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CONSTRUCTION OF SYMPLECTIC-HAANTJES MANIFOLD OF CERTAIN HAMILTONIAN SYSTEMS

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Abstract. Symplectic-Haantjes manifolds are constructed for several Hamiltonian systems following Tempesta-Tondo [5], which yields the complete integrability of systems.

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1. Introduction

Tempesta-Tondo [5] introduces a concept of symplectic-Haantjes manifolds or $\omega\mathcal{H}$ manifolds and Lenard-Haantjes chain to treat completely integrable Hamiltonian system by means of the Haantjes tensor [2]. For a (1,2)-tensor field L, the Haantjes torsion $\mathcal{H}_L$ is given by Definition 1 below. If $\mathcal{H}_L$ vanishes, the tensor is called a *Haantjes operator*. In [5], Tempesta and Tondo showed that the existence of an $\omega\mathcal{H}$ manifold is a necessary and sufficient condition for a non-degenerate Hamiltonian system to be completely integrable. They showed an algorithm for solving the inverse problem, that is, for a given set of involutive functions, a Haantjes structure of the involutive functions is constructed by using Lenard-Haantjes chains.
In this note, using their method we construct $\omega\mathcal{H}$ manifolds for several Hamiltonian systems of two degrees of freedom such as so-called Fukaya system [1], a geodesic flow of two-dimensional Minkowski space and a system given by the

Hamiltonian [6]

$$H = \frac{1}{2}\frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} + \frac{1}{q_1^2 + q_2^2}.$$

2. Haantjes Operator

In this section, we recall basic concepts of Haantjes chain, Haantjes manifolds and recursion operators (see for details, e.g., [5]).

Let M be a differentiable manifold and $L : TM \to TM$ be a $(1,1)$ tensor field, i.e., a field of linear operators on the tangent space at each point of M.

Definition 1. *The Nijenhuis torsion of L is the skew-symmetric $(1,2)$ tensor field defined by*

$$\mathcal{N}_L(X,Y) = L^2[X,Y] + [LX, LY] - L\left([X, LY] + [LX, Y]\right)$$

and the Haantjes tensor associated with L is the $(1,2)$ tensor field defined by

$$\mathcal{H}_L(X,Y) = L^2\mathcal{N}_L(X,Y) + \mathcal{N}_L(LX, LY) - L\left(\mathcal{N}_L(X, LY) + \mathcal{N}_L(LX, Y)\right)$$

where X, Y are vector fields on M and $[\ ,\ ]$ denotes the commutator of two vector fields.

In local coordinates $x = (x_1, \cdots, x_n)$, the Nijenhuis torsion and the Haantjes tensor can be written in the form

$$(\mathcal{N}_L)^i_{jk} = \sum_{\alpha=1}^{n}\left(\frac{\partial L^j_k}{\partial x^\alpha}L^\alpha_j - \frac{\partial L^i_j}{\partial x^\alpha}L^\alpha_k + \left(\frac{\partial L^\alpha_j}{\partial x^k} - \frac{\partial L^\alpha_k}{\partial x^j}\right)L^i_\alpha\right)$$

and

$$(\mathcal{H}_L)^i_{jk} = \sum_{\alpha,\beta=1}^{n}\left(L^i_\alpha L^\alpha_\beta(\mathcal{N}_L)^\beta_{jk} + (\mathcal{N}_L)^i_{\alpha\beta}L^\alpha_j L^\beta_k - L^i_\alpha\left((\mathcal{N}_L)^\alpha_{\beta k}L^\beta_j + (\mathcal{N}_L)^\alpha_{j\beta}L^\beta_k\right)\right)$$

respectively.

We remark that the skew-symmetry of the Nijenhuis torsion implies that of the Haantjes tensor.

Definition 2. *A* (1,1)*-tensor is called Haantjes operator when its Haantjes tensor vanishes.*

Proposition 3. *Let L be a* (1,1)*-tensor. If there exists a local coordinate system on an open set $U \subseteq M$ such that*

$$L = \sum_{i=1}^{n}\ell_i(x)\frac{\partial}{\partial x_i}\otimes \mathrm{d}x_i$$

then the Haantjes tensor of L vanishes on U.

