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GEOMETRIC ASPECTS OF MULTIPLE FOURIER SERIES CONVERGENCE ON THE SYSTEM OF CORRECTLY COUNTED SETS

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Abstract. For multiple Fourier series the convergence of partial sums essentially depends on the type of integer sets, to which the sequence numbers of their terms belong. The problem on the general form of such sets is studying in u-convergence theory ($u(K)$ - convergence) for multiple Fourier series. An alternative method of summation is based on the concept of the so-called correctly denumarable sets. In the paper some results describing the u-convergence relations and convergence on the system or correctly denumarable sets are presented. It is shown that the system of $U(K)$-sets containing a sphere of infinitely increasing radius for fixed K is correctly denumarable. It is established that for the functions satisfying the Lipschitz condition and having a certain growthing $p-k$-variation, the coefficients of multiple Fourier series decrease at the average on the system of $U(K)$-sets faster than it is predicted by their ordinary estimations. It is shown the accurate estimation of the Fourier coefficients of functions of several variables is achieved at a very "poor" set of elements of the integer lattice.

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1. Introduction

Fourier series are the most effective tool for solving many technical problems, for example, in mechanics [1, 10] and control [13]. Let us consider a set of integer numbers which consists of coefficients of multiple Fourier series. Depending on its structure we distinguish convergence over spheres, over rectangles (according to Pringsheim), over triangles, over hyperbolic crosses and others [6]. Kuznetsova and Bahvalov in their papers [2–5, 8, 9] considered the convergence over triangles. Temlyakov [14] investigated the convergence of multiple Fourier series on convex sets. The theory of so called *u*-convergence studies the general form of such sets for multiple Fourier series. Dyachenko [6] studied in detail the two-dimensional case of *u*-convergence and also *u(K)*-convergence. Kotlyar in [7] proposed the alternative summation method, which is based on the definition of series of correctly counted sets. In this paper we present results describing the relation of *u(K)*-convergence and convergence on the system of series of correctly counted sets.

2. Basic Definitions and Statement of the Problem

Let us consider the class of periodic functions $f\colon \mathbb{R}^m \to \mathbb{R}$ with the parallelepiped $\overline{Q}^m$ of its periods

$$f(y_1 + 2\pi l_1, \ldots, y_m + 2\pi l_m) = f(y_1, \ldots, y_m), \quad f(y_1, \ldots, y_m) \subset \tilde{L}^2\left(\overline{Q}^m\right)$$

$$l \equiv (l_1, \ldots, l_m) \in \mathbb{Z}^m \subset \mathbb{R}^m, \qquad \overline{Q}^m = \prod\nolimits_{j=1}^{m} [-\pi, \pi].$$

Coefficients of multiple Fourier series of f in the trigonometric system we denote as $\{c_l(f)\}_{l\in\mathbb{Z}^m}$. They can be represented in follows form

$$\left\{\mathrm{e}^{2\pi\mathrm{i}(l,t)}\right\}_{l\in\mathbb{Z}^m} = \left\{\exp 2\pi\mathrm{i}\sum_{j=1}^{m} l_j t_j\right\}_{l\in\mathbb{Z}^m}.$$

Let the sequence $\{c_l^*(f)\}_{l\in\mathbb{Z}^m}$ of modules $\{|c_l(f)|\}_{l\in\mathbb{Z}^m}$ be rearranged in ascending order. From [11] we know the order of decrease to zero of this sequence and conditions of convergence of multiple series for functions, which belong to different classes, of the form

$$\sum_{l\in\mathbb{Z}^m} |c_{l_j}(f)|^\beta = \sum_{\substack{l_j\in\mathbb{N},\\ j=1,\ldots,m}} |c_{l_1,\ldots,l_m}(f)|^\beta. \tag{1}$$

These results significantly improves traditional data which obtained directly by estimation of the Fourier coefficients.

