbf{uv}$ resolves into scalar $(\mathbf{u}, \mathbf{v})$ plus the bi-vector $\mathbf{u} \wedge \mathbf{v}$. Note that the wedge symbol $\wedge$, used also in differential forms (see the next Chapter) conveys similar idea. For 3-D vectors this product is related to the cross-product by: $\mathbf{u} \wedge \mathbf{v} = I\mathbf{u} \times \mathbf{v}$, where $I = \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}}$ is a three-vector referred to as the unit pseudo-scalar. While I takes a role analogous to the imaginary unit j, it has the additional property that when it multiples a vector it creates a bi-vector. In this fashion, therefore

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + I\mathbf{u} \times \mathbf{v}.$$

This leads straight to the following equations

$$\nabla\mathbf{E} = \nabla \cdot \mathbf{E} + I\nabla \times \mathbf{E} = \frac{1}{\varepsilon_0}\rho - \frac{1}{c}\partial_t(Ic\mathbf{B})$$

$$\nabla(Ic\mathbf{B}) = Ic\nabla \cdot \mathbf{B} + I^2 c\nabla \times \mathbf{B} = -Z_0\mathbf{J} - \frac{1}{c}\partial_t\mathbf{E}.$$

Since the addition of different grades is permitted, we find that we may put both of these results together so as to render Maxwell's equations in free space as a single (3 + 1)-D equation

$$(\nabla + \partial_t)F = J$$

in which the entire electromagnetic field is expressed as the multi-vector

$$F = \mathbf{E} + Ic\mathbf{B}.$$

Likewise, the total electromagnetic source density is expressed as the multi-vector $J = \rho/\varepsilon_0 - Z_0\mathbf{J}$. We have to define a multi-vector of the auxiliary electromagnetic field, $G = \mathbf{D}/\varepsilon_0 + IZ_0\mathbf{H}$, the role of which is the macroscopic treatment of physical media. In "space-time" we get similar but far more effectual expression of these equations, namely

$$\nabla \wedge F = 0, \qquad \nabla \cdot G = J^{\text{free}} \tag{4}$$

(it should be noted here that use of bold symbols is usually dropped with these space-time quantities). Since time is now embodied as a vector in this 4-D space, ∇ is now equivalent to $\nabla = \partial_x \hat{\mathbf{x}} + \partial_y \hat{\mathbf{y}} + \partial_z \hat{\mathbf{z}} - \frac{1}{c}\partial_t \hat{\mathbf{t}}$ in which $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{t}})$ are the unit basis vectors of some space-time frame. In the second equation (4) J^{free} (the free sources vector density) is now a pure vector, and F and G are pure bi-vectors. Electric and magnetic fields are just the time-like and space-like parts of the same bi-vector field, i.e., of F and G as appropriate, and because the last equations work in any frame, it is essentially $covariant$. When all is stripped back to a fundamental setting devoid of phenomenological representations for physical media, the auxiliary fields vanish, and Maxwell's equations are in single equation

$$\nabla F = J$$

in which J now comprises all sources of charge and current. This equation is very simple and is also covariant. It defines an important class of equation that seems from concepts more abstract and physical. In the more-familiar case of complex functions in 2-D, $\nabla F = 0$ corresponds to the pair of Cauchy-Riemann conditions $\partial_x F_y + \partial_y F_x = 0$, meaning that F must be an analytic function with singularities wherever $J \neq 0$. In space-time, where there are two extra dimensions, solutions of $\nabla F = 0$ are called meromorphic functions, but otherwise the situation is analogous to two dimensions. Finally, since all non-null vectors in a geometric algebra have inverses, we may write down a particular solution of the last equation in closed form, by simple inversion shown in equation (5) below

$$F = \nabla^{-1} J \tag{5}$$

where ∇^{-1} turns out to be an integral operator with a time-dependent Green's function as its kernel.