Let us consider Hamiltonian systems with two degrees of freedom. In [5], Tempesta and Tondo proposed a general procedure to compute a Haantjes operator adapted to the Lenard-Haantjes chain formed by two integrals of motion in involution. Let (M, ω) be a four dimensional symplectic manifold. They searched for a Haantjes operator K whose minimal polynomial should be of degree two, namely, the maximum degree allowed by their assumptions

$$m_K(\lambda) = \lambda^2 - c_1(x)\lambda_1 - c_2(x)\lambda_2.$$

We construct the Haantjes operator K according to the conditions in [5]

$$K^T \Omega = \Omega K \tag{1}$$

$$K^T \mathrm{d}H = \mathrm{d}H_2 \tag{2}$$

$$(K^T)^2 \mathrm{d}H = (c_1 K^T + c_2 I)\mathrm{d}H \tag{3}$$

$$\mathcal{H}_K(X, Y) = 0 \qquad X, Y \in TM \tag{4}$$

where $\Omega = \omega^\flat$ and I denotes the identity operator.

3. Construction of Symplectic-Haantjes Manifold for Certain Hamiltonian Systems

Example 4. *Let us consider the Hamiltonian system* [1]

$$H = \frac{p_1^2}{2} + (p_1^2 + q_1^2)p_2^2 + \frac{q_1^2}{2} - q_2 \tag{5}$$

with the independent integrals of motion

$$H_2 = p_1^2 + q_1^2. \tag{6}$$

We construct a Haantjes operator K for H in the following way.

H_2 is functionally independent of H and satisfies

$$\{H, H_2\} = \sum_{i=1}^{2} \left(\frac{\partial H}{\partial p_i} \frac{\partial H_2}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial H_2}{\partial p_i} \right) = (p_1 + 2p_1 p_2^2) 2q_1 - (q_1 + 2q_1 p_2^2) 2p_1$$
$$= 0.$$

From condition (1), we put a four-dimensional square matrix

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is an arbitrary matrix, B, C are skew-symmetric matrices, and $D = A^T$.

Also, total derivative of (5) and (6) are as following

$$\mathrm{d}H = (p_1 + 2p_1p_2^2)\mathrm{d}p_1 + 2(p_1^2 + q_1^2)p_2\mathrm{d}p_2 + (2q_1p_2^2 + q_1)\mathrm{d}q_1 - \mathrm{d}q_2 \tag{7}$$

$$\mathrm{d}H_2 = 2p_1\mathrm{d}p_1 + 2q_1\mathrm{d}q_1. \tag{8}$$

By equations (7), (8) and condition (2), we get the following relation

$$\begin{pmatrix} a & b & 0 & \alpha \\ c & d & -\alpha & 0 \\ 0 & \beta & a & c \\ -\beta & 0 & b & d \end{pmatrix} \begin{pmatrix} p_1 + 2p_1p_2^2 \\ 2(p_1^2 + q_1^2) \\ 2q_1p_2^2 + q_1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2p_1 \\ 0 \\ 2q_1 \\ 0 \end{pmatrix}. \tag{9}$$

From relation (9), we see that

$$c = aq_1(2p_2^2 + 1) + 2\beta p_2(p_1^2 + q_1^2) - 2q_1$$
$$d = (bq_1 - \beta p_1)(2p_2^2 + 1)$$
$$\alpha = ap_1(2p_2^2 + 1) + 2bp_2(p_1^2 + q_1^2) - 2p_1$$

where a, b and β are constants.
Further, we put

$$c_1 = a + d = a + (bq_1 - \beta p_1)(2p_2^2 + 1)$$
$$c_2 = -ad - \alpha\beta + bc = 2(\beta p_1 - bq_1).$$

In this case, condition (3) is satisfied. Condition (3) yields the semisimplicity of K, and then (4) holds.