Let $\tilde{\Lambda}_{\alpha,p}$ denotes the class of defined and periodical on $\mathbb{R}^m$ functions $f\colon \mathbb{R}^m \to \mathbb{R}$ with limited p-variation and the cube of periods $\overline{Q}^m = \prod_{j=1}^{m}[-\pi, \pi]$, which satisfy the condition $\mathrm{Lip}\,\alpha$ for $0 < \alpha \le 1$. Respectively $\tilde{\Lambda}_\alpha$ is the class of defined and periodical on $\mathbb{R}^m$ functions $f\colon \mathbb{R}^m \to \mathbb{R}$ with the cube of periods $\overline{Q}^m = \prod_{j=1}^{m}[-\pi,\pi]$ which satisfy the condition $\mathrm{Lip}\,\alpha$ for $0 < \alpha \le 1$.

Definition 1. *Let $\Lambda = \mathbb{Z}^m \subset \mathbb{R}^m$ be an integer lattice in $\mathbb{R}^m$ and $\mathfrak{B} = \{B_j\}_1^\infty$ be a system of finite subsets Λ that $\bigcup\limits_{j=1}^{\infty} B_j = \Lambda$, $B_1 \subset B_2 \subset \ldots$, and $|B_j|$ be the number of elements in B_j. The limit (if it exists) $\mathrm{mes}\,\mathfrak{N} = (\mathfrak{B})\mathrm{mes}\,\mathfrak{N} = \lim\limits_{N\to\infty} \frac{1}{|B_N|} \sum\limits_{\mathfrak{N}\cap B_N} 1$ is called asymptotic density of the set $\mathfrak{N} \subset \Lambda$ according to the system $\mathfrak{B}$.*

Definition 2. *Let $\Lambda = \mathbb{Z}^m \subset \mathbb{R}^m$ be an integer lattice in $\mathbb{R}^m$ and $\mathfrak{B} = \{B_j\}_1^\infty$ be a system of finite subsets Λ that $\bigcup\limits_{j=1}^{\infty} B_j = \Lambda$, $B_1 \subset B_2 \subset \ldots$, and $|B_j|$ be the number of elements in B_j. Let the system of sets $\mathfrak{B} = \{B_j\}_1^\infty$ meets the following requirement. All points $l = (l_1,\ldots,l_m) \in \lambda$ can be numbered by the natural numbers $\varphi(l)$ so that the point's number would be not less than the maximum of modulus of any of its coordinates. Initially we number the elements of the set B_1 then the elements of the set $B_2 \setminus B_1$, etc. Then the elements of the set have numbers $B_j \setminus B_{j-1}$ smaller than the elements of the set $B_{j+1} \setminus B_j$. System $\mathfrak{B}$ to which the points Λ can be numbered in such a way is called correctly counted. The correctly counted system means the existence of such a function $\varphi\colon l \in \Lambda \to \varphi(l) \in \mathbb{N}$ which perform one-to-one mapping Λ on $\mathbb{N}$ that $\varphi(l) \ge \max\limits_{1\le j\le m} |l_j|$ and if $l^{(j)} \in B_j \setminus B_{j-1}$ $(B_0 = \varnothing)$ then $\varphi\left(l^{(j+1)}\right) > \varphi\left(l^{(j)}\right)$. In* [7] *were considered various systems $\mathfrak{B}$ and was shown that such a restriction is satisfied by all natural systems (parallelotopics with limited ratio of edges lengths, homotheties of fixed set, etc.).*

At the same time, [6] provides a definition of u-convergence and $u(K)$-convergence for the two-dimensional case. In [6] it is stated, that they can be easily extended to the multidimensional case. Let us do it.

Definition 3. *A limited set $u \in \mathbb{Z}^m \subset \mathbb{R}^m$ belongs to the set U if*

$$\prod_{j=1}^{m}[-|l_j|,|l_j|] \bigcap \mathbb{Z}^m \subseteq u$$

for any $l = (l_1,\ldots,l_m) \in u$.