2.4. Differential Forms Model

Still early in the 20-th century the French mathematician Elie Cartan was a leading light in the development of differential forms, first published in his book [2]. A close link exists between differential forms and the integral form of Maxwell's equations. However, before we expand on that premise we give a simplified and very basic sketch of how they work. Starting from the basic portion of a differential, e.g. $\mathrm{d}f = \partial_x f \mathrm{d}x + \partial_y f \mathrm{d}y + \partial_z f \mathrm{d}z$, the similar-looking one-form, $f = f_x \mathrm{d}x + f_y \mathrm{d}y + f_z \mathrm{d}z$, is quite distinct. In fact, in the generalized way, it corresponds to an ordinary vector to the extent that the infinitesimal scalar quantities, like dx, dy, and dz may also be treated as independent symbols, like $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, but which entirely different connotations. Provided the meaning of $\leftrightarrow$ is limited to this sort correspondence, we may write in the form

$$f = f_x \mathrm{d}x + f_y \mathrm{d}y + f_z \mathrm{d}z \quad \longleftrightarrow \quad \mathbf{f} = f_x \hat{\mathbf{x}} + f_y \hat{\mathbf{y}} + f_z \hat{\mathbf{z}}$$

but note that while unit vectors are dimensionless the differentials are not. Extending the idea, a two-form corresponds to an axial vector (or to a bi-vector)

$$U = U_x \mathrm{d}y\mathrm{d}z + U_y \mathrm{d}z\mathrm{d}x + U_z \mathrm{d}x\mathrm{d}y \quad \longleftrightarrow \quad \mathbf{U} = U_x \hat{\mathbf{x}} + U_y \hat{\mathbf{y}} + U_z \hat{\mathbf{z}}$$

where $\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$, $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ and $\hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$. Here $\hat{\mathbf{x}} \leftrightarrow \mathrm{d}y\mathrm{d}z$, $\hat{\mathbf{y}} \leftrightarrow \mathrm{d}z\mathrm{d}x$, $\hat{\mathbf{z}} \leftrightarrow \mathrm{d}x\mathrm{d}y$. While fg, the *direct product* of two one-forms f and g, will clearly include a two-form in the result, it is their exterior product, denoted by $f \wedge g$. That produces exclusively a two-form. This is defined as being anti-symmetric, $\mathrm{d}u \wedge \mathrm{d}v = -\mathrm{d}v \wedge \mathrm{d}u$. In particular, if $\mathrm{d}u$ and $\mathrm{d}v$ are two of the differentials $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, then, where $\mathrm{d}u \wedge \mathrm{d}u = \mathrm{d}v \wedge \mathrm{d}v = 0$. However, it is customary to drop the $\wedge$ sign in these products and to simply write $\mathrm{d}x\mathrm{d}y, \mathrm{d}y\mathrm{d}z, \mathrm{d}z\mathrm{d}x$.

Along similar lines, the exterior product of a one-form with a two-form yields three-form, but in the case the product is symmetric, as may be interpreted from the following example

$$\begin{aligned} \mathrm{d}x \wedge (\mathrm{d}y \wedge \mathrm{d}z) &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = -\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z \\ &= \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x = (\mathrm{d}y \wedge \mathrm{d}z) \wedge \mathrm{d}x. \end{aligned}$$

The general commutation is therefore that the exterior product is $symmetric$ when the sum of the degrees of the forms involved is odd, $anti-symmetric$ when it is even. A key one-form is the *exterior vector derivative*, which in 3-D space takes the form

$$\mathrm{d} = \partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z. \tag{6}$$

By way of example, the exterior derivative of U is $\mathrm{d} \wedge U$ commonly written as $\mathrm{d}U$, so that

$$\mathrm{d}U \equiv \mathrm{d} \wedge U \equiv (\partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z) \wedge U.$$

For example

$$\mathrm{d} \wedge x = (\partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z) \wedge x = \mathrm{d}x$$

and

$$\mathrm{d} \wedge (xy) = y\mathrm{d}x + x\mathrm{d}y.$$

It is therefore clear that the exterior derivative of a scalar function is a prescription for its differential. However, applying it to a one-form, we find

$$\begin{aligned} \mathrm{d}E &= (\partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z) \wedge (E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z) \\ &= (\partial_x E_y - \partial_y E_x)\mathrm{d}x\mathrm{d}y + (\partial_y E_z - \partial_z E_y)\mathrm{d}y\mathrm{d}z + (\partial_z E_x - \partial_x E_z)\mathrm{d}z\mathrm{d}x \\ &\Longleftrightarrow \mathrm{d}E \leftrightarrow \nabla \times \mathbf{E}. \end{aligned}$$

That is to say, in differential forms, $\mathrm{d}E$ *takes the place of* $\nabla \times \mathbf{E}$. However, in the case of a two-form by applying the computation rules we find

$$\begin{aligned} \mathrm{d}D &= (\partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z) \wedge (D_x \mathrm{d}y\mathrm{d}z + D_y \mathrm{d}z\mathrm{d}x + D_z \mathrm{d}x\mathrm{d}y) \\ &= (\partial_x D_x + \partial_y D_y + \partial_z D_z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\Longleftrightarrow \mathrm{d}D \leftrightarrow \nabla \cdot \mathbf{D}. \end{aligned}$$