Example 5. *Let us consider the Hamiltonian system of the geodesic flow of Minkowski space* (*cf.* [4])

$$H = \frac{1}{2}(-p_1^2 + p_2^2) \tag{10}$$

with a independent integral of motion

$$H_2 = \frac{1}{2}p_1^2. \tag{11}$$

On the other hand, H *has also an independent integral of motion*

$$H_3 = p_2q_1 + p_1q_2. \tag{12}$$

Thus, H *has Haantjes operators* K *and* K' *in the following way.*

We consider $G = G(q, p)$ which is functionally independent of H. We assume the Poisson bracket $\{H, G\}$ vanishes, that is

$$\{H, G\} = \sum_{i=1}^{2}\left(\frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial G}{\partial p_i}\right) = -p_1\frac{\partial G}{\partial q_1} + p_2\frac{\partial G}{\partial q_2} = 0.$$

Then we get the following condition

$$p_1\frac{\partial G}{\partial q_1} = p_2\frac{\partial G}{\partial q_2}. \tag{13}$$

The functions (11) and (12) are satisfying condition (13) of G.

Under condition (1), we put the matrix (9). In addition, we calculate the total derivatives of (10), (11) and (12)

$$\mathrm{d}H = -p_d p_1 + p_2\mathrm{d}p_2 \tag{14}$$

$$\mathrm{d}H_2 = p_1\mathrm{d}p_1 \tag{15}$$

$$\mathrm{d}H_3 = q_2\mathrm{d}p_1 + q_1\mathrm{d}p_2 + p_2\mathrm{d}q_1 + p_1\mathrm{d}q_2. \tag{16}$$

By equations (14), (15) and condition (2), we get the following relation

$$\begin{pmatrix} a & b & 0 & \alpha \\ c & d & -\alpha & 0 \\ 0 & \beta & a & c \\ -\beta & 0 & b & d \end{pmatrix}\begin{pmatrix} -p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we see that

$$a = \frac{1}{p_1}(bp_2 - 1), \qquad c = \frac{dp_1}{p_2}, \qquad \beta = 0$$

where b, d and α are constants.

Further, we put

$$c_1 = a + d = \frac{1}{p_1}(bp_2 - 1) + d$$

$$c_2 = -ad + bc = \frac{d}{p_1p_2}\left\{b(p_1+p_2)(p_1-p_2) + p_2\right\}.$$

Then condition (3) is satisfied.

On the other hand, by equations (14), (16) and condition (2), we get the following relation

$$\begin{pmatrix} a' & b' & 0 & \alpha' \\ c' & d' & -\alpha' & 0 \\ 0 & \beta' & a' & c' \\ -\beta' & 0 & b' & d' \end{pmatrix}\begin{pmatrix} -p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} q_2 \\ q_1 \\ p_2 \\ p_1 \end{pmatrix}. \tag{17}$$

From the relation (17), we see that

$$b' = \frac{1}{p_2}(a'p_1 + q_2), \qquad d' = \frac{1}{p_2}(c'p_1 + q_1), \qquad \beta' = 1$$

where a', c' and α' are constants.

Further, we put

$$c_1' = a' + d' = a' + \frac{1}{p_2}(c'p_1 + q_1)$$
$$c_2' = -a'd' - \alpha'\beta' + b'c' = \frac{1}{p_2}(-a'q_1 - \alpha'p_2 + c'q_2).$$

Then condition (3) is satisfied. We have that

$$K = \begin{pmatrix} a & b & 0 & \alpha \\ c & d & -\alpha & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}, \qquad K' = \begin{pmatrix} a' & b' & 0 & \alpha' \\ c' & d' & -\alpha' & 0 \\ 0 & 1 & a' & c' \\ -1 & 0 & b' & d' \end{pmatrix}$$

satisfy (4).

Example 6. *Let us consider a Hamiltonian system* [6]

$$H = \frac{1}{2}\frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} + \frac{1}{q_1^2 + q_2^2} \tag{18}$$

with an independent integral of motion

$$H_2 = \frac{1}{q_1^2 + q_2^2}\left\{q_1^2(1 + p_2^2) + q_2^2(1 + p_1^2)\right\}. \tag{19}$$

Then H has the Haantjes operators K in the following form.