Definition 4. *A limited set $u \in \mathbb{Z}^m \subset \mathbb{R}^m$ belongs to the set $U(K)$ for $K > 1$ if $u \in U$ and for any system of vectors*

$$(L_1,0,\ldots,0),(0,L_2,\ldots,0),\ldots,(0,0,\ldots,L_m) \subseteq u$$

integer polyhedron

$$\left\{ l = (l_1, l_2, \dots, l_m) \in \mathbb{Z}^m \; ; \; K|l_j| + \sum_{\substack{i=1, \\ i \neq j}}^{m} |l_j| \leq L_j \right\}_{j=1}^{m} \subseteq u.$$

Definition 5. *We say that the series* (1) *u-converges ($u(K)$- converges) to the number α if for any $\varepsilon > 0$ exists such N that for each set $u \in U$ ($u \in U(K)$) containing a ball $\{(l_1, \dots, l_m) \in \mathbb{Z}^m ; \, |(l_1, \dots, l_m)| \leq N\}$*

$$\left| \sum_{l \in u} |c_{l_1,\dots,l_m}(f)|^{\beta} - \alpha \right| < \varepsilon.$$

In [6] Dyachenko noted that "different partial sums of Fourier series behave differently, for example, for cubic and spherical partial amounts. Especially their properties vary greatly if the sets, for which amounts are taken, can be extended along the coordinate axes. A classic example is the situation with the cubic and rectangular sums of multiple Fourier series of continuous functions. First of them converge almost everywhere, and the latter may diverge at each point. It is interesting to identify patterns of behavior of the partial sums, taken over the sets, which prohibited "crawl" along the coordinate axis. The $u(K)$- convergence is one of the most broad generalizations of this kind of ways of the wording of convergence though perhaps this definition is not optimal for describing the behavior of multiple Fourier series".

Thus convergence and the correct counting represent two different, but fairly common approaches to the summation of multiple series. Various properties of series can be described with their help. Establishing of a connection between these two approaches allows us to expand the description of the object being studied.

We define a coherent set of points of integer lattice which included in the partial sums. We reformulate the Definition 5.

Definition 6. *We say that the series* (1) *u-converges ($u(K)$-converges) to the number α if for any $\varepsilon > 0$ there exists such N, that for any $M \geq N$ and for each set $u \in U$ ($u \in U(K)$ for some fixed finite value K), containing a sphere $\{(l_1, \dots, l_m) \in \mathbb{Z}^m ; \, |(l_1, \dots, l_m)| \leq M\}$, the following formula is true*

$$\left| \sum_{l \in u} |c_{l_1,\dots,l_m}(f)|^{\beta} - \alpha \right| < \varepsilon.$$

The $u(K)$-convergence does not describe the summation over rectangles (according to Pringsheim). At the same time the rectangles is a correctly counted system of sets. Therefore, the correct counting is not a special case of $u(K)$-convergence.

Let us show that the system of sets from Definition 6 is the correctly counted system and consider the properties of coefficients of multiple series derived from this fact.

3. Determination of the Relation Between Systems Summation

Theorem 7. *Let $\Lambda = \mathbb{Z}^m \subset \mathbb{R}^m$ be an integer lattice in $\mathbb{R}^m$, $\mathfrak{B} = \{B_j\}_1^\infty$ be a system of finite subsets Λ with some fixed finite value K such that for any value j $B_j \in U(K)$ and contains a sphere $\{(l_1, \dots, l_m) \in \mathbb{Z}^m;\ |(l_1, \dots, l_m)| \leq j\}$. Then $\mathfrak{B}$ is the system of the correctly counted sets.*

Proof: The system $U(K)$ is sufficiently definite. This is a set of nested multidimensional “starlike” polyhedrons with tapering thorns. Then the proof can build constructively by a specific recalculation of elements.

Let us consider the part of the multidimensional space having positive values of all the coordinates. We will start numbering in the positive direction along the axis that contains the most distant point from the origin. In this part we have $\varphi(l) = \max\limits_{1 \leq j \leq m} |l_j| + 1$. Then consider the following axis which contains the most distant point, etc. After going through all the lines in this plane we shift it on unit and we shift by the next axis with the following length. After that we renew the numbering. The procedure is repeated until there will be counted all points of the set.

For other $2^m - 1$ areas the recount is made similarly, but with considering the growth modules direction along the coordinate axes. Condition $\varphi(l) \geq \max\limits_{1 \leq j \leq m} |l_j|$ is satisfied automatically because the regions are symmetrical, and the numbering starts with delay on the number of points in the first subregion.

Completeness of the system $\mathfrak{B}$ in Λ is define by the equality

$$\bigcup_{j=1}^{\infty} B_j = \Lambda.$$

This follows from inserting into the B_j the sphere whose radius tends to infinity. The sequence of inserted of sets B_j

$$B_1 \subset B_2 \subset \dots$$

follows from their construction.