In contrast to the case with a one-form such as E, $\mathrm{d}D$ *takes place of* $\nabla \cdot \mathbf{D}$ *rather than* $\nabla \times \mathbf{D}$. Nothing that the differential 3-D volume element $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ appears in the result, this is an example of a three-form, a class that corresponds to a scalar volume density. In 3-D, we are then left with one other sort of form, the 0-form, a form of scalar that is free from any association with a volume density.

Conventionally, the ubiquitous electromagnetic quantities and source densities are represented by different degrees forms, as shown in the Table 1 below. In each case, the associated differential elements are shown in the column to the right of the given symbol. The physical quantity like q is a 0-form or a scalar. The physical significance of these becomes clearer we note that $E_x \mathrm{d}x$ is the decrease in the potential ϕ, of a unit charge when, in vector forms, it is moved through an electric field, E, by infinitesimal displacement $\mathrm{d}x\hat{\mathbf{x}}$. We also that note that $E = -\mathrm{d}\phi \leftrightarrow \mathbf{E} = -\nabla\phi$, which is a one-form, related to linear elements $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$.The one-form vector E is called also a polar vector. Similarly, the two-form quantities may be associated with a flux, so that, for example, $D_z \mathrm{d}x\mathrm{d}y$ is the total current that flows through the orientated element of area $\mathrm{d}x\mathrm{d}y$, i.e., the area $\mathrm{d}x\mathrm{d}y$ is in the xy-plane such a positive flow is along $\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = \hat{\mathbf{z}}$. The two-form vector D is called also an axial vector. Finally, the three-form is $\mathrm{d}x\mathrm{d}y\mathrm{d}z$. Here the quantity ρ is a charge density, which is a pseudo-scalar.

Following these preliminaries, it should be clear that the result of applying the operator d depends on the degree of the form on which it acts, so that it means that $\nabla\times$ and $\nabla\cdot$ are all replaced by the single operator d (meaning $\mathrm{d}\wedge$) on its own. The expression of Maxwell's equations in terms of differential forms is therefore very straightforward, for we can use this rule to write the following equations

$$\mathrm{d}D = \rho^{(\mathrm{free})}, \qquad \mathrm{d}B = 0, \qquad \mathrm{d}E = -\partial_t B, \qquad \mathrm{d}H = J^{(\mathrm{free})} + \partial_t D \quad (7)$$

fairly self-evident, so $q = \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is the total charge contained within the volume element. These are then a direct source for the integral equations, which are exactly

Table 1. The electromagnetic quantities and source densities represented by differential degrees of forms.

Forms	**Electromagnetic Quantities**	**Differential Elements**
0-forms	q, ϕ	$(\mathrm{d}u)^0 \equiv 1$
1-forms	E, H, A	$\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$
2-forms	D, B, J	$\mathrm{d}y\mathrm{d}z, \mathrm{d}z\mathrm{d}x, \mathrm{d}x\mathrm{d}y$
3-forms	ρ	$\mathrm{d}x\mathrm{d}y\mathrm{d}z$

mirrored by the system of equations, given below

$$\begin{aligned} \int_{\partial V} \mathrm{d}D &= \int_V \rho^{(\text{free})}, \qquad & \int_{\partial V} \mathrm{d}B &= \int_V 0 = 0 \\ \int_{\partial A} \mathrm{d}E &= -\partial_t \int_A B, \qquad & \int_{\partial A} \mathrm{d}H &= \int_A (J^{(\text{free})} + \partial_t D). \end{aligned} \tag{8}$$

It has been necessary only to write integral signs on both sides of these equations with the degree of the form telling us what sort of integral is involved: line, surface, or volume. The integrals on the left-hand side are taken over the closed boundary of the volume or area associated with the integrals on the right-hand side, that is to say, ∂A is the closed path taken around the outside of the area A, and ∂V is the surface enclosing the volume V. Because the integrands are differential forms, not only are the requisite differentials for the integration are already in place, they also provide the orientation of the paths and surfaces, e.g. $\mathrm{d}x$ is along $\hat{\mathbf{x}}$ and normal to $\mathrm{d}x\mathrm{d}y$ is $\hat{\mathbf{z}}$.