From condition (1), we put a four-dimensional square matrix

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{20}$$

where A is an arbitrary matrix, B, C are skew-symmetric matrices, and $D = A^T$. Also, total derivative of (18) and (19) are as follows

$$\begin{aligned} \mathrm{d}H = {} & \frac{p_1}{q_1^2 + q_2^2}\mathrm{d}p_1 + \frac{p_2}{q_1^2 + q_2^2}\mathrm{d}p_2 \\ & - \frac{q_1(p_1^2 + p_2^2 + 2)}{(q_1^2 + q_2^2)^2}\mathrm{d}q_1 - \frac{q_2(p_1^2 + p_2^2 + 2)}{(q_1^2 + q_2^2)^2}\mathrm{d}q_2 \end{aligned} \tag{21}$$

$$\begin{aligned} \mathrm{d}H_2 = {} & -\frac{2q_2^2p_1}{q_1^2 + q_2^2}\mathrm{d}p_1 + \frac{2q_1^2p_2}{q_1^2 + q_2^2}\mathrm{d}p_2 \\ & + \frac{2q_1q_2^2(p_1^2 + p_2^2 + 2)}{(q_1^2 + q_2^2)^2}\mathrm{d}q_1 - \frac{2q_1^2q_2(p_1^2 + p_2^2 + 2)}{(q_1^2 + q_2^2)^2}\mathrm{d}q_2. \end{aligned} \tag{22}$$

By equations (21), (22) and condition (2), we get the following relation

$$\begin{pmatrix} a & b & 0 & \alpha \\ c & d & -\alpha & 0 \\ 0 & \beta & a & c \\ -\beta & 0 & b & d \end{pmatrix} \begin{pmatrix} \frac{p_1}{q_1^2+q_2^2} \\ \frac{p_2}{q_1^2+q_2^2} \\ -\frac{q_1(p_1^2+p_2^2+2)}{(q_1^2+q_2^2)^2} \\ -\frac{q_2(p_1^2+p_2^2+2)}{(q_1^2+q_2^2)^2} \end{pmatrix} = \begin{pmatrix} -\frac{2q_2^2p_1}{q_1^2+q_2^2} \\ \frac{2q_1^2p_2}{q_1^2+q_2^2} \\ \frac{2q_1q_2^2(p_1^2+p_2^2+2)}{(q_1^2+q_2^2)^2} \\ -\frac{2q_1^2q_2(p_1^2+p_2^2+2)}{(q_1^2+q_2^2)^2} \end{pmatrix}. \tag{23}$$

From the relation (9), we see that

$$a = \frac{-2p_1q_2^4 + (-2p_1q_1^2 - bp_2)q_2^2 + \alpha q_2(p_1^2 + p_2^2 + 2) - bp_2q_1^2}{p_1(q_1^2 + q_2^2)}$$

$$\beta = -\frac{(p_1^2 + p_2^2 + 2)\{(-2q_1^2 + d)q_2 + bq_1)}{p_1(q_1^2 + q_2^2)}$$

$$c = \frac{2p_2q_1^4 - p_2(-2q_2^2 + d)q_1^2 - \alpha q_1(p_1^2 + p_2^2 + 2) - dp_2q_2^2}{p_1(q_1^2 + q_2^2)}$$

where b, d and α are constants. Further, we put

$$\begin{aligned} c_1 &= a + d \\ &= \frac{-2p_1q_2^4 + \{(-2q_1^2 + d)p_1 - bp_2\}q_2^2 + \alpha q_2(p_1^2 + p_2^2 + 2) - q_1^2(bp_2 - dp_1)}{p_1(q_1^2 + q_2^2)} \\ c_2 &= -ad - \alpha\beta + bc \\ &= \frac{2bp_2q_1^4 + 2\{(bp_2 + dp_1)q_2 - \alpha(p_1^2 + p_2^2 + 2)\}q_2q_1^2 + 2dp_1q_2^4}{p_1(q_1^2 + q_2^2)}. \end{aligned}$$

Then conditions (3) and (4) are satisfied.

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