For multidimensional areas the number of points in the first set B_1 always will be greater than the maximum values of their coordinates (including degenerate one-dimensional case). Then the numbering B_2 begins with the number which is more of the minimum coordinate values of the points included in it. Thus recount of B_2 and of the subsequent sets should start from the point with the minimum values of the coordinates and move consistently as they increase. This algorithm provides a value of point number greater than the value of coordinates of this point. Then all numbers of points from the set with higher index numbers are more points of the sets with a smaller index. Thus, the system of sets from Definition 6 is a correctly counted system. ■

Well-known the connection between the smoothness of the kernel of the integral operator of Fredholm and order of convergence to zero of eigenvalues and singular values of the operator [7]. The paper [12] is devoted to the dissemination of these results on the function of many variables. In this paper obtained an upper bound for the singular numbers of the operator based on such properties of its nucleus as the limitations of its variations and continuity module behavior. In this work is introduced and used a notion of $p-k$-variation of a function of many variables (equivalent definitions for the one-dimensional case are given in [7]). From the results about the order of decrease of singular values are obtained [12] the results of the order of the Fourier coefficients of the function of many variables, if the function has a given modulus of continuity and given behavior of $p-k$-variation. In [11] it is shown that an accurate assessment of the Fourier coefficients of the function of several variables is achieved on a very “poor” set of elements of the integer lattice $\Lambda = \mathbb{Z}^m$. In [12] established that for a function, satisfying to the condition $\operatorname{Lip}\alpha$, $0 < \alpha \leq 1$ and which has a certain growth of $p-k$-variation, Fourier coefficients in average decrease faster than in conventional estimates.

Thus, taking into account the results of [12] we can formulate and prove following theorems and their corollaries.

Theorem 8. *Let $\Lambda = \mathbb{Z}^m \subset \mathbb{R}^m$ be an integer lattice in $\mathbb{R}^m$, $\mathfrak{B} = \{B_j\}_1^\infty$ be a system of finite subsets Λ with some fixed finite value K such that for any value j $B_j \in U(K)$ and contains a sphere $\{(l_1,\dots,l_m) \in \mathbb{Z}^m\,;\ |(l_1,\dots,l_m)| \leq j\}$. Let the function $f\colon \mathbb{R}^m \to \mathbb{R}$ be specified and periodic on $\mathbb{R}^m$ with the cube of periods $\overline{Q}^m = \prod_{j=1}^m [-\pi,\pi]$, satisfies by the condition $\operatorname{Lip}\alpha$, $0 < \alpha \leq 1$ and has bounded variation ($f \in \tilde{\Lambda}_{\alpha,p}$), $q > 0$, $0 < \beta < 1 + \alpha(2-q)/2m$, $C > 0$ and a set of points of the integer lattice $\mathfrak{M}$ is defined as follows*

$$\mathfrak{M} = \mathfrak{M}(f; C, \beta) = \left\{ l \in \Lambda\ ;\ |c_l| \geq C \left(\max_{1 \leq j \leq m} |l_j|^{-\beta} \right) \right\}.$$

Then for any $\varepsilon > 0$

$$\sum_{\mathfrak{M} \bigcap B_N} 1 = \mathcal{O}\left(|B_N|^{\varepsilon + \beta(1+\alpha(2-q)/2m)^{-1}} \right).$$

Corollary 9. *Let $\Lambda = \mathbb{Z}^m \subset \mathbb{R}^m$ be an integer lattice in $\mathbb{R}^m$, $\mathfrak{B} = \{B_j\}_1^\infty$ be a system of finite subsets Λ with some fixed finite value K such that for any value j*