Equations (8) may be put into 4-D space-time form by making the following expression taken as a generalization of (6)

$$\mathrm{d} = (\partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z + \partial_t \mathrm{d}t).$$

As in Minkowski's matrix, the single two-form F now represents the complete electromagnetic field. Likewise, the auxiliary fields combine into separate two-form, G, while the three current and charge densities combining into a single three-form, J. The system of equations (7) may now be written as

$$\mathrm{d}F = 0, \qquad \mathrm{d}G = J \tag{9}$$

where

$$F = E \wedge \mathrm{d}t + B, \qquad G = -H \wedge \mathrm{d}t + D, \qquad J = -J^{(\mathrm{free})} \wedge \mathrm{d}t + \rho^{(\mathrm{free})}.$$

By applying the same simple rules as before and writing d as $(\mathrm{d} + \partial_t \mathrm{d}t)\wedge$, where d inside the brackets represents the original 3-D exterior derivative, $(\partial_x \mathrm{d}x + \partial_y \mathrm{d}y + \partial_z \mathrm{d}z)$, this may be decoded in the following manner as in the equations (10) below

$$\begin{aligned}
\mathrm{d}F &= (\mathrm{d} + \mathrm{d}t\,\partial_t) \wedge E \wedge \mathrm{d}t + (\mathrm{d} + \mathrm{d}t\,\partial_t) \wedge B \\
&= \mathrm{d}E \wedge \mathrm{d}t - \partial_t E \wedge \mathrm{d}t \wedge \mathrm{d}t + \mathrm{d}B + \partial_t B \wedge \mathrm{d}t \\
&= (\mathrm{d}E + \partial_t B) \wedge \mathrm{d}t + \mathrm{d}B = 0 + 0 \\
\mathrm{d}G &= -(\mathrm{d} + \mathrm{d}t\,\partial_t) \wedge H \wedge \mathrm{d}t + (\mathrm{d} + \mathrm{d}t\,\partial_t)\wedge D \\
&= -\mathrm{d}H \wedge \mathrm{d}t + \partial_t H \wedge \mathrm{d}t \wedge \mathrm{d}t + \mathrm{d}D + \partial_t D \wedge \mathrm{d}t \\
&= (-\mathrm{d}H + \partial_t D) \wedge \mathrm{d}t + \mathrm{d}D = -J^{(\mathrm{free})} + \rho^{(\mathrm{free})}.
\end{aligned} \tag{10}$$

Given the operator $*$ (so-called *star-operator*) that in 3-D converts a one-form into a two-form (its dual), and vice versa, the substitutions

$$B = \mu * H, \qquad D = \varepsilon * E, \qquad J = \sigma * E$$

is possible to be made, but this still leaves us with three separate equations. The same star operator also converts a 0-form into a three-form and vice versa.

3. Conclusion

In this article four different models of Classical Electrodynamics proposed by James Maxwell are presented and compared. 1) The usual model used nowadays is the "Vector model" of Heaviside - equations (1). Another three more contemporary models, not so widely known, were introduced in the beginning of twenty century. These more contemporary models are: 2) "Tensor model" of Ricci (used extensively by Einstein in Special Theory of Relativity) - here equations (3); 3) "Geometric algebra" of Clifford (which is an extension of the quaternions of Hamilton) - equations (4); 4) "Differential forms" of Cartan (a simple generalization of differential and integral operators) - equations (9). Some specific applications of the last two models in Classical Electrodynamics are considered recently in the engineering textbooks [10] and [14]. The serious advantages of the last three models are: 1) brief notations; 2) coordinate-free definitions; 3) easy check of "covariance". In the modern theory of Electromagnetism (so-called Quantum Electrodynamics) the first model (tensor model) became very popular for solution of electromagnetic problems. The main contributions in this area are made by the two Nobel Laureates: the British physicist Paul Dirac and the American physicist Richard Feynman. One important difference between the classical and quantum theories is the fact that the quantum theory uses for the photons the D'Alambert equations for the four-vector potential A^{μ} instead of the Maxwell's equations (3) for the four-tensor electromagnetic field $F^{\mu\nu}$ and this quantity becomes there a kind of an "wave function". Quantum theory uses for the moving electrons the so called Dirac's equation for the wave-function "ψ" instead of the Newton's equation. The electrons and the photons are described now similarly by using a probabilistic interpretation. However, the explanation of this very interesting and new physical theory lies out of our scopes here.

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