$B_j \in U(K)$ *and contains a sphere* $\{(l_1,\dots,l_m)\in\mathbb{Z}^m;\ |(l_1,\dots,l_m)|\le j\}$. *Let the function* $f\colon \mathbb{R}^m\to\mathbb{R}$ *be specified and periodic on* $\mathbb{R}^m$ *with the cube of periods* $\overline{Q}^m = \prod_{j=1}^m[-\pi,\pi]$, *satisfies by the condition* $\operatorname{Lip}\alpha$, $0<\alpha\le 1$ *and has bounded variation* ($f\in\tilde{\Lambda}_{\alpha,p}$), $q>0$, $0<\beta<1+\alpha(2-q)/2m$, $C>0$ *and a set of points of the integer lattice* $\mathfrak{M}$ *is defined as follows*

$$\mathfrak{M}=\mathfrak{M}(f;C,\beta)=\left\{l\in\Lambda;\ |c_l|\ge C\left(\max_{1\le j\le m}|l_j|^{-\beta}\right)\right\}.$$

Then

$$(\mathfrak{B})\mathrm{mes}\mathfrak{M}=0.$$

Theorem 10. *Let* $\Lambda=\mathbb{Z}^m\subset\mathbb{R}^m$ *be an integer lattice in* $\mathbb{R}^m$, $\mathfrak{B}=\{B_j\}_1^\infty$ *be a system of finite subsets* Λ *with some fixed finite value* K *such that for any value* j $B_j\in U(K)$ *and contains a sphere* $\{(l_1,\dots,l_m)\in\mathbb{Z}^m;\ |(l_1,\dots,l_m)|\le j\}$. *Let the function* $f\colon\mathbb{R}^m\to\mathbb{R}$ *be specified and periodic on* $\mathbb{R}^m$ *with the cube of periods* $\overline{Q}^m=\prod_{j=1}^m[-\pi,\pi]$, *satisfies by the condition* $\operatorname{Lip}\alpha$, $0<\alpha\le1$ ($f\in\tilde{\Lambda}_\alpha$), $0<\beta<0,5+\alpha/2m$, $C>0$ *and a set of points of the integer lattice* $\mathfrak{M}$ *is defined as follows*

$$\mathfrak{M}=\mathfrak{M}(f;C,\beta)=\left\{l\in\Lambda;\ |c_l|\ge C\left(\max_{1\le j\le m}|l_j|^{-\beta}\right)\right\}.$$

Then for any $\varepsilon>0$

$$\sum_{\mathfrak{M}\bigcap B_N}1=\mathcal{O}\left(|B_N|^{\varepsilon+\beta(0,5+\alpha/2m)^{-1}}\right).$$

Corollary 11. *Let* $\Lambda=\mathbb{Z}^m\subset\mathbb{R}^m$ *be an integer lattice in* $\mathbb{R}^m$, $\mathfrak{B}=\{B_j\}_1^\infty$ *be a system of finite subsets* Λ *with some fixed finite value* K *such that for any value* j $B_j\in U(K)$ *and contains a sphere* $\{(l_1,\dots,l_m)\in\mathbb{Z}^m;\ |(l_1,\dots,l_m)|\le j\}$. *Let the function* $f\colon\mathbb{R}^m\to\mathbb{R}$ *be specified and periodic on* $\mathbb{R}^m$ *with the cube of periods* $\overline{Q}^m=\prod_{j=1}^m[-\pi,\pi]$, *satisfies by the condition* $\operatorname{Lip}\alpha$, $0<\alpha\le1$ ($f\in\tilde{\Lambda}_\alpha$), $0<\beta<0,5+\alpha/2m$, $C>0$ *and a set of points of the integer lattice* $\mathfrak{M}$ *is defined as follows*

$$\mathfrak{M}=\mathfrak{M}(f;C,\beta)=\left\{l\in\Lambda;\ |c_l|\ge C\left(\max_{1\le j\le m}|l_j|^{-\beta}\right)\right\}.$$

Then

$$(\mathfrak{B})\mathrm{mes}\mathfrak{M}=0.$$

Proof of Theorems 8, 10 and the consequences of them is made by means of a direct application of Theorem 1 to the results of [12].

4. Conclusions

It is shown that the system of $U(K)$-sets which contains the sphere with increasing of radius to infinity for fixed K is a correctly counted system. It was found that for function satisfying the condition $\mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$ and having a certain growth of the $p-k$- variation, multiple Fourier series coefficients decrease faster than average on the system of the $U(K)$-sets and than it is dictated by their regular assessments. It is shown that an accurate assessment of the Fourier coefficients of the function of several variables is achieved on a very “poor” set of elements of the integer lattice.